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ON DEMAZURE AND LOCAL WEYL MODULES FOR AFFINE HYPERALGEBRAS

ANGELO BIANCHI, TIAGO MACEDO AND ADRIANO MOURA

We establish the existence of Demazure flags for graded local Weyl modules for hyper current algebras in positive characteristic. If the underlying simple Lie algebra is simply laced, the flag has length one; that is, the graded local Weyl modules are isomorphic to Demazure modules. This extends to the positive characteristic setting results of Chari and Loktev, Fourier and Littelmann, and Naoi for current algebras in characteristic zero. Using this result, we prove that the character of local Weyl modules for hyper loop algebras depend only on the highest weight, but not on the (algebraically closed) ground field, and deduce a tensor product factorization for them.

Introduction

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over the complex numbers and, given an algebraically closed field $\mathbb{F}$, let $G_{\mathbb{F}}$ be a connected, simply connected, semisimple algebraic group over $\mathbb{F}$ of the same Lie type as $\mathfrak{g}$. The category of finite-dimensional $G_{\mathbb{F}}$-modules is equivalent to that of the hyperalgebra $U_{\mathbb{F}}(\mathfrak{g})$. The hyperalgebra is a Hopf algebra obtained from the universal enveloping algebra of $\mathfrak{g}$ by first choosing a certain integral form and then changing scalars to $\mathbb{F}$ (this process is often referred to as reduction modulo p). If the characteristic of $\mathbb{F}$ is positive, the category of finite-dimensional $G_{\mathbb{F}}$-modules is not semisimple, and the modules obtained by reduction modulo p of simple $\mathfrak{g}$-modules — called Weyl modules — provide examples of indecomposable, reducible modules. The Weyl modules have several interesting properties which are independent of $\mathbb{F}$ such as: a description in terms of generators and relations, being the universal highest-weight modules of the category of finite-dimensional $G_{\mathbb{F}}$-modules, their characters are given by the Weyl character formula.

Consider now the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. The finite-dimensional representation theory of $\tilde{\mathfrak{g}}$ was initiated by Chari and Presley [1986], where the

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simple modules were classified in terms of tensor products of evaluation modules. Differently from the category of finite-dimensional $\mathfrak{g}$-modules, the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules is not semisimple. Therefore, it is natural to ask if there is a notion analogue to that of Weyl modules for $\tilde{\mathfrak{g}}$. Chari and Presley [2001] proved that the simple finite-dimensional $\tilde{\mathfrak{g}}$-modules are highest-weight in an appropriate sense and introduced the Weyl modules for $\tilde{\mathfrak{g}}$ in terms of generator and relations which are the natural analogues of the relations for the original Weyl modules. The highest-weight vector is now an eigenvector for the action of the loop algebra $\tilde{\mathfrak{h}}$ over the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Because of this, it eventually became common practice to use the terms ℓ-weight and highest-ℓ-weight. In particular, it was shown in [Chari and Pressley 2001] that the just-introduced Weyl modules share a second property with their older relatives: they are the universal finite-dimensional highest-ℓ-weight modules. These results were immediately quantized and, still in the same paper, the notion of Weyl modules for the quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ was introduced. Chari and Presley conjectured (and proved for $\mathfrak{g} = \mathfrak{sl}_2$) that the Weyl modules for $\tilde{\mathfrak{g}}$ were classical limits of quantum Weyl modules. Moreover, all Weyl modules for $\tilde{\mathfrak{g}}$ could be obtained as classical limits of quantum Weyl modules which are actually irreducible. This can be viewed as the analogue of the property that the original Weyl modules are obtained by reduction modulo p from simple $\mathfrak{g}$-modules.

Motivated by bringing the discussion of the last paragraph to the positive characteristic setting, [Jakelić and Moura 2007] initiated the study of the finite-dimensional representation theory of the hyperalgebras associated to $\tilde{\mathfrak{g}}$, which we refer to as hyper loop algebras. Several basic properties of the underlying abelian category were established and, in particular, the notion of Weyl modules was introduced. Moreover, it was shown that certain Weyl modules for $\tilde{\mathfrak{g}}$ can be reduced modulo p. In analogy with the previous paragraphs, it is natural to conjecture that the reduction modulo p of a Weyl module is again a Weyl module (the difference is that now we cannot restrict attention to Weyl modules which are irreducible since there are too few of these).

In the meantime, two partial proofs of Chari and Presley's conjecture appeared [Chari and Loktev 2006; Fourier and Littelmann 2007]. Namely, it follows from a tensor product factorization of the Weyl modules for $\tilde{\mathfrak{g}}$ proved in [Chari and Pressley 2001] together with the fact that the irreducible quantum Weyl modules are tensor products of fundamental modules, that it suffices to compute the dimension of graded analogues of Weyl modules for the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. These graded analogues of Weyl modules were introduced in [Feigin and Loktev 2004] as a particular case of a class of modules (named local Weyl modules) for algebras of the form $\mathfrak{g} \otimes A$, where A is a commutative associative algebra (see also [Chari et al. 2010; Fourier et al. 2012] and references therein for more on the recent development of the representation theory of such algebras). For $\mathfrak{g}$ of type A, the dimensions of the graded Weyl modules were computed in [Chari and Loktev 2006] by explicitly exhibiting a vector space basis. As a consequence, it was observed that they are isomorphic to certain Demazure modules. For a general simply-laced Lie algebra, this isomorphism was proved in [Fourier and Littelmann 2007] by using a certain presentation of Demazure modules by generators and relations as well as by studying fusion products. In particular, the dimension of the graded Weyl modules could be computed resulting in a proof of the conjecture. It was also shown in [Fourier and Littelmann 2007] that such isomorphisms do not exist in general in the non-simply laced case. It was pointed out by Nakajima that the general case could be deduced by using global bases theory (this proof remains unpublished, but a brief sketch is given in the introduction of [Fourier and Littelmann 2007]). The relation with Demazure modules in the nonsimply laced case was finally established in completely generality in [Naoi 2012] where it was shown that the graded Weyl modules for $\mathfrak{g}[t]$ admit Demazure flags, that is, filtrations whose quotients are Demazure modules. Such flags are actually obtained from results of Joseph [2003; 2006] (see also [Littelmann 1998]) on global bases for tensor products of Demazure modules. Therefore, in the nonsimply laced case, the relation between Weyl and Demazure modules is, so far, dependent on the theory of global bases, although in a different manner than Nakajima's proposed proof.

The goal of the present paper is to extend to the positive characteristic context the results of [Fourier and Littelmann 2007; Naoi 2012] and prove the conjecture of [Jakelić and Moura 2007] on reduction modulo p of Weyl modules for hyper loop algebras. Moreover, we prove a tensor product factorization of Weyl modules — the hyperalgebraic analogue of that proved in [Chari and Pressley 2001]. However, due to the extra technical difficulties which arise when dealing with hyperalgebras in positive characteristic, there are several differences in our proofs from those used in the characteristic zero setting. For instance, the tensor product factorization was originally used to restrict the study to computing the dimension of the graded Weyl modules for current algebras. In the positive characteristic setting, we actually deduce the tensor product factorization from the computation of the dimension. Also, for proving the existence of the Demazure flags, some arguments used in [Naoi 2012] do not admit a hyperalgebraic analogue. Our approach to overcome these issues actually makes use of the characteristic-zero version of the same statements. We also use the fact proved in [Mathieu 1988; 1989] that the characters of Demazure modules do not depend on the ground field. Different presentations of Demazure modules in terms of generator and relations are needed for different parts of the argument. For $\mathfrak{g}$ of type G_2, technical issues for proving one of these presentations require that we restrict ourselves to characteristic different than 2 and 3. Outside type G_2, there is no restriction in the characteristic of the ground field.

