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DETERMINANT RANK OF C^*-ALGEBRAS

GUIHUA GONG, HUAXIN LIN AND YIFENG XUE

Dedicated to George A. Elliott on his seventieth birthday

Let A be a unital C^*-algebra and let $U_0(A)$ be the group of unitaries of A which are path-connected to the identity. Denote by $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. Let $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ be the homomorphism defined by sending u to $\mathrm{diag}(u, 1_{n-1})$. We study the problem of when the map $i_A^{(1,n)}$ is an isomorphism for all n. We show that it is always surjective and that it is injective when A has stable rank one. It is also injective when A is a unital C^*-algebra of real rank zero, or A has no tracial state. We prove that the map is an isomorphism when A is Villadsen's simple AH-algebra of stable rank $k > 1$. We also prove that the map is an isomorphism for all Blackadar's unital projectionless separable simple C^*-algebras. Let $A = \mathrm{M}_n(C(X))$, where X is any compact metric space. We note that the map $i_A^{(1,n)}$ is an isomorphism for all n. As a consequence, the map $i_A^{(1,n)}$ is always an isomorphism for any unital C^*-algebra A that is an inductive limit of the finite direct sum of C^*-algebras of the form $\mathrm{M}_n(C(X))$ as above. Nevertheless we show that there is a unital C^*-algebra A such that $i_A^{(1,2)}$ is not an isomorphism.

1. Introduction

Let A be a unital C^*-algebra and let $U(A)$ be the unitary group. Denote by $U_0(A)$ the normal subgroup which is the connected component of $U(A)$ containing the identity of A. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and by $CU(A)$ the closure of $DU(A)$. We will study the group $U_0(A)/CU(A)$. Recently this group has become an important invariant for the structure of C^*-algebras. It plays an important role in the classification of C^*-algebras (see [Elliott and Gong 1996; Nielsen and Thomsen 1996; Elliott 1997; Thomsen 1997; Gong 2002; Elliott et al. 2007; Lin 2007; 2011; Gong et al. 2015], for example). It was shown in [Lin 2007] that the map $U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is an isomorphism for all $n \geq 1$ if A is a unital simple C^*-algebra of tracial rank at most one (see also [Lin

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2010b, Corollary 3.5]). In general, when A has stable rank k, it was shown by Rieffel [1987] that the map $U(\mathrm{M}_k(A))/U_0(\mathrm{M}_k(A)) \to U(\mathrm{M}_{k+m}(A))/U_0(\mathrm{M}_{k+m}(A))$ is an isomorphism for all integers $m \geq 1$. In this case $U(\mathrm{M}_k(A))/U_0(\mathrm{M}_k(A)) = K_1(A)$. This fact plays an important role in the study of the structure of C^*-algebras, in particular those C^*-algebras of stable rank one, since it simplifies computations when K-theory involved. Therefore it seems natural to ask when the map $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is an isomorphism. It will also greatly simplify our understanding and usage of the group when $i_A^{(1,n)}$ is an isomorphism for all n. The main tool to study $U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is the de la Harpe–Skandalis determinant, studied early by K. Thomsen [1995] (henceforth abbreviated [Th]), which involves the tracial state space $T(A)$ of A. On the other hand, we observe that when $T(A) = \varnothing$, $U_0(A)/CU(A) = \{0\}$. So we focus our attention on the case $T(A) \neq \varnothing$. One of the authors was asked repeatedly if the map $i_A^{(1,n)}$ is an isomorphism when A has stable rank one.

It turns out that it is easy to see that the map $i_A^{(1,n)}$ is always surjective for all n. Therefore the issue is when $i_A^{(1,n)}$ is injective.

Definition 1.1. Let A be a unital C^*-algebra. Consider the homomorphism

$$i_A^{(m,n)}\colon U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

(induced by $u \mapsto \mathrm{diag}(u, 1_{n-m})$) for integers $n \geq m \geq 1$. The determinant rank of A is defined to be

$$\mathrm{Dur}\, A = \min\{m \in \mathbb{N} \mid i_A^{(m,n)} \text{ is isomorphism for all } n > m\}.$$

If no such integer exists, we set $\mathrm{Dur}\, A = \infty$.

We show that if $A = \lim_{n\to\infty} A_n$, then $\mathrm{Dur}\, A \leq \sup_{n\geq 1}\{\mathrm{Dur}\, A_n\}$. We prove that $\mathrm{Dur}\, A = 1$ for all C^*-algebras of stable rank one, which answers the question mentioned above. We also show that $\mathrm{Dur}\, A = 1$ for any unital C^*-algebra A with real rank zero. A closely related and repeatedly used fact is that the map $u \to u + (1-e)$ is an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$ when A is a unital simple C^*-algebra of tracial rank at most one and $e \in A$ is a projection (see [Lin 2007, Theorem 6.7; 2010b, Theorem 3.4]). We show in this note that this holds for any simple C^*-algebra of stable rank one.

Given Rieffel's early result mentioned above, one might be led to think that, when A has higher stable rank, or at least when $A = C(X)$ for higher-dimensional finite CW complexes, $\mathrm{Dur}\, A$ is perhaps large. On the other hand it was suggested (see [Th, Section 3]) that $\mathrm{Dur}\, A = 1$ may hold for most unital simple separable C^*-algebras. We found out, somewhat surprisingly, that the determinant rank of $\mathrm{M}_n(C(X))$ is always 1 for any compact metric space X and for any integer $n \geq 1$. This, together with previous mentioned result, shows that if $A = \lim_{n\to\infty} A_n$, where A_n is a finite direct sum of C^*-algebras of the form $\mathrm{M}_n(C(X))$, then $\mathrm{Dur}\, A = 1$. Furthermore, we found out that $\mathrm{Dur}\, A = 1$ for all of Villadsen's examples of unital simple AH-algebras A with higher stable rank. This research suggests that when A has an abundant amount of projections then $\mathrm{Dur}\, A$ is likely to be 1 (see Proposition 3.6(3)). In fact, we prove that if A is a unital simple AH-algebra with property (SP), then $\mathrm{Dur}\, A = 1$. On the other hand, however, we show that if A is a unital projectionless simple C^*-algebra and $\rho_A(K_0(A)) = \mathbb{Z}$, then $\mathrm{Dur}\, A = 1$. Furthermore, if A is one of Blackadar's examples of unital projectionless simple separable C^*-algebras with infinite many extremal tracial states, then $\mathrm{Dur}\, A = 1$. Indeed, it seems that it is difficult to find any example of unital separable simple C^*-algebras for which $\mathrm{Dur}\, A$ is larger than 1. Nevertheless Proposition 3.12 below provides a necessary condition for $\mathrm{Dur}\, A = 1$. In fact, we find that a certain unital separable C^*-algebra violates this condition, which, in turn, provides an example of a unital separable C^*-algebra A such that $\mathrm{Dur}\, A > 1$.

2. Preliminaries

In this section, we list some notation and basic known facts for convenience, many of which are taken from [Th] and other sources.

Definition 2.1. Let A be a C^*-algebra. Denote by $\mathrm{M}_n(A)$ the $n \times n$ matrix algebra of over A. If A is not unital, we will use $\widetilde{A}$, the unitization of A, so suppose that A is unital. For u in $U_0(A)$, let $[u]$ be the class of u in $U_0(A)/CU(A)$.

We view A^n as the set of all $n \times 1$ matrices over A. Set

$$S_n(A) = \left\{ (a_1, \dots, a_n)^T \in A^n \;\middle|\; \sum_{i=1}^{n} a_i^* a_i = 1 \right\},$$

$$\mathrm{Lg}_n(A) = \left\{ (a_1, \dots, a_n)^T \in A^n \;\middle|\; \sum_{i=1}^{n} b_i a_i = 1 \text{ for some } b_1, \dots, b_n \in A \right\}.$$

According to [Rieffel 1983; 1987], the topological stable rank and the connected stable rank of A are defined as

$$\mathrm{tsr}\, A = \min\{n \in \mathbb{N} \mid \mathrm{Lg}_m(A) \text{ is dense in } A^m \text{ for all } m \geq n\}$$

$$\mathrm{csr}\, A = \min\{n \in \mathbb{N} \mid U_0(\mathrm{M}_m(A)) \text{ acts transitively on } S_m(A) \text{ for all } m \geq n\}.$$

If no such integer exists, we set $\mathrm{tsr}\, A = \infty$ and $\mathrm{csr}\, A = \infty$. These notions are very useful tools in computing K-groups of C^*-algebras (see, e.g., [Rieffel 1987; Xue 2000; 2001; 2010]).

Definition 2.2. Let A be a C^*-algebra. Denote by A_{sa} (resp. A_+) the set of all self-adjoint (resp. positive) elements in A. Denote by $T(A)$ the tracial state space of A. Let $\tau \in T(A)$. We will also use the notation τ for the unnormalized trace

$\tau \otimes \mathrm{Tr}_n$ on $\mathrm{M}_n(A)$, where Tr_n is the standard trace for $\mathrm{M}_n(\mathbb{C})$. Every tracial state on $\mathrm{M}_n(A)$ has the form $(1/n)\tau$.

Definition 2.3. For $a, b \in A$, set $[a, b] = ab - ba$. Furthermore, set

$$[A, A] = \left\{ \sum_{j=1}^{n} [a_j, b_j] \,\middle|\, a_j, b_j \in A,\ j = 1, \dots, n,\ n \geq 1 \right\}.$$

Now, let A_0 denote the subset of A_{sa} consisting of elements of the form $x - y$ for $x, y \in A_{\mathrm{sa}}$ with $x = \sum_{j=1}^{\infty} c_j c_j^*$ and $y = \sum_{j=1}^{\infty} c_j^* c_j$ (convergent in norm) for some sequence $\{c_j\}$ in A. By [Cuntz and Pedersen 1979], A_0 is a closed subspace of A_{sa}.

Proposition 2.4 [Cuntz and Pedersen 1979; Thomsen 1995, Section 3]. *Let A be a C^*-algebra with unit* 1. *The following statements are equivalent*:

(1) $A_0 = A_{\mathrm{sa}}$.

(2) $1 \in A_0$.

(3) $T(A) = \varnothing$.

(4) $A = \overline{[A, A]}$.

(5) $A_{\mathrm{sa}} = \overline{\mathrm{span}\{[a^*, a] \mid a \in A\}}$.

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). If $T(A) \neq \varnothing$, then there is a tracial state τ on A. Since $1 \in A_0$, it follows that there is a sequence $\{a_j\}$ in A such that $b = \sum_{j=1}^{\infty} a_j^* a_j$ and $c = \sum_{j=1}^{\infty} a_j a_j^*$ are convergent in A and $1 = b - c$. Thus, $\tau(b) = \sum_{j=1}^{\infty} \tau(a_j^* a_j) = \tau(c)$ and $\tau(1) = \tau(b - c) = 0$, a contradiction since $\tau(1) = 1$.

(3) $\Longrightarrow$ (1). This follows from the proof of [Th, Lemma 3.1].

(4) $\Longleftrightarrow$ (5). Let $a, b \in A$ and write $a = a_1 + i a_2$ and $b = b_1 + i b_2$, where $a_1, a_2, b_1, b_2 \in A_{\mathrm{sa}}$. Then

$$[a, b] = [a_1, b_1] - [a_2, b_2] + i[a_2, b_1] + i[a_1, b_2]. \tag{2-1}$$

Put $c_1 = a_1 + i b_1$, $c_2 = a_2 + i b_2$, $c_3 = a_2 + i b_1$ and $c_4 = a_1 + i b_2$. Then, from (2-1), we get that

$$[a, b] = \frac{1}{2i}[c_1^*, c_1] - \frac{1}{2i}[c_2^*, c_2] + \frac{1}{2}[c_3^*, c_3] + \frac{1}{2}[c_4^*, c_4]. \tag{2-2}$$

So, by (2-2), (4) and (5) are equivalent.

(5) $\Longrightarrow$ (1). Let $x \in \mathrm{span}\{[a^*, a] \mid a \in A\}$. Then there are elements $a_1, \dots, a_k \in A$ and positive numbers $\lambda_1, \dots, \lambda_k$ such that $x = \sum_{i=1}^{j} \lambda_i [a_i^*, a_i] - \sum_{i=j+1}^{k} \lambda_i [a_i^*, a_i]$ for some $j \in \{1, \dots, k\}$. Put $c_i = \sqrt{\lambda_i}\, a_i$, $i = 1, \dots, j$ and $c_i^* = \sqrt{\lambda_i}\, a_i^*$ when

$i = j+1, \dots, k$. Then $x = \sum_{i=1}^k c_i^* c_i - \sum_{i=1}^k c_i c_i^* \in A_0$. Since A_0 is closed, we get that

$$A_{sa} = \overline{\text{span}\{[a^*, a] \mid a \in A\}} \subset \overline{A_0} = A_0 \subset A_{sa}.$$

(1) $\Longrightarrow$ (5). According to the definition of A_0, every element $x \in A_0$ has the form $x = x_1 - x_2$, where $x_1 = \sum_{i=1}^\infty z_i^* z_i$ and $x_2 = \sum_{i=1}^\infty z_i z_i^*$. Thus, $x \in \overline{\text{span}\{[a^*, a] \mid a \in A\}}$ and hence $A_{sa} = \overline{\text{span}\{[a^*, a] \mid a \in A\}}$. □

Combining Proposition 2.4 with Definition 2.2, we have:

Corollary 2.5. *Let A be a unital C^*-algebra with $A_0 = A_{sa}$. Then* $(\mathrm{M}_n(A))_0 = (\mathrm{M}_n(A))_{sa}$.

Let $a, b \in A_{sa}$. Then, for any $n \geq 1$,

$$\exp(ia)\exp(ib)\Big(\exp\Big(-i\frac{a}{n}\Big)\exp\Big(-i\frac{b}{n}\Big)\Big)^n \in DU(A)$$

and $\exp(-i(a+b)) = \lim_{n\to\infty}(\exp(-ia/n)\exp(-ib/n))^n$ by the Trotter product formula [Masani 1981, Theorem 2.2]. So $\exp(ia)\exp(ib)\exp(-i(a+b)) \in CU(A)$. Consequently,

$$[\exp(ia)][\exp(ib)] = [\exp(i(a+b))] \quad \text{in } U_0(A)/CU(A). \tag{2-3}$$

The following is taken from the proof of [Th, Lemma 3.1].

