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David Hartley

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MOTION BY MIXED VOLUME PRESERVING CURVATURE FUNCTIONS NEAR SPHERES

DAVID HARTLEY

In this paper we investigate the flow of hypersurfaces by a class of symmetric functions of the principal curvatures with a mixed volume constraint. We consider compact hypersurfaces without boundary that can be written as a graph over a sphere. The linearisation of the resulting fully nonlinear PDE is used to prove a short-time existence theorem for hypersurfaces that are sufficiently close to a sphere and, using centre manifold analysis, the stability of the sphere as a stationary solution to the flow is determined. We will find that for initial hypersurfaces sufficiently close to a sphere, the flow will exist for all time and the hypersurfaces will converge exponentially fast to a sphere. This result was shown for the case where the symmetric function is the mean curvature and the constraint is on the $(n+1)$-dimensional enclosed volume by Escher and Simonett (1998).

1. Introduction

Given a sufficiently smooth hypersurface $\Omega_0 = X_0(M^n) \subset \mathbb{R}^{n+1}$ that is compact without boundary, where M^n is an n-dimensional manifold, we are interested in finding a family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ such that

(1)
$$\frac{\partial X}{\partial t} = (h_k - F(\kappa))\nu_{\Omega_t}, \quad X(\,\cdot\,, 0) = X_0, \quad h_k = \frac{1}{\int_{M^n} E_{k+1}\, d\mu_t} \int_{M^n} F(\kappa) E_{k+1}\, d\mu_t,$$

where $\kappa = (\kappa_1, \dots, \kappa_n)$, κ_i are the principal curvatures of the hypersurface $\Omega_t = X(M^n, t) = X_t(M^n)$, ν_{Ω_t} and $d\mu_t$ are the outward pointing unit normal and induced measure of Ω_t, respectively, and k is a fixed integer between -1 and $n-1$. Here E_l denotes the l-th elementary symmetric function of the principal curvatures:

$$E_l = \begin{cases} 1 & \text{if } l = 0, \\ \sum_{1 \le i_1 < \cdots < i_l \le n} \kappa_{i_1}\kappa_{i_2} \dots \kappa_{i_l} & \text{if } l = 1, \dots, n, \end{cases}$$

and $F(\kappa)$ is a given smooth, symmetric function that satisfies $(\partial F/\partial \kappa_i)(\kappa_0) > 0$, where $\kappa_0 = (1/R, \dots, 1/R)$ for some fixed $R \in \mathbb{R}^+$. The flow can be seen to

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preserve the $(n-k)$-th mixed volume of the hypersurface (see Corollary 2.2). Note that while such a quantity is usually only defined for convex hypersurfaces, there is an obvious extension to all hypersurfaces (see Section 2).

This flow has been studied previously in [McCoy 2005]. There it was proved that under some additional conditions on F, for example homogeneity of degree one and convexity or concavity, initially convex hypersurfaces admit a solution for all time and the hypersurfaces converge to a sphere as $t \to \infty$. This result had previously been proved for the specific case where $F(\kappa) = H$, the mean curvature, in [McCoy 2004] and, if in addition, $k = -1$ (in which case the flow is the well-known volume preserving mean curvature flow), in [Huisken 1987]. Other results for the volume preserving mean curvature flow include average mean convex hypersurfaces with initially small traceless second fundamental form converging to spheres (see [Li 2009]) and hypersurfaces that are graphs over spheres with a height function close to zero, in a certain function space, converging to spheres (see [Escher and Simonett 1998b]). Techniques similar to those in this paper were used to study volume preserving mean curvature flow for hypersurfaces close to a cylinder in [Hartley 2013] and spherical caps in [Abels et al. 2015].

The situation where F has homogeneity greater than one has been considered in [Cabezas-Rivas and Sinestrari 2010]. There it was proved that if $k = -1$ and $F(\kappa) = H_m^\beta$, with $m\beta > 1$ and $H_m = \binom{n}{m}^{-1} E_m$ the m-th mean curvature, the flow takes initially convex hypersurfaces that satisfy a pinching condition to spheres; the pinching condition is of the form $E_n > CH^n > 0$, where C is a constant depending on the parameters of the flow.

The main result of this paper is:

Theorem 1.1. *Let F be a smooth, symmetric function of the principal curvatures satisfying $(\partial F/\partial \kappa_a)(\kappa_0) > 0$ for $a = 1, \dots, n$ and some $R \in \mathbb{R}^+$. If Ω_0 is a graph over the sphere $\mathcal{S}_R^n$ with height function sufficiently small in $h^{2+\alpha}(\mathcal{S}_R^n)$, $0 < \alpha < 1$* (see Section 2), *then its flow by* (1) *exists for all time and converges exponentially fast to a sphere as $t \to \infty$, with respect to the $h^{2+\alpha}(\mathcal{S}_R^n)$-topology.*

Part (c) of the main result in [Escher and Simonett 1998b] proves a similar result for the specific case of the volume preserving mean curvature flow. Some differences include that Escher and Simonett are able to use the quasilinear nature of the flow to prove the hypersurfaces are smooth after the initial time. This also allows them to obtain convergence with respect to the $C^l(\mathcal{S}_R^n)$-topology, for any fixed l, and only requires the initial height function to be small in $h^{1+\alpha}(\mathcal{S}_R^n)$. In contrast, the current paper deals with flows that are, in general, fully nonlinear, so the methods we use require the initial height function to be small in $h^{2+\alpha}(\mathcal{S}_R^n)$, which gives a condition on its curvature, and only give convergence in the $h^{2+\alpha}(\mathcal{S}_R^n)$-topology. The key theorems for nonlinear flow appear in [Lunardi 1995] and have been included in the Appendix for the reader's ease.

