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# HYPERSURFACES WITH CONSTANT CURVATURE QUOTIENTS IN WARPED PRODUCT MANIFOLDS

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We study rigidity problems for hypersurfaces with constant curvature quotients  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  in the warped product manifolds. Here  $\mathcal{H}_{2k}$  is the  $k$ -th Gauss–Bonnet curvature and  $\mathcal{H}_{2k+1}$  arises from the first variation of the total integration of  $\mathcal{H}_{2k}$ . Hence the quotients considered here are in general different from  $\sigma_{2k+1}/\sigma_{2k}$ , where  $\sigma_k$  are the usual mean curvatures. We prove several rigidity and Bernstein-type results for compact or noncompact hypersurfaces corresponding to such quotients.

## 1. Introduction

Let  $\Sigma^{n-1}$  be a closed smooth hypersurface isometrically immersed in an  $n$ -dimensional Riemannian manifold  $(M^n, g)$ . Assume that  $\Sigma_t$  is a variation of  $\Sigma$  with the unit outward normal vector field  $\nu_t$  as the variational vector field. It is well known that the first variation of the area functional  $\text{Area}(\Sigma_t)$  is given by

$$\left. \frac{d}{dt} \right|_{t=0} \text{Area}(\Sigma_t) = \int_{\Sigma} H \, d\mu,$$

where  $H$  is the mean curvature of  $\Sigma$  with respect to the inner normal and  $d\mu$  is the area element of  $\Sigma$ . On the other hand, it is well known that the first variation of the total scalar curvature functional  $\int_{\Sigma} R \, d\mu$  is given by

$$\left. \frac{d}{dt} \right|_{t=0} \int_{\Sigma_t} R \, d\mu_t = \int_{\Sigma} -2 \sum_{i,j=1}^{n-1} E^{ij} h_{ij} \, d\mu,$$

where  $E^{ij} = R^{ij} - \frac{1}{2} R g^{ij}$  and  $h_{ij}$  are respectively the Einstein tensor and the second fundamental form of  $\Sigma$  with respect to the inner normal in the local coordinates.

There is a natural generalization of scalar curvature, called Gauss–Bonnet curvatures  $L_k$  for an integer  $1 \leq k \leq \frac{1}{2}(n-1)$  for  $(n-1)$ -dimensional Riemannian

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manifolds.  $L_k$  are intrinsic curvature functions. When  $n - 1$  is even, the highest order Gauss–Bonnet curvature  $L_{(n-1)/2}$  is exactly the Pfaffian in the Gauss–Bonnet–Chern formula.  $L_2$  appeared first in [Lanczos 1938] and has been intensively studied in the theory of Gauss–Bonnet gravity, which is a generalization of Einstein gravity.

The first variation of the total Gauss–Bonnet curvature functional  $\int_{\Sigma} L_k d\mu$  has been considered long time ago by Lovelock [1971]. Li [1985] also computed the first variation of these functionals as well as the second variation for submanifolds in the general ambient Riemannian manifolds. Recently an alternative computation was given by Labbi [2008b]:

$$\frac{d}{dt} \Big|_{t=0} \int_{\Sigma_t} L_k d\mu_t = \int_{\Sigma} -2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij} d\mu,$$

where  $E_{(k)}^{ij}$  is the generalized Einstein tensor defined by (2-1). Labbi [2008a] referred to the critical point of  $\int_{\Sigma} L_k d\mu$  as  $2k$ -minimal submanifolds. In this sense, the ordinary minimal submanifolds are referred as 0-minimal submanifolds.

For the ambient space  $M^n = \mathbb{R}^n$ , by the Gauss equation, one can verify that  $L_k = (2k)! \sigma_{2k}$  and  $-2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij} = (2k + 1)! \sigma_{2k+1}$ , where  $\sigma_k$  are the usual mean curvatures defined by the elementary symmetric functions of the principal curvatures of associated hypersurfaces. Hence the Gauss–Bonnet curvatures  $L_k$  as well as the integrand  $-2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij}$  appear like higher order mean curvatures.

Throughout this paper, we use the notation

$$\mathcal{H}_{2k} := L_k, \quad \mathcal{H}_{2k+1} := -2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij},$$

and call them  $2k$ -mean curvature and  $(2k + 1)$ -mean curvature. By convention, we use  $L_0 = 1$ . We emphasize here that in general these mean curvatures are different from the usual ones defined by  $\sigma_k$  except  $\mathcal{H}_0$  and  $\mathcal{H}_1$ . The 0-mean curvature  $\mathcal{H}_0$  is equal to 1 and the 1-mean curvature  $\mathcal{H}_1$  is equal to the usual mean curvature  $H$ .

We will consider some rigidity problems related to  $\mathcal{H}_{2k}$  and  $\mathcal{H}_{2k+1}$  in a class of Riemannian manifolds: warped product manifolds. A warped product manifold  $(M, \bar{g})$  is the product manifold of an interval and an  $(n - 1)$ -dimensional Riemannian manifold with some smooth positive warping function. Precisely,

$$M = [0, \bar{r}) \times_{\lambda} N^{n-1} \quad (0 < \bar{r} \leq \infty)$$

is equipped with

$$\bar{g} = dr^2 + \lambda(r)^2 g_N,$$

where  $\lambda : [0, \bar{r}) \rightarrow \mathbb{R}_+$  is a smooth positive function and  $(N^{n-1}, g_N)$  is an  $(n - 1)$ -dimensional Riemannian manifold.

The rigidity problems for hypersurfaces in Riemannian manifolds with constant curvature functions are one of the central problems in the classical differential geometry. Historically, the rigidity problems for hypersurfaces in the Euclidean space was studied by Liebmann [1899], Hsiung [1954], Süss [1952], Alexandrov [1956; 1957; 1958a; 1958b], Alexandrov and Volkov [1958], Reilly [1977], Ros [1988], Korevaar [1988], etc. Recently, many works concerning the rigidity for hypersurfaces in warped product manifolds have appeared, see, for example, [Montiel 1999; Alías et al. 2013; Brendle 2013; Brendle and Eichmair 2013; Wu and Xia 2014] and the references therein.