While this paper was being finished, new ideas for studying Demazure, local Weyl modules, and Kirillov–Reshetikhin modules are introduced. In particular, several results of [Chari and Pressley 2001; Fourier and Littelmann 2007; Naoi 2012] are recovered and generalized. Moreover, new (and simpler) presentations in terms of generators and relations for Demazure modules are obtained. It will be interesting to study if the ideas and results of [Chari and Venkatesh 2014] can be brought to the positive characteristic setting as well.

The paper is organized as follows. We start Section 1 fixing the notation regarding finite and affine types, Kac–Moody algebras and reviewing the construction of the hyperalgebras. Next, using generators and relations, we define the Weyl modules for hyper loop algebras, their graded analogues for hyper current algebras, and the subclass of the class of Demazure modules which is relevant for us. We then state our main result (Theorem 1.5.2) and recall the precise statement (1.5.4) of the conjecture in [Jakelić and Moura 2007]. Theorem 1.5.2 is stated in 4 parts. Part (a) states the isomorphism between graded Weyl modules and Demazure modules for simply laced $\mathfrak{g}$. Part (b) states the existence of Demazure flags for graded Weyl modules. Part (c) establishes an isomorphism between a given graded Weyl module and a twist of certain Weyl module for the hyper loop algebra. Finally, part (d) is the aforementioned tensor product factorization. In Section 2, we fix some further notation and establish a few technical results needed in the proofs.

Section 3.1 brings a review of the finite-dimensional representation theory of the finite-type hyperalgebras while Section 3.2 gives a very brief account of the relevant results from [Jakelić and Moura 2007]. Section 3.3 is concerned with the category of finite-dimensional graded modules for the hyper current algebras. The main results of this subsection are Theorem 3.3.4, where the basic properties of the category are established, and Corollary 3.3.3 which states that the graded Weyl modules for $\mathfrak{g}[t]$ admit integral forms. Assuming Theorem 1.5.2(b), we prove (1.5.4) in Section 3.4. The proof actually makes use of the characteristic-zero version of all parts of Theorem 1.5.2 as well as [Naoi 2012, Corollary A] (stated here as Proposition 3.4.1). In Section 3.5, we prove a second presentation of Demazure modules in terms of generator and relations. It basically replaces a highest-weight generator by a lowest-weight one. This is the presentation which allows us to use the results of [Mathieu 1988; 1989] on the independence of the characters of Demazure modules on the ground field.

In the first three subsections of Section 4 we collect the results of [Joseph 2003; 2006] on crystal and global bases which we need to prove Theorem 4.4.1 which is an integral analogue of [Naoi 2012, Corollary 4.16] on the existence of higher level Demazure flags for Demazure modules when the underlying simple Lie algebra $\mathfrak{g}$ is simply laced. We remark that the proof of Theorem 4.4.1 is the only one where the theory of global bases is used. We further remark that, in order to prove

Theorem 1.5.2(b), we only need the statement of Theorem 4.4.1 for $\mathfrak{g}$ of type A. We observe that the only other place quantum groups are being used here is in the proof of the characteristic-zero version of Theorem 1.5.2(c); [Fourier and Littelmann 2007, Lemmas 1 and 3 and Equation (15)].

Theorem 1.5.2 is proved in Section 5. In particular, in Section 5.2, we prove a positive characteristic analogue of [Naoi 2012, Proposition 4.1] which is a third presentation of Demazure modules in terms of generator and relations in the case that $\mathfrak{g}$ is not simply laced. This is where the restriction on the characteristic of the ground field for type G_2 appears. Parts (b) and (c) of Theorem 1.5.2 are proved in Sections 5.3 and 5.4, respectively. Finally, in Sections 5.5 and 5.6, we prove that the tensor product of finite-dimensional highest-ℓ-weight modules for hyper loop algebras with relatively prime highest ℓ-weights is itself a highest-ℓ-weight module and deduce Theorem 1.5.2(d). As an application of Theorem 1.5.2, we end the paper proving that the graded Weyl modules are fusion products of Weyl modules with "smaller" highest weights (Proposition 5.7.1).

1. The main results

1.1. ***Finite-type data.*** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The associated root system will be denoted by $R \subset \mathfrak{h}^*$. We fix a simple system $\Delta = \{\alpha_i : i \in I\} \subset R$ and denote the corresponding set of positive roots by R^+. The Borel subalgebra associated to R^+ will be denoted by $\mathfrak{b}^+ \subset \mathfrak{g}$ and the opposite Borel subalgebra will be denoted by $\mathfrak{b}^- \subset \mathfrak{g}$. We fix a Chevalley basis of the Lie algebra $\mathfrak{g}$ consisting of $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, for each $\alpha \in R^+$, and $h_i \in \mathfrak{h}$, for each $i \in I$. We also define $h_\alpha \in \mathfrak{h}, \alpha \in R^+$, by $h_\alpha = [x_\alpha^+, x_\alpha^-]$ (in particular, $h_i = h_{\alpha_i}, i \in I$) and set $R^\vee = \{h_\alpha \in \mathfrak{h} : \alpha \in R\}$. We often simplify notation and write $x_i^\pm$ in place of $x_{\alpha_i}^\pm$, $i \in I$. Let $(\ ,\ )$ denote the invariant symmetric bilinear form on $\mathfrak{g}$ such that $(h_\theta, h_\theta) = 2$, where θ is the highest root of $\mathfrak{g}$. Let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the linear isomorphism induced by $(\ ,\ )$ and keep denoting by $(\ ,\ )$ the nondegenerate bilinear form induced by ν on $\mathfrak{h}^*$. Notice that

$$(x_\alpha^+, x_\alpha^-) = \frac{2}{(\alpha, \alpha)} \quad \text{for all } \alpha \in R^+ \tag{1.1.1}$$

and

$$(\alpha, \alpha) = \begin{cases} 2 & \text{if } \alpha \text{ is long,} \\ 2/r^\vee & \text{if } \alpha \text{ is short,} \end{cases} \tag{1.1.2}$$

where $r^\vee \in \{1, 2, 3\}$ is the lacing number of $\mathfrak{g}$. For notational convenience, set

$$r_\alpha^\vee = \frac{2}{(\alpha, \alpha)} = \begin{cases} 1, & \text{if } \alpha \text{ is long,} \\ r^\vee, & \text{if } \alpha \text{ is short.} \end{cases} \tag{1.1.3}$$

We shall need the following fact [Carter 1972, Section 4.2]. Given $\alpha \in R$, let $x_\alpha = x^{\pm}_{\pm\alpha}$ according to whether $\alpha \in \pm R^+$. For $\alpha, \beta \in R$ let $p = \max\{n : \beta - n\alpha \in R\}$. Then there exists $\varepsilon \in \{-1, 1\}$ such that

$$[x_\alpha, x_\beta] = \varepsilon(p+1)x_{\alpha+\beta}. \tag{1.1.4}$$

We define the weight lattice $P = \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R\}$, the subset of dominant weights $P^+ = \{\lambda \in P : \lambda(h_\alpha) \in \mathbb{N} \text{ for all } \alpha \in R^+\}$, the coweight lattice $P^\vee = \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{Z} \text{ for all } \alpha \in R\}$, and the subset of dominant coweights $P^{\vee+} = \{h \in P^\vee : \alpha(h) \in \mathbb{N} \text{ for all } \alpha \in R^+\}$. We denote the fundamental weights by ω_i, $i \in I$, the root lattice of $\mathfrak{g}$ by Q, and we let $Q^+ = \mathbb{Z}_{\geq 0}R^+$. We consider the usual partial order on $\mathfrak{h}^*$: $\mu \leq \lambda$ if and only if $\lambda - \mu \in Q^+$. The Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is the subgroup of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ generated by the simple reflections s_i, $i \in I$, defined by $s_i(\mu) = \mu - \mu(h_i)\alpha_i$ for all $\mu \in \mathfrak{h}^*$. As usual, w_0 denotes the longest element in $\mathcal{W}$.