Lemma 2.6. *Let $a \in A_{sa}$.*

(1) *If $a \in A_0$, then $[\exp(ia)] = 0$ in $U_0(A)/CU(A)$;*

(2) *If $T(A) \neq \varnothing$ and $\tau(a) = \tau(b)$ for all $\tau \in T(A)$, then $a - b \in A_0$ and $[\exp(ia)] = [\exp(ib)]$ in $U_0(A)/CU(A)$.*

Combining Lemma 2.6(1) with Corollary 2.5, we have

Corollary 2.7. *If $T(A) = \varnothing$, then $U_0(\mathrm{M}_n(A)) = CU(\mathrm{M}_n(A))$ for $n \geq 1$.*

Definition 2.8. Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Let $PU_0^n(A)$ denote the set of all piecewise smooth maps $\xi : [0, 1] \to U_0(\mathrm{M}_n(A))$ with $\xi(0) = 1_n$, where 1_n is the unit of $\mathrm{M}_n(A)$. For $\tau \in T(A)$, the de la Harpe–Skandalis function Δ_τ^n on $PU_0^n(A)$ is given by

$$\Delta_\tau^n(\xi(t)) = \frac{1}{2\pi i}\int_0^1 \tau(\xi'(t)(\xi(t))^*)\,\mathrm{d}t \quad \text{for all } \xi \in PU_0^n(A).$$

Note that we use an unnormalized trace $\tau = \tau \otimes \mathrm{Tr}_n$ on $\mathrm{M}_n(A)$. This gives a homomorphism $\Delta^n : PU_0^n(A) \to \mathrm{Aff}(T(A))$, the space of all real affine continuous functions on $T(A)$.

We list some properties of $\Delta_\tau^n(\cdot)$:

Lemma 2.9 [de la Harpe and Skandalis 1984, Lemmas 1 and 3]. *Let A be a unital C^*-algebra with $T(A)\neq\varnothing$. Let $\xi_1,\xi_2,\xi\in PU_0^n(A)$. Then*:

(1) $\Delta_\tau^n(\xi_1(t))=\Delta_\tau^n(\xi_2(t))$ *for all* $\tau\in T(A)$, *if* $\xi_1(1)=\xi_2(1)$ *and*

$$\xi_1\xi_2^*\in U_0(\widetilde{(C_0(S^1,\mathrm{M}_n(A))}).$$

(2) *There are* $y_1,\dots,y_k\in\mathrm{M}_n(A)_{\mathrm{sa}}$ *such that* $\Delta_\tau^n(\xi(t))=\sum_{j=1}^k\tau(y_j)$ *for all* $\tau\in T(A)$ *and* $\xi(1)=\exp(i2\pi y_1)\cdots\exp(i2\pi y_k)$.

Definition 2.10. Let A be a C^*-algebra with $T(A)\neq\varnothing$. Let $\mathrm{Aff}(T(A))$ be the set of all real continuous affine functions on $T(A)$. Define $\rho_A:K_0(A)\to\mathrm{Aff}(T(A))$ by

$$\rho_A([p])(\tau)=\tau(p)\quad\text{for all }\tau\in T(A),$$

where $p\in\mathrm{M}_n(A)$ is a projection.

Define $P_n(A)$ to be the subgroup of $K_0(A)$ generated by projections in $\mathrm{M}_n(A)$. Denote by $\rho_A^n(K_0(A))$ the subgroup $\rho_A(P_n(A))$ of $\rho_A(K_0(A))$. In particular, $\rho_A^1(K_0(A))$ is the subgroup of $\rho_A(K_0(A))$ generated by the images of projections in A under the map ρ_A.

Definition 2.11. Let A be a unital C^*-algebra. Denote by $LU_0^n(A)$ the set of piecewise smooth loops in

$$U(\widetilde{C_0(S^1,\mathrm{M}_n(A))}).$$

Then, by Bott periodicity, $\Delta^n(LU_0^n(A))\subset\rho_A(K_0(A))$. Denote by

$$\mathfrak{q}^n:\mathrm{Aff}(T(A))\to\mathrm{Aff}(T(A))/\overline{\Delta^n(LU_0^n(A))}$$

the quotient map. Put $\overline{\Delta^n}=\mathfrak{q}^n\circ\Delta^n$. Since $\overline{\Delta^n}$ vanishes on $LU_0^n(A)$, we also use $\overline{\Delta^n}$ for the homomorphism from $U_0(\mathrm{M}_n(A))$ into $\mathrm{Aff}(T(A))/\overline{\Delta^n(LU_0^n(A))}$. An important fact that we will repeatedly use is that *the kernel of* $\overline{\Delta^n}$ *is exactly* $CU(\mathrm{M}_n(A))$, by [Th, Lemma 3.1]. In other words, if $u\in U_0(\mathrm{M}_n(A))$ and $\overline{\Delta^n}(u)=0$, then $u\in CU(\mathrm{M}_n(A))$.

Corollary 2.12. *Let A be a unital C^*-algebra and let $u\in U_0(\mathrm{M}_n(A))$ for $n\geq 1$. Then there are an $a\in A_{\mathrm{sa}}$ and a $v\in CU(\mathrm{M}_n(A))$ such that*

$$u=\mathrm{diag}(\exp(i2\pi a),1_{n-1})v$$

(*in the case $n=1$, we define* $\mathrm{diag}(\exp(i2\pi a),1_{n-1})=\exp(i2\pi a)$).

Moreover, if there is a $u\in PU_0^n(A)$ with $u(1)=u$, we can choose a self-adjoint element a so that $\hat a=\Delta^n(u(t))$, where $\hat a(\tau)=\tau(a)$ for all $\tau\in T(A)$.

Proof. Fix a piecewise smooth path $u(t) \in PU_0^n(A)$ with $u(0) = 1$ and $u(1) = u$. By Lemma 2.9(2), there are $a_1, a_2, \dots, a_m \in \mathrm{M}_n(A)_{\mathrm{sa}}$ such that

$$u = \prod_{j=1}^{m} \exp(i2\pi a_j) \quad \text{and} \quad \Delta_\tau^n(u(t)) = \tau \sum_{j=1}^{m} a_j \quad \text{for all } \tau \in T(A).$$

Put $a_0 = \sum_{j=1}^{n} a_j$. Write $a_0 = (b_{i,j})_{n\times n}$. Define $a = \sum_{i=1}^{n} b_{i,i}$. Then $a \in A_{\mathrm{sa}}$. Moreover,

$$\overline{\Delta^n}(\mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u) = 0.$$

Thus, by [Th, Lemma 3.1], $\mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u \in CU(\mathrm{M}_n(A))$. Put $v = \mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u$. Then $u = \mathrm{diag}(\exp(i2\pi a), 1_{n-1})v$. □

3. Determinant rank

Let A be a unital C^*-algebra. Consider the homomorphism

$$i_A^{(m,n)} : U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

for integers $n \geq m \geq 1$.

Proposition 3.1. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Then*

$$i_A^{(m,n)} : U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

is surjective for $n \geq m \geq 1$.

Proof. It suffices to show that $i_A^{(1,n)}$ is surjective. Let $u \in U_0(\mathrm{M}_n(A))$. It follows from Corollary 2.12 that $u = \mathrm{diag}(\exp(i2\pi a), 1_{n-1})v$ for some $a \in A_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_n(A))$. Then $i_A^{(1,n)}([\exp(i2\pi a)]) = [u]$. □

Lemma 3.2. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Assume $u \in U_0(\mathrm{M}_m(A))$.*

(1) *If $\Delta^n(\mathrm{diag}(u(t), 1_{n-m}) \in \overline{\Delta^n(LU_0^n(A))}$ for some $n > m$, where $\{u(t) : t \in [0,1]\}$ is a piecewise smooth path with $u(0) = 1_m$ and $u(1) = u$, then, for any $\epsilon > 0$, there exist $a \in \mathrm{M}_m(A)_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $b \in \mathrm{M}_m(A)_{\mathrm{sa}}$, $v \in CU(\mathrm{M}_m(A))$ and $w \in LU_0^n(A)$ such that*

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \textit{and} \quad \tau(b) = \Delta_\tau^n(w(t)) \ \textit{for all } \tau \in T(A). \tag{3-1}$$

(2) *If $\Delta^m(u(t)) \in \overline{\rho_A(K_0(A))}$ for some $u \in PU_0^m(A)$ with $u(1) = u$, then, for any $\epsilon > 0$, there exist $a \in \mathrm{M}_m(A)_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $b \in \mathrm{M}_m(A)_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_m(A))$ such that*

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \textit{and} \quad \hat{b} \in \rho_A(K_0(A)), \tag{3-2}$$

where $\hat{b}(\tau) = \tau(b)$ for all $\tau \in T(A)$.

Proof. Let $\epsilon > 0$. For (1), there is a $w \in LU_0^n(A)$ such that

$$\sup\{|\Delta_\tau^n(u(t)) - \Delta_\tau^n(w(t))| : \tau \in T(A)\} < \epsilon/3\pi. \tag{3-3}$$

There is an $a_1 \in M_m(A)_{\text{sa}}$ by Corollary 2.12 such that

$$\tau(a_1) = \Delta_\tau^n(u(t)) - \Delta_\tau^n(w(t)) \quad \text{for all } \tau \in T(A). \tag{3-4}$$

Combining (3-3) with [Cuntz and Pedersen 1979] and the proof of [Th, Lemma 3.1], we can find $a \in \mathrm{M}_m(A)_{\text{sa}}$ such that $\tau(a) = \tau(a_1)$ for all $\tau \in T(A)$ and $\|a\| < \epsilon/2\pi$. There is also a $b \in A_{\text{sa}}$ such that $\tau(b) = -\Delta_\tau^n(w(t))$ for all $\tau \in T(A)$. Put

$$v(t) = \exp(-i2\pi bt)\exp(-i2\pi at)u(t) \quad \text{for } t \in [0,1] \tag{3-5}$$

and $v = v(1)$. Then $\Delta^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(A)$. Then $u = \exp(i2\pi a)\exp(i2\pi b)v$.

For (2), there are an integer $n \geq m$ and projections $p, q \in \mathrm{M}_n(A)$ such that (for a piecewise smooth path $\{u(t) : t \in [0,1]\}$ with $u(0) = 1_n$ and $u(1) = u$)

$$\|\Delta_\tau^m(u(t)) - \tau(p) + \tau(q)\| < \epsilon \quad \text{for all } \tau \in T(A). \tag{3-6}$$

Let $b \in \mathrm{M}_m(A)_{\text{sa}}$ such that $\tau(b) = \tau(p) - \tau(q)$ for all $\tau \in T(A)$ (see the proof above); there is an $a \in \mathrm{M}_m(A)_{\text{sa}}$ with $\|a\| < \epsilon$ such that

$$\tau(a) = \Delta_\tau^m(u(t)) - \tau(p) + \tau(q) \quad \text{for all } \tau \in T(A). \tag{3-7}$$

Let $v = u\exp(-i2\pi a)\exp(-i2\pi b)$ and $v(t) = u(t)\exp(-i2\pi at)\exp(-i2\pi bt)$. Then $\Delta_\tau^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(\mathrm{M}_m(A))$. □

Let A be a unital C^*-algebra. Let Dur A be defined as in Definition 1.1. It follows from Corollary 2.7 that if $T(A) = \varnothing$ then Dur $A = 1$.

Proposition 3.3. *Let A be a unital C^*-algebra. Then, for any integer $n \geq 1$,*

$$\mathrm{Dur}(\mathrm{M}_n(A)) \leq \left\lfloor \frac{\mathrm{Dur}\, A - 1}{n} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is the integer part of x.

Proof. Note that $n(\lfloor(\mathrm{Dur}\, A - 1)/n\rfloor + 1) \geq \mathrm{Dur}\, A$. □

Theorem 3.4. *Let A be a unital C^*-algebra, and $I \subset A$ a closed ideal of A such that the quotient map $\pi : A \to A/I$ induces the surjective map from $K_0(A)$ onto $K_0(A/I)$. Then* $\mathrm{Dur}(A/I) \leq \mathrm{Dur}\, A$.

Proof. Let $m = \mathrm{Dur}\, A$ and $n > m$. Let $u \in U_0(\mathrm{M}_m(A/I))$ be a unitary such that $\mathrm{diag}(u, 1_{n-m}) \in CU(\mathrm{M}_n(A/I))$. We will show that $u \in CU(\mathrm{M}_m(A/I))$.

Let $\epsilon > 0$. By Lemma 3.2, without loss of generality we may assume that there are $a_1,\ b_1 \in (\mathrm{M}_m(A/I))_{\mathrm{sa}}$ such that

$$u = \exp(i2\pi a_1)\exp(i2\pi b_1)v,$$
$$v \in CU(\mathrm{M}_m(A/I)),\quad \|a_1\| < \epsilon \quad\text{and}\quad \tau(b_1) = \tau(q_1) - \tau(q_2), \tag{3-8}$$

where $q_1, q_2 \in M_K(A/I)$ are projections for some large $K \geq m$, for all $\tau \in T(A/I)$. By the assumption, without loss of generality we may assume further that there are projections $p_1, p_2 \in \mathrm{M}_K(A)$ such that $\pi_*([p_1 - [p_2]) = [q_1] - [q_2]$, where $\pi_* : K_0(A) \to K_0(A/I)$ is induced by π. Let $b_2 \in (\mathrm{M}_m(A))_{\mathrm{sa}}$ such that $\tau(b_2) = \tau(p_1) - \tau(p_2)$ for all $\tau \in T(A)$. There exists an $a \in (\mathrm{M}_m(A))_{\mathrm{sa}}$ such that $\pi_m(a) = a_1$, where $\pi_m\colon \mathrm{M}_m(A) \to \mathrm{M}_m(A/I)$ is the map induced by π. Then, by (3-8),

$$\pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2))u^* \in CU(\mathrm{M}_m(A/I)). \tag{3-9}$$

Put $u_1 = \pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2))$. Let $w = \exp(i2\pi b_2)$. Then $\bar{\Delta}(w) = 0$. Since $m = \mathrm{Dur}\, A$, this implies that $w \in CU(M_m(A))$. It follows that $\pi_m(w) \in CU(\mathrm{M}_m(A/I))$, which implies by (3-9) that $\mathrm{dist}\big(u, CU(\mathrm{M}_m(A/I))\big) < \epsilon$. □

Theorem 3.5. *Let $A = \lim_{n\to\infty}(A_n, \phi_n)$ be a unital C^*-algebra, where each A_n is unital. Suppose that* Dur $A_n \leq r$ *for all n. Then* Dur $A \leq r$.