In Section 2 of this paper we convert the flow (1) to a PDE for the graph function and also introduce the spaces and notation that will be used throughout the paper. The section ends with a corollary proving the flow preserves a certain mixed volume. In Section 3 we consider the problem as an ODE on Banach spaces and determine the linearisation of the speed. This leads to a new short-time existence theorem for the flow that includes some initially $h^{2+\alpha}(\mathcal{S}^n_R)$ hypersurfaces. In the final section the eigenvalues of the linearised operator are determined and a centre manifold is constructed. The proof of the main result is finished by showing that the centre manifold consists entirely of spheres and is exponentially attractive.

We note here that the $(n-k)$-th mixed volumes, for $k \geq 1$, are only well defined for convex hypersurfaces (see [Andrews 2001]). However, we will refer to the flow (1) as mixed volume preserving for any Ω_0, with the understanding that it preserves a quantity that coincides with the $(n-k)$-th mixed volume when Ω_0 is convex (see Corollary 2.2).

2. Notation and preliminaries

In this paper we consider $M^n = \mathcal{S}^n_R$, a given sphere of radius R, and hypersurfaces that are normal graphs over $\mathcal{S}^n_R$, $\boldsymbol{X}_\rho(\boldsymbol{p}) = \boldsymbol{p} + \rho(\boldsymbol{p})\nu_{\mathcal{S}^n_R}(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{S}^n_R$. The volume form on such a hypersurface will be denoted by $d\mu_\rho$ and we let $\mu(\rho)$ be the function such that $d\mu_\rho = \mu(\rho)\, d\mu_0$. We now proceed as in [Escher and Simonett 1998b] and convert the flow to an evolution equation for the height function $\rho : \mathcal{S}^n_R \times [0, T) \to \mathbb{R}$. Up to a tangential diffeomorphism the flow (1) is equivalent to solving the PDE

$$\frac{\partial \rho}{\partial t} = \sqrt{1 + \frac{R^2}{(R+\rho)^2} |\nabla \rho|^2}\big(h_k(\rho) - F(\kappa_\rho)\big), \quad \rho(\,\cdot\,, 0) = \rho_0, \tag{2}$$

where $h_k(\rho) = \int_{\mathcal{S}^n_R} E_{k+1}(\rho) F(\kappa_\rho)\, d\mu_\rho \big/ \int_{\mathcal{S}^n_R} E_{k+1}(\rho)\, d\mu_\rho$, κ_ρ is the principal curvature vector of the hypersurface defined by $\rho(\,\cdot\,, t)$, and ∇ denotes the gradient on $\mathcal{S}^n_R$ (see [McCoy 2005]).

The graph functions ρ are chosen in the little Hölder spaces, $h^{l+\alpha}(\mathcal{S}^n_R)$, for $\alpha \in (0, 1)$, $l \in \mathbb{N}$. These spaces are defined for an open set $U \subset \mathbb{R}^n$ and a multi-index $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| = \sum_{i=1}^n \beta_i$ as follows:

$$h^\alpha(\bar{U}) = \left\{ \rho \in C^\alpha(\bar{U}) : \lim_{r\to 0} \sup_{\substack{x,y\in\bar{U} \\ 0<|x-y|<r}} \frac{|\rho(x) - \rho(y)|}{|x-y|^\alpha} = 0 \right\},$$

$$h^{l+\alpha}(\bar{U}) = \big\{ \rho \in C^{l+\alpha}(\bar{U}) : D^\beta \rho \in h^\alpha(\bar{U}) \text{ for all } \beta, |\beta| = l \big\},$$

where D is the derivative operator on $\mathbb{R}^n$ and C^α, $C^{l+\alpha}$ are the Hölder spaces (see [Lunardi 1995]). The norm on the little Hölder space $h^{l+\alpha}$ is inherited from $C^{l+\alpha}$.

The little Hölder spaces can be extended to $\mathcal{S}_R^n$ by means of an atlas. In addition, it is known that the little Hölder spaces are the continuous interpolation spaces between themselves (see [Guenther et al. 2002, Equation 19]), that is, for real numbers $0 < \alpha < \beta$ we have

$$\left(h^\alpha(\mathcal{S}_R^n), h^\beta(\mathcal{S}_R^n)\right)_\theta = h^{(\beta-\alpha)\theta+\alpha}(\mathcal{S}_R^n), \tag{3}$$

provided $(\beta-\alpha)\theta+\alpha \notin \mathbb{Z}$, where $(\cdot\,,\cdot)_\theta$ is an interpolation functor for each $\theta \in (0,1)$, defined for $Y \subset X$ as

$$\begin{gathered}(X,Y)_\theta = \left\{x \in X : \lim_{t\to 0^+} t^{-\theta} K(t,x,X,Y) = 0\right\},\\ \text{where } K(t,x,X,Y) = \inf_{a\in Y}(\|x-a\|_X + t\|a\|_Y).\end{gathered}$$

We will often abuse notation and use ρ to represent both a function on $\mathcal{S}_R^n \times [0,T)$ and the mapping from $[0,T)$ to a space of functions such that $\rho(t) = \rho(\cdot\,,t)$ for all $t \in [0,T)$. In this regard we define the spaces $C(I,X)$ and $C^k(I,X)$ consisting of continuous and continuously k-differentiable functions from an interval $I \subset \mathbb{R}$ to a Banach space X. They have the norms $\|\rho\|_{C^k(I,X)} = \sum_{j=0}^k \sup_{t\in I}\|\rho^{(j)}(t)\|_X$.

For an operator between function spaces $G : Y \to \tilde{Y}$ we denote the Fréchet derivative by ∂G. A linear operator, $A : Y \subset X \to X$, is called *sectorial* if there exist $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $\omega \in \mathbb{R}$ and $M > 0$ such that

(i) $\rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega,\ |\arg(\lambda-\omega)| < \theta\}$,

(ii) $\|R(\lambda,A)\|_{\mathcal{L}(X,X)} \leq \dfrac{M}{|\lambda-\omega|}$ for all $\lambda \in S_{\theta,\omega}$.

Here $\rho(A)$ is the resolvent set, $R(\lambda,A) = (\lambda I - A)^{-1}$ is the resolvent operator, and $\|\cdot\|_{\mathcal{L}(X,X)}$ is the standard linear operator norm (see [Lunardi 1995]).