In all above works, the curvature functions are related to the elementary symmetric functions  $\sigma_k$  of the principal curvatures of hypersurfaces. Our concern in this paper is the curvature functions  $\mathcal{H}_{2k}$  and  $\mathcal{H}_{2k+1}$ . In view of the Gauss equation, for hypersurfaces in general ambient Riemannian manifolds,  $\mathcal{H}_{2k}$  and  $\mathcal{H}_{2k+1}$  depend not only on  $\sigma_k$  but also on the Riemannian curvature tensor of the ambient manifolds. Therefore, except for the case that the ambient spaces are the space forms, for which  $\mathcal{H}_{2k}$  and  $\mathcal{H}_{2k+1}$  can be written as linear combinations of  $\sigma_k$ , one cannot express them as pure functions on the principal curvatures of hypersurfaces.

The first attempt in which we succeed is the rigidity on the curvature quotients  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  in a class of warped product manifolds. These quotients can be viewed as a generalization of the usual mean curvature  $H$  since the case  $k = 0$  corresponds to  $H$ . We remark that the rigidity on the quotients of  $\sigma_k$  in a class of warped product manifolds has been considered in [Wu and Xia 2014]. However, as mentioned before, these two kinds of quotients have large differences in general. Many techniques seem to be difficult to apply for the quotients  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  considered here.

The first main result of this paper is stated as:

**Theorem 1.1.** *Define  $(M^n, \bar{g})$  to be an  $n$ -dimensional warped product manifold  $[0, \bar{r}) \times_{\lambda} N^{n-1}$  whose warped product function satisfies*

$$(1-1) \quad \lambda\lambda'' - (\lambda')^2 \geq 0 \quad (\text{i.e., } \log \lambda \text{ is convex}).$$

*Let  $\Sigma^{n-1}$  be a closed star-shaped hypersurface in  $M$  such that the generalized Einstein tensor  $E_{(k)}$  is semidefinite on  $\Sigma$ . For any integer  $k$  with  $0 \leq k < \frac{1}{2}(n - 1)$  and  $\mathcal{H}_{2k}$  not vanishing on  $\Sigma$ , if the curvature quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  is a constant, then  $\Sigma$  is a slice  $\{r_0\} \times N$  for some  $r_0 \in [0, \bar{r})$  and the constant is  $(n - 1 - 2k) \log \lambda(r_0)$ .*

The star-shapedness means that  $\Sigma$  can be written as a graph over  $N$ , alternatively,  $\langle \partial/\partial r, \nu \rangle \geq 0$ , where  $\nu$  is the outer normal of  $\Sigma$ . The method to prove Theorem 1.1 is to apply the maximum principle to an elliptic equation. This method was previously indicated by Montiel [1999] and was used widely in [Alías and Colares 2007; Alías et al. 2012].

The condition (1-1) imposed on  $M$  only depends on the warped product function  $\lambda$ , not the fiber manifold  $N$ . We notice that the condition excludes the usual space forms  $\mathbb{R}^n$ ,  $\mathbb{S}_+^n$  (hemisphere) and  $\mathbb{H}^n$  (hyperbolic space) in which cases  $\lambda\lambda'' - (\lambda')^2 = -1$ . For  $\mathbb{R}^n$ , since the quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  is equal to  $\sigma_{2k+1}/\sigma_{2k}$ , the result still holds; see [Korevaar 1988; Koh 2000]. We will consider the case  $\mathbb{S}_+^n$  and  $\mathbb{H}^n$  elsewhere since the proof has a different flavor. We also notice that the condition (1-1) is satisfied by some local space forms such as  $[0, \infty) \times_{e^r} \mathbb{R}^{n-1}$  or  $[0, \infty) \times_{\cosh r} \mathbb{R}^{n-1}$ . There are also nonconstant curvature manifolds which satisfy (1-1). A typical example for which the condition (1-1) is satisfied is the so-called Kottler–Schwarzschild spaces  $[0, \infty) \times_\lambda N(\kappa)$ , whose warped product fact  $\lambda$  satisfies  $\lambda'(r) = \sqrt{\kappa + \lambda(r)^2 - 2m\lambda(r)^{2-n}}$  and  $N(\kappa)$  is a closed space form of the constant sectional curvature  $\kappa = 0$  or  $-1$ . See Appendix for a detailed explanation.

Note that  $E_{(1)}^{ij} = R^{ij} - \frac{1}{2}Rg^{ij}$  is the Einstein tensor, so that in  $k = 1$  case, the semidefinite condition of  $E_{(1)}$  is just the semidefiniteness of the Einstein tensor. In particular, if  $M = \mathbb{R}^n$ , one readily sees that  $-E_{(k)} = \frac{1}{2}(2k)!T_{2k}$ , where  $T_{2k}$  is the  $2k$ -Newton tensor associated to the hypersurface  $\Sigma$ , and the seminegative definite condition of  $E_{(k)}$  relates to  $2k$ -convexity.

In order to extend the above result to noncompact hypersurfaces, we need a generalization of the Omori–Yau maximum principle for the trace-type semi-elliptic operators. The classical Omori–Yau maximum principle is initially stated for the Laplacian  $\Delta$ . A Riemannian manifold  $\Sigma$  is said to satisfy the Omori–Yau maximum principle if for any function  $u \in C^2(\Sigma)$  with  $\sup_\Sigma u < +\infty$ , there exists a sequence  $\{p_i\}_{i \in \mathbb{N}} \subset \Sigma$  such that for each  $i$ , the following inequalities hold:

$$u(p_i) > \sup_\Sigma u - \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad \Delta u(p_i) < \frac{1}{i}.$$

This principle was first proved by Omori [1967] and later generalized by Yau [1975] under the condition that the Ricci curvature is bounded from below. It has proved to be very useful in the framework of noncompact manifolds and attracted considerable extending works. For example, it was improved by Chen and Xin [1992] and Ratto, Rigoli and Setti [Ratto et al. 1995] by assuming that the radial curvature decays slower than a certain decreasing function. Recently, the essence of the Omori–Yau maximum principle was captured by Pigola, Rigoli and Setti (see [Pigola et al. 2005, Theorem 1.9]) that the validity of the Omori–Yau maximum principle is assured by the existence of some nonnegative  $C^2$  function satisfying some appropriate requirements, and thus may not necessarily depend on the curvature bounds. Also, they discussed the generalizations for the trace-type differential operators (see Definition 3.1) which will be used in this paper. For a detailed discussion of the

sufficient condition to guarantee the Omori–Yau maximum principle for the trace-type differential operators to hold in the warped product manifolds, see [Alías et al. 2013] or Section 3 below.