Proof. We write $\phi_{n_1,n_2} : A_{n_1} \to A_{n_2}$ for $\phi_{n_2} \circ \phi_{n_2-1} \circ \cdots \circ \phi_{n_1}$ and $\phi_{n_1,\infty} : A_{n_1} \to A$ for the map induced by the inductive limit system. Let $u \in U_0(\mathrm{M}_r(A))$ such that $u_1 = \mathrm{diag}(u, 1_{n-r}) \in CU(\mathrm{M}_n(A))$ for some $n > r$. Let $\epsilon > 0$. There is a $v \in DU(\mathrm{M}_n(A))$ such that

$$\|u_1 - v\| < \frac{\epsilon}{8n}. \tag{3-10}$$

Write $v = \prod_{j=1}^{K} v_j$, where $v_j = x_j y_j x_j^* y_j$ and $x_j, y_j \in U_0(\mathrm{M}_n(A))$ for $j = 1, 2, \ldots, K$. Choose a large $N \geq 1$ such that there are $v' \in U_0(\mathrm{M}_r(A_N))$ and $x_j', y_j' \in U_0(\mathrm{M}_n(A_N))$ such that

$$\|u - \phi_{N,\infty}(u')\| < \frac{\epsilon}{8nK} \quad\text{and}\quad \|\phi_{N,\infty}(x_j') - x_j\| < \frac{\epsilon}{8nK} \tag{3-11}$$

for $j = 1, 2, \ldots, K$. Then we have by (3-10) and (3-11)

$$\left\|\phi_{N,\infty}(u_1') - \prod_{j=1}^{K} \phi_{N,\infty}(v_j')\right\| < \frac{\epsilon}{4n}, \tag{3-12}$$

for $j = 1, 2, \ldots, K$, where $u_1' = \mathrm{diag}(u', 1_{n-r})$ and $v_j' = x_j' y_j' (x_j')^* (y_j')^*$. Then (3-12) implies that there is an $N_1 > N$ such that

$$\left\|\phi_{N,N_1}(u_1') - \prod_{j=1}^{K} \phi_{N,N_1}(v_j')\right\| < \frac{\epsilon}{2n}. \tag{3-13}$$

Put $U=\phi_{N,N_1}(u')$, $U_1=\mathrm{diag}(U,1_{n-r})$ and $w_j=\phi_{N,N_1}(v'_j)$, $j=1,2,\dots,K$. Note that $\phi_{N_1,\infty}(U)=\phi_{N,\infty}(u')$. There is an $a\in(\mathrm{M}_n(A_{N_1}))_{\mathrm{sa}}$ (by (3-13)) such that

$$U_1=\exp(i2\pi a)\prod_{j=1}^{K} w_j \quad\text{and}\quad \|a\|<2\arcsin\frac{\epsilon}{8n}. \tag{3-14}$$

There is a $b\in(\mathrm{M}_r(A_{N_1}))_{\mathrm{sa}}$ such that

$$\tau(b)=\tau(a)\quad\text{for all }\ \tau\in T(A)\quad\text{and}\quad \|b\|<2n\arcsin\frac{\epsilon}{8n}. \tag{3-15}$$

Put $W=\mathrm{diag}(U\exp(-i2\pi b),1_{n-r})$; then $W\in CU(\mathrm{M}_n(A_{N_1}))$. Since $\mathrm{Dur}\,A_{N_1}\le r$, we conclude that $U\exp(-i2\pi b)\in CU(\mathrm{M}_r(A_{N_1}))$. It follows that

$$\phi_{N_1,\infty}(U\exp(-i2\pi b))\in CU(\mathrm{M}_r(A)).$$

However, by (3-10), (3-11), (3-15),

$$\begin{aligned}\|u-\phi_{N_1,\infty}(U\exp(-i2\pi b))\|&\\ &\le\|u-\phi_{N,\infty}(u')\|+\|\phi_{N_1,\infty}(U)-\phi_{N_1,\infty}(U\exp(-i2\pi b))\|\\ &<\frac{\epsilon}{8nK}+\|1-\exp(-i2\pi\phi_{N_1,\infty}(b))\|<\frac{\epsilon}{8nK}+\epsilon/4<\epsilon.\end{aligned}$$

Therefore, $\mathrm{Dur}\,A\le r$. □

Proposition 3.6. *Let A be a unital C*-algebra with $T(A)\ne\varnothing$. Let $a\in A_{\mathrm{sa}}$ and put $\hat a(\tau)=\tau(a)$ for all $\tau\in T(A)$.*

(1) *If* $\exp(2\pi i a)\in CU(A)$, *then* $\hat a\in\overline{\rho_A(K_0(A))}$.

(2) *If $u\in U_0(A)$ and for some piecewise smooth path $\{u(t):t\in[0,1]\}$ with $u(0)=1$ and $u(1)=u$, $\Delta^1(u(t))\in\rho_A^k(K_0(A))$ for some $k\ge1$, then* $\mathrm{diag}(u,1_{k-1})\in CU(\mathrm{M}_k(A))$.

(3) *If* $\overline{\rho_A^1(K_0(A))}=\overline{\rho_A(K_0(A))}$, *then* $\mathrm{Dur}\,A=1$.

Proof. Part (1) follows from [Th].

(2) By applying Corollary 2.12, there exists a $v\in CU(A)$ such that

$$u=\exp(i2\pi a)v\qquad\text{and}\qquad\tau(a)=\Delta^1_\tau(u(t))\ \text{ for all }\tau\in T(A).$$

So for any $\epsilon\in(0,1)$, there are projections $p_1,\dots,p_{m_1},q_1,\dots,q_{m_2}\in\mathrm{M}_k(A)$ such that

$$\sup\left\{\left|\sum_{j=1}^{m_1}\tau(p_j)-\sum_{j=1}^{m_2}\tau(q_j)-\tau(a)\right|:\tau\in T(A)\right\}<\frac{\arcsin(\epsilon/4)}{\pi}. \tag{3-16}$$

Set $b=\sum_{j=1}^{m_1} p_j - \sum_{j=1}^{m_2} q_j$ and $a_0=\mathrm{diag}(a,\overbrace{0,0,\dots,0}^{(k-1)})$. Then $a_0, b\in \mathrm{M}_k(A)_{\mathrm{sa}}$ and

$$|\tau(a_0)-\tau(b)|<\frac{\arcsin(\epsilon/4)}{k\pi}\quad \text{for all } \tau\in T(\mathrm{M}_k(A))$$

by (3-16). Thus, by the proof of [Th, Lemma 3.1], we have

$$\inf\{\|a_0-b-x\| \mid x\in(\mathrm{M}_k(A))_0\} = \sup\{|\tau(a_0-b)| \mid \tau\in T(\mathrm{M}_k(A))\}\le \frac{\arcsin(\epsilon/4)}{k\pi}.$$

Choose $x_0\in(\mathrm{M}_k(A))_0$ such that $\|a_0-b-x_0\|<2\arcsin(\epsilon/4)/k\pi$. Put $y_0=a_0-b-x_0$. Then $\|y_0\|\le 2\arcsin(\epsilon/4)/k\pi$. Put $u_1=\mathrm{diag}(u,1_{k-1})\exp(-i2\pi y_0)$. Define

$$w(t)=\mathrm{diag}(u(t),1_{k-1})\exp(-i2\pi y_0 t)\prod_{j=1}^{m_1}\exp(-i2\pi p_j t)\prod_{j=1}^{m_2}\exp(i2\pi q_j t)$$

for $t\in[0,1]$. Then $w(0)=1$, $w(1)=u(1)\exp(-i2\pi y_0)=u_1$ and, moreover,

$$\begin{aligned}\Delta_\tau^k(w(t)) &= \tau(a)-\tau(y_0)-\Big(\sum_{j=1}^{m_1}\tau(p_j)-\sum_{j=1}^{m_2}\tau(q_j)\Big)\\ &= \tau(a)-\tau(a_0)+\tau(b)-\tau(x_0)-\tau(b)\\ &= \tau(a)-\tau(a_0)=0 \qquad\qquad \text{for all } \tau\in T(A).\end{aligned}$$

It follows that $w(1)=u_1\in CU(\mathrm{M}_k(A))$. Then

$$\|\mathrm{diag}(u,1_{k-1})-u_1\| = \|\exp(i2\pi y_0)-1_k\|<\epsilon.$$

(3) Let $u\in U_0(A)$ such that $\mathrm{diag}(u,1_{n-1})\in CU(\mathrm{M}_n(A))$. Let $u(t)$ be a piecewise smooth path with $u(0)=1$ and $u(1)=u$. Then

$$\Delta^1(u(t))\in\overline{\rho_A(K_0(A))}=\overline{\rho_A^1(K_0(A))}.$$

By Part (2), $u\in CU(A)$. This implies that Dur $A=1$. ☐

Proposition 3.7. *Let X be a compact metric space. Then* $\mathrm{Dur}\big(\mathrm{M}_n(C(X))\big)=1$ *for all $n\ge 1$.*

Proof. By Proposition 3.3, it suffices to consider the case $A=C(X)$. One has

$$\rho_A^1(K_0(A))=C(X,\mathbb{Z})=\rho_A(K_0(A)).$$

It follows from Proposition 3.6(3) that Dur $A=1$. ☐

Combining Theorem 3.5 with Proposition 3.7, we have:

Corollary 3.8. *Let* $A = \lim_{n\to\infty}(A_n, \phi_n)$, *where* $A_m = \bigoplus_{j=1}^{m(n)} \mathrm{M}_{k(n,j)}(X_{n,j})$ *and each* $X_{n,j}$ *is a compact metric space. Then* Dur $A = 1$.

Theorem 3.9. *Let* A *be a unital* C^**-algebra with real rank zero. Then* $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ *and* Dur $A = 1$.

Proof. By Corollary 2.7, we may assume that $T(A) \neq \varnothing$. Since A is of real rank zero, by [Zhang 1990, Theorem 3.3], for any $n \geq 2$ and any nonzero projection $p \in \mathrm{M}_n(A)$, there are projections $p_1, \dots, p_n \in A$ such that $p \sim \mathrm{diag}(p_1, \dots, p_n)$ in $\mathrm{M}_n(A)$. Thus, $\tau(p) = \sum_{j=1}^n \tau(p_j)$ for all $\tau \in T(A)$ and, consequently, $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$. It follows from Proposition 3.6(3) that Dur $A = 1$. □

Theorem 3.10. *Let* A *be a unital* C^**-algebra with* $T(A) \neq \varnothing$. *If* $\mathrm{csr}(C(S^1, A)) \leq n+1$ *for some* $n \geq 1$, *then* Dur $A \leq n$.

Proof. Let $u \in U_0(\mathrm{M}_n(A))$ such that $\mathrm{diag}(u, 1_k) \in CU(\mathrm{M}_{n+k}(A))$ for some integer $k \geq 1$. Let $\{u(t) : t \in [0,1]\}$ be a piecewise smooth path with $u(0) = 1_n$ and $u(1) = u$. By [Th], $\Delta^{n+k}(\mathrm{diag}(u(t), 1_k)) \in \overline{\Delta^{n+k}(LU_0^{n+k}(A))}$. It follows from Lemma 3.2(1) that, for any $\epsilon > 0$, there are $a, b \in \mathrm{M}_n(A)_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_n(A))$ with $\|a\| < 2\arcsin(\epsilon/4)/\pi$ such that

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \text{and} \quad \tau(b) = \Delta_\tau^{n+k}(w(t)) \ \text{ for all } \tau \in T(A), \tag{3-17}$$

where $w \in LU_0^{n+k}(A)$. Since $\mathrm{csr}(C(S^1, A)) \leq n+1$, by Proposition 2.6 of [Rieffel 1987] there is a $w_1 \in LU_0^n(A)$ such that $\mathrm{diag}(w_1, 1_{n+k})$ is homotopy to w. In particular, $\Delta_\tau^n(w_1(t)) = \Delta_\tau^{n+k}(w(t))$ for all $\tau \in T(A)$. Consider the piecewise smooth path

$$U(t) = \exp(-i2\pi at)\exp(i2\pi bt)w_1^*(t), \quad t \in [0,1].$$

Then $U(0) = 1_n$ and $U(1) = \exp(i2\pi b)$. We compute that $\Delta_\tau^n(U(t)) = 0$ for all $\tau \in T(A)$. It follows by [Th, Lemma 3.1] that $\exp(i2\pi b) \in CU(\mathrm{M}_n(A))$. By (3-17),

$$[u] = [\exp(i2\pi a)] \quad \text{in } U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A)),$$

Therefore $\mathrm{dist}\big(u, CU(\mathrm{M}_n(A))\big) \leq \|\exp(i2\pi a) - 1_n\| < \epsilon$. □

Corollary 3.11. *Let* A *be a unital* C^**-algebra of stable rank one. Then* Dur $A = 1$.

Proof. This follows from the inequality $\mathrm{csr}(C(S^1, A)) \leq \mathrm{tsr}\, A + 1$ (see [Rieffel 1983, Corollary 8.6]) and Theorem 3.10. □

We end this section with the following:

Proposition 3.12. *Let* A *be a unital* C^**-algebra. Suppose that there is a projection* $p \in \mathrm{M}_2(A)$ *such that, for any* $x \in K_0(A)$ *with* $\rho_A(x) = \rho_A([p])$, *no unitary in* $U(\widetilde{C})$ *represents* x, *where* $C = C_0((0,1), A)$. *Then* Dur $A > 1$.