For all closed, compact hypersurfaces $\Omega \subset \mathbb{R}^{n+1}$, we define the quantity

$$V_l(\Omega) = \begin{cases}\left((n+1)\binom{n}{l}\right)^{-1}\int_M E_{n-l}\,d\mu & \text{if } l = 0,\dots,n,\\ \mathrm{Vol}(\Phi) & \text{if } l = n+1,\end{cases}$$

where Φ is the $(n+1)$-dimensional region contained inside Ω; for convex hypersurfaces this agrees with the mixed volumes.

Lemma 2.1. *For a family of hypersurfaces Ω_t satisfying* (1), *the Weingarten map, volume form and mixed volumes satisfy the evolution equations*

$$\begin{gathered}\frac{\partial h^i_j}{\partial t} = g^{im}\nabla_m\nabla_j F - (h_k - F)h^i_m h^m_j, \qquad \frac{\partial(d\mu)}{\partial t} = (h_k - F)H\,d\mu,\\ \frac{dV_l}{dt} = \begin{cases} 0 & \textit{if } l = 0,\\ \binom{n+1}{l}^{-1}\int_M E_{n+1-l}(h_k - F)\,d\mu & \textit{if } l = 1,\dots,n+1.\end{cases}\end{gathered}$$

Proof. The first two equations are well known; see [Andrews 1994], for example. The last equation, except for the $l = n+1$ case, which can be found in [Cabezas-Rivas and Sinestrari 2010], is given in Lemma 4.3 of [McCoy 2005] for the case where the Ω_t are convex hypersurfaces. McCoy uses the definition of mixed volumes of convex hypersurfaces (see [Andrews 2001]), which is not valid unless the hypersurface is convex. To obtain the result for all solutions to the flow we use the following identity, found in Equation (5.86) of [Gerhardt 2008]:

$$\frac{\partial E_{a+1}}{\partial h^i_j} = E_a \delta^j_i - h^j_q \frac{\partial E_a}{\partial h^i_q}, \tag{4}$$

where $a = 0, \dots, n$ (in the $a = n$ case we use the convention $E_{n+1} = 0$). Now if we take the divergence of this identity, we obtain

$$\begin{aligned} g^{im}\nabla_m\left(\frac{\partial E_{a+1}}{\partial h^i_j}\right) &= g^{jm}\nabla_m E_a - g^{im}\nabla_m h^j_q \frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right) \\ &= g^{jm}\nabla_m h^p_q \frac{\partial E_a}{\partial h^p_q} - g^{im} g^{jp}\nabla_m h_{pq} \frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right) \\ &= g^{jm} g^{pi}\nabla_m h_{iq} \frac{\partial E_a}{\partial h^p_q} - g^{im} g^{jp}\nabla_p h_{mq} \frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right) \\ &= -h^j_q g^{im}\nabla_i\left(\frac{\partial E_a}{\partial h^m_q}\right), \end{aligned}$$

using the Codazzi equation to get to the second last line. Since $g^{im}\nabla_i\left(\frac{\partial E_0}{\partial h^m_j}\right)$ vanishes, we see this equation implies

$$g^{im}\nabla_i\left(\frac{\partial E_a}{\partial h^m_j}\right) = 0 \quad \text{for all } a = 0, \dots, n.$$

We can now derive the evolution equation:

$$\begin{aligned} &(n+1)\binom{n}{l}\frac{dV_l}{dt} \\ &\quad= \int_M \frac{\partial E_{n-l}}{\partial t} + (h_k - F) H E_{n-l}\, d\mu \\ &\quad= \int_M \frac{\partial E_{n-l}}{\partial h^i_j}\frac{\partial h^i_j}{\partial t} + (h_k - F) H E_{n-l}\, d\mu \\ &\quad= \int_M \frac{\partial E_{n-l}}{\partial h^i_j} g^{im}\nabla_m\nabla_j F - (h_k - F)\frac{\partial E_{n-l}}{\partial h^i_j} h^i_m h^m_j + (h_k - F) H E_{n-l}\, d\mu \end{aligned}$$

$$
\begin{aligned}
&= \int_M \nabla_m \left(\frac{\partial E_{n-l}}{\partial h^i_j} g^{im} \nabla_j F \right) \\
&\qquad + (h_k - F) h^i_m \left(\frac{\partial E_{n+1-l}}{\partial h^i_m} - E_{n-l} \delta^m_i \right) + (h_k - F) H E_{n-l} \, d\mu \\
&= (n+1-l) \int_M (h_k - F) E_{n+1-l} \, d\mu,
\end{aligned}
$$

where the second last line is due to (4) and the last line is due to the homogeneity of E_{n+1-l}. □

Corollary 2.2. *For a compact hypersurface without boundary*, Ω_0, *the flow* (1) *preserves the value of* V_{n-k}, *i.e.*, $V_{n-k}(\Omega_t) = V_{n-k}(\Omega_0)$ *as long as the flow exists.*

3. Graphs over spheres

The flow in (2) can be considered as an ordinary differential equation between Banach spaces. Set $0 < \alpha < 1$ and define

$$
\begin{aligned}
&G : h^{2+\alpha}(\mathcal{S}^n_R) \to h^\alpha(\mathcal{S}^n_R), \\
&\qquad\qquad G(\rho) := L(\rho)(h_k(\rho) - F(\kappa_\rho)), \quad L(\rho) := \sqrt{1 + \frac{R^2}{(R+\rho)^2} |\nabla \rho|^2}.
\end{aligned}
$$

The flow (2) is then rewritten as

$$
\rho'(t) = G(\rho(t)), \quad \rho(0) = \rho_0 \in h^{2+\alpha}(\mathcal{S}^n_R). \tag{5}
$$

Lemma 3.1. *The linearisation of G about zero is given by*

$$
\partial G(0) u = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \left(\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R} \right) u - \frac{n}{R^2} \fint_{\mathcal{S}^n_R} u \, d\mu_0 \right),
$$

for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$.