We have a rigidity result for noncompact hypersurfaces:

**Theorem 1.2.** *Define  $(M^n, \bar{g})$  to be an  $n$ -dimensional warped product manifold  $[0, \bar{r}] \times_{\lambda} N^{n-1}$  whose warped product function satisfies  $\lambda\lambda'' - (\lambda')^2 \geq 0$  with equality only at isolated points. Let  $(\Sigma^{n-1}, g)$  be a complete noncompact star-shaped hypersurface in  $M$ , which is contained in a slab  $[r_1, r_2] \times N$ ,  $0 \leq r_1 < r_2 < \bar{r}$ , such that the generalized Einstein tensor  $E_{(k)}$  being semidefinite on  $\Sigma$ . Assume the Omori–Yau maximum principle holds for the trace-type operator  $\text{tr}_g(-2E_{(k)}\nabla_g^2)$  on  $\Sigma$ . For an integer  $k$  with  $0 \leq k < \frac{1}{2}(n-1)$  and  $\mathcal{H}_{2k}$  not vanishing on  $\Sigma$ , if the curvature quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  is a constant, then  $\Sigma$  is a slice  $\{r_0\} \times N$  for some  $r_0 \in [r_1, r_2]$  and the constant is  $(n-1-2k) \log \lambda(r_0)$ .*

Motivated by the analogous Bernstein type result on the quotient of the usual mean curvatures [Aquino and de Lima 2014], we can establish a corresponding result in our case. More precisely, instead of assuming the curvature quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  being constant, we can establish the rigidity result via assuming a natural comparison inequality between  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  and its value on the slices.

**Theorem 1.3.** *Define  $(M^n, \bar{g})$  to be an  $n$ -dimensional warped product manifold  $[0, \bar{r}] \times_{\lambda} N$ . Let  $(\Sigma^{n-1}, g)$  be a complete, star-shaped hypersurface in  $M$ , which is contained in a slab  $[r_1, r_2] \times N$ ,  $0 \leq r_1 < r_2 < \bar{r}$ , such that the generalized Einstein tensor  $E_{(k)}$  is semidefinite on  $\Sigma$ . Assume that the Omori–Yau maximum principle holds for the trace-type operator  $\text{tr}_g(-2E_{(k)}\nabla_g^2)$  on  $\Sigma$  and that the Gauss–Bonnet curvature  $\mathcal{H}_{2k}$  is bounded by two positive constants, i.e.,  $0 < C_1 \leq \mathcal{H}_{2k} \leq C_2$ . If*

$$\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \leq (n-1-2k) \frac{\lambda'(r)}{\lambda(r)} \quad \text{and} \quad |\nabla_g r|_g \leq \inf_{\Sigma} \left( (n-1-2k) \frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right),$$

then the hypersurface  $\Sigma$  is a slice  $\{r_0\} \times M$  for some  $r_0 \in [r_1, r_2]$ .

We remark that we do not assume the log-convexity of the warped product function for Theorem 1.3.

## 2. Preliminaries

In this section, we first recall the work of [Lovelock 1971] on the generalized Einstein tensors and Gauss–Bonnet curvatures. Throughout this paper, we use the notation  $R_{ijkl}$ ,  $R_{ij}$  and  $R$  to indicate the Riemannian 4-tensor, the Ricci tensor in local coordinates and the scalar curvature respectively. We use the metric  $g$  to lower or raise an index and adopt the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

For an  $(n - 1)$ -dimensional Riemannian manifold  $(\Sigma^{n-1}, g)$ , the Einstein tensor  $E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$  is very important in theoretical physics. It is a conserved quantity, i.e.,

$$\nabla_j E_i^j = 0,$$

where  $\nabla$  is the covariant derivative with respect to the metric  $g$ .

Lovelock [1971] studied the classification of tensors  $A$  satisfying

- (i)  $A^{ij} = A^{ji}$ , i.e.  $A$  is symmetric.
- (ii)  $A^{ij} = A^{ij}(g, \partial g, \partial^2 g)$ .
- (iii)  $\nabla_j A^{ij} = 0$ , i.e.,  $A$  is divergence-free.
- (iv)  $A^{ij}$  is linear in the second derivatives of  $g$ .

It is clear that the Einstein tensor  $E_{ij}$  satisfies all above conditions. Lovelock classified all 2-tensors satisfying (i)–(iii). For an integer  $0 \leq k \leq \frac{1}{2}(n - 1)$ , let us define a 2-tensor  $E_{(k)}$  locally by

$$(2-1) \quad E_{(k)}^{ij} := -\frac{1}{2^{k+1}} g^{lj} \delta_{l j_1 j_2 \dots j_{2k-1} j_{2k}}^{i i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}}.$$

Here the generalized Kronecker delta is defined by

$$\delta_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_r} = \det \begin{pmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \dots & \delta_{i_1}^{j_r} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \dots & \delta_{i_2}^{j_r} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_r}^{j_1} & \delta_{i_r}^{j_2} & \dots & \delta_{i_r}^{j_r} \end{pmatrix}.$$

One can check that  $E_{(k)}$  satisfies (i)–(iii). Lovelock proved that any 2-tensor satisfying (i)–(iii) has the form

$$\sum_k \alpha^k E_{(k)},$$

with certain constants  $\alpha^k$ ,  $k \geq 0$ . The  $E_{(k)}$  are called the generalized Einstein tensors.