1.2. ***Affine-type data.*** Consider the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with Lie bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$, for any $x, y \in \mathfrak{g}$, $r, s \in \mathbb{Z}$. We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of $\tilde{\mathfrak{g}}$. The subalgebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the current algebra of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, let $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ and $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$. Let also $\mathfrak{a}[t]_\pm := \mathfrak{a} \otimes (t^{\pm 1}\mathbb{C}[t^{\pm 1}])$. In particular, as vector spaces,

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+ \quad \text{and} \quad \mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t].$$

The affine Kac–Moody algebra $\hat{\mathfrak{g}}$ is the 2-dimensional extension $\hat{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ of $\tilde{\mathfrak{g}}$ with Lie bracket given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r,-s}(x, y)c,$$
$$[c, \hat{\mathfrak{g}}] = \{0\}, \quad \text{and} \quad [d, x \otimes t^r] = rx \otimes t^r$$

for any $x, y \in \mathfrak{g}$, $r, s \in \mathbb{Z}$. Observe that if $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ is the derived subalgebra of $\hat{\mathfrak{g}}$, then $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C}c$, and we have a nonsplit short exact sequence of Lie algebras $0 \to \mathbb{C}c \to \hat{\mathfrak{g}}' \to \tilde{\mathfrak{g}} \to 0$.

Set $\hat{\mathfrak{h}}' = \mathfrak{h} \oplus \mathbb{C}c$. Notice that $\mathfrak{g}$, $\mathfrak{g}[t]$, and $\mathfrak{g}[t]_\pm$ remain subalgebras of $\hat{\mathfrak{g}}$. Set

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{n}}^\pm = \mathfrak{n}^\pm \oplus \mathfrak{g}[t]_\pm, \quad \text{and} \quad \hat{\mathfrak{b}}^\pm = \hat{\mathfrak{n}}^\pm \oplus \hat{\mathfrak{h}}.$$

The root system, positive root system, and set of simple roots associated to the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$ will be denoted by $\hat{R}$, $\hat{R}^+$ and $\hat{\Delta}$, respectively. Let $\hat{I} = I \sqcup \{0\}$ and $h_0 = c - h_\theta$, so that $\{h_i : i \in \hat{I}\} \cup \{d\}$ is a basis of $\hat{\mathfrak{h}}$. Identify $\mathfrak{h}^*$ with the subspace $\{\lambda \in \hat{\mathfrak{h}}^* : \lambda(c) = \lambda(d) = 0\}$. Let also $\delta \in \hat{\mathfrak{h}}^*$ be such that $\delta(d) = 1$ and $\delta(h_i) = 0$ for all $i \in \hat{I}$ and define $\alpha_0 = \delta - \theta$. Then $\hat{\Delta} = \{\alpha_i : i \in \hat{I}\}$, $\hat{R}^+ = R^+ \cup \{\alpha + r\delta : \alpha \in R \cup \{0\}, r \in \mathbb{Z}_{>0}\}$, and $\hat{\mathfrak{g}}_{\alpha+r\delta} = \mathfrak{g}_\alpha \otimes t^r$ if $\alpha \in R$, $r \in \mathbb{Z}$, and $\hat{\mathfrak{g}}_{r\delta} = \mathfrak{h} \otimes t^r$, if $r \in \mathbb{Z} \setminus \{0\}$. Observe that

$$\alpha(c) = 0 \ \text{ for all } \alpha \in \hat{R}. \tag{1.2.1}$$

A root $\gamma \in \hat{R}$ is called real if $\gamma = (\alpha + r\delta)$ with $\alpha \in R, r \in \mathbb{Z}$, and imaginary if $\gamma = r\delta$ with $r \in \mathbb{Z} \setminus \{0\}$. Set $x^{\pm}_{\alpha,r} = x^{\pm}_{\alpha} \otimes t^r, h_{\alpha,r} = h_\alpha \otimes t^r, \alpha \in R^+, r \in \mathbb{Z}$. We often simplify notation and write $x^{\pm}_{i,r}$ and $h_{i,r}$ in place of $x^{\pm}_{\alpha_i,r}$ and $h_{\alpha_i,r}, i \in I, r \in \mathbb{Z}$. Observe that $\{x^{\pm}_{\alpha,r}, h_{i,r} : \alpha \in R^+, i \in I, r \in \mathbb{Z}\}$ is a basis of $\tilde{\mathfrak{g}}$. Given $\alpha \in R^+$ and $r \in \mathbb{Z}_{>0}$, set

$$x^{+}_{\pm\alpha+r\delta} = x^{\pm}_{\alpha,r}, \quad x^{-}_{\pm\alpha+r\delta} = x^{\mp}_{\alpha,-r}, \quad h_{\pm\alpha+r\delta} = [x^{+}_{\pm\alpha+r\delta}, x^{-}_{\pm\alpha+r\delta}] = \pm h_\alpha + r r^\vee_\alpha c.$$

Define also $\Lambda_i \in \hat{\mathfrak{h}}^*, i \in \hat{I}$, by the requirement $\Lambda_i(d) = 0$, $\Lambda_i(h_j) = \delta_{ij}$ for all $i, j \in \hat{I}$. Set $\hat{P} = \mathbb{Z}\delta \oplus \bigoplus_{i\in\hat{I}} \mathbb{Z}\Lambda_i$, $\hat{P}^+ = \mathbb{Z}\delta \oplus \bigoplus_{i\in\hat{I}} \mathbb{N}\Lambda_i$, $\hat{P}' = \bigoplus_{i\in\hat{I}} \mathbb{Z}\Lambda_i$, and $\hat{P}'^+ = \hat{P}' \cap \hat{P}^+$. Notice that

$$\Lambda_0(h) = 0 \iff h \in \mathfrak{h} \oplus \mathbb{C}d \text{ and } \Lambda_i - \omega_i = \omega_i(h_\theta)\Lambda_0 \text{ for all } i \in I.$$

Hence, $\hat{P} = \mathbb{Z}\Lambda_0 \oplus P \oplus \mathbb{Z}\delta$. Given $\Lambda \in \hat{P}$, the number $\Lambda(c)$ is called the level of Λ. By (1.2.1), the level of Λ depends only on its class modulo the root lattice $\hat{Q}$. Set also $\hat{Q}^+ = \mathbb{Z}_{\geq 0}\hat{R}^+$ and let $\widehat{\mathcal{W}}$ denote the affine Weyl group, which is generated by the simple reflections $s_i, i \in \hat{I}$. Finally, observe that $\{\Lambda_0, \delta\} \cup \Delta$ is a basis of $\hat{\mathfrak{h}}^*$.