Note that only the derivative of $F(\kappa)$ with respect to κ_1 appears in this formula for convenience, since $(\partial F / \partial \kappa_1)(\kappa_0) = (\partial F / \partial \kappa_i)(\kappa_0)$ for all $i = 1, \ldots, n$. We also use the notation $\fint_{M^n} f \, d\mu := \int_{M^n} f \, d\mu \big/ \int_{M^n} d\mu$.

Proof. Firstly note $L(0) = 1$ and $\partial L(0) = 0$. By linearising the curvature function, we find

$$
\partial F(\kappa_\rho)\big|_{\rho=0} = \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i}(\kappa_\rho) \partial \kappa_i(\rho) \bigg|_{\rho=0} = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \sum_{i=1}^n \partial \kappa_i(0) = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \partial H(0).
$$

It follows that for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$,

$$
\begin{aligned}
&\partial h_k(0)u \\
&= \partial\left(\frac{1}{\int_{\mathcal{S}^n_R} E_{k+1}(\rho)\mu(\rho)\,d\mu_0}\int_{\mathcal{S}^n_R} E_{k+1}(\rho)F(\kappa_\rho)\mu(\rho)\,d\mu_0\right)\Big|_{\rho=0} u \\
&= \frac{1}{\left(\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0\right)^2}\left(\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0\,\partial\left(\int_{\mathcal{S}^n_R} E_{k+1}(\rho)F(\kappa_\rho)\mu(\rho)\,d\mu_0\right)\Big|_{\rho=0} u\right. \\
&\qquad\qquad \left. - \int_{\mathcal{S}^n_R} E_{k+1}(0)F(\kappa_0)\,d\mu_0\,\partial\left(\int_{\mathcal{S}^n_R} E_{k+1}(\rho)\mu(\rho)\,d\mu_0\right)\Big|_{\rho=0} u\right) \\
&= \frac{1}{\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0} \\
&\qquad \times \left(\int_{\mathcal{S}^n_R}\left(E_{k+1}(0)\,\partial F(\kappa_\rho)\big|_{\rho=0}u + F(\kappa_0)\,\partial\big(E_{k+1}(\rho)\mu(\rho)\big)\big|_{\rho=0}u\right)d\mu_0\right. \\
&\qquad\qquad \left. - F(\kappa_0)\int_{\mathcal{S}^n_R}\partial\big(E_{k+1}(\rho)\mu(\rho)\big)\big|_{\rho=0}u\,d\mu_0\right) \\
&= \frac{\partial F}{\partial\kappa_1}(\kappa_0) ⨍_{\mathcal{S}^n_R}\partial H(0)u\,d\mu_0.
\end{aligned}
$$

It was shown in [Escher and Simonett 1998a] that

$$\partial H(0) = -\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right),$$

so combining these results gives, for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$,

$$\partial G(0)u = \frac{\partial F}{\partial\kappa_1}(\kappa_0)\left(\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right)u - ⨍_{\mathcal{S}^n_R}\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right)u\,d\mu_0\right). \tag{6}$$

The divergence theorem gives the result. □

Lemma 3.2. *For any α_0 such that $0 < \alpha_0 < \alpha$, there exists a neighbourhood, O_1, of $0 \in h^{2+\alpha}(\mathcal{S}^n_R)$ such that the operator $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of a sectorial operator $A_\rho : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$ for all $\rho \in O_1$.*

Proof. We set $\bar{G} : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$ with $\bar{G}(\rho) := L(\rho)(h_k(\rho) - F(\kappa_\rho))$ so that with $A_\rho = \partial\bar{G}(\rho)$ it is clear that $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of A_ρ. It remains to show that there exists O_1 such that A_ρ is sectorial for $\rho \in O_1$.

As $\partial H(0) = -(n/R^2 + \Delta_{\mathcal{S}^n_R})$ is a uniformly elliptic operator on a compact manifold without boundary, its negative is sectorial as a map from $h^{2+\alpha_0}(\mathcal{S}^n_R)$; see [Guenther et al. 2002, Lemma 3.4], for example. Now the operator $A_0 : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$, defined by

$$A_0 u = \left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right)u - \frac{n}{R^2}⨍_{\mathcal{S}^n_R} u\,d\mu_0,$$

is sectorial by the perturbation result in Proposition 2.4.1(ii) of [Lunardi 1995], since the map $u \mapsto -(n/R^2) \fint_{\mathcal{S}^n_R} u \, d\mu_0$ is in $\mathcal{L}(h^{2+\alpha_0}(\mathcal{S}^n_R), h^{2+\alpha_0}(\mathcal{S}^n_R))$. This then implies, by Proposition 2.4.2 of [Lunardi 1995], that $A_\rho = A_0 + (\partial\bar{G}(\rho) - \partial\bar{G}(0))$ is sectorial for all ρ in a neighbourhood of zero, $O_2 \subset h^{2+\alpha_0}(\mathcal{S}^n_R)$. The result follows by setting $O_1 = O_2 \cap h^{2+\alpha}(\mathcal{S}^n_R)$. □

Theorem 3.3. *There are constants $\delta, r > 0$ such that if $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)} \le r$, then* (5) *has a unique maximal solution*:

$$\rho \in C\big([0,\delta), h^{2+\alpha}(\mathcal{S}^n_R)\big) \cap C^1\big([0,\delta), h^{\alpha}(\mathcal{S}^n_R)\big).$$

Proof. This existence theorem is a result of Theorem A.1, which is Theorem 8.4.1 in [Lunardi 1995], by setting $\bar{u} = 0$. In order to satisfy the assumption of the theorem it must be shown that there exists a neighbourhood of zero, $O \subset h^{2+\alpha}(\mathcal{S}^n_R)$, such that G and ∂G are continuous on O and for every $\rho \in O$ the operator $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of a sectorial operator $A : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$.