For an integer  $0 \leq k \leq \frac{1}{2}(n - 1)$ , the Gauss–Bonnet curvatures  $L_k$  are defined by

$$(2-2) \quad L_k := \frac{1}{2^k} \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}}.$$

When  $2k = n - 1$ ,  $L_k$  is the Euler density. When  $k < \frac{1}{2}(n - 1)$ ,  $L_k$  is called the dimensional continued Euler density in physics. We set  $E_{(0)} = -\frac{1}{2}g$  and  $L_0 = 1$ . It is clear from the definitions (2-1) and (2-2) that

$$(2-3) \quad \text{tr}_g(E_{(k)}) := E_{(k)}^{ij} g_{ij} = -\frac{n - 1 - 2k}{2} L_k.$$

It is easy to see that  $(E_{(1)})_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$  is the Einstein tensor and  $L_1 = R$  is the scalar curvature. One can also check that

$$E_{(2)}^{ij} = 2RR^{ij} - 4R^is R_s^j - 4R_{sl}R^{silj} + 2R^i{}_{klm}R^{jklm} - \frac{1}{2}g^{ij}L_2,$$

and

$$L_2 = \frac{1}{4}\delta_{j_1i_2j_3j_4}^{i_1i_2i_3i_4}R^{j_1j_2}{}_{i_1i_2}R^{j_3j_4}{}_{i_3i_4} = R_{ijsl}R^{ijsl} - 4R_{ij}R^{ij} + R^2.$$

In [Lovelock 1971], the author proved that the first variational formula for the total Gauss–Bonnet curvature functional is given in terms of the generalized Einstein tensor. It was also presented in [Li 1985; Labbi 2008b], although with different notation and formalism. For the convenience of readers, we include a proof here.

**Proposition 2.1** [Lovelock 1971]. *Let  $(\Sigma^{n-1}, g)$  be a smooth closed manifold. Assume that  $g_t$  is a variation of  $g$  with  $\frac{\partial}{\partial t}\Big|_{t=0}g_{ij} = v_{ij}$  for a symmetric 2-tensor  $v$ , then*

$$(2-4) \quad \frac{d}{dt}\Big|_{t=0} \int_{\Sigma_t} L_k d\mu_t = \int_{\Sigma} -E_{(k)}^{ij}v_{ij} d\mu.$$

*In particular, if  $(\Sigma^{n-1}, g)$  is a closed, smooth hypersurface immersed in an  $n$ -dimensional Riemannian manifold  $(M^n, \bar{g})$  and the variational vector field is given by the outward unit normal  $\nu$ , then*

$$(2-5) \quad \frac{d}{dt}\Big|_{t=0} \int_{\Sigma_t} L_k d\mu_t = \int_{\Sigma} -2E_{(k)}^{ij}h_{ij} d\mu.$$

where  $h_{ij}$  denotes the second fundamental form of  $\Sigma$  with respect to  $-\nu$ .

*Proof.* By the simple fact that  $\frac{d}{dt}\Big|_{t=0}d\mu_t = \frac{1}{2}\text{tr}_g v d\mu$  and the definition of  $L_k$ , we compute

$$(2-6) \quad \begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_{\Sigma_t} L_k d\mu_t &= \int_{\Sigma} \frac{d}{dt}\Big|_{t=0} L_k d\mu + \int_{\Sigma} \frac{1}{2}L_k \text{tr}_g v d\mu \\ &= \int_{\Sigma} k P_{(k)sl}^{ij} \frac{d}{dt}\Big|_{t=0} R_{ij}{}^{sl} d\mu + \int_{\Sigma} \frac{1}{2}L_k \text{tr}_g v d\mu, \end{aligned}$$

where the 4-tensor  $P_{(k)}$  is given by

$$(2-7) \quad P_{(k)}^{stlj} := \frac{1}{2^k} \delta_{j_1j_2\cdots j_{2k-3}j_{2k-2}j_{2k-1}j_{2k}}^{i_1i_2\cdots i_{2k-3}i_{2k-2}st} R_{i_1i_2}{}^{j_1j_2} \cdots R_{i_{2k-3}i_{2k-2}}{}^{j_{2k-3}j_{2k-2}} g^{j_{2k-1}l} g^{j_{2k}j},$$

and

$$P_{(k)sl}^{ij} = P_{(k)}^{ijpq} g_{sp}g_{lq}.$$

We remark that  $P_{(k)}$  shares the same symmetry as the Riemann curvature tensor, that is,

$$(2-8) \quad P_{(k)}^{stjl} = -P_{(k)}^{tsjl} = -P_{(k)}^{stlj} = P_{(k)}^{jlst}.$$



Furthermore, by applying the second Bianchi identity of the curvature tensor, one can check that  $P_{(k)}$  has the crucial property of being divergence-free (see [Ge et al. 2014, Lemma 2.2] for a proof)

$$(2-9) \quad \nabla_s P_{(k)}^{stjl} = 0.$$

To calculate the first term in (2-6), we recall that if  $(\partial/\partial t)g = v$ , then the evolution equation of the curvature tensor is given by (see [Chow et al. 2006, Equation (2.66)])

$$\frac{d}{dt} R_{ijst} = -\frac{1}{2}(\nabla_i \nabla_j v_{st} - \nabla_i \nabla_l v_{sj} - \nabla_s \nabla_j v_{il} + \nabla_s \nabla_l v_{ij} - R_{ijsm} v^m{}_l - R_{ijml} v^m{}_s).$$

Then we use (2-8) and (2-9) to compute that

$$(2-10) \quad \begin{aligned} & \int_{\Sigma} k P_{(k)sl}^{ij} \left( \frac{d}{dt} \Big|_{t=0} R_{ij}{}^{sl} \right) d\mu \\ &= \int_{\Sigma} k P_{(k)sl}^{ij} \left( \frac{1}{2}(-\nabla_i \nabla_j v^{sl} + \nabla_i \nabla^l v_j^s + \nabla^s \nabla_j v_i^l - \nabla^s \nabla^l v_{ij}) \right. \\ &\quad \left. + \frac{1}{2}(R_{ijm}{}^l v^{ms} - R_{ijm}{}^s v^{ml}) + (-R_{ijp}{}^l v^{sp} - R_{ijq}{}^s v^{lq}) \right) d\mu \\ &= - \int_{\Sigma} k P_{(k)sl}^{ij} R_{ijm}{}^l v^{sm} d\mu, \end{aligned}$$

where in the last equality we used (2-9), (2-8) and the simple observation that  $(R_{ijm}{}^l v^{ms} - R_{ijm}{}^s v^{ml})$  and  $(-R_{ijp}{}^l v^{sp} - R_{ijq}{}^s v^{lq})$  are both antisymmetric with respect to the pair  $(s, l)$ .