1.3. ***Integral forms and hyperalgebras.*** We use the following notation. Given a $\mathbb{Q}$-algebra U with unity, an element $x \in U$, and $k \in \mathbb{N}$, set

$$x^{(k)} = \frac{1}{k!}x^k \quad \text{and} \quad \binom{x}{k} = \frac{1}{k!}x(x-1)\cdots(x-k+1).$$

In the case $U = U(\tilde{\mathfrak{g}})$, we also introduce elements $\Lambda_{x,\pm r} \in U(\tilde{\mathfrak{g}}), x \in \mathfrak{g}, r \in \mathbb{N}$, by the following identity of power series in the variable u:

$$\Lambda^{\pm}_x(u) := \sum_{r\geq 0} \Lambda_{x,\pm r} u^r = \exp\biggl(-\sum_{s>0} \frac{x \otimes t^{\pm s}}{s} u^s\biggr).$$

Most of the time we will work with such elements with $x = h_\alpha$ for some $\alpha \in R^+$. We then simplify notation and write $\Lambda^{\pm}_\alpha(u) = \Lambda^{\pm}_{h_\alpha}(u)$, and if $\alpha = \alpha_i$ for some $i \in I$, we simply write $\Lambda^{\pm}_i(u) = \Lambda^{\pm}_{h_i}(u)$. To shorten notation, we also set $\Lambda_x(u) = \Lambda^{+}_x(u)$.

Consider the $\mathbb{Z}$-subalgebra $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ of $U(\hat{\mathfrak{g}}')$ generated by the set

$$\{(x^{\pm}_{\alpha,r})^{(k)} : \alpha \in R^+, r \in \mathbb{Z}, k \in \mathbb{N}\}.$$

By [Garland 1978, Theorem 5.8], it is a free $\mathbb{Z}$-submodule of $U(\hat{\mathfrak{g}}')$ and satisfies $\mathbb{C}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\hat{\mathfrak{g}}') = U(\hat{\mathfrak{g}}')$. In other words, $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ is an integral form of $U(\hat{\mathfrak{g}}')$. Moreover, the image of $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ in $U(\tilde{\mathfrak{g}})$ is an integral form of $U(\tilde{\mathfrak{g}})$ denoted by $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$. For a Lie subalgebra $\mathfrak{a}$ of $\hat{\mathfrak{g}}'$ set

$$U_{\mathbb{Z}}(\mathfrak{a}) = U(\mathfrak{a}) \cap U_{\mathbb{Z}}(\hat{\mathfrak{g}}'),$$

and similarly for subalgebras of $\tilde{\mathfrak{g}}$. The subalgebra $U_{\mathbb{Z}}(\mathfrak{g})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\hat{\mathfrak{g}})$ generated by $\{(x_\alpha^\pm)^{(k)} : \alpha \in R^+, k \in \mathbb{N}\}$. The subalgebra $U_{\mathbb{Z}}(\mathfrak{n}^\pm)$ of $U_{\mathbb{Z}}(\mathfrak{g})$ is generated, as a $\mathbb{Z}$-subalgebra, by the set $\{(x_\alpha^\pm)^{(k)} : \alpha \in R^+, k \in \mathbb{N}\}$ while $U_{\mathbb{Z}}(\mathfrak{h})$ is generated, as a $\mathbb{Z}$-subalgebra, by $\left\{\binom{h_i}{k} : i \in I, k \in \mathbb{N}\right\}$. Similarly, the subalgebra $U_{\mathbb{Z}}(\mathfrak{n}^\pm[t])$ of $U_{\mathbb{Z}}(\mathfrak{g}[t])$ is generated, as a $\mathbb{Z}$-subalgebra, by the set $\{(x_{\alpha,r}^\pm)^{(k)} : \alpha \in R^+, k \in \mathbb{N}, r \in \mathbb{Z}_{\geq 0}\}$ while $U_{\mathbb{Z}}(\mathfrak{h}[t]_+)$ is generated by $\{\Lambda_{i,r} : i \in I, r \in \mathbb{Z}_{>0}\}$. In fact, the latter is free commutative over the given set. The Poincaré–Birkhoff–Witt (PBW) theorem implies that multiplication establishes isomorphisms of $\mathbb{Z}$-modules

$$U_{\mathbb{Z}}(\hat{\mathfrak{g}}') \cong U_{\mathbb{Z}}(\hat{\mathfrak{n}}^-) \otimes U_{\mathbb{Z}}(\hat{\mathfrak{h}}') \otimes U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+),$$
$$U_{\mathbb{Z}}(\tilde{\mathfrak{g}}) \cong U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^-) \otimes U_{\mathbb{Z}}(\tilde{\mathfrak{h}}) \otimes U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+),$$
$$U_{\mathbb{Z}}(\mathfrak{g}[t]) \cong U_{\mathbb{Z}}(\mathfrak{n}^-[t]) \otimes U_{\mathbb{Z}}(\mathfrak{h}[t]) \otimes U_{\mathbb{Z}}(\mathfrak{n}^+[t]).$$

Moreover, restricted to $U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$ this gives rise to an isomorphism of $\mathbb{Z}$-algebras

$$U_{\mathbb{Z}}(\tilde{\mathfrak{h}}) \cong U_{\mathbb{Z}}(\mathfrak{h}[t]_-) \otimes U_{\mathbb{Z}}(\mathfrak{h}) \otimes U_{\mathbb{Z}}(\mathfrak{h}[t]_+).$$

In general, it may not be true that $U_{\mathbb{Z}}(\mathfrak{a})$ is an integral form of $U(\mathfrak{a})$. However, if $\mathfrak{a}$ has a basis consisting of real root vectors, an elementary use of the PBW theorem implies that this is true. We shall make use of algebras of this form later on.

Given a field $\mathbb{F}$, define the $\mathbb{F}$-hyperalgebra of $\mathfrak{a}$ by $U_{\mathbb{F}}(\mathfrak{a}) = \mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{a})$, where $\mathfrak{a}$ is any of the Lie algebras with $\mathbb{Z}$-forms defined above. Clearly, if the characteristic of $\mathbb{F}$ is zero, the algebra $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ is naturally isomorphic to $U(\tilde{\mathfrak{g}}_{\mathbb{F}})$ where $\tilde{\mathfrak{g}}_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} \tilde{\mathfrak{g}}_{\mathbb{Z}}$ and $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of the Chevalley basis of $\tilde{\mathfrak{g}}$, and similarly for all algebras $\mathfrak{a}$ we have considered. For fields of positive characteristic we just have an algebra homomorphism $U(\mathfrak{a}_{\mathbb{F}}) \to U_{\mathbb{F}}(\mathfrak{a})$ which is neither injective nor surjective. We will keep denoting by x the image of an element $x \in U_{\mathbb{Z}}(\mathfrak{a})$ in $U_{\mathbb{F}}(\mathfrak{a})$. Notice that we have $U_{\mathbb{F}}(\tilde{\mathfrak{g}}) = U_{\mathbb{F}}(\tilde{\mathfrak{n}}^-)U_{\mathbb{F}}(\tilde{\mathfrak{h}})U_{\mathbb{F}}(\tilde{\mathfrak{n}}^+)$.

Given an algebraically closed field $\mathbb{F}$, let $\mathbb{A}$ be a Henselian discrete valuation ring of characteristic zero having $\mathbb{F}$ as its residue field. Set $U_{\mathbb{A}}(\mathfrak{a}) = \mathbb{A} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{a})$. Clearly $U_{\mathbb{F}}(\mathfrak{a}) \cong \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{a})$. We shall also fix an algebraic closure $\mathbb{K}$ of the field of fractions of $\mathbb{A}$. For an explanation why we shall need to move from integral forms to $\mathbb{A}$-forms, see Remark 1.5.5 (and [Jakelić and Moura 2007, Section 4C]). As mentioned in the introduction, we assume the characteristic of $\mathbb{F}$ is either zero or at least 5 if $\mathfrak{g}$ is of type G_2.

Notice that the Hopf algebra structure of the universal enveloping algebras induce such structure on the hyperalgebras. For any Hopf algebra H, denote by H^0 its augmentation ideal.