As in [Andrews and McCoy 2012, Remark 1], since F is a smooth symmetric function of the principal curvatures, it is also a smooth function of the elementary symmetric functions, which depend smoothly on the components of the Weingarten map. We now consider a neighbourhood of zero, O_3, such that if $\rho \in O_3$, then $\int_{\mathcal{S}^n_R} E_{k+1}(\rho) \, d\mu_\rho > 0$ and $\rho(\boldsymbol{p}) > -R$ for all $\boldsymbol{p} \in \mathcal{S}^n_R$ (note if $k = -1$ the former is always satisfied). It is easily seen that the Weingarten map depends smoothly on $\rho \in O_3 \subset h^{2+\alpha}(\mathcal{S}^n_R)$, so that G depends smoothly on $\rho \in O_3$. The sectorial condition was established in Lemma 3.2 for a neighbourhood O_1, so the proof is complete by setting $O = O_3 \cap O_1$. □

4. Stability around spheres

As we are considering the flow locally about $\rho = 0$, it is convenient to rewrite (5) highlighting the dominant linear part:

$$\rho'(t) = \partial G(0)\rho(t) + \tilde{G}(\rho(t)), \qquad \tilde{G}(u) := G(u) - \partial G(0)u. \tag{7}$$

Lemma 4.1. *The spectrum*, $\sigma(\partial G(0))$, *of* $\partial G(0)$ *consists of a sequence of isolated nonpositive eigenvalues where the multiplicity of the* 0 *eigenvalue is* $n+2$.

Proof. This follows from [Escher and Simonett 1998b], as $\partial G(0)$ is a positive constant multiple of the linear operator in that paper. To be exact, we calculate all the elements of the spectrum. Since $h^{2+\alpha}(\mathcal{S}^n_R)$ is compactly embedded in $h^\alpha(\mathcal{S}^n_R)$, the spectrum consists entirely of eigenvalues. To characterise the spectrum we first look at the spectrum of the L^2-self adjoint operator:

$$\tilde{A}u = \frac{\partial F}{\partial \kappa_1}(\kappa_0)\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right)u.$$

The eigenvalues of the spherical Laplacian are well known to be $-l(l+n-1)/R^2$ for $l \in \mathbb{N} \cup \{0\}$ with eigenfunctions the spherical harmonics of order l, denoted by $Y_{l,p}$, $1 \le p \le M_l$, where

$$M_l = \begin{cases} \binom{l+n}{n} - \binom{l+n-2}{n} & \text{if } l \ge 2, \\ \binom{l+n}{n} & \text{if } l \in \{0, 1\}. \end{cases}$$

Therefore the eigenfunctions of $\tilde{A}$ are also the spherical harmonics with eigenvalues

$$\xi_l = \frac{\partial F}{\partial \kappa_1}(\kappa_0)\left(\frac{n}{R^2} - \frac{l(l+n-1)}{R^2}\right) = -\frac{\partial F}{\partial \kappa_1}(\kappa_0)\frac{(l-1)(l+n)}{R^2}.$$

Returning to the spectrum of $\partial G(0)$, $Y_{0,1} = 1$ is still an eigenfunction but with eigenvalue $\lambda_0 = 0$. The operator $\partial G(0)$ is also self-adjoint with respect to the L^2 inner product on $h^{2+\alpha}(\mathcal{S}^n_R)$. Therefore we need only consider eigenfunctions orthogonal to $Y_{0,1}$ in order to characterise the remainder of the spectrum. This means that for an eigenfunction u we assume

$$\int_{\mathcal{S}^n_R} u \, d\mu_0 = 0,$$

and hence by Lemma 3.1, $\partial G(0)u = \tilde{A}u$. The remaining eigenfunctions of $\partial G(0)$ are then the remaining eigenfunctions of $\tilde{A}$, with the same eigenvalues. So the spectrum of $\partial G(0)$ consists of the eigenvalues

$$\lambda_l = \begin{cases} 0 & \text{if } l = 0, \\ -\frac{\partial F}{\partial \kappa_1}(\kappa_0)\frac{l(l+n+1)}{R^2} & \text{if } l \in \mathbb{N}, \end{cases}$$

with eigenfunctions

$$u_{l,p} = \begin{cases} Y_{0,1} & \text{if } l = p = 0, \\ Y_{l+1,p} & \text{if } l \in \mathbb{N} \cup \{0\},\ 1 \le p \le M_{l+1}. \end{cases}$$

The multiplicity of the 0 eigenvalue is then $M_1 + 1 = n + 2$. □

In what follows, we set P to be the projection from $h^\alpha(\mathcal{S}^n_R)$ onto the $\lambda = 0$ eigenspace given by

$$Pu := \sum_{p=0}^{n+1} \langle u, u_{0,p} \rangle u_{0,p},$$

where we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product on $h^\alpha(\mathcal{S}^n_R)$. Because $\partial G(0)$ is self-adjoint with respect to this inner product, clearly $P\partial G(0)u = \partial G(0)Pu = 0$ for every $u \in h^{2+\alpha}(\mathcal{S}^n_R)$. Due to this, $h^{2+\alpha}(\mathcal{S}^n_R)$ can be split into the subspaces

$X^c = P(h^\alpha(\mathcal{S}^n_R))$ and $X^s = (I - P)(h^{2+\alpha}(\mathcal{S}^n_R))$, called the *centre subspace* and *stable subspace*, respectively. We are now in a position to apply Theorem A.3, which is Theorem 9.2.2 in [Lunardi 1995].