Going back to (2-6), we obtain that

$$\frac{d}{dt} \Big|_{t=0} \int_{\Sigma_t} L_k d\mu_t = \int_{\Sigma} (-k P_{(k)l}^{ijs} R_{ij}{}^{ml} + \frac{1}{2} L_k g^{ms}) v_{ms} d\mu.$$

On the other hand, from definitions (2-1), (2-2) and (2-7), it is direct to check that

$$\begin{aligned} E_{(k)}^{ms} &= -\frac{1}{2^{k+1}} g^{ls} \delta_{l j_1 j_2 \dots j_{2k-1} j_{2k}}^{m i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= -\frac{1}{2^{k+1}} g^{ms} \delta_{m j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &\quad - \frac{2k}{2^{k+1}} g^{i_1 s} \delta_{i_1 j_1 j_2 \dots j_{2k-1} j_{2k}}^{m i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= -\frac{1}{2} L_k g^{ms} + k P_{(k)l}^{ijs} R_{ij}{}^{ml}. \end{aligned}$$

Hence we complete the proof of (2-4).

In the case that  $\Sigma$  is a hypersurface, one only needs to note that  $\frac{\partial}{\partial t} g_{ij} = 2h_{ij}$  for the evolving hypersurfaces.  $\square$

The second aim of this section is to give several simple facts on the warped product manifolds. Let  $M^n = [0, \bar{r}) \times_{\lambda} N^{n-1}$  ( $0 < \bar{r} \leq \infty$ ) be a warped product

manifold equipped with a Riemannian metric

$$\bar{g} = dr^2 + \lambda(r)^2 g_N.$$

where  $\lambda : [0, \bar{r}) \rightarrow \mathbb{R}$  is a smooth positive function. Let  $\Sigma$  be a smooth hypersurface in  $(M, \bar{g})$  with induced metric  $g$ . We denote by  $\bar{\nabla}$  and  $\nabla$  the covariant derivatives with respect to  $\bar{g}$  and  $g$  respectively. We define a vector field  $X$  on  $M$  by

$$X(r) = \lambda(r) \frac{\partial}{\partial r}.$$

Let  $\{e_1, \dots, e_{n-1}\}$  be a local frame on  $\Sigma$ , it is well known that  $X$  is a conformal Killing vector field satisfying

$$(2-11) \quad \bar{\nabla}_{e_i} X(r) = \lambda'(r) e_i.$$

We denote by  $r$  the height function which is obtained by the projection of  $\Sigma$  in  $M$  onto the first factor  $[0, \bar{r})$ . Let  $\phi(r)$  be a primitive of  $\lambda(r)$ .

**Proposition 2.2.** *The restriction of  $\phi$  on  $\Sigma$ , still denoted by  $\phi$ , satisfies*

$$(2-12) \quad \nabla_i \nabla_j \phi(r) = \lambda'(r) g_{ij} - \langle X, v \rangle h_{ij}.$$

The height function  $r$  on  $\Sigma$  satisfies

$$(2-13) \quad \nabla_i \nabla_j r = \frac{\lambda'(r)}{\lambda(r)} g_{ij} - \frac{\lambda'(r)}{\lambda(r)} \nabla_i r \nabla_j r - \langle \partial_r, v \rangle h_{ij}.$$

Consequently, we have

$$(2-14) \quad -2E_{(k)}^{ij} \nabla_i \nabla_j \phi(r) = (n - 1 - 2k) \lambda'(r) \mathfrak{H}_{2k} - \langle X, v \rangle \mathfrak{H}_{2k+1}.$$

$$(2-15) \quad -2E_{(k)}^{ij} \nabla_i \nabla_j r = (n - 1 - 2k) \frac{\lambda'(r)}{\lambda(r)} \mathfrak{H}_{2k} + \frac{2\lambda'(r)}{\lambda(r)} E_{(k)}^{ij} \nabla_i r \nabla_j r - \langle \partial_r, v \rangle \mathfrak{H}_{2k+1}.$$

*Proof.* Using (2-11), we have

$$\begin{aligned} \nabla_i \nabla_j \phi(r) &= \bar{\nabla}_i \bar{\nabla}_j \phi - \langle \bar{\nabla} \phi(r), v \rangle h_{ij} = \bar{\nabla}_i X_j - \langle X, v \rangle h_{ij} \\ &= \lambda'(r) g_{ij} - \langle X, v \rangle h_{ij}. \end{aligned}$$

Equation (2-13) follows from (2-12) and

$$\nabla_i \nabla_j r = \nabla_i \left( \frac{1}{\lambda(r)} \nabla_j \phi(r) \right) = \frac{1}{\lambda(r)} \nabla_i \nabla_j \phi(r) - \frac{\lambda'(r)}{\lambda(r)} \nabla_i r \nabla_j r.$$

For equations (2-14) and (2-15), we only need to notice that

$$-2E_{(k)}^{ij} g_{ij} = (n - 1 - 2k) L_k = (n - 1 - 2k) \mathfrak{H}_{2k}$$

and

$$-2E_{(k)}^{ij} h_{ij} = \mathfrak{H}_{2k+1}.$$

□

### 3. rigidity for the quotient $\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}$

In this section, we prove our main theorems.