Proof. There exists an $a \in A_+$ such that $\tau(a) = \rho_A([p])(\tau)$ for all $\tau \in T(A)$. Put $u = \exp(i2\pi a)$ and $v = \mathrm{diag}(u, 1)$. Then it follows from Proposition 3.6(2) that $v \in CU(\mathrm{M}_2(A))$. This implies that $i_A^{(1,2)}([u]) = 0$. Now we show that $u \notin CU(A)$. Let

$$w(t) = \exp(2i(1-t)\pi a) \quad \text{for all } t \in [0, 1].$$

Then $w(0) = u$ and $w(1) = 1_A$. If $u \in CU(A)$, then, by [Th, Lemma 3.1], there is a continuous and piecewise smooth path of unitaries $\xi \in \widetilde{C}$, where $C = C_0((0, 1), A)$, such that

$$\Delta_\tau(\xi(t)) = \tau(p) \quad \text{for all } \tau \in T(A). \tag{3-18}$$

The Bott map shows that the unitary ξ is homotopic to a projection loop which corresponds to some $x \in K_0(A)$ with $\rho_A(x) = \rho_A([p])$, which contradicts the assumption. □

4. Simple C^*-algebras

Let us begin with the following:

Theorem 4.1. *Let A be a unital infinite-dimensional simple C^*-algebra of real rank zero with $T(A) \neq \varnothing$. Then*

$$\overline{\rho_A^1(K_0(A))} = \mathrm{Aff}(T(A)) \quad \textit{and} \quad U_0(A) = CU(A).$$

Proof. Let $p \in A$ be a nonzero projection, let $\lambda = n/m$ with $n, m \in \mathbb{N}$ and let $\epsilon > 0$. Then by Zhang's half theorem (see [Lin 2010a, Lemma 9.4]), there is a projection $e \in A$ such that $\max_{\tau \in T(A)} |\tau(p) - n\tau(e)| < n\epsilon/m$. Thus,

$$\max_{\tau \in T(A)} |\lambda\tau(p) - m\tau(e)| < \epsilon,$$

and consequently $r\rho_A(p) \in \overline{\rho_A^1(K_0(A))}$ for all $r \in \mathbb{R}$.

Let $a \in A_{\mathrm{sa}}$. Since A has real rank zero, a is a limit of the form $\sum_{j=1}^k \lambda_j p_j$, where $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in A and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$. Therefore $\hat{a} \in \overline{\rho_A^1(K_0(A))}$ by the above argument, where $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$. Since $\mathrm{Aff}(T(A)) = \{\hat{a} \mid a \in A_{\mathrm{sa}}\}$ by [Lin 2007, Theorem 9.3], it follows from Theorem 3.9 that

$$\mathrm{Aff}(T(A)) \subset \overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))} \subset \mathrm{Aff}(T(A)),$$

that is, $\mathrm{Aff}(T(A)) = \overline{\rho_A^1(K_0(A))}$.

Note that

$$\rho_A^1(K_0(A)) \subset \Delta^1(LU_0^1(A)) \subset \rho_A(K_0(A)) = \rho_A^1(K_0(A)).$$

So $\overline{\Delta^1(LU_0^1(A))} = \overline{\rho_A^1(K_0(A))} = \mathrm{Aff}(T(A))$. Thus, $\overline{\Delta^1} = 0$ (see Definition 2.11), and the assertion follows. □

For unital simple C^*-algebras, we have:

Theorem 4.2. *Let A be a unital infinite-dimensional simple C^*-algebra. Then* $\mathrm{Dur}\, A = 1$ *if one of the following holds*:

(1) *A is not stably finite.*

(2) *A has stable rank one.*

(3) *A has real rank zero.*

(4) *A is projectionless and $\rho_A(K_0(A)) = \mathbb{Z}$ (with $\rho_A([1_A]) = 1$).*

(5) *A has property* (SP) *and has a unique tracial state.*

Proof. (1) In this case, there is a nonunitary isometry $u \in \mathrm{M}_k(A)$ for some $k \geq 2$. Since $\mathrm{M}_k(A)$ is also simple, every tracial state on $\mathrm{M}_k(A)$ is faithful if $T(A) \neq \varnothing$. This implies that $T(A) = \varnothing$. The assertion follows from Corollary 2.7.

(2) This follows from Corollary 3.11.

(3) This follows from Theorem 4.1 or Theorem 3.9.

(4) By the assumption, we have $\rho_A^1(K_0(A)) = \rho_A(K_0(A)) = \mathbb{Z}$. By Proposition 3.6, $\mathrm{Dur}\, A = 1$.

(5) Let $\epsilon > 0$ and let $\tau \in T(A)$ be the unique tracial state. Let $k \geq 1$ be an integer and $p \in \mathrm{M}_k(A)$ a projection. Since A has (SP), there is a nonzero projection $q \in A$ such that $0 < \tau(q) < \frac{1}{2}\epsilon$ (see, for example, [Lin 2001, Lemma 3.5.7]). Then, there is an integer $m \geq 1$ such that $|m\tau(q) - \tau(p)| < \epsilon$. This implies that $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$. Therefore, by Proposition 3.6, $\mathrm{Dur}\, A = 1$. □

For a unital simple C^*-algebra A, Theorem 4.2 indicates that the only case when $\mathrm{Dur}\, A$ might not be 1 is when A is stably finite and has stable rank greater than 1. The only example of this that we know so far is given by Villadsen [1999].

However, we have the following:

Theorem 4.3. *For each integer $n \geq 1$, there is a unital simple* AH*-algebra A with* $\mathrm{tsr}\, A = n$ *such that* $\mathrm{Dur}\, A = 1$.

Proof. Fix an integer $n > 1$. Let $A = \lim_{k\to\infty}(A_k, \phi_k)$ be the unital simple AH-algebra with $\mathrm{tsr}\, A = n$ constructed by Villadsen [1999]. Then $A_1 = C(D^n)$. The connecting maps ϕ_k are "diagonal" maps. More precisely, $\phi_k(f) = \sum_{j=1}^{n(k)} f(\gamma_{k,j}) \otimes p_{k,j}$ for all $f \in A_k$, where $p_{k,1}$ is a trivial rank-1 projection, $A_{k+1} = \phi_k(\mathrm{id}_{A_k})\mathrm{M}_{(r(k)}(C(X_k))\phi_k(\mathrm{id}_{A_k})$ (for some large $r(n)$) for some spaces X_k, and $\gamma_{k,j} : X_{k+1} \to X_k$ is a continuous map (these are π_{i+1}^1 and some point evaluations as denoted in [Villadsen 1999, p. 1092]). Clearly A_1 contains a rank-1 projection. Suppose that A_k, as a unital hereditary C^*-subalgebra of

$M_{r(k)}(C(X_k))$, contains a rank-1 projection e_k (of $M_{r(k)}(C(X_k))$). Then, since $(\mathrm{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \leq \phi_k(\mathrm{id}_{A_k})$, we have $(\mathrm{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \in A_{k+1}$. Then $e_k \circ \gamma_{k,1} \otimes p_{k,1} \in A_{k+1}$, which is a rank-1 projection.

The above shows every A_k contains a rank-1 projection.

Now let $p \in M_m(A)$ be a projection. We may assume that there is a projection $q \in M_m(A_{k_0+1})$ such that $\phi_{k_0+1,\infty}(q) = p$. Let $e_{k_0} \in A_{k_0+1}$ be a rank-1 projection. Then there is an integer $L \geq 1$ such that $L\tau(e_{k_0}) = \tau(q)$ for all $\tau \in T(A_{k_0+1})$. It follows that

$$L\tau(\phi_{k_0+1,\infty}(e_{k_0})) = \tau(p) \quad \text{for all } \tau \in T(A).$$

So $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ and hence $\mathrm{Dur}\, A = 1$ by Proposition 3.6. □

Theorem 4.4. *Let A be a unital simple* AH-*algebra with* (SP) *property. Then* $\mathrm{Dur}\, A = 1$.

Proof. By Proposition 3.1, it suffices to show that $i_A^{(1,n)}$ is injective, and by Proposition 3.6 it suffices to show that $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$.

Let p be a projection in $M_n(A)$. Since A is simple, $\inf\{\tau(p) \mid \tau \in T(A)\} = d > 0$. Given a positive number $\epsilon < \min\{\frac{1}{2}, \frac{1}{2}d\}$. Choose an integer $K \geq 1$ such that $1/K < \frac{1}{2}\epsilon$. Since A is a simple unital C^*-algebra with (SP), it follows from [Lin 2001, Lemma 3.5.7] that there are mutually orthogonal and mutually equivalent nonzero projections $p_1, p_2, \dots, p_K \in A$ such that $\sum_{j=1}^K p_j \leq p$. We compute that

$$\tau(p_1) < \epsilon/2 \quad \text{and} \quad \tau(p_1) < d/K \quad \text{for all } \tau \in T(A). \tag{4-1}$$

Since A is simple and unital, there are $x_1, x_2, \dots, x_N \in A$ such that

$$\sum_{j=1}^{N} x_j^* p_1 x_j = 1_A.$$

Let $A = \varinjlim(A_m, \phi_m)$, where $A_m = \bigoplus_{i=1}^{r(m)} P_{m,j} M_{R(m,j)}(C(X_{m,j}))P_{n,j}$ for each m, $X_{n,j}$ is a connected finite CW-complex and $P_{m,j} \in M_{R(m,j)}(C(X_{m,j}))$ is a projection. Without loss of generality, we may assume that, there are projections $p_1' \in A_m$, $p' \in M_n(A_m)$ and elements $y_1, y_2, \dots, y_N \in A_m$ such that $\phi_{m,\infty}(p_1') = p_1$, $\phi_{m,\infty}(y_j) = x_j$, $(\phi_{m,\infty} \otimes \mathrm{id}_{M_n})(p') = p$ and

$$\left\| \sum_{j=1}^{N} y_j^* p_1' y_j - 1_A \right\| < 1. \tag{4-2}$$

Write p_1' and p' as

$$p_1' = p_{1,1}' \oplus p_{1,2}' \oplus \cdots \oplus p_{1,r(m)}' \quad \text{and} \quad p' = q_1 \oplus q_2 \oplus \cdots \oplus q_{r(m)},$$

where, for each $j = 1, \dots, r(m)$, $p'_{1,j} \in P_{m,j} \mathrm{M}_{R(m,j)}(C(X_{m,j}))P_{m,j}$ and $q_j \in \mathrm{M}_n(P_{m,j}\mathrm{M}_{R(m,j)}(C(X_{m,j}))P_{m,j})$ are projections. Note that (4-2) implies that $p'_{1,j} \neq 0$ for $j = 1, 2, \dots, r(m)$. Define

$$r_{1,j} = \operatorname{rank} p'_{1,j} \quad \text{and} \quad r_j = \operatorname{rank} q_j \qquad \text{for } j = 1, 2, \dots, r(m).$$

Then $r_j = l_j r_{1,j} + s_j$, where $l_j, s_j \geq 0$ are integers and $s_j < r_{1,j}$. It follows that

$$\left| t(p') - \sum_{j=1}^{r(m)} l_j t(p'_{1,j}) \right| < t(p'_1) \quad \text{for all } t \in T(A_m). \tag{4-3}$$

Define $q_{1,j} = \phi_{m,\infty}(p'_{1,j})$ for $j = 1, \dots, r(m)$. Then each $q_{1,j}$ is a projection in A. Note that for each $\tau \in T(A)$, $\tau \circ \phi_{m,\infty}$ is a tracial state on A_m. So, by (4-3),

$$\left| \tau(p) - \sum_{j=1}^{r(m)} l_j \tau(q_{1,j}) \right| < \tau(p_1) < \epsilon \quad \text{for all } \tau \in T(A).$$

This implies that $\overline{\rho^1_A(K_0(A))} = \overline{\rho_A(K_0(A))}$. □

Lemma 4.5. *Let A be a unital simple C^*-algebra with $T(A) \neq \varnothing$, and let $a \in A_+ \setminus \{0\}$. Then, for any $b \in A_{\mathrm{sa}}$, there is a $c \in \mathrm{Her}\, a$ such that $b - c \in A_0$.*

Proof. Since A is simple and unital, there are $x_1, x_2, \dots, x_m \in A$ such that $\sum_{j=1}^m x_j^* a x_j = 1_A$. Set $c = \sum_{j=1}^m a^{1/2} x_j b x_j^* a^{1/2}$. Then $c \in \mathrm{Her}\, a$ and

$$\tau(c) = \sum_{j=1}^m \tau(a^{1/2} x_j b x_j^* a^{1/2}) = \sum_{j=1}^m \tau(b x_j^* a x_j) = \tau(b) \quad \text{for all } \tau \in T(A).$$

It follows from Lemma 2.6(2) that $b - c \in A_0$. □

A special case of the following can be found in [Lin 2010b, Theorem 3.4]:

Theorem 4.6. *Let A be a unital simple C^*-algebra and let $e \in A$ be a nonzero projection. Consider the map $U_0(eAe)/CU(eAe) \to U_0(A)/CU(A)$ given by $i_e([u]) = [u + (1-e)]$. This map is always surjective, and is also injective if* $\mathrm{tsr}\, A = 1$.

Proof. To see that i_e is surjective, let $u \in U_0(A)$. Write $u = \prod_{k=1}^n \exp(i a_k)$ for $a_k \in A_{\mathrm{sa}}$, $k = 1, 2, \dots, n$. By Lemma 4.5, there are $b_1, \dots, b_n \in eAe$ such that $b_k - a_k \in A_0$. Put $w = e \prod_{k=1}^n \exp(i b_k)$. Then $w \in U_0(eAe)$. Set $v = w + (1-e)$. Then $v = \prod_{k=1}^n \exp(i b_k)$. Thus, by Lemma 2.6(1),

$$i_e([w]) = [v] = \sum_{k=1}^n [\exp(i b_k)] = \sum_{k=1}^n [\exp(i a_k)] = [u] \quad \text{in } U_0(A)/CU(A),$$

that is, i_e is surjective.