1.4. ***The ℓ-weight lattice.*** For a ring A, we shall denote by $A^\times$ its set of unities. Consider the set $\mathcal{P}_{\mathbb{F}}^+$ consisting of $|I|$-tuples $\boldsymbol{\omega} = (\boldsymbol{\omega}_i)_{i \in I}$, where $\boldsymbol{\omega}_i \in \mathbb{F}[u]$ and

$\boldsymbol{\omega}_i(0) = 1$ for all $i \in I$. Endowed with coordinatewise polynomial multiplication, $\mathcal{P}_{\mathbb{F}}^+$ is a monoid. We denote by $\mathcal{P}_{\mathbb{F}}$ the multiplicative abelian group associated to $\mathcal{P}_{\mathbb{F}}^+$ which will be referred to as the ℓ-weight lattice associated to $\mathfrak{g}$. One can describe $\mathcal{P}_{\mathbb{F}}$ in another way. Given $\mu \in P$ and $a \in \mathbb{F}^\times$, let $\boldsymbol{\omega}_{\mu,a}$ be the element of $\mathcal{P}_{\mathbb{F}}$ defined as

$$(\boldsymbol{\omega}_{\mu,a})_i(u) = (1-au)^{\mu(h_i)} \text{ for all } i \in I.$$

If $\mu = \omega_i$ is a fundamental weight, we simplify notation and write $\boldsymbol{\omega}_{\omega_i,a} = \boldsymbol{\omega}_{i,a}$. We refer to $\boldsymbol{\omega}_{i,a}$ as a fundamental ℓ-weight, for all $i \in I$ and $a \in \mathbb{F}^\times$. Notice that $\mathcal{P}_{\mathbb{F}}$ is the free abelian group on the set of fundamental ℓ-weights. One defines $\mathcal{P}_{\mathbb{K}}$ in the obvious way. Let also $\mathcal{P}_{\mathbb{A}}^\times$ be the submonoid of $\mathcal{P}_{\mathbb{K}}^+$ generated by $\boldsymbol{\omega}_{i,a}$, $i \in I$, $a \in \mathbb{A}^\times$.

Let $\mathrm{wt} : \mathcal{P}_{\mathbb{F}} \to P$ be the unique group homomorphism such that $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$ for all $i \in I$, $a \in \mathbb{F}^\times$. Let also $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}^-$ be the unique group automorphism of $\mathcal{P}_{\mathbb{F}}$ mapping $\boldsymbol{\omega}_{i,a}$ to $\boldsymbol{\omega}_{i,a^{-1}}$ for all $i \in I$, $a \in \mathbb{F}^\times$. For notational convenience we set $\boldsymbol{\omega}^+ = \boldsymbol{\omega}$.

The abelian group $\mathcal{P}_{\mathbb{F}}$ can be identified with a subgroup of the monoid of $|I|$-tuples of formal power series with coefficients in $\mathbb{F}$ by identifying the rational function $(1-au)^{-1}$ with the corresponding geometric formal power series $\sum_{n\geq 0}(au)^n$. This allows us to define an inclusion $\mathcal{P}_{\mathbb{F}} \hookrightarrow U_{\mathbb{F}}(\tilde{\mathfrak{h}})^*$. Indeed, if $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}$ is such that $\boldsymbol{\omega}_i^\pm(u) = \sum_{r\geq 0} \omega_{i,\pm r} u^r \in \mathcal{P}_{\mathbb{F}}$, set

$$\boldsymbol{\omega}\left(\binom{h_i}{k}\right) = \binom{\mathrm{wt}(\boldsymbol{\omega})(h_i)}{k}, \quad \boldsymbol{\omega}(\Lambda_{i,r}) = \omega_{i,r}, \text{ for all } i \in I,\, r, k \in \mathbb{Z},\, k \geq 0,$$

and $\boldsymbol{\omega}(xy) = \boldsymbol{\omega}(x)\boldsymbol{\omega}(y)$, for all $x, y \in U_{\mathbb{F}}(\tilde{\mathfrak{h}})$.

1.5. ***Demazure and local Weyl modules.*** Given $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$, the local Weyl module $W_{\mathbb{F}}(\boldsymbol{\omega})$ is the quotient of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ by the left ideal generated by

$$U_{\mathbb{F}}(\tilde{\mathfrak{n}}^+)^0, \quad h - \boldsymbol{\omega}(h), \quad (x_\alpha^-)^{(k)} \text{ for all } h \in U_{\mathbb{F}}(\tilde{\mathfrak{h}}), \alpha \in R^+, k > \mathrm{wt}(\boldsymbol{\omega})(h_\alpha).$$

It is known that the local Weyl modules are finite-dimensional (see Theorem 3.2.1(c)).

For $\lambda \in P^+$, the graded local Weyl module $W_{\mathbb{F}}^c(\lambda)$ is the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}^c(\lambda)$ generated by

$$(1.5.1) \quad U_{\mathbb{F}}(\mathfrak{n}^+[t])^0, \quad U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0, \quad h - \lambda(h), \quad (x_\alpha^-)^{(k)} \quad \text{for all } h \in U_{\mathbb{F}}(\mathfrak{h}), \alpha \in R^+, k > \lambda(h_\alpha).$$

Also, given $\ell \geq 0$, let $D_{\mathbb{F}}(\ell, \lambda)$ denote the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}(\ell, \lambda)$ generated by $I_{\mathbb{F}}^c(\lambda)$ together with

$$(1.5.2) \qquad (x_{\alpha,s}^-)^{(k)} \text{ for all } \alpha \in R^+,\ s, k \in \mathbb{Z}_{\geq 0},\ k > \max\{0, \lambda(h_\alpha) - s\ell r_\alpha^\vee\}.$$

In particular, $D_{\mathbb{F}}(\ell, \lambda)$ is a quotient of $W_{\mathbb{F}}^c(\lambda)$.

The algebra $U_{\mathbb{F}}(\mathfrak{g}[t])$ inherits a $\mathbb{Z}$-grading from the grading on the polynomial algebra $\mathbb{C}[t]$. The ideals $I_{\mathbb{F}}^c(\lambda)$ and $I_{\mathbb{F}}(\ell,\lambda)$ are clearly graded and, hence, the modules $W_{\mathbb{F}}^c(\lambda)$ and $D_{\mathbb{F}}(\ell,\lambda)$ are graded. If V is a graded module, let $V[r]$ be its r-th graded piece. Given $m\in\mathbb{Z}$, let $\tau_m(V)$ be the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module such that $\tau_s(V)[r]=V[r-m]$ for all $r\in\mathbb{Z}$. Set

$$D_{\mathbb{F}}(\ell,\lambda,m)=\tau_m(D_{\mathbb{F}}(\ell,\lambda)).$$

Remark 1.5.1. Local Weyl modules were simply called Weyl modules in [Chari and Pressley 2001]. Certain infinite-dimensional modules, which were called maximal integrable modules in the same work, are now called global Weyl modules. The modern names, local and global Weyl modules were coined in [Feigin and Loktev 2004], where the authors introduced these modules in the context of generalized current algebras. We will not consider the global Weyl modules in this paper. We refer the reader to [Chari et al. 2010; Fourier et al. 2012; Fourier et al. 2014] and references therein for recent developments in the theory of global and local Weyl modules for (equivariant) map algebras. See also [Chamberlin 2013] for the initial steps in the study of the hyperalgebras of (equivariant) map algebras.

We are ready to state the main theorem of the paper.