*Proof of Theorem 1.1.* Since  $\Sigma$  is compact, there exist points  $p_{\min}, p_{\max} \in \Sigma$  such that the height function  $r$  attains its maximum and minimum values, i.e.,

$$\min_{\Sigma} r = r(p_{\min}), \quad \max_{\Sigma} r = r(p_{\max}).$$

At these points,

$$(3-1) \quad \nabla r(p_{\min}) = \nabla r(p_{\max}) = 0,$$

$$(3-2) \quad \nabla^2 r(p_{\min}) \geq 0, \quad \nabla^2 r(p_{\max}) \leq 0.$$

It follows from (3-1) and the star-shapedness of  $\Sigma$  that

$$(3-3) \quad \langle \partial_r, \nu \rangle(p_{\min}) = \langle \partial_r, \nu \rangle(p_{\max}) = 1.$$

By using (3-1) and (3-3) in (2-15), we obtain

$$(3-4) \quad -2E_{(k)}^{ij} \nabla_i \nabla_j r(p_{\min}) = (n-1-2k)(\log \lambda)'(\min_{\Sigma} r) \mathcal{H}_{2k}(p_{\min}) - \mathcal{H}_{2k+1}(p_{\min}),$$

$$(3-5) \quad -2E_{(k)}^{ij} \nabla_i \nabla_j r(p_{\max}) = (n-1-2k)(\log \lambda)'(\max_{\Sigma} r) \mathcal{H}_{2k}(p_{\max}) - \mathcal{H}_{2k+1}(p_{\max}).$$

We claim that the quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  satisfies

$$(3-6) \quad \min_{\Sigma} \left( \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right) \leq (n-1-2k)(\log \lambda)'(\min_{\Sigma} r),$$

$$(n-1-2k)(\log \lambda)'(\max_{\Sigma} r) \leq \max_{\Sigma} \left( \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right).$$

Consider first the case that  $-2E_{(k)}^{ij}$  is positive semidefinite. It follows from (3-2), (3-4) and (3-5) that

$$(3-7) \quad (n-1-2k)(\log \lambda)'(\min_{\Sigma} r) \mathcal{H}_{2k}(p_{\min}) - \mathcal{H}_{2k+1}(p_{\min}) \geq 0,$$

$$(3-8) \quad (n-1-2k)(\log \lambda)'(\max_{\Sigma} r) \mathcal{H}_{2k}(p_{\max}) - \mathcal{H}_{2k+1}(p_{\max}) \leq 0.$$

From the fact that

$$-2E_{(k)}^{ij} g_{ij} = (n-1-2k) \mathcal{H}_{2k},$$

together with the assumption that  $\mathcal{H}_{2k}$  is nonvanishing on  $\Sigma$ , we know that  $\mathcal{H}_{2k} > 0$ . Hence the claim in this case follows from (3-7) and (3-8) immediately. For the second case that  $-2E_{(k)}^{ij}$  is negative semidefinite, similar argument applies by taking  $\mathcal{H}_{2k} < 0$  into account. We finish the proof of the claim.

Now using the assumption that  $\log \lambda$  is convex, we obtain from (3-6) that

$$\begin{aligned} \min_{\Sigma} \left( \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right) &\leq (n - 1 - 2k)(\log \lambda)'(\min_{\Sigma} r) \\ &\leq (n - 1 - 2k)(\log \lambda)'(\max_{\Sigma} r) \leq \max_{\Sigma} \left( \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right). \end{aligned}$$

Since the quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  is constant, we have from above that

$$(3-9) \quad \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} = (n - 1 - 2k)(\log \lambda)'(\min_{\Sigma} r) = (n - 1 - 2k)(\log \lambda)'(\max_{\Sigma} r),$$

which yields that  $(\log \lambda)'(r)$  is a constant function on  $\Sigma$ . Substituting (3-9) into (2-14), we have

$$(3-10) \quad -2E_{(k)}^{ij} \nabla_i \nabla_j \phi(r) = \lambda(1 - \langle \partial_r, \nu \rangle) \mathcal{H}_{2k+1}.$$

Notice that  $\langle \partial_r, \nu \rangle \leq 1$  and  $\mathcal{H}_{2k+1} = c\mathcal{H}_{2k}$  does not change sign on  $\Sigma$ . Applying the classical maximum principle to the elliptic equation (3-10), we conclude that  $\phi(r)$  is a constant function on  $\Sigma$ . Since  $\phi$  is an increasing function with respect to  $r$  due to the fact  $\phi' = \lambda > 0$ , we conclude that the height function  $r$  is a constant function on  $\Sigma$ , i.e.,  $\Sigma$  is a slice  $\{r_0\} \times N$ . □

To extend the previous result to noncompact hypersurfaces, we will apply a generalization of the Omori–Yau maximum principle for trace-type differential operators. Consider a Riemannian manifold  $(\Sigma, g)$  and a semi-elliptic operator  $L = \text{tr}_g(T \circ \nabla_g^2)$ , where  $T : T\Sigma \rightarrow T\Sigma$  is a positive semidefinite symmetric tensor. For simplicity we will omit the subscription  $g$ .

**Definition 3.1.** We say that the Omori–Yau maximum principle holds on  $\Sigma$  for  $L$ , if for any function  $u \in C^2(\Sigma)$  with  $\sup_{\Sigma} u < +\infty$ , there exists a sequence  $\{p_i\}_{i \in \mathbb{N}} \subset \Sigma$  such that for each  $i$ , the following holds:

$$u(p_i) > \sup_{\Sigma} u - \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad Lu(p_i) < \frac{1}{i}.$$

Since  $\inf_{\Sigma} u = -\sup_{\Sigma}(-u)$ , the above is equivalent to that for any function  $u \in C^2(\Sigma)$  with  $\inf_{\Sigma} u > -\infty$ , there exists a sequence  $\{p_i\}_{i \in \mathbb{N}} \subset \Sigma$  such that for each  $i$ , the following holds:

$$u(p_i) < \inf_{\Sigma} u + \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad Lu(p_i) > -\frac{1}{i}.$$

Assume the generalized Omori–Yau maximum principle holds for the trace-type operator  $L = \text{tr}(-2E_{(k)} \nabla^2)$ , one can prove the analogous result for noncompact hypersurfaces.

*Proof of Theorem 1.2.* Due to the same argument as in the proof of [Theorem 1.1](#), we only need to prove the theorem in the case that  $-2E_{(k)}^{ij}$  is positive semidefinite. By the generalized Omori–Yau maximum principle, we have two sequences  $\{p_i\}$  and  $\{q_i\}$  in  $\Sigma$  with properties

- (i)  $\lim_{i \rightarrow +\infty} \phi(r(p_i)) = \sup_{\Sigma} \phi(r)$ ,  $\lim_{i \rightarrow +\infty} \phi(r(q_i)) = \inf_{\Sigma} \phi(r)$ ;
- (ii)  $|\nabla \phi(r)|(p_i) = \lambda(r(p_i))|\nabla r|(p_i) < \frac{1}{i}$ ,  $|\nabla \phi(r)|(q_i) = \lambda(r(p_i))|\nabla r|(q_i) < \frac{1}{i}$ ;
- (iii)  $\text{tr}(-2E_{(k)}\nabla^2\phi(r))(p_i) < \frac{1}{i}$ ,  $\text{tr}(-2E_{(k)}\nabla^2\phi(r))(q_i) > -\frac{1}{i}$ .