To see that i_e is injective when A has stable rank one, let $w \in U_0(eAe)$ such that $w + (1-e) \in CU(A)$. Since A is simple, there are $z_1, \dots, z_n \in A$ such that $1 - e = \sum_{j=1}^n z_j^* e z_j$. Set

$$X = \begin{bmatrix} ez_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ez_n & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{M}_n(A).$$

Then

$$\text{(4-4)} \qquad \mathrm{diag}(1-e, \overbrace{0, \dots, 0}^{n-1}) = X^*X, \quad XX^* \le \mathrm{diag}(\overbrace{e, e, \dots, e}^{n}).$$

Equation (4-4) indicates that $[1-e] \le n[e]$ in $K_0(A)$. Since tsr $A = 1$, we can find a projection $p \in \mathrm{M}_s(A)$ for some $s \ge n$ and a unitary $U \in \mathrm{M}_{s+1}(A)$ such that

$$\text{(4-5)} \qquad \mathrm{diag}(\overbrace{e, \dots, e}^{n}, \overbrace{0, \dots, 0}^{r}) = U \mathrm{diag}(1-e, p) U^*,$$

where $r = s - n + 1$. Write $v = w + (1-e)$ as $v = \left[\begin{smallmatrix} w & \\ & 1-e \end{smallmatrix}\right]$, and set

$$W = \begin{bmatrix} e & \\ & U \end{bmatrix} \quad \text{and} \quad Q = \mathrm{diag}(\overbrace{e, \dots, e}^{n}, \overbrace{0, \dots, 0}^{r}).$$

Then $W \mathrm{diag}(e, 1-e, p) \mathrm{M}_{s+2}(A) \,\mathrm{diag}(e, 1-e, p) W^* \subset \mathrm{M}_{n+1}(eAe) \oplus 0$ and

$$\text{(4-6)} \qquad W \begin{bmatrix} v & \\ & p \end{bmatrix} W^* = \begin{bmatrix} w & \\ & U \mathrm{diag}(1-e, p)\, U^* \end{bmatrix} = \mathrm{diag}(w, Q),$$

by (4-5). Note that $\mathrm{diag}(v, p) \in CU(\mathrm{diag}(e, 1-e, p) \mathrm{M}_{s+2}(A) \,\mathrm{diag}(e, 1-e, p))$. So, by (4-6),

$$\mathrm{diag}(w, \overbrace{e, \dots, e}^{n}) \in CU(\mathrm{M}_{n+1}(eAe)).$$

Since $\mathrm{tsr}(eAe) = 1$, it follows from Theorem 4.2(2) that $w \in CU(eAe)$. □

Lemma 4.7. *Let C be a nonunital C^*-algebra and $B = \widetilde{C}$. Assume $u_1, u_2, \dots, u_n \in U(\mathrm{M}_k(B))$ for some $k \ge 2$. Then, there are unitaries $u_1', u_2', \dots, u_n' \in \mathrm{M}_k(\widetilde{C})$ with $\pi_k(u_j') = 1_k$ and $w, z_j, \bar{u}_j \in U(\mathrm{M}_k(\mathbb{C}))$ for $j = 1, \dots, n$ such that*

$$\prod_{j=1}^n u_j = \left(\prod_{j=1}^n u_j' \right) w, \quad \textit{with } u_j' = z_j^* u_j \bar{u}_j^* z_j \textit{ for } j = 1, \dots, n,$$

$$w = \pi_k \prod_{j=1}^n u_j,$$

where $\pi(x+\lambda)=\lambda$ for all $x\in C$ and $\lambda\in\mathbb{C}$ and π_k is the induced homomorphism of π on $\mathrm{M}_k(B)$.

Moreover, if $u_j\in U_0(\mathrm{M}_k(B))$, then we may assume that each $u_j'\in U_0(\widetilde{\mathrm{M}_k(C)})$ for $j=1,\dots,n$.

Proof. Put $\bar u_j=\pi_k(u_j)\in U(\mathrm{M}_k(\mathbb{C}))$. If $n=2$, then

$$\begin{aligned}u_1u_2&=u_1\bar u_1^*(\bar u_1u_2\bar u_1^*)(\bar u_1\bar u_2^*\bar u_1^*)(\bar u_1\bar u_2\bar u_1^*\bar u_1)\\&=u_1\bar u_1^*(\bar u_1u_2\bar u_1^*)(\bar u_1\bar u_2^*\bar u_1^*)(\bar u_1\bar u_2).\end{aligned}$$

Put $u_1'=u_1\bar u_1^*$, $u_2'=\bar u_1u_2\bar u_1^*\bar u_1\bar u_2^*\bar u_1^*$, $w_1=\bar u_1\bar u_2$, $z_1=1_k$, $z_2=\bar u_1$. Then

$$\pi_k(u_1')=1_k,\quad \pi_k(u_2')=\pi_k(\bar u_1(u_2\bar u_2^*)\bar u_1^*)=1_k,\quad w_1=\pi_k(u_1u_2).$$

Thus the lemma holds if $n=2$. Suppose that the lemma holds for s. Then

$$u_1u_2\cdots u_su_{s+1}=(u_1'u_2'\cdots u_s')w_su_{s+1},$$

where $u_j'\in\mathrm{M}_k(\widetilde C)$ are unitaries with $\pi_k(u_j')=1_k$ and $u_j'=z_j^*u_j\bar u_j^*z_j$, where $z_j,\bar u_j\in U(\mathrm{M}_k(\mathbb{C}))$, $j=1,\dots,s$ and $w_s=\pi_k\prod_{j=1}^s u_j$. It follows that

$$\prod_{j=1}^{s+1}u_j=\biggl(\prod_{j=1}^{s}u_j'\biggr)w_su_{s+1}w_s^*(w_s\bar u_{s+1}^*w_s^*)(w_s\bar u_{s+1}).$$

Put $u_{s+1}'=w_su_{s+1}w_s^*(w_s\bar u_{s+1}^*w_s^*)=w_s(u_{s+1}\bar u_{s+1}^*)w_s^*$, $z_{s+1}=w_s^*$ and $w_{s+1}=w_s\bar u_{s+1}$. Then

$$\begin{aligned}\pi_s(u_{s+1}')&=\pi_k(w_s)\pi(u_{s+1}\bar u_{s+1}^*)\pi_k(w_s^*)=1_k,\\ w_{s+1}=w_s\bar u_{s+1}&=\pi_k\biggl(\biggl(\prod_{j=1}^{s}u_j\biggr)u_{s+1}\biggr)=\pi_k\prod_{j=1}^{s+1}u_j.\end{aligned}$$

The first part of the lemma follows.

To see the second part, we first assume that $u_j=\exp(ia_j)$ for some $a_j\in(\mathrm{M}_k(B))_{\mathrm{sa}}$. Note that $\bar u_j=\exp(i\bar a_j)$, where $\bar a_j=\pi_k(a_j)\in(\mathrm{M}_k(\mathbb{C}))_{\mathrm{sa}}$, $j=1,\dots,n$. Consider the path $u_j'(t)=\exp(ita_j)\exp(-it\bar a_j)$ for $t\in[0,1]$. Note that, for each $t\in[0,1]$ and $j=1,\dots,n$,

$$\pi_k(\exp(ita_j)\exp(-it\bar a_j))=\exp(it\pi_k(a_j))\exp(-it\pi_k(a_j))=1_k.$$

It follows that $u_j'(t)\in\widetilde{\mathrm{M}_k(\mathbb{C})}$ for all $t\in[0,1]$ and $j=1,\dots,n$. The case that $u_j=\exp\bigl(\prod_{k=1}^{m_j}(ia_k)\bigr)$ follows from this and what has been proved. □

Lemma 4.8. *Let C be a nonunital C^*-algebra and $B = \widetilde{C}$. Suppose that $z = aba^*b^*$, where $a, b \in U_0(\mathrm{M}_k(B))$. Then $z = yw$, where $y \in CU(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $w \in CU(\mathrm{M}_k(\mathbb{C}))$. Moreover, if $u = \prod_{j=1}^n z_j$, where each $z_j \in CU(\mathrm{M}_k(B))$, then $u = yv$, where $y \in CU(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $v \in CU(\mathrm{M}_k(\mathbb{C}))$.*

Proof. Let $\bar{a} = \pi_k(a)$ and $\bar{b} = \pi_k(b)$. Then $\bar{a}, \bar{b} \in U(\mathrm{M}_k(\mathbb{C}))$. It follows from Lemma 4.7 that for $j = 1, 2$ there are $a_j, b_j \in U_0(\widetilde{\mathrm{M}_k(\mathbb{C})})$ with $\pi_k(a_j) = \pi_k(b_j) = 1_k$ and $z_j \in U(\mathrm{M}_k(\mathbb{C}))$ such that

$$ab = a_1 b_1 w_1, \quad a_1 = a\bar{a}^*, \quad b_1 = z_1^* b\bar{b}^* z_1, \quad w_1 = \bar{a}\bar{b}, \tag{4-7}$$

$$ba = b_2 a_2 w_2, \quad b_2 = b\bar{b}^*, \quad a_2 = z_2^* a\bar{a}^* z_2, \quad w_2 = \bar{b}\bar{a}. \tag{4-8}$$

Set $x_1 = w_1 w_2^* z_2^*$ and $x_2 = w_1 w_2^* z_1$. Then $x_1, x_2 \in U_0(\mathrm{M}_k(\mathbb{C}))$ and

$$\begin{aligned} aba^*b^* &= a_1 b_1 (w_1 w_2^* z_2^* (a\bar{a}^*) z_2 w_2 w_1^*)(w_1 w_2^* (b\bar{b}^*) w_2 w_1^*)) w_1 w_2^* \\ &= a_1 b_1 (x_1 a_1^* x_1^*)(x_2^* b_1^* x_2) w_1 w_2^* \end{aligned}$$

by (4-7) and (4-8).

Write $a_1 = \prod_{j=1}^{m_1} \exp(i y_{1j})$ and $b_1 = \prod_{k=1}^{m_2} \exp(i y_{2k})$, where $y_{1j}, y_{2k} \in (\mathrm{M}_k(C))_{\mathrm{sa}}$, $j = 1, \ldots, m_1$, $k = 1, \ldots, m_2$. Let

$$y_{1j} = y_{1j}^+ - y_{1j}^- \quad \text{and} \quad y_{2k} = y_{2k}^+ - y_{2k}^-,$$

with $y_{1j}^+, y_{1j}^-, y_{2k}^+, y_{2k}^- \in (\mathrm{M}_k(C))_+$ for $j = 1, \ldots, m_1$ and $k = 1, \ldots, m_2$. Set

$$\begin{aligned} c_1 &= \sum_{j=1}^{m_1} (y_{1j}^+ + x_1 y_{1j}^- x_1^*) + \sum_{k=1}^{m_2} (y_{2k}^+ + x_2 y_{2k}^- x_2^*), \\ c_2 &= \sum_{j=1}^{m_1} (y_{1j}^- + x_1 y_{1j}^+ x_1^*) + \sum_{k=1}^{m_2} (y_{2k}^- + x_2 y_{2k}^+ x_2^*), \\ d_1 &= \sum_{j=1}^{m_1} (y_{1j}^+ + y_{1j}^-) + \sum_{k=1}^{m_2} (y_{2k}^+ + y_{2k}^-), \\ d_2 &= \sum_{j=1}^{m_1} (y_{1j}^- + y_{1j}^+) + \sum_{k=1}^{m_2} (y_{2k}^- + y_{2k}^+). \end{aligned}$$

Then $c_1, c_2, d_1, d_2 \in (\mathrm{M}_2(C))_+$ and clearly $c_1 - d_1, c_2 - d_2 \in (\mathrm{M}_k(C))_0$. Therefore, $(c_1 - c_2) - (d_1 - d_2) \in (\mathrm{M}_k(C))_0$. Put $y = a_1 b_1 (x_1 a_1^* x_1^*)(x_2^* b_1^* x_2)$ and $w = w_1 w_2^*$. Then $y \in U_0(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $w = \bar{a}\bar{b}\bar{a}^*\bar{b}^* \in DU_k(\mathbb{C})$. Moreover, in $U_0(\widetilde{\mathrm{M}_k(C)})/CU(\widetilde{\mathrm{M}_k(C)})$,

$$[y] = [\exp(i(c_1 - c_2))] = [\exp(i(d_1 - d_2))] = [a_1][b_1][a_1^*][b_1^*] = 0.$$

This proves the first part of the lemma. The second part follows. □

Theorem 4.9. *Let A be an infinite-dimensional unital simple C^*-algebra with $T(A) \neq \varnothing$ such that there is an $m \geq 1$, for every hereditary C^*-subalgebra C, with* $\operatorname{Dur} \widetilde{C} \leq m$. *Then* $\operatorname{Dur} A = 1$.

Proof. Let $n \geq 1$. By Proposition 3.1, it suffices to show that $i_A^{(1,n)}$ is injective. Let $u \in U_0(A)$ with $\operatorname{diag}(u, 1_{n-1}) \in CU(\mathrm{M}_n(A))$. Since A is simple and infinite-dimensional, we can find nonzero mutually orthogonal positive elements $c_1, \dots, c_m \in A$ and $x_1, \dots, x_m \in A$ such that

$$x_j^* x_j = c_1 \quad \text{and} \quad x_j x_j^* = c_j, \qquad j = 2, 3, \dots, m.$$

Put $\operatorname{Her} c_1 = C$ and $B = \widetilde{C}$. Then $\operatorname{Her}(c_1 + c_2 + \cdots + c_m) \cong \mathrm{M}_m(C)$. Note that $\mathrm{M}_m(B)$ is not isomorphic to a subalgebra of $\mathrm{M}_m(A)$.