Theorem 1.5.2. *Let $\lambda\in P^+$.*

(a) *If $\mathfrak{g}$ is simply laced, then $D_{\mathbb{F}}(1,\lambda)$ and $W_{\mathbb{F}}^c(\lambda)$ are isomorphic $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules.*

(b) *There exist $k\geq 1$, $m_j\in\mathbb{Z}_{\geq 0}$, and $\lambda_j\in P^+$, $j=1,\dots,k$, (independent of $\mathbb{F}$) such that the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module $W_{\mathbb{F}}^c(\lambda)$ admits a filtration $(0)=W_0\subset W_1\subset\cdots\subset W_{k-1}\subset W_k=W_{\mathbb{F}}^c(\lambda)$, with*

$$W_j/W_{j-1}\cong D_{\mathbb{F}}(1,\lambda_j,m_j).$$

(c) *For any $a\in\mathbb{F}^\times$, there exists an automorphism φ_a of $U_{\mathbb{F}}(\mathfrak{g}[t])$ such that the pull-back of $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$ by φ_a is isomorphic to $W_{\mathbb{F}}^c(\lambda)$.*

(d) *If $\boldsymbol{\omega}=\prod_{j=1}^m\boldsymbol{\omega}_{\lambda_j,a_j}$ for some $m\geq 0$, $\lambda_j\in P^+$, $a_j\in\mathbb{F}^\times$, $j=1,\dots,m$, with $a_i\neq a_j$ for $i\neq j$, then*

$$W_{\mathbb{F}}(\boldsymbol{\omega})\cong\bigotimes_{j=1}^m W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda_j,a_j}).$$

Assume the characteristic of $\mathbb{F}$ is zero. Then part (a) of this theorem was proved in [Chari and Pressley 2001] for $\mathfrak{g}=\mathfrak{sl}_2$, in [Chari and Loktev 2006] for type A, and in [Fourier and Littelmann 2007] for types ADE. Part (b) was proved in [Naoi 2012]. Part (c) for simply-laced $\mathfrak{g}$ was proved in [Fourier and Littelmann 2007] using part (a) (see Lemmas 1 and 3 and Equation (15) of that reference). The same proof works in the nonsimply laced case once part (b) is established. The last part was proved in [Chari and Pressley 2001]. We will make use of Theorem 1.5.2 in the characteristic zero setting for extending it to the positive characteristic context. Both [Chari and Loktev 2006] and [Fourier and Littelmann 2007] use the $\mathfrak{sl}_2$-case of part (a) in the proofs. A characteristic-free proof of Theorem 1.5.2(a) for $\mathfrak{sl}_2$ was given in [Jakelić and Moura 2014].

We will see in Section 3.5 that the class of modules $D_{\mathbb{F}}(\ell, \lambda)$ form a subclass of the class of Demazure modules. In particular, it follows from [Mathieu 1988, Lemme 8] that $\dim(D_{\mathbb{F}}(\ell, \lambda))$ depends only on ℓ and λ, but not on $\mathbb{F}$ (see also the remark on page 56 of [Mathieu 1989] and references therein). Together with Theorem 1.5.2(b), this implies the following corollary.

Corollary 1.5.3. *For all* $\lambda \in P^+$, *we have* $\dim W^c_{\mathbb{F}}(\lambda) = \dim W^c_{\mathbb{C}}(\lambda)$. □

As an application of this corollary, we will prove a conjecture of Jakelić and Moura, which we recall after quoting a theorem of theirs.

Theorem 1.5.4 [Jakelić and Moura 2007]. *Suppose* $\boldsymbol{\omega} \in \mathcal{P}^\times_{\mathbb{A}}$ *and let* $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, v *be the image of* 1 *in* $W_{\mathbb{K}}(\boldsymbol{\omega})$, *and* $L_{\mathbb{A}}(\boldsymbol{\omega}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. *Then* $L_{\mathbb{A}}(\boldsymbol{\omega})$ *is a free* $\mathbb{A}$*-module such that* $\mathbb{K} \otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega}) \cong W_{\mathbb{K}}(\boldsymbol{\omega})$. □

Let $\boldsymbol{\varpi}$ be the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. It easily follows that $\mathbb{F} \otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega})$ is a quotient of $W_{\mathbb{F}}(\boldsymbol{\varpi})$ and, hence,

$$\dim W_{\mathbb{K}}(\boldsymbol{\omega}) \leq \dim W_{\mathbb{F}}(\boldsymbol{\varpi}). \tag{1.5.3}$$

It was conjectured in [Jakelić and Moura 2007] that

$$\mathbb{F} \otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega}) \cong W_{\mathbb{F}}(\boldsymbol{\varpi}). \tag{1.5.4}$$

We will prove (1.5.4) in Section 3.4. In particular, it follows that

$$\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) = \dim W^c_{\mathbb{C}}(\lambda). \tag{1.5.5}$$

Remark 1.5.5. Theorem 1.5.2(d) was also conjectured in [Jakelić and Moura 2007] and it is false if $\mathbb{F}$ were not algebraically closed (see [Jakelić and Moura 2010] in that case). Observe that for all $\boldsymbol{\varpi} \in \mathcal{P}^+_{\mathbb{F}}$ there exists $\boldsymbol{\omega} \in \mathcal{P}^\times_{\mathbb{A}}$ such that $\boldsymbol{\varpi}$ is the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. This is the main reason for considering $\mathbb{A}$-forms instead of $\mathbb{Z}$-forms. The block decomposition of the categories of finite-dimensional representations of hyper loop algebras was established in [Jakelić and Moura 2007; 2010] assuming (1.5.4) and Theorem 1.5.2(d). The proof of one part of [Bianchi and Moura 2014, Theorem 4.1] also relies on these two results. Therefore, by proving (1.5.4) and Theorem 1.5.2(d), we confirm these results of [Bianchi and Moura 2014; Jakelić and Moura 2007; 2010]. A version of Theorem 1.5.2 for twisted affine Kac–Moody algebras was obtained in [Fourier and Kus 2013] in the characteristic-zero setting. We will consider the characteristic-free twisted version of Theorem 1.5.2 elsewhere.

2. Further notation and technical lemmas

2.1. ***Some commutation relations.*** We begin recalling the following well-known relation in $U_{\mathbb{Z}}(\mathfrak{g})$

$$(x_\alpha^+)^{(l)}(x_\alpha^-)^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x_\alpha^-)^{(k-m)}\binom{h_\alpha - k - l + 2m}{m}(x_\alpha^+)^{(l-m)} \tag{2.1.1}$$

for all $\alpha \in R^+, l, k \in \mathbb{Z}_{\geq 0}$. Since for all $\alpha \in R^+, s \in \mathbb{Z}$, the span of $x^{\pm}_{\alpha,\pm s}, h_\alpha$ is a subalgebra isomorphic to $\mathfrak{sl}_2$, we get the following relation in $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$

$$(x_{\alpha,s}^+)^{(l)}(x_{\alpha,-s}^-)^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x_{\alpha,-s}^-)^{(k-m)}\binom{h_\alpha - k - l + 2m}{m}(x_{\alpha,s}^+)^{(l-m)}. \tag{2.1.2}$$

Next, we consider the case when the grades of the elements in the left-hand side is not symmetric.