Since  $\phi(r)$  is strictly increasing due to  $\phi'(r) = f(r) > 0$ , we have

$$\lim_{i \rightarrow +\infty} r(p_i) = \sup_{\Sigma} r, \quad \lim_{i \rightarrow \infty} r(q_i) = \inf_{\Sigma} r,$$

and thus

$$\lim_{i \rightarrow +\infty} \langle \partial_r, \nu \rangle(p_i) = \lim_{i \rightarrow +\infty} \langle \partial_r, \nu \rangle(q_i) = 1.$$

Using the above facts in [\(2-14\)](#) and letting  $i \rightarrow +\infty$ , we get

$$(3-11) \quad (n - 1 - 2k)(\log \lambda)'(\sup_{\Sigma} r) \leq \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \leq (n - 1 - 2k)(\log \lambda)'(\inf_{\Sigma} r).$$

By the assumption that  $(\log \lambda)'' \geq 0$  with equality only at isolated points, we obtain the desired result that  $r$  is constant. That is,  $\Sigma$  is a slice  $\{r_0\} \times M$ . □

In the following, we discuss some sufficient condition to guarantee the generalized Omori–Yau maximum principle to hold for  $\Sigma$ . Inspired by [\[Pigola et al. 2005\]](#), Alías et al. [\[2013, Theorem 1 and Corollary 3\]](#) proved that the Omori–Yau maximum principle holds for a trace-type elliptic operator  $L = \text{tr}(T \circ \nabla^2)$  with positive semidefinite  $T$  satisfying  $\sup_{\Sigma} \text{tr} T < \infty$  on a Riemannian manifold  $\Sigma$ , provided that the radial sectional curvature (the sectional curvature of the 2-planes containing  $\nabla \rho$ , where  $\rho$  is the distance function on  $\Sigma$  from a fixed point in  $\Sigma$ ) of  $\Sigma$  satisfies the condition

$$(3-12) \quad K_{\Sigma}^{\text{rad}}(\nabla \rho, \nabla \rho) > -G(\rho),$$

where  $G : [0, +\infty) \rightarrow \mathbb{R}$  is a smooth function satisfying

$$(3-13) \quad G(0) > 0, \quad G'(t) \geq 0, \quad \int_0^{+\infty} \frac{1}{\sqrt{G(t)}} = +\infty, \quad \limsup_{t \rightarrow +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty.$$

A special case for which [\(3-12\)](#) holds is that the sectional curvature of  $\Sigma$  is bounded from below (one can choose  $G(\rho) = C(1 + \rho^2)$ , where  $C$  is a constant).

In the case of warped product manifolds, Alías et al. gave a detailed discussion of [\(3-12\)](#). More precisely, they proved [\[ibid., Corollary 4\]](#) that for a hypersurface  $\Sigma$  in a slab of a warped product manifold  $[r_1, r_2] \times N$ , [\(3-12\)](#) holds for  $L$  with

positive semidefinite  $T$  satisfying  $\sup_{\Sigma} \operatorname{tr} T < \infty$ , provided that the radial sectional curvature of the fiber manifold  $N$  satisfies

$$(3-14) \quad K_N^{\text{rad}}(\nabla^N \hat{\rho}, \nabla^N \hat{\rho}) > -G(\hat{\rho}),$$

where  $\hat{\rho}$  is the distance function on the fiber  $N$  from a fixed point in  $N$  and  $G : [0, +\infty) \rightarrow \mathbb{R}$  is a smooth function satisfying the conditions listed in (3-13), together with  $\sup_{\Sigma} \|h\|^2 < +\infty$  on  $\Sigma$ . Geometrically, the condition (3-14) means that the radial sectional curvature of the fiber manifold  $N$  has a strong quadratic decay at infinity, that is, one can choose  $G(\rho) = C(1 + \rho^2 \log^2(2 + \rho))$  as shown in [Chen and Xin 1992]. In particular, when  $N$  has sectional curvature bounded from below or  $N$  is compact, (3-14) holds. As a direct result of Theorem 1.2, we have:

**Corollary 3.2.** *Let  $(M^n, \bar{g})$  be as in Theorem 1.2. Assume that the radial sectional curvature of  $N$  satisfies (3-14). Let  $\Sigma^{n-1}$  be a complete, noncompact star-shaped hypersurface in  $M$  which is contained in a slab  $[r_1, r_2] \times N$  with  $\sup_{\Sigma} \|h\|^2 < +\infty$ . Assume  $-2E_{(k)}$  is semidefinite on  $\Sigma$  and  $\sup_{\Sigma} \mathcal{H}_{2k} < \infty$  on  $\Sigma$ . If the quotient  $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$  is constant, then the hypersurface is a slice  $\{r_0\} \times N$ .*

Following the argument close to the proof of Theorem 1.2, one may prove the Bernstein-type result in this case.