By Lemma 4.5, we may assume, without loss of generality, that $u = \exp(2\pi i b)$ for some $b \in C_{\mathrm{sa}}$. Then, by Proposition 3.6(1), $\hat{b} \in \overline{\rho_A(K_0(A))}$.

Since A is simple and C is σ-unital, it follows from [Brown 1977, Theorem 2.8] that there is a unitary element W in $M(A \otimes \mathcal{K})$ (the multiplier algebra of $A \otimes \mathcal{K}$) such that $W^*(C \otimes \mathcal{K})W = A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra consisting of all compact operators on l^2. Note that since A is a unital simple C^*-algebra, every tracial state τ on C is the normalization of a tracial state restricted on C. Therefore

$$\hat{b} \in \overline{\rho_A(K_0(A))} = \overline{\rho_B(K_0(C))} \subset \overline{\rho_B(K_0(B))}. \tag{4-9}$$

Viewing b in $B_{s.a}$, consider $v = \exp(i2\pi b) \in U_0(B)$ and $v(t) = \exp(i2\pi t b)$, $t \in [0, 1]$. Then (4-9) implies that $\Delta^1(v(t)) \in \overline{\rho_B(K_0(B))}$. By Lemma 3.2(2), for any $\epsilon > 0$, there are $a \in B_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $d \in B_{\mathrm{sa}}$ with $\hat{d} \in \rho_B(K_0(B))$ and $v_0 \in CU(B)$ such that

$$v = \exp(i2\pi a)\exp(i2\pi d)v_0. \tag{4-10}$$

Choose projections $p, q \in \mathrm{M}_n(B)$ for some $n > m$ such that for all $\tau \in T(B)$, $\tau(\operatorname{diag}(d, 0_{(n-1)\times(n-1)})) = \tau(p) - \tau(q)$. So $\operatorname{diag}(\exp(i2\pi d), 1_{n-1}) \in CU(\mathrm{M}_n(B))$ by Lemma 2.6(2). By assumption, $i_B^{(m,k)}$ is injective for all $k > m$. Therefore, we have $\operatorname{diag}(v, 1_{m-1}) \in CU(\mathrm{M}_m(B))$ by (4-10).

Let $\epsilon > 0$. Then there is a $v_1 \in DU(\mathrm{M}_m(B))$ such that $\|\operatorname{diag}(v, 1_{m-1}) - v_1\| < \frac{1}{2}\epsilon$. We may write $v_1 = \prod_{j=1}^r z_j$, where $z_j \in \mathrm{M}_m(B)$ is a commutator. It follows from Lemma 4.8 that there are $y \in CU(\widetilde{\mathrm{M}_m(C)})$ with $\pi_m(y) = 1_m$ and $w \in DU(\mathrm{M}_m(\mathbb{C}))$ such that $v_1 = yw$. Noting that $w = \pi_m(w) = \pi_m(v_1)$ and $\pi(v) = 1$, we have $\|1_m - w\| < \frac{1}{2}\epsilon$. Thus $\|\operatorname{diag}(v, 1_{m-1}) - y\| < \epsilon$. Set $v_0 = v - 1$ and $y_0 = y - 1_m$. Then

$$\begin{gathered} \operatorname{diag}(v_0, 0_{(m-1)\times(m-1)}), y_0 \in \mathrm{M}_m(C), \\ \|\operatorname{diag}(v_0, 0_{(m-1)\times(m-1)}) - y_0\| < \epsilon. \end{gathered} \tag{4-11}$$

By identifying $1_m + \mathrm{M}_m(C)$ with a unital C^*-subalgebra $1_A + \mathrm{Her}(c_1 + c_2 + \cdots + c_m)$ of A, we get that $\|\exp(i2\pi b) - y\| < \epsilon$ by (4-11). Since $y \in CU(\widetilde{\mathrm{M}_m(C)}) \subset CU(A)$ and hence $u \in CU(A)$, we have $\mathrm{Dur}\, A = 1$. ☐

Corollary 4.10. *Let A be a unital simple C^*-algebra. Suppose that there is an integer $K \geq 1$ such that* $\mathrm{csr}(C(S^1, C)) \leq K$ *for every hereditary C^*-subalgebra C. Then* $\mathrm{Dur}\, A = 1$.

Proof. It follows from Theorem 3.10 that $\mathrm{Dur}\, \widetilde{C} \leq \max\{K-1, 1\}$. Theorem 4.9 then applies. ☐

Definition 4.11. Let A be a C^*-algebra with $T(A) \neq \varnothing$. Define

$$\begin{aligned} D\big(\rho_A^1(K_0(A)), \rho_A(K_0(A))\big) &= \sup\big\{\mathrm{dist}\big(x, \rho_A^1(K_0(A))\big) \mid x \in \overline{\rho_A(K_0(A))}\big\} \\ &= \sup\big\{\mathrm{dist}\big(x, \rho_A^1(K_0(A))\big) \mid x \in \rho_A(K_0(A))\big\}. \end{aligned}$$

Theorem 4.12. *Let A be a unital simple C^*-algebra with $T(A) \neq \varnothing$ such that there is an $M > 0$ with $D\big(\rho_C^1(K_0(C)), \rho_C(K_0(C))\big) < M$ for all nonzero hereditary C^*-subalgebras C of A. Then* $\mathrm{Dur}\, A = 1$.

Proof. Let $u \in U_0(A)$ such that $\mathrm{diag}(u, 1_{n-1}) \in CU(\mathrm{M}_n(A))$. By Corollary 2.12, we may assume that $u = \exp(i2\pi a)$ for some $a \in A_{\mathrm{sa}}$. Then $\hat{a} \in \overline{\rho_A(K_0(A))}$ by Proposition 3.6(1).

Given $\epsilon > 0$, choose an integer $N \geq 1$ such that $M/N < \epsilon/2\pi$. There are mutually orthogonal nonzero positive elements $c_1, c_2, \ldots, c_N$ in A and elements $x_1, x_2, \ldots, x_N \in A$ such that

$$x_j^* x_j = c_1 \quad \text{and} \quad x_j x_j^* = c_j, \qquad j = 2, 3, \ldots, N. \tag{4-12}$$

Let $C = \mathrm{Her}\, c_1$ and $B = \widetilde{C}$. It follows from Lemma 4.5 that there is a $b \in C_{\mathrm{sa}}$ such that $a - b$ is in A_0, i.e., $\tau(a) = \tau(b)$ for all $\tau \in T(A)$. Therefore $[\exp(i2\pi a)] = [\exp(i2\pi b)]$ in $U_0(A)/CU(A)$ by Lemma 2.6(2).

Since A is a unital simple C^*-algebra and C is σ-unital, it follows from the proof of Theorem 4.9 that $\rho_C(b) \in \overline{\rho_C(K_0(C))}$. Therefore, by assumption, there are projections $p_1, p_2, \ldots, p_{k_1}, q_1, q_2, \ldots, q_{k_2} \in C$ such that

$$\sup_{\tau \in T(C)} \left| \tau(b) - \left(\sum_{i=1}^{k_1} \tau(p_i) - \sum_{j=1}^{k_2} \tau(q_j) \right) \right| < M.$$

Put $d = \sum_{i=1}^{k_1} p_i - \sum_{j=1}^{k_2} q_j$ and $f = b - d$. Then $\exp(i2\pi d) \in CU(A)$ by (2-3) and $[\exp(i2\pi f)] = [\exp(i2\pi b] \in U_0(A)/CU(A)$. Moreover, from

$$\inf\{\|f - x\| \mid x \in C_0\} = \sup\{|\tau(f)| \mid \tau \in T(C)\} < M$$

(see the proof of [Th, Lemma 3.1]), there are $f_0 \in C_0$ and $f_1 \in C_{\mathrm{sa}}$ with $\|f_1\| < M$ such that $f = f_1 + f_0$. By Lemma 2.6(1), $\exp(i2\pi f_0) \in CU(A)$. Since $f_1 \in C_{\mathrm{sa}}$, by (4-12), for $i = 1, 2, \ldots, N$ there are $g_i \in \mathrm{Her}\, c_i$ with

$$\|g_i\| \le \|f_1\|/N \quad \text{and} \quad \tau(g_i) = \tau(f_1/N) \ \text{ for all } \tau \in T(A). \tag{4-13}$$

Set $g = \sum_{i=1}^{n} g_i \in A$. Then, by (4-13),

$$\|\exp(i2\pi g) - 1_A\| < M/N < \epsilon \quad \text{and} \quad \overline{\Delta^1}(\exp(i2\pi f)\exp(-i2\pi g)) = 0. \tag{4-14}$$

So $\exp(i2\pi f)\exp(-i2\pi g) \in CU(A)$ and consequently $\mathrm{dist}(e^{i2\pi a}, CU(A)) < \epsilon$. □

Bruce Blackadar [1981] constructed three examples of unital simple separable nuclear C^*-algebras $A, A_\triangle, A_H$ with no nontrivial projections. By [Blackadar 1981, Theorem 4.9], $K_0(A) = \mathbb{Z}$ with a unique tracial state. It follows from Theorem 4.2(4) that $\mathrm{Dur}\, A = 1$. We turn to his examples $A_\triangle$ and A_H, which may have rich tracial spaces. It should be also noted that, as Blackadar showed, when $\triangle$ is not trivial (for example), $\mathrm{M}_2(A_\triangle)$ has a projection p with $\tau(p) = 1$ for all $\tau \in T(A_\triangle)$. In particular, this implies that

$$\overline{\rho^1_{A_\triangle}(K_0(A_\triangle))} \ne \bar{\rho}_{A_\triangle}(K_0(A_\triangle)).$$

However, $\mathrm{Dur}\, A_\triangle = 1$ as shown below. It follows that there is a unitary $u \in \widetilde{C}$, where $C = C_0((0,1), A)$, which represents a projection q with $\tau(q) = 1$ for all $\tau \in T(A_\triangle)$.

Proposition 4.13. *Let B be a unital AF-algebra and σ an automorphism of B. Put $M_\sigma = \{f \in C([0,1], B) \mid f(1) = \sigma(f(0))\}$. Then* $\mathrm{Dur}\, M_\sigma = 1$.

Proof. Clearly, $T(M_\sigma) \ne \varnothing$. From the exact sequence of C^*-algebras

$$0 \longrightarrow C_0((0,1), B) \longrightarrow M_\sigma \longrightarrow B \longrightarrow 0,$$

we obtain the exact sequence of C^*-algebras

$$0 \longrightarrow C_0((0,1) \times S^1, B) \longrightarrow C(S^1, M_\sigma) \longrightarrow C(S^1, B) \longrightarrow 0. \tag{4-15}$$

Since B is an AF-algebra, it follows from [Nistor 1986, Corollary 2.11] that

$$\mathrm{csr}(C(S^1, B)) = \mathrm{csr}(C(S^1)) = 2,$$
$$\mathrm{csr}(C_0((0,1) \times S^1, B)) = \mathrm{csr}(C_0((0,1) \times S^1)) = 2,$$

and consequently, applying [Nagy 1987, Lemma 2] to (4-15), we get

$$\mathrm{csr}(C(S^1, M_\sigma)) \le \max\{\mathrm{csr}(C(S^1, B)), \mathrm{csr}(C_0((0,1) \times S^1, B))\} \le 2.$$

Therefore $\mathrm{Dur}\, A = 1$ by Theorem 3.10. □

Corollary 4.14. $\mathrm{Dur}\, A_\triangle = 1$ *and* $\mathrm{Dur}\, A_H = 1$.

Proof. Both C^*-algebras are of the form $\lim_{n\to\infty} A_n$, where each $A_n \cong M_\sigma$, where M_σ is as in Proposition 4.13, and thus $\operatorname{Dur} A_n = 1$. By Theorem 3.5, $\operatorname{Dur} A_\triangle = 1$ and $\operatorname{Dur} A_H = 1$. □

5. C^*-algebras with Dur $A > 1$

In this section, we will present a unital C^*-algebra C such that $\operatorname{Dur} C = 2$. In particular, we will show that there are C^*-algebras which satisfy the condition described in Proposition 3.12.

5.1. We first list some standard facts from elementary topology. We will give a brief proof of each fact for the reader's convenience.

Fact 1. *Let*

$$B_d(0) = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \le d\right\}.$$

Let $f : B_d(0) \times S^1 \to S^3 = \mathrm{SU}(2)$ *be a continuous map which is not surjective. Then there is a homotopy*

$$F : B_d(0) \times S^1 \times [0,1] \to S^3 = \mathrm{SU}(2)$$

such that $F(x, e^{i\theta}, 0) = f(x, e^{i\theta})$, $F(x, e^{i\theta}, s) = f(x, e^{i\theta})$ *if* $\|x\| = d$ *(i.e., if* $x \in \partial B_d(0)$*) and* $g(x, e^{i\theta}) = F(x, e^{i\theta}, 1)$ *satisfies*

$$g(0, e^{i\theta}) = F(0, e^{i\theta}, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{SU}(2) = S^3.$$

Proof. Assume that f misses a point $z \in S^3 = \mathrm{SU}(2)$ and that $z \neq \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SU}(2)$. Then $S^3 \setminus \{z\}$ is homeomorphic to $D^3 = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$, with the identity matrix mapping to $(0, 0, 0)$. Without loss of generality, we can assume that f is a map from $B_d(0) \times S^1$ to D^3. Let $F : B_d(0) \times S^1 \times [0,1] \to D^3$ be defined by

$$F(x, e^{i\theta}, s) = f(x, e^{i\theta}) \max\{1 - s, \|x\|/d\},$$

which satisfies the condition. □

Fact 2. *Let* $f, g : S^4 \times S^1 \to \mathrm{SU}(n) \subset U(n) = U_n(\mathbb{C})$ *(where* $n \ge 2$*) be continuous maps. If* f *is homotopic to* g *in* $U(n)$*, then they are also homotopic in* $\mathrm{SU}(n)$.