Given $m > 0$, consider the Lie algebra endomorphism τ_m of $\tilde{\mathfrak{g}}$ induced by the ring endomorphism of $\mathbb{C}[t, t^{-1}]$, $t \mapsto t^m$. Notice that the restriction of τ_m to $\mathfrak{g}[t]$ gives rise to an endomorphism of $\mathfrak{g}[t]$. Moreover, denoting by τ_m its extension to an algebra endomorphism of $U(\tilde{\mathfrak{g}})$, notice that $U_{\mathbb{Z}}(\mathfrak{a})$ is invariant under τ_m for $\mathfrak{a} = \mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, \tilde{\mathfrak{n}}^{\pm}, \tilde{\mathfrak{h}}, \mathfrak{n}^{\pm}[t], \mathfrak{h}[t], \mathfrak{h}[t]_+$. In fact $\tau_m((x^{\pm}_{\alpha,r})^{(k)}) = (x^{\pm}_{\alpha,mr})^{(k)}$ and $\tau_m(\Lambda_{\alpha,r})$ satisfies $\sum_{i\geq 0}\tau_m(\Lambda_{\alpha,r})u^r = \exp\bigl(-\sum_{s\geq 1}\frac{h_{\alpha,ms}}{s}u^s\bigr)$ for all $r, m \in \mathbb{Z}$ and $\alpha \in R^+$. Consider the power series

$$X^-_{\alpha,m,s}(u) = \sum_{r=1}^{\infty} x^-_{\alpha,m(r-1)+s}u^r \quad \text{and} \quad \Lambda^{\pm}_{\alpha,m}(u) = \tau_m(\Lambda^{\pm}_\alpha(u)).$$

Lemma 2.1.1. *Let $\alpha \in R^+, k, l \geq 0, m > 0, s \in \mathbb{Z}$. Then*

$$(x^+_{\alpha,m-s})^{(l)}(x^-_{\alpha,s})^{(k)} = (-1)^l\bigl((X^-_{\alpha,m,s}(u))^{(k-l)}\Lambda^+_{\alpha,m}(u)\bigr)_k \mod U_{\mathbb{Z}}(\tilde{\mathfrak{g}})U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+)^0,$$

where the subindex k denotes the coefficient of u^k of the above power series. Moreover, if $0 \leq s \leq m$, the same holds modulo $U_{\mathbb{Z}}(\mathfrak{g}[t])U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0_{\mathbb{Z}}$.

Proof. The case $m = 1, s = 0$ was proved in [Garland 1978, Lemma 7.5] (see [Jakelić and Moura 2007, Equation (1-11)]). Consider the Lie algebra endomorphism $\sigma_s : \tilde{\mathfrak{sl}}_\alpha \to \tilde{\mathfrak{sl}}_\alpha$ given by $x^{\pm}_{\alpha,r} \mapsto x^{\pm}_{\alpha,r\mp s}$. The first statement of the lemma is obtained from the case $m = 1, s = 0$ by applying $(\sigma_s \circ \tau_m)$. The second statement is then clear. □

Sometimes it will be convenient to work with a smaller set of generators for the hyperalgebras.

Proposition 2.1.2 [Mitzman 1985, Corollary 4.4.12]. *The ring $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ is generated by $(x_i^{\pm})^{(k)}, i \in \hat{I}, k \geq 0$ and $U_{\mathbb{Z}}(\mathfrak{g})$ is generated by $(x_i^{\pm})^{(k)}, i \in I, k \geq 0$.* □

2.2. ***On certain automorphisms of hyper current algebras.*** Let $\mathfrak{a}$, $\mathfrak{b}$ be such that $U_{\mathbb{Z}}(\mathfrak{a})$ have been defined. Then, given a homomorphism of $\mathbb{A}$-algebras $f: U_{\mathbb{A}}(\mathfrak{a}) \to U_{\mathbb{A}}(\mathfrak{b})$, we have an induced homomorphism $U_{\mathbb{F}}(\mathfrak{a}) \to U_{\mathbb{F}}(\mathfrak{b})$. We will now use this procedure to define certain homomorphism between hyperalgebras. As a rule, we shall use the same symbol to denote the induced homomorphism in the hyperalgebra level.

Recall that there exists a unique involutive Lie algebra automorphism ψ of $\mathfrak{g}$ such that $x_i^{\pm} \mapsto x_i^{\mp}$ and $h_i \mapsto -h_i$ for all $i \in I$. It admits a unique extension to an automorphism of $\mathfrak{g}[t]$ such that $\psi(x \otimes f(t)) = \psi(x) \otimes f(t)$ for all $x \in \mathfrak{g}$, $f \in \mathbb{C}[t]$. Keep denoting by ψ its extension to an automorphism of $U(\mathfrak{g}[t])$. In particular, it easily follows that

$$\psi((x_{\alpha,r}^{\pm})^{(k)}) = (x_{\alpha,r}^{\mp})^{(k)} \text{ for all } \alpha \in R^+, r, k \geq 0. \tag{2.2.1}$$

Since $U_{\mathbb{Z}}(\mathfrak{g}[t])$ is generated by the elements $(x_{\alpha,r}^{\pm})^{(k)}$, it follows that the restriction of ψ to $U_{\mathbb{Z}}(\mathfrak{a})$ induces an automorphism of $U_{\mathbb{Z}}(\mathfrak{a})$, for $\mathfrak{a} = \mathfrak{g}, \mathfrak{h}, \mathfrak{g}[t], \mathfrak{h}[t], \mathfrak{h}[t]_+$. We have an inclusion $P \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(U_{\mathbb{Z}}(\mathfrak{h}), \mathbb{Z})$ determined by

$$\mu\left(\binom{h_i}{k}\right) = \binom{\mu(h_i)}{k} \quad \text{and} \quad \mu(xy) = \mu(x)\mu(y) \quad \text{for all } i \in I, k \geq 0, x, y \in U_{\mathbb{Z}}(\mathfrak{h}). \tag{2.2.2}$$

Therefore,

$$\mu\left(\psi\left(\binom{h_i}{k}\right)\right) = \binom{-\mu(h_i)}{k} \text{ for all } i \in I, k > 0, \mu \in P. \tag{2.2.3}$$

Suppose now that γ is a Dynkin diagram automorphism of $\mathfrak{g}$ and keep denoting by γ the $\mathfrak{g}$-automorphism determined by $x_i^{\pm} \mapsto x_{\gamma(i)}^{\pm}$, $h_i \mapsto h_{\gamma(i)}$, $i \in I$. It admits a unique extension to an automorphism of $\mathfrak{g}[t]$ such that $\gamma(x \otimes f(t)) = \gamma(x) \otimes f(t)$ for all $x \in \mathfrak{g}$, $f \in \mathbb{C}[t]$. Keep denoting by γ its extension to an automorphism of $U(\mathfrak{g}[t])$. Let γ also denote the associated automorphism of P determined by $\gamma(\omega_i) = \omega_{\gamma(i)}$, $i \in I$. In particular, $\gamma(\alpha_i) = \alpha_{\gamma(i)}$, $i \in I$. It then follows that for each $\alpha \in R^+, k > 0$, there exist $\varepsilon_{\alpha,k}^{\pm} \in \{-1, 1\}$ (depending on how the Chevalley basis was chosen) such that

$$\gamma\left((x_{\alpha,r}^{\pm})^{(k)}\right) = \varepsilon_{\alpha,k}^{\pm}(x_{\gamma(\alpha),r}^{\pm})^{(k)} \text{ for all } r \geq 0. \tag{2.2.4}$$

This implies that the restriction of γ to $U_{\mathbb{Z}}(\mathfrak{a})$ induces an automorphism of $U_{\mathbb{Z}}(\mathfrak{a})$, for any $\mathfrak{a}$ in the set $\{\mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, \mathfrak{g}[t], \mathfrak{n}^{\pm}[t], \mathfrak{h}[t], \mathfrak{h}[t]_+\}$. It is also easy to see that

$$\mu\left(\gamma\left(\binom{h_i}{k}\right)\right) = \binom{(\gamma^{-1}(\mu))(h_i)}{k} \text{ for all } i \in I, k > 0, \mu \in P. \tag{2.2.5}$$