Theorem 4.2. *For any $l \in \mathbb{N}$, there is a function $\gamma \in C^{l-1}(X^c, X^s)$ such that $\gamma^{(l-1)}$ is Lipschitz continuous*, $\gamma(0) = \partial\gamma(0) = 0$, *and $\mathcal{M}^c = \operatorname{graph}(\gamma)$ is a locally invariant manifold for* (7) *of dimension $n + 2$.*

Note that by *locally invariant* it is meant that there exists a ball around zero, $B_r(0) \subset X^c$ with $r > 0$, such that if $\rho_0 \in \operatorname{graph}(\gamma|_{B_r(0)})$ then the solution to (7) is in $\operatorname{graph}(\gamma|_{B_r(0)})$ for all time or until $P\rho(t) \notin B_r(0)$. We now set

$$\mathcal{S} := \{\rho \in h^{2+\alpha}(\mathcal{S}^n_R) : \operatorname{graph}(\rho) \text{ is a sphere}\}.$$

Lemma 4.3. *$\mathcal{M}^c$ coincides with the set $\mathcal{S}$ in a neighbourhood of zero, $\Lambda \subset h^{2+\alpha}(\mathcal{S}^n_R)$.*

Proof. By Theorem 2.4 in [Simonett 1995], the equation $y'(t) = \partial G(0)|_{X^s} y(t) + f(t)$ has a unique continuous, bounded solution for any continuous, bounded $f : \mathbb{R} \to (I - P)(h^\alpha(\mathcal{S}^n_R))$. Furthermore the solution is given by $y(t) = (Kf)(t)$, with $K \in \mathcal{L}\big(BC_\eta\big(\mathbb{R}, (I - P)(h^\alpha(\mathcal{S}^n_R))\big), BC_\eta(\mathbb{R}, X^s)\big)$ for any $\eta \in [0, -\lambda_1)$, where

$$BC_\eta(\mathbb{R}, X) := \Big\{g \in C(\mathbb{R}, X) : \|g\|_\eta := \sup_{t\in\mathbb{R}} \exp(-\eta|t|)\|g(t)\|_X < \infty\Big\}.$$

This is the key condition that allows us to apply Theorem 2.3 in [Vanderbauwhede and Iooss 1992] and conclude that $\mathcal{M}^c$ contains all equilibria of (7) with $P\rho_0 \in B_r(0)$. It was shown in [Escher and Simonett 1998b] that (along with $\mathcal{M}^c$) $\mathcal{S}$ is locally a graph over X^c, so since $\mathcal{S} \cap (B_r(0) \times X^s) \subset \mathcal{M}^c$, we conclude that $\mathcal{S}$ and $\mathcal{M}^c$ coincide locally. Note that while [Vanderbauwhede and Iooss 1992] proves the existence of a centre manifold differently than [Lunardi 1995], the two manifolds can be seen to be equal over $B_r(0)$, possibly making r smaller. □

We now prove the main result.

Proof of Theorem 1.1. By Proposition A.4, which is Proposition 9.2.4 in [Lunardi 1995], when $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ is small enough we obtain the decay in (11), with $x(t) = P\rho(t)$ and $y(t) = (I - P)\rho(t)$, for any $\omega \in (0, -\lambda_1)$ and as long as $P\rho(t) \in B_r(0)$. However, by using (11) evaluated at $t = 0$, we obtain

$$\begin{aligned}\|\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)} &\le \|P\rho_0\|_{h^\alpha(\mathcal{S}^n_R)} + \|P\rho_0 - \bar{x}\|_{h^\alpha(\mathcal{S}^n_R)}\\ &\le \|P\rho_0\|_{h^\alpha(\mathcal{S}^n_R)} + C(\omega)\|(I - P)\rho_0 - \gamma(P\rho_0)\|_{h^{2+\alpha}(\mathcal{S}^n_R)},\end{aligned}$$

and since γ is Lipschitz and P is bounded, this leads to a bound of the form $\|\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)} \le C(\omega)\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$. Therefore we can ensure that $\bar{x} \in P(\Lambda) \cap B_r(0)$ by taking $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ small enough, and Lemma 4.3 then implies that the function

$\bar{x}+\gamma(\bar{x})$ defines a sphere. Hence $\bar{x}+\gamma(\bar{x})$ is a stationary solution to (2), which in turn means that $z(t)=\bar{x}$ is the solution to (12). So we can restate (11) as

$$\|P\rho(t)-\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)}+\|(I-P)\rho(t)-\gamma(\bar{x})\|_{h^{2+\alpha}(\mathcal{S}^n_R)} \leq C(\omega)e^{-\omega t}\|(I-P)\rho_0-\gamma(P\rho_0)\|_{h^{2+\alpha}(\mathcal{S}^n_R)}, \tag{8}$$

for as long as $P\rho(t)\in B_r(0)$. However, using this bound and our bound for $\bar{x}$, it follows that $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<C(\omega)\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ as long as $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<r$. By choosing $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ small enough, we can therefore ensure $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<r/2$ for all $t\geq 0$. Thus (8) is true for all $t\geq 0$, and this proves that $\rho(t)$ converges to $\bar{x}+\gamma(\bar{x})$ as $t\to\infty$, which is the height function of a sphere. □

Corollary 4.4. *Let Ω_0 be a graph over a sphere with height ρ_0 such that the solution, $\rho(t)$, to the flow* (2) *with initial condition ρ_0 exists for all time and converges to zero. Suppose further that $(\partial F/\partial\kappa_i)|_{\kappa_{\rho(t)}}>0$ for all $t\in[0,\infty)$ and $i=1,\dots,n$. Then there exists a neighbourhood, O, of ρ_0 in $h^{2+\alpha}(\mathcal{S}^n_R)$, $0<\alpha<1$, such that for every $u_0\in O$ the solution to* (2) *with initial condition u_0 exists for all time and converges to a function near zero whose graph is a sphere.*