Proof. This follows from the fact that there is a continuous map $\pi : U(n) \to \mathrm{SU}(n)$ with $\pi \circ i = \mathrm{id}|_{\mathrm{SU}(n)}$, where $i : \mathrm{SU}(n) \to U(n)$ is inclusion. □

Fact 3. *Let $\xi \in S^4$ be the north pole. Suppose that $f, g : S^4 \times S^1 \to \mathrm{SU}(n)$ are two continuous maps such that*

$$f(\xi, e^{i\theta}) = 1_n = g(\xi, e^{i\theta})$$

for all $e^{i\theta} \in S^1$. If f and g are homotopic in $\mathrm{SU}(n)$, then there is a homotopy

$$F : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$$

such that $F(x, e^{i\theta}, 0) = f(x, e^{i\theta})$, $F(x, e^{i\theta}, 1) = g(x, e^{i\theta})$ for all $x \in S^4$, $e^{i\theta} \in S^1$ and $F(\xi, e^{i\theta}, t) = 1_n$ for all $e^{i\theta} \in S^1$.

Proof. Let $G : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$ be a homotopy between f and g. That is, $G(\cdot, \cdot, 0) = f$ and $G(\cdot, \cdot, 1) = g$. Let $F : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$ be defined by

$$F(x, e^{i\theta}, t) = G(x, e^{i\theta}, t)(G(\xi, e^{i\theta}, t))^*.$$

Then F satisfies the condition. □

5.2. We will describe the projection $P \in \mathrm{M}_4(C(S^4))$ of rank two which represents the class of $(2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(S^4))$ as follows: One can regard S^4 as the quotient space $D^4/\partial D^4$, where

$$D^4 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \leq 1\}.$$

It is standard to construct a unitary

$$\alpha : D^4 \to U_4(\mathbb{C}) = U(\mathrm{M}_4(\mathbb{C}))$$

such that $\alpha(0) = 1_4$ and such that, for any $(z, w) \in \partial D^4$ (i.e., $|z|^2 + |w|^2 = 1$),

$$\alpha(z, w) := \begin{bmatrix} z & w & 0 & 0 \\ -\bar{w} & \bar{z} & 0 & 0 \\ 0 & 0 & \bar{z} & -w \\ 0 & 0 & \bar{w} & z \end{bmatrix} \triangleq \begin{bmatrix} \beta(z, w) & 0 \\ 0 & \beta(z, w)^* \end{bmatrix},$$

where $\beta(z, w) = \left[\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right]$, for $(z, w) \in \partial D^4 = S^3$, represents the generator of $K_1(C(S^3))$. Define $P : S^4 \to U_4(\mathbb{C})$ by

$$P(z, w) \triangleq \alpha(z, w) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w).$$

Note that α is not defined as a function from $S^4 = D^4/\partial D^4$ to $U(4)$, but P is, since

$$P(z, w) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z, w) \in \partial D^4$$

and ∂D^4 is identified with the north pole $\xi \in S^4$. Hence $P(\xi) = \left[\begin{smallmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{smallmatrix}\right]$.

5.3. In the rest of the paper, for a compact metric space X with a given base point and a C^*-algebra A, by $C_0(X, A)$ we mean the C^*-algebra of the continuous functions from X to A which vanish at the base point (and $C_0(X, \mathbb{C})$ will be denoted by $C_0(X)$). (Most spaces we used here have an obvious base point, which we will not mention afterward.) Let $A = C_0(S^1, P\mathrm{M}_4C(S^4)P)$. Let $\widetilde{A}$ be the unitization of A. Let $B = C_0(S^1, C(S^4))$. Since A is a corner of $\mathrm{M}_4(B)$ and B is a corner of $\mathrm{M}_2(A)$ (note that a trivial projection of rank 1 is equivalent to a subprojection of $P \oplus P$), A is stably isomorphic to B. Let $\widetilde{B}$ be a unitization of B. Then $\widetilde{B} = C(S^4 \times S^1)$ and

$$K_1(\widetilde{A}) \cong K_1(A) \cong K_1(B) \cong K_1(\widetilde{B}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

5.4. For a unitary $u \in \mathrm{M}_4(C(S^4 \times S^1))$, in the identification of $[u] \in K_1(C(S^4 \times S^1))$ with $\mathbb{Z} \oplus \mathbb{Z}$, the first component corresponds to the winding number of

$$S^1 \hookrightarrow S^4 \times S^1 \xrightarrow{\det u} S^1 \subset \mathbb{C},$$

that is, the winding number of the map

$$e^{i\theta} \to \det u(\xi, e^{i\theta}),$$

where ξ is the north pole of S^4. Hence, if $u : S^4 \times S^1 \to \mathrm{SU}(n)$, then the first component of $[u] \in K_1(C(S^4 \times S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is automatically zero.

Lemma 5.5. *Let $u : S^4 \times S^1 \to \mathrm{SU}(2)$. Then $u \in \mathrm{M}_2(C(S^4 \times S^1))$ represents the zero element in $K_1(C(S^4 \times S^1))$. In other words, if $u \in \mathrm{SU}_n(S^4 \times S^1)$ represents a nonzero element in K-theory, then $n \geq 3$.*

Proof. Let $f : S^4 \times S^1 \to S^5$ be the standard quotient map sending $\{\xi\} \times S^1 \cup S^4 \times \{1\}$ to a single point. Consider $u : S^4 \times S^1 \to \mathrm{SU}(2)$. Without loss of generality, assume $u(\xi, 1) = 1_2 \in \mathrm{SU}(2)$. Then $u|_{S^4 \times \{1\}} : S^4 \to \mathrm{SU}(2) = S^3$ represents an element in $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $u^2|_{S^4 \times \{1\}} : S^4 \to \mathrm{SU}(2) = S^3$ is homotopically trivial, with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Evidently, $u^2|_{\{\xi\} \times S^1} : S^1 \to S^3 = \mathrm{SU}(2)$ is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Consequently

$$u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1} : S^4 \times \{1\} \cup \{\xi\} \times S^1 \to S^3$$

is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed base point. There is a homotopy

$$F : (S^4 \times \{1\} \cup \{\xi\} \times S^1) \times [0, 1] \to S^3$$

with $F(\cdot, 0) = u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1}$ and

$$F(x, 1) = 1_2 \quad \text{for all } x \in S^4 \times \{1\} \cup \{\xi\} \times S^1.$$

The following is a well-known easy fact: For any relative CW complex (X, Y) $(Y \subset X)$, any continuous map $Y \times I \cup X \times \{0\} \to Z$ (where Z is any other CW complex) can be extended to a continuous map $X \times I \to Z$.

Hence, there is a homotopy $G : (S^4 \times S^1) \times [0,1] \to S^3$ with $G(\cdot, 0) = u^2$, and $G|_{S^4 \times \{1\} \cup \{\xi\} \times S^1 \times [0,1]} = F$. Let $v : S^4 \times S^1 \to \mathrm{SU}(2)$ be defined by $v(x) = G(x, 1)$; then $[v] = [u^2] \in K_1(C(S^4 \times S^1))$ and v maps $S^4 \times \{1\} \cup \{\xi\} \times S^1$ to $1_2 \in \mathrm{SU}(2)$. Consequently, v passes to a map

$$v_1 : S^5 \triangleq S^4 \times S^1 / S^4 \times \{1\} \cup \{\xi\} \times S^1 \to S^3 = \mathrm{SU}(2)$$

and represents an element in $\pi_5(S^3) = \mathbb{Z}/2\mathbb{Z}$. Hence $v_1^2 : S^5 \to S^3$ is homotopically trivial, and therefore v^2 is as well. So we have

$$4[u] = 2[u^2] = 2[v] = [v^2] = 0 \in K_1(C(S^4 \times S^1)),$$

which implies $[u] = 0 \in K_1(C(S^4 \times S^1))$. □

Remark 5.6. In the proof of Lemma 5.5, we in fact proved the following fact: For any $u : S^4 \times S^1 \to \mathrm{SU}(2)$, the map $u^4 : S^4 \times S^1 \to \mathrm{SU}(2)$ is homotopically trivial.

5.7. Note that $P \in \mathrm{M}_4(C(S^4))$ can be regarded as a projection in $\mathrm{M}_4(C(S^4 \times S^1))$, still denoted by P, i.e., for fixed $x \in S^4$, $P(x, \cdot)$ is a constant projection along the S^1 direction. Then

$$K_1(A) \cong K_1(\widetilde{A}) \cong K_1(C(S^4 \times S^1)) \cong K_1(P\mathrm{M}_4(C(S^4 \times S^1))P), \tag{5-1}$$

where $A = C_0(S^1, P\mathrm{M}_4(C(S^4))P)$ is defined in Section 5.2. Let

$$E = \{(\zeta, u) : \zeta \in S^4 \times S^1, u \in \mathrm{M}_4(\mathbb{C}) \text{ with } P(x)uP(x) = u, u^*u = uu^* = P(x)\},$$
$$SE = \{(\zeta, u) \in E : \det(P(x)uP(x) + (1_4 - P(x)) = 1\}.$$

Then $E \to S^4 \times S^1$ and $SE \to S^4 \times S^1$ are fiber bundles with fibers $U(2)$ and $\mathrm{SU}(2)$, respectively. Also the unitaries in $P\mathrm{M}_4(C(S^4 \times S^1))P$ correspond bijectively to the cross-sections of a bundle $E \to S^4 \times S^1$. For this reason, we will call a unitary (of $P\mathrm{M}_4(C(S^4 \times S^1))P$) with determinant 1 everywhere a cross-section of a bundle $SE \to S^4 \times S^1$.

Theorem 5.8. *If $u \in P\mathrm{M}_4(C(S^4 \times S^1))P$ has determinant 1 everywhere, i.e., if u is a cross-section of $SE \to S^4 \times S^1$, then $[u] = 0$ in $K_1(P\mathrm{M}_4(C(S^4 \times S^1))P)$.*

Proof. Note that $SE \to S^4 \times S^1$ is a smooth fiber bundle over the smooth manifold $S^4 \times S^1$. By a standard result in differential topology, u is homotopic to a C^∞-section. Without loss of generality, we may assume that u itself is smooth. Identify the north pole $\xi \in S^4$ with $0 \in \mathbb{R}^4$ and a neighborhood of ξ with $B_\epsilon(0) \subset \mathbb{R}^4$ for $\epsilon > 0$. Since $B_\epsilon(0)$ is contractible, $SE|_{B_\epsilon(0) \times S^1}$ is a trivial bundle. Note that the projection $P \in \mathrm{M}_4(C(S^4 \times S^1))$ is constant along S^1, hence $SE \cong SE|_{S^4 \times \{1\}} \times S^1$

and $SE|_{B_\epsilon(0)\times S^1} \cong SE|_{B_\epsilon(0)\times\{1\}} \times S^1$; in other words, the fiber is constant along S^1 and $SE|_{B_\epsilon(0)\times\{1\}}$ is trivial and isomorphic to $(B_\epsilon(0)\times\{1\})\times \mathrm{SU}(2)$. There is a smooth bundle isomorphism

$$\gamma : SE|_{B_\epsilon(0)\times S^1} \to (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2). \tag{5-2}$$

Then

$$\gamma\circ u|_{B_\epsilon(0)\times S^1} : B_\epsilon(0)\times S^1 \to (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2)$$

is a smooth map with

$$\pi_1\circ(\gamma\circ u)|_{B_\epsilon(0)\times S^1} = \mathrm{id}_{B_\epsilon(0)\times S^1},$$

where $\pi_1 : (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2) \to B_\epsilon(0)\times S^1$ is the projection onto the first coordinate. Define $\phi = \pi_2\circ(\gamma\circ u|_{B_\epsilon(0)\times S^1})$, where $\pi_2 : (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2) \to \mathrm{SU}(2)$ is the projection onto the second coordinate. Since ϕ is smooth, $\phi|_{\{\xi\}\times S^1}$ is not onto SU(2) (note $\dim(\mathrm{SU}(2)) = 3$ and $\dim(S^1) = 1$). Therefore, if ϵ is small enough, $\phi|_{B_\epsilon(0)\times S^1}$ is not onto. By Fact 1 of Section 5.1, ϕ is homotopic to a constant map $\phi_1 : B_\epsilon(0)\times S^1 \to \mathrm{SU}(2)$ with

$$\phi_1(\{\xi\}\times S^1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad\text{and}\quad \phi|_{\partial B_\epsilon(0)\times S^1} = \phi_1|_{\partial B_\epsilon(0)\times S^1}, \tag{5-3}$$

via a homotopy $F : (B_\epsilon(0)\times S^1)\times[0,1] \to \mathrm{SU}(2)$ with $F(x, e^{i\theta}, t)$ constant with respect to t if $x \in \partial B_\epsilon(0)$.

Let $u_1 : B_\epsilon(0)\times S^1 \to SE$ be the cross-section defined by

$$u_1(x, e^{i\theta}) = \gamma^{-1}((x, e^{i\theta}), \phi_1(x, e^{i\theta})) \in SE.$$

Then $u_1(x, e^{i\theta}) = u(x, e^{i\theta})$ if $x \in \partial B_\epsilon(0)$. We can extend u_1 to $S^4\times S^1$ by defining

$$u_1(x, e^{i\theta}) = u(x, e^{i\theta}) \quad \text{if } (x, e^{i\theta}) \notin B_\epsilon(0)\times S^1.$$

Hence u_1 is a section of SE with

$$u_1(\xi, e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} = P(\xi) \quad \text{for all } e^{i\theta} \in S^1.$$

Moreover, u_1 is homotopic to u by a homotopy that is constant on $(S^4\setminus B_\epsilon(0))\times S^1$ (on which $u_1 = u$) and that agrees with F on $B_\epsilon(0)\times S^1$. Hence $[u] = [u_1] \in K_1(P\mathrm{M}_4(C(S^4\times S^1))P)$. Recall that S^4 is obtained from

$$D^4 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \le 1\}$$

by identifying

$$\partial D^4 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

with the north pole $\xi \in S^4$. Recall that $P \in \mathrm{M}_4(C(S^4))$ (viewed as a projection in $\mathrm{M}_4(C(S^4 \times S^1))$ constant along the S^1 direction) is defined as

$$P(z,w) = \alpha(z,w) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z,w),$$

where $\alpha(z,w)$ is defined as in Section 5.2.