*Proof of Theorem 1.3.* By the generalized Omori–Yau maximum principle to the height function  $r$ , there exists a sequence  $\{p_i\} \subset \Sigma$  such that

$$\lim_{i \rightarrow \infty} r(p_i) = \sup_{\Sigma} r, \quad \lim_{i \rightarrow \infty} |\nabla r|(p_i) = 0, \quad \lim_{i \rightarrow \infty} \sup \operatorname{tr}(-2E_{(k)} \nabla^2 r)(p_i) \leq 0.$$

It follows from the semidefiniteness of  $-2E_{(k)}$  and the positivity of  $\mathcal{H}_{2k}$  that

$$0 \leq \langle -2E_{(k)} \nabla r, \nabla r \rangle \leq \operatorname{tr}(-2E_{(k)}) |\nabla r|^2 \leq (n - 1 - 2k) C_2 |\nabla r|^2.$$

From the fact  $\langle \partial_r, \nu \rangle^2 = 1 - |\nabla r|^2$ , we have

$$\lim_{i \rightarrow \infty} \langle \partial_r, \nu \rangle(p_i) = 1,$$

and thus

$$\lim_{i \rightarrow \infty} \langle -2E_{(k)} \nabla r, \nabla r \rangle(p_i) = 0.$$

Combining all the above facts together into (2-15), we have

$$0 \geq \lim_{i \rightarrow \infty} \sup \operatorname{tr}(-2E_{(k)} \nabla^2 r)(p_i) \geq C_1 \lim_{i \rightarrow \infty} \left( (n - 1 - 2k) \frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right)(p_i) \geq 0,$$

so that

$$\lim_{i \rightarrow \infty} \left( (n - 1 - 2k) \frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \right)(p_i) = 0.$$

From the hypothesis, we have  $\inf_{\Sigma} ((n - 1 - 2k)\lambda'(r)/\lambda(r) - \mathcal{H}_{2k+1}/\mathcal{H}_{2k}) = 0$ , and thus  $|\nabla r| \equiv 0$  on  $\Sigma$ , which yields that  $\Sigma$  is a slice  $\{r_0\} \times M$  for some  $r_0 \in [0, \bar{r})$ .  $\square$

### Appendix: Kottler–Schwarzschild manifolds

The Kottler manifolds, or Kottler–Schwarzschild manifolds, are analogues of the Schwarzschild space in the setting of asymptotically locally hyperbolic manifolds. For  $\kappa = 1, 0$  or  $-1$ , let  $(N(\kappa), \hat{g})$  be a closed space form of constant sectional curvature  $\kappa$ . An  $n$ -dimensional Kottler–Schwarzschild manifold

$$P_{\kappa,m} = [\rho_{\kappa,m}, \infty) \times N(\kappa)$$

is equipped with the metric

$$(A-1) \quad g_{\kappa,m} = \frac{d\rho^2}{V_{\kappa,m}^2(\rho)} + \rho^2 \hat{g}, \quad V_{\kappa,m} = \sqrt{\rho^2 + \kappa - \frac{2m}{\rho^{n-2}}}.$$

Let  $\rho_0 := \rho_{\kappa,m}$  be the largest positive root of

$$\phi(\rho) := \rho^2 + \kappa - \frac{2m}{\rho^{n-2}} = 0.$$

Remark that in (A-1), in order to have a positive root  $\rho_0$ , if  $\kappa = 0$  or  $1$ , the parameter  $m$  should be always positive; if  $\kappa = -1$ , the parameter  $m$  can be negative. In fact, in this case,  $m \in [m_c, +\infty)$  and

$$m_c = -\frac{(n-2)^{(n-2)/2}}{n^{n/2}}.$$

Here the certain critical value  $m_c$  comes from the following. If  $m \leq 0$ , one can solve the equation

$$\phi'(\rho) = 2\rho + (n-2)\frac{2m}{\rho^{n-1}} = 0,$$

to get the root  $\rho_1 = (-(n-2)m)^{1/n}$ . Note the fact that  $\phi(\rho_1) \leq 0$ , which yields

$$m \geq -\frac{(n-2)^{(n-2)/2}}{n^{n/2}}.$$

By a change of variable  $r = r(\rho)$  with

$$r'(\rho) = \frac{1}{V_{\kappa,m}(\rho)}, \quad r(\rho_{\kappa,m}) = 0,$$

we can rewrite  $P_{\kappa,m}$  as a warped product manifold  $P_{\kappa,m} = [0, \infty) \times_{\lambda_\kappa} N(\kappa)$  equipped with the metric

$$g_{\kappa,m} := \bar{g} := dr^2 + \lambda_\kappa(r)^2 \hat{g},$$

where  $\lambda_\kappa : [0, \infty) \rightarrow [\rho_{\kappa,m}, \infty)$  is the inverse of  $r(\rho)$ , i.e.,  $\lambda_\kappa(r(\rho)) = \rho$ .

It is easy to check

$$\begin{aligned} \lambda'_\kappa(r) &= V_{\kappa,m}(\rho) = \sqrt{\kappa + \lambda_\kappa(r)^2 - 2m\lambda_\kappa(r)^{2-n}}, \\ \lambda''_\kappa(r) &= \lambda_\kappa(r) + (n-2)m\lambda_\kappa(r)^{1-n}. \end{aligned}$$

Hence

$$\lambda_\kappa \lambda''_\kappa - (\lambda'_\kappa)^2 = -\kappa + nm\lambda_\kappa^{2-n}.$$

For the case  $\kappa = 0, m \geq 0$  and hence  $\lambda_\kappa \lambda''_\kappa - (\lambda'_\kappa)^2 = nm\lambda_\kappa^{2-n} \geq 0$ . For the case  $\kappa = -1$ , if  $m \geq 0$ , then  $\lambda_\kappa \lambda''_\kappa - (\lambda'_\kappa)^2 = 1 + nm\lambda_\kappa^{2-n} > 0$ . If

$$m \in \left[ -\frac{(n-2)^{(n-2)/2}}{n^{n/2}}, 0 \right),$$

then

$$\begin{aligned} \lambda_\kappa \lambda''_\kappa - (\lambda'_\kappa)^2 &= 1 + nm\lambda_\kappa^{2-n} \geq 1 + nm\rho_0^{2-n} \geq 1 + nm\rho_1^{2-n} \\ &= 1 + nm(-n-2)m^{(2-n)/n} = 1 - n(n-2)^{(2-n)/n}(-m)^{2/n} \\ &\geq 1 - n(n-2)^{(2-n)/n} \left( \frac{(n-2)^{(n-2)/2}}{n^{n/2}} \right)^{2/n} = 0. \end{aligned}$$

As a conclusion, the condition on the log convexity of  $\lambda$  holds for the Kottler–Schwarzschild manifolds with  $\kappa = 0$  and  $-1$ . We remark that the log convexity of  $\lambda$  does not hold for the Kottler–Schwarzschild manifolds when  $\kappa = 1$ .

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