Proof. This follows by the same arguments given in [Guenther et al. 2002] for the Ricci flow. First we set $U\subset h^{2+\alpha}(\mathcal{S}^n_R)$ to be the neighbourhood of zero given in Theorem 1.1. Since $\rho(t)$ converges to zero in the $h^{2+\alpha}$-topology, there exists a time T such that $\rho(T)\in U$ and, as U is open, there exists an open ball $B_\epsilon(\rho(T))\subset U$ of radius ϵ centred at $\rho(T)$. The condition that $(\partial F/\partial\kappa_i)|_{\kappa_{\rho(t)}}>0$ for all $t\in[0,\infty)$ and $i=1,\dots,n$ ensures that the operator $L(\rho)F(\kappa_\rho)$ is elliptic around the point $\rho(t)$ for every $t\in[0,\infty)$ (see [Andrews 1994]). As the global term is in $\mathcal{L}(h^{2+\beta}(\mathcal{S}^n_R),h^\alpha(\mathcal{S}^n_R))$ for any $\beta<\alpha$, we can use Proposition 2.4.1(i) in [Lunardi 1995] to conclude that the linear operator $\partial G(\rho(t))$ is sectorial for all $t\in[0,T]$, and hence in a neighbourhood of each point. By Theorem A.2, which is Theorem 8.4.4 in [Lunardi 1995], the flow depends continuously on the initial condition in a neighbourhood of ρ_0. Therefore there exists a ball $B_\delta(\rho_0)$ such that if $u_0\in B_\delta(\rho_0)$, then the solution, $u(t)$, to (2) with initial condition u_0 exists for $t\in[0,T]$ and $u(T)\in B_\epsilon(\rho(T))$. Since $u(T)$ is in U, by Theorem 1.1, the solution to (2) with initial condition $u(T)$ converges to a function near zero that defines a sphere. By uniqueness of the flow we get the result. □

Appendix: Key theorems

In this appendix we restate the key theorems from [Lunardi 1995] using the notation of this paper. In the following, E_1, E_0 and E will represent Banach spaces with $E_1\subset E_0\subset E$.

Theorem A.1 [Lunardi 1995, Theorem 8.4.1]. *Let $O_1 \subset E_1$ be a neighbourhood of 0 and let $G : O_1 \to E_0$ and $\partial G : O_1 \to \mathscr{L}(E_1, E_0)$ be continuous. Assume that for every $v \in O_1$, the operator $\partial G(v) : E_1 \to E_0$ is the part in E_0 of a sectorial operator $A : D \subset E \to E$ such that $E_0 \simeq (E, D)_\theta$ and $E_1 \simeq \{x \in D : Ax \in (E, D)_\theta\}$, for some $\theta \in (0, 1)$. Then for every $\bar{u} \in O_1$ there are $\delta > 0$, $r > 0$ such that if $\|u_0 - \bar{u}\|_{E_1} \le r$, then the problem*

$$u'(t) = G(u(t)), \quad 0 \le t \le \delta, \quad u(0) = u_0 \tag{9}$$

has a unique solution $u \in C([0, \delta], E_1) \cap C^1([0, \delta], E_0)$.

Theorem A.2 [Lunardi 1995, Theorem 8.4.4]. *Let G be as in Theorem A.1. For every $\bar{u} \in O_1$ and for every $\bar{\tau} \in (0, \tau(\bar{u}))$, where $\tau(v)$ is the maximal time of a solution to (9) with $u_0 = v$, there is $r > 0$ such that if $\|u_0 - \bar{u}\|_D \le r$, then $\tau(u_0) \ge \bar{\tau}$ and the mapping*

$$\Phi : B_r(\bar{u}) \to C([0, \bar{\tau}], E_1) \cap C^1([0, \bar{\tau}], E_0), \quad \Phi(v) = u(\,\cdot\,; v),$$

where $u(\,\cdot\,; v)$ solves (9) with $u_0 = v$, is continuously differentiable with respect to v. If in addition G is k times continuously differentiable or analytic, then so is Φ.

We now set $E_0 = (E, D)_\theta$, $E_1 = \{x \in D : Ax \in (E, D)_\theta\}$ for some $\theta \in (0, 1)$, and let O_1 be a neighbourhood of $0 \in E_1$. For a finite-dimensional space X we also define $\eta : X \to \mathbb{R}$ to be a cutoff function such that $0 \le \eta(x) \le 1$ for all $x \in X$, $\eta(x) = 1$ if $\|x\|_X \le 1$, and $\eta(x) = 0$ if $\|x\|_X \ge 2$.

Theorem A.3 [Lunardi 1995, Theorem 9.2.2]. *Let $A : D \subset E \to E$ be a sectorial operator such that $\sigma(A)\backslash\mathbb{R}_-$ consists of a finite number of isolated eigenvalues, each with finite algebraic multiplicity. Let $\tilde{G} \in C^1(O_1, E_0)$ be a nonlinear function such that $\tilde{G}(0) = 0$ and $\partial\tilde{G}(0) = 0$. Then there exists $r_1 > 0$ such that for $r \le r_1$ there is a Lipschitz continuous function $\gamma : P(E_0) \to (I - P)(E_1)$ such that the graph of γ is invariant for the system*

$$\begin{aligned} x'(t) &= A|_{P(E_0)}x(t) + P\tilde{G}\left(\eta\left(\frac{x(t)}{r}\right)x(t) + y(t)\right), \quad x(0) = x_0 \in P(E_0), \\ y'(t) &= A|_{(I-P)(E_1)}y(t) + (I - P)\tilde{G}\left(\eta\left(\frac{x(t)}{r}\right)x(t) + y(t)\right), \\ &\qquad\qquad y(0) = y_0 \in (I - P)(E_1), \end{aligned} \tag{10}$$

where P is the spectral projection associated with the set of nonnegative eigenvalues. If in addition $\tilde{G}$ is k times continuously differentiable, with $k \ge 2$, then there exists

$r_k > 0$ *such that if* $r < r_k$, *then* $\gamma \in C^{k-1,1}$ *and for* $x \in P(E_0)$,

$$\partial\gamma(x)\left(A|_{P(E_0)}x + P\tilde{G}\left(\eta\left(\frac{x}{r}\right)x + \gamma(x)\right)\right) = A|_{(I-P)(E_1)}\gamma(x) + (I-P)\tilde{G}\left(\eta\left(\frac{x}{r}\right)x + \gamma(x)\right).$$