Define

$$v(z,w,e^{i\theta}) = \alpha^*(z,w) u_1(z,w,e^{i\theta}) \alpha(z,w).$$

Then we have that

(i) $$v(z,w,e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z,w) \in \partial D^4,$$

and therefore v can be regarded as a map from $S^4 \times S^1$ to $\mathrm{M}_4(\mathbb{C})$. Moreover,

(ii) $$v(z,w,e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} v(z,w,e^{i\theta}) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z,w,e^{i\theta}) \in S^4 \times S^1.$$

By considering the upper-left corner of v (still denoted by v), we obtain a unitary $v : S^4 \times S^1 \to \mathrm{SU}(2)$. By Lemma 5.5 and Remark 5.6, v^4 is homotopically trivial. Furthermore, by Fact 3 of Section 5.1, there is a homotopy $F : S^4 \times S^1 \times [0,1] \to \mathrm{SU}(2)$ such that

(iii) $$F(z,w,e^{i\theta},0) = v^4(z,w,e^{i\theta}) \qquad \text{for all } (z,w) \in S^4, e^{i\theta} \in S^1,$$

(iv) $$F(\xi, e^{i\theta}, t) = 1_2 \qquad \text{for all } e^{i\theta} \in S^1,$$

(v) $$F(z,w,e^{i\theta},1) = 1_2 \qquad \text{for all } (z,w) \in S^4, e^{i\theta} \in S^1.$$

Define $G : D^4 \times S^1 \times [0,1] \to \mathrm{M}_4(\mathbb{C})$ by

$$G(z,w,e^{i\theta},t) = \alpha(z,w) \begin{bmatrix} F(z,w,e^{i\theta},t) & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z,w).$$

Then, by (iv), for $(z,w) \in \partial D^4$ we have

$$G(z,w,e^{i\theta},t) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}.$$

Hence G defines a map (still denoted by G) from $S^4 \times S^1 \times [0,1] \to \mathrm{M}_4(\mathbb{C})$. Furthermore $G(z,w,e^{i\theta},t) \in P(z,w) M_4(\mathbb{C}) P(z,w)$, and

$$G(z,w,e^{i\theta},0) = \alpha(z,w) \begin{bmatrix} v^4 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z,w) = u_1^4.$$

That is, G defines a homotopy between u_1^4 and the unit $P \in P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$. Consequently $[u_1^4] = 0$ and $[u_1] = 0 \in K_1\big(P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P\big)$. Moreover, $[u] = 0 \in K_1(C(S^4 \times S^1))$, as desired. ☐

5.9. We identify $P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$ as a corner of $\mathrm{M}_4(C(S^4 \times S^1))$; then $K_1\big(P(\mathrm{M}_4(C(S^4 \times S^1)))P\big)$ is isomorphic to $K_1(C(S^4 \times S^1)) = \mathbb{Z} \oplus \mathbb{Z}$ naturally. Let $a \in P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$ be defined by

$$a(x, e^{i\theta}) = e^{i\theta} P(x).$$

On the other hand, a could also be regarded as a unitary in $\mathrm{M}_4(C(S^4 \times S^1))$ as $a(x, e^{i\theta}) = e^{i\theta} P(x) + (1_4 - P(x))$. Then $[a] = (2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_1(C(S^4 \times S^1))$, since $[a]$ is the image of $[P] \in K_0(C(S^4))$ under the exponential map

$$K_1(C(S^4)) \to K_1\big(C_0(S^1, C(S^4))\big),$$

and $[P] = (2, 1) \in K_0(C(S^4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 5.10. *No element* $(1, k) \in K_1(C(S^4 \times S^1))$ *can be realized by a unitary* $b \in P\mathrm{M}_4(C(S^4 \times S^1))P$.

Proof. We argue by contradiction. Assume $b \in P\mathrm{M}_4(C(S^4 \times S^1))P$ satisfies $[b] = (1, k) \in K_1(P\mathrm{M}_4(C(S^4 \times S^1)P))$. Without loss of generality, we assume that $b(\xi, 1) = P$. Then

$$[b^2 a^*] = (0, 2k - 1) \in K_1(PM_4(C(S^4 \times S^1))P).$$

In particular, the map

$$e^{i\theta} \to \det \begin{bmatrix} P(\xi)(b^2 a^*)(\xi, e^{i\theta}) P(\xi) & 0 \\ 0 & 1_4 - P(\xi) \end{bmatrix}_{8\times 8}$$

has winding number 0. That is, it is homotopically trivial. Hence

$$(x, e^{i\theta}) \xrightarrow{h} \det \begin{bmatrix} P(\xi)(b^2 a^*)(x, e^{i\theta}) P(\xi) & 0 \\ 0 & 1_4 - P(\xi) \end{bmatrix}_{8\times 8}$$

defines a map $h : S^4 \times S^1 \to S^1$ such that $h_* : \pi_1(S^4 \times S^1) \to \pi_1(S^1)$ is the zero map. Hence there is a lifting $\tilde{h} : S^4 \times S^1 \to \mathbb{R}$ with $h(x, e^{i\theta}) = \exp(i\tilde{h}(x, e^{i\theta}))$. Define a unitary $b_1 \in P\mathrm{M}_4(C(S^4 \times S^1))P$ by $b_1(x, e^{i\theta}) = \exp(i\tfrac{1}{2}\tilde{h}(x, e^{i\theta}))P(x)$. Then $[b_1] = 0 \in K_1(C(S^4 \times S^1))$, and $b^2 a^* b_1^* \in U(P\mathrm{M}_4 C(S^4 \times S^1)P)$ has determinant 1 everywhere. By Theorem 5.8, $[b^2 a^* b_1^*] = 0 \in K_1(C(S^4 \times S^1))$. On the other hand,

$$[b^2 a^* b_1^*] = [b^2 a^*] = (0, 2k - 1) \neq 0 \in K_1(C(S^4 \times S^1)),$$

which is a contradiction. ☐

Remark 5.11. Similarly, we can show that for any unitary $u \in P\mathrm{M}_4(C(S^4 \times S^1))P$, $[u] = l[a] = (2l, l) \in K_1(C(S^4 \times S^1))$ for some $l \in \mathbb{Z}$.

Corollary 5.12. *Let $A = C_0(S^1, PC(S^4)P)$, and let $\widetilde{A}$ be the unitization of A. Then there is no unitary $u \in \widetilde{A}$ such that $[u] = (1, k) \in K_1(A)$. In particular, no unitary u can correspond to a rank-1 projection in $\mathrm{M}_4(C(S^4))$.*

Proof. Note that we may view P as a projection in $\mathrm{M}_4(C(S^4 \times S^1))$ which is constant along the direction of S^1 (Section 5.7). So we may view $\widetilde{A}$ as a unital C^*-subalgebra of $P\mathrm{M}_4(C(S^4 \times S^1))P$. Thus, by the identification (5-1), Theorem 5.10 applies. □

Theorem 5.13. *Let $A = P\mathrm{M}_4(C(S^4))P$. Then* $\mathrm{Dur}\, A = 2$.

Proof. There is a projection $e \in \mathrm{M}_2(A)$ which is unitarily equivalent to a rank-1 projection in $\mathrm{M}_8(C(S^4))$ corresponding to $(1, 0) \in K_0(C(S^4))$. Let $C = C_0((0, 1), A)$. By Corollary 5.12, there is no unitary in $\widetilde{C}$ which represents a rank-1 projection. It follows from Proposition 3.12 that $\mathrm{Dur}\, A > 1$.

However, since $\rho_C\big(K_0(\mathrm{M}_2(C))\big) = \frac{1}{2}\mathbb{Z}$ and $\mathrm{M}_2(C)$ contains a rank-1 projection (with trace $\frac{1}{2}$), by Proposition 3.6(3), $\mathrm{Dur}(\mathrm{M}_2(C)) = 1$. It follows that $\mathrm{Dur}\, C = 2$. □

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References

[Blackadar 1981] B. E. Blackadar, "A simple unital projectionless C^*-algebra", *J. Operator Theory* **5**:1 (1981), 63–71. MR 82h:46076 Zbl 0494.46056

[Brown 1977] L. G. Brown, "Stable isomorphism of hereditary subalgebras of C^*-algebras", *Pacific J. Math.* **71**:2 (1977), 335–348. MR 56 #12894 Zbl 0362.46042

[Cuntz and Pedersen 1979] J. Cuntz and G. K. Pedersen, "Equivalence and traces on C^*-algebras", *J. Funct. Anal.* **33**:2 (1979), 135–164. MR 80m:46053 Zbl 0427.46042

[Elliott 1997] G. A. Elliott, "A classification of certain simple C^*-algebras, II", *J. Ramanujan Math. Soc.* **12**:1 (1997), 97–134. MR 98j:46060 Zbl 0954.46035

[Elliott and Gong 1996] G. A. Elliott and G. Gong, "On the classification of C^*-algebras of real rank zero, II", *Ann. of Math.* (2) **144**:3 (1996), 497–610. MR 98j:46055 Zbl 0867.46041

[Elliott et al. 2007] G. A. Elliott, G. Gong, and L. Li, "On the classification of simple inductive limit C^*-algebras, II: The isomorphism theorem", *Invent. Math.* **168**:2 (2007), 249–320. MR 2010g:46102 Zbl 1129.46051

[Gong 2002] G. Gong, "On the classification of simple inductive limit C^*-algebras, I: The reduction theorem", *Doc. Math.* **7** (2002), 255–461. MR 2007h:46069 Zbl 1024.46018

[Gong et al. 2015] G. Gong, H. Lin, and Z. Niu, "Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras", preprint, 2015. arXiv 1501.00135 [de la Harpe and Skandalis 1984] P. de la Harpe and G. Skandalis, "Produits finis de commutateurs dans les C^*-algèbres", *Ann. Inst. Fourier* (*Grenoble*) **34**:4 (1984), 169–202. MR 87i:46146b Zbl 0536.46044

[Lin 2001] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific, River Edge, NJ, 2001. MR 2002k:46141 Zbl 1013.46055

[Lin 2007] H. Lin, "Simple nuclear C^*-algebras of tracial topological rank one", *J. Funct. Anal.* **251**:2 (2007), 601–679. MR 2008k:46164 Zbl 1206.46052

[Lin 2010a] H. Lin, *Approximate homotopy of homomorphisms from $C(X)$ into a simple C^*-algebra*, Memoirs of the American Mathematical Society **205**:963, American Mathematical Society, Providence, RI, 2010. MR 2011g:46101 Zbl 1205.46037

[Lin 2010b] H. Lin, "Homotopy of unitaries in simple C^*-algebras with tracial rank one", *J. Funct. Anal.* **258**:6 (2010), 1822–1882. MR 2011g:46100 Zbl 1203.46038

[Lin 2011] H. Lin, "Asymptotic unitary equivalence and classification of simple amenable C^*-algebras", *Invent. Math.* **183**:2 (2011), 385–450. MR 2012c:46157 Zbl 1255.46031

[Masani 1981] P. Masani, "Multiplicative partial integration and the Trotter product formula", *Adv. in Math.* **40**:1 (1981), 1–9. MR 82m:47030a Zbl 0485.47026

[Nagy 1987] G. Nagy, "Stable rank of C^*-algebras of Toeplitz operators on polydisks", pp. 227–235 in *Operators in indefinite metric spaces, scattering theory and other topics* (Bucharest, 1985), edited by H. Helson et al., Oper. Theory Adv. Appl. **24**, Birkhäuser, Basel, 1987. MR 89i:47045 Zbl 0642.47014

[Nielsen and Thomsen 1996] K. E. Nielsen and K. Thomsen, "Limits of circle algebras", *Exposition. Math.* **14**:1 (1996), 17–56. MR 97e:46097 Zbl 0865.46037

[Nistor 1986] V. Nistor, "Stable range for tensor products of extensions of $\mathcal{K}$ by $C(X)$", *J. Operator Theory* **16**:2 (1986), 387–396. MR 88b:46085

[Rieffel 1983] M. A. Rieffel, "Dimension and stable rank in the K-theory of C^*-algebras", *Proc. London Math. Soc.* (3) **46**:2 (1983), 301–333. MR 84g:46085 Zbl 0533.46046

[Rieffel 1987] M. A. Rieffel, "The homotopy groups of the unitary groups of non-commutative tori", *J. Operator Theory* **17**:2 (1987), 237–254. MR 88f:22018 Zbl 0656.46056

[Thomsen 1995] K. Thomsen, "Traces, unitary characters and crossed products by $\mathbb{Z}$", *Publ. Res. Inst. Math. Sci.* **31**:6 (1995), 1011–1029. MR 97a:46074 Zbl 0853.46073

[Thomsen 1997] K. Thomsen, *Limits of certain subhomogeneous C^*-algebras*, Mémoires de la Société Mathématique de France **71**, Société Mathématique de France, Paris, 1997. MR 2000c:46110 Zbl 0922.46055

[Villadsen 1999] J. Villadsen, "On the stable rank of simple C^*-algebras", *J. Amer. Math. Soc.* **12**:4 (1999), 1091–1102. MR 2000f:46075 Zbl 0937.46052

[Xue 2000] Y. Xue, "The general stable rank in nonstable K-theory", *Rocky Mountain J. Math.* **30**:2 (2000), 761–775. MR 2001h:46125 Zbl 0980.46053

[Xue 2001] Y. Xue, "The K-groups of $C(M) \times_\theta \mathbb{Z}_p$ for certain pairs (M, θ)", *J. Operator Theory* **46**:2 (2001), 337–354. MR 2003a:46098 Zbl 0998.46037

[Xue 2010] Y. Xue, "Approximate diagonalization of self-adjoint matrices over $C(M)$", *Funct. Anal. Approx. Comput.* **2**:1 (2010), 53–65. MR 2012b:46112 Zbl 1289.46083 arXiv 1002.3962

[Zhang 1990] S. Zhang, "Diagonalizing projections in multiplier algebras and in matrices over a C^*-algebra", *Pacific J. Math.* **145**:1 (1990), 181–200. MR 92h:46088 Zbl 0673.46049

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