We conclude this subsection by constructing the automorphism mentioned in

Theorem 1.5.2(c). Thus, let $a \in \mathbb{F}$ and $\tilde{a} \in \mathbb{A}$ be such that the image of $\tilde{a}$ in $\mathbb{F}$ is a, and $\varphi_{\tilde{a}}$ the Lie algebra automorphism of $\mathfrak{g}[t]_{\mathbb{K}}$ given by $x \otimes t \mapsto x \otimes (t - \tilde{a})$. Keep denoting by $\varphi_{\tilde{a}}$ the induced automorphism of $U_{\mathbb{K}}(\mathfrak{g}[t])$ and observe that $\varphi_{\tilde{a}}$ is the identity on $U_{\mathbb{K}}(\mathfrak{g})$. One easily checks that

$$\varphi_{\tilde{a}}\big((x^{\pm}_{\alpha,r})^{(k)}\big) = \sum_{k_0+\cdots+k_r=k} \prod_{s=0}^{r} \binom{r}{s}^{k_s} (-\tilde{a})^{k_s(r-s)} (x^{\pm}_{\alpha,s})^{(k_s)} \in U_{\mathbb{A}}(\mathfrak{g}[t]).$$

Hence, $\varphi_{\tilde{a}}$ induces an automorphism of $U_{\mathbb{A}}(\mathfrak{g}[t])$. Notice that in the hyperalgebra level we have

$$(x^{\pm}_{\alpha,r})^{(k)} \mapsto \sum_{k_0+\cdots+k_r=k} \prod_{s=0}^{r} \binom{r}{s}^{k_s} (-a)^{k_s(r-s)} (x^{\pm}_{\alpha,s})^{(k_s)}. \tag{2.2.6}$$

This justifies a change of notation from $\varphi_{\tilde{a}}$ to φ_a.

2.3. ***Subalgebras of rank 1 and 2.*** For any $\alpha \in R^+$, consider the Lie subalgebra of $\mathfrak{g}$ generated by $x^{\pm}_{\alpha}$ which is isomorphic to $\mathfrak{sl}_2$. Denote this subalgebra by $\mathfrak{sl}_{\alpha}$. Consider also $\mathfrak{n}^{\pm}_{\alpha} = \mathbb{C}x^{\pm}_{\alpha}$, $\mathfrak{h}_{\alpha} = \mathbb{C}h_{\alpha}$ and $\mathfrak{b}^{\pm}_{\alpha} = \mathbb{C}h_{\alpha} \oplus \mathbb{C}x^{\pm}_{\alpha}$. Notice that $U_{\mathbb{Z}}(\mathfrak{g}) \cap U(\mathfrak{sl}_{\alpha})$ coincides with the $\mathbb{Z}$-subalgebra $U_{\mathbb{Z}}(\mathfrak{sl}_{\alpha})$ of $U(\mathfrak{g})$ generated by $(x^{\pm}_{\alpha})^{(k)}, k \geq 0$ (see details in [Macedo 2013]). This implies that $U_{\mathbb{Z}}(\mathfrak{g}) \cap U(\mathfrak{sl}_{\alpha})$ is naturally isomorphic to $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ and, hence, the corresponding subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_{\alpha})$ of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ is naturally isomorphic to $U_{\mathbb{F}}(\mathfrak{sl}_2)$. Similarly, for any $\alpha \in R^+, r \in \mathbb{Z}$, the Lie subalgebra $\mathfrak{sl}_{\alpha,r}$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,\pm r}$ is isomorphic to $\mathfrak{sl}_2$ and $U_{\mathbb{Z}}(\tilde{\mathfrak{g}}) \cap U(\mathfrak{sl}_{\alpha,r})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\tilde{\mathfrak{g}})$ generated by $(x^{\pm}_{\alpha,\pm r})^{(k)}, k \geq 0$. We shall denote the corresponding subalgebra of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ by $U_{\mathbb{F}}(\mathfrak{sl}_{\alpha,r})$. We also consider the subalgebra $\tilde{\mathfrak{sl}}_{\alpha}$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,r}, r \in \mathbb{Z}$ and the subalgebra $\mathfrak{sl}_{\alpha}[t]$ of $\mathfrak{g}[t]$ generated by $x^{\pm}_{\alpha,r}, r \geq 0$. The corresponding subalgebras $U_{\mathbb{F}}(\tilde{\mathfrak{sl}}_{\alpha})$ and $U_{\mathbb{F}}(\mathfrak{sl}_{\alpha}[t])$ of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ are naturally isomorphic to $U_{\mathbb{F}}(\tilde{\mathfrak{sl}}_2)$ and $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$.

We will also need to work with root subsystems of rank 2. Suppose $\alpha, \beta \in R^+$ form a simple system of a root subsystem R' of rank 2 and let $\mathfrak{t}$ denote a simple Lie algebra of type R'. Denote by $\mathfrak{g}_{\alpha,\beta}$ the subalgebra of $\mathfrak{g}$ generated by $x^{\pm}_{\alpha}$ and $x^{\pm}_{\beta}$, which is isomorphic to $\mathfrak{t}$. Notice that, for $r, s \in \mathbb{Z}$, the subalgebra $\mathfrak{g}^{r,s}_{\alpha,\beta}$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,\pm r}$ and $x^{\pm}_{\beta,\pm s}$ is also isomorphic to $\mathfrak{t}$. Let $U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$ be the subalgebra of $U_{\mathbb{Z}}(\mathfrak{g})$ generated by $(x^{\pm}_{\alpha})^{(k)}$, $(x^{\pm}_{\beta})^{(k)}$, $k \geq 0$, and $U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})$ the subalgebra of $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$ generated by $(x^{\pm}_{\alpha,\pm r})^{(k)}$, $(x^{\pm}_{\beta,\pm s})^{(k)}$, $k \geq 0$. Proposition 2.1.2 implies that $U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$ and $U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})$ are naturally isomorphic to $U_{\mathbb{Z}}(\mathfrak{t})$. Recall that if $\mathfrak{a}$ is a subalgebra of $U(\tilde{\mathfrak{g}})$, then $U_{\mathbb{Z}}(\mathfrak{a}) = U(\mathfrak{a}) \cap U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$. As in the rank-1 case, we have

$$U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta}) \quad \text{and} \quad U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta}). \tag{2.3.1}$$

The details can be found in [Macedo 2013]. It follows from (2.3.1) that $U_{\mathbb{F}}(\mathfrak{g}_{\alpha,\beta}) = \mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta}) \subseteq U_{\mathbb{F}}(\mathfrak{g})$ and $U_{\mathbb{F}}(\mathfrak{g}^{r,s}_{\alpha,\beta}) = \mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta}) \subseteq U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ are isomorphic to $U_{\mathbb{F}}(\mathfrak{t})$.

2.4. ***The algebra*** $\mathfrak{g}_{\mathrm{sh}}$**.** Another important subalgebra used in the proof of our main result is the subalgebra $\mathfrak{g}_{\mathrm{sh}}$ generated by the root vectors associated to short simple roots.

Let $\Delta_{\mathrm{sh}} = \{\alpha \in \Delta : (\alpha, \alpha) < 2\}$ denote the set of simple short roots. In particular, if $\mathfrak{g}$ is simply laced, $\Delta_{\mathrm{sh}} = \varnothing$. Let $R^+_{\mathrm{sh}} = \mathbb{Z}\Delta_{\mathrm{sh}} \cap R^+$ and $R_{\mathrm{sh}} = \mathbb{Z}\Delta_{\mathrm{sh}} \cap R$ (and notice that if $\mathfrak{g}$ is not simply laced, $R_{\mathrm{sh}} \neq \{\alpha \in R : (\alpha, \alpha) < 2