Proposition A.4 [Lunardi 1995, Proposition 9.2.4]. *Take A and $\tilde{G}$ as in Theorem A.3. For every $\omega \in (0, \omega_-)$, where $\omega_- = -\sup\{\Re(\lambda) : \lambda \in \sigma(A) \cap \mathbb{R}_-\}$, there is $C(\omega) > 0$ such that if $\|x_0\|_{E_0}$ and $\|y_0\|_{E_1}$ are small enough, then there exists $\bar{x} \in P(E_0)$ such that for all $t \geq 0$,*

$$\|x(t) - z(t)\|_{E_0} + \|y(t) - \gamma(z(t))\|_{E_1} \leq C(\omega)\exp(-\omega t)\|y_0 - \gamma(x_0)\|_{E_1}, \tag{11}$$

where $(x(t), y(t))$ is the solution to (10) *and $z(t)$ is the solution to*

$$z'(t) = A|_{P(E_0)}z(t) + P\tilde{G}\left(\eta\left(\frac{z(t)}{r}\right)z(t) + \gamma(z(t))\right), \quad z(0) = \bar{x}. \tag{12}$$

Note that throughout the paper we considered, for $0 < \alpha_0 < \alpha < 1$, the spaces $E_1 = h^{2+\alpha}(\mathcal{S}^n_R)$, $E_0 = h^{\alpha}(\mathcal{S}^n_R)$ and $E = h^{\alpha_0}(\mathcal{S}^n_R)$, with D, the domain of a linear operator A, given by $h^{2+\alpha_0}(\mathcal{S}^n_R)$. The characterisation of $h^{\alpha}(\mathcal{S}^n_R)$ as an interpolation space between $h^{\alpha_0}(\mathcal{S}^n_R)$ and $h^{2+\alpha_0}(\mathcal{S}^n_R)$ is given in (3).

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References

[Abels et al. 2015] H. Abels, H. Garcke, and L. Müller, "Stability of spherical caps under the volume-preserving mean curvature flow with line tension", *Nonlinear Anal. Theory Methods Appl.* **117** (2015), 8–37. arXiv 1403.1139

[Andrews 1994] B. Andrews, "Contraction of convex hypersurfaces in Euclidean space", *Calc. Var. Partial Differ. Equations* **2**:2 (1994), 151–171. MR 97b:53012 Zbl 0805.35048

[Andrews 2001] B. Andrews, "Volume-preserving anisotropic mean curvature flow", *Indiana Univ. Math. J.* **50**:2 (2001), 783–827. MR 2002m:53105 Zbl 1047.53037

[Andrews and McCoy 2012] B. Andrews and J. A. McCoy, "Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature", *Trans. Amer. Math. Soc.* **364**:7 (2012), 3427–3447. MR 2901219 Zbl 1277.53061

[Cabezas-Rivas and Sinestrari 2010] E. Cabezas-Rivas and C. Sinestrari, "Volume-preserving flow by powers of the *m*th mean curvature", *Calc. Var. Partial Differ. Equations* **38**:3-4 (2010), 441–469. MR 2011g:53141 Zbl 1197.53082

[Escher and Simonett 1998a] J. Escher and G. Simonett, "A center manifold analysis for the Mullins–Sekerka model", *J. Differ. Equations* **143**:2 (1998), 267–292. MR 98m:35228 Zbl 0896.35142

[Escher and Simonett 1998b] J. Escher and G. Simonett, "The volume preserving mean curvature flow near spheres", *Proc. Amer. Math. Soc.* **126**:9 (1998), 2789–2796. MR 98m:53054 Zbl 0909.53043

[Gerhardt 2008] C. Gerhardt, "Curvature flows in semi-Riemannian manifolds", pp. 113–165 in *Geometric flows*, edited by H.-D. Cao and S.-T. Yau, Surveys in Differential Geometry **12**, International Press, Somerville, MA, 2008. MR 2010f:53111 Zbl 1162.53047

[Guenther et al. 2002] C. Guenther, J. Isenberg, and D. Knopf, "Stability of the Ricci flow at Ricci-flat metrics", *Comm. Anal. Geom.* **10**:4 (2002), 741–777. MR 2003g:53118 Zbl 1028.53043

[Hartley 2013] D. Hartley, "Motion by volume preserving mean curvature flow near cylinders", *Comm. Anal. Geom.* **21**:5 (2013), 873–889. MR 3152966 Zbl 06293093

[Huisken 1987] G. Huisken, "The volume preserving mean curvature flow", *J. Reine Angew. Math.* **382** (1987), 35–48. MR 89d:53015 Zbl 0621.53007

[Li 2009] H. Li, "The volume-preserving mean curvature flow in Euclidean space", *Pacific J. Math.* **243**:2 (2009), 331–355. MR 2010k:53100 Zbl 1182.53061

[Lunardi 1995] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser, Basel, 1995. MR 96e:47039 Zbl 0816.35001

[McCoy 2004] J. A. McCoy, "The mixed volume preserving mean curvature flow", *Math. Z.* **246**:1-2 (2004), 155–166. MR 2004k:53101 Zbl 1062.53057

[McCoy 2005] J. A. McCoy, "Mixed volume preserving curvature flows", *Calc. Var. Partial Differ. Equations* **24**:2 (2005), 131–154. MR 2006g:53098 Zbl 1079.53099

[Simonett 1995] G. Simonett, "Center manifolds for quasilinear reaction-diffusion systems", *Differ. Integral Equations* **8**:4 (1995), 753–796. MR 96a:35091 Zbl 0815.35054

[Vanderbauwhede and Iooss 1992] A. Vanderbauwhede and G. Iooss, "Center manifold theory in infinite dimensions", pp. 125–163 in *Dynamics Reported* (*N. S.*), vol. 1, edited by C. K. R. T. Jones et al., Springer, Berlin, 1992. MR 93f:58174 Zbl 0751.58025

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DAVID HARTLEY
INSTITUTO DE CIENCIAS MATEMÁTICAS
CALLE NICOLÁS CABRERA, 13–15
CAMPUS DE CANTOBLANCO, UAM
28049 MADRID
SPAIN
david.hartley@icmat.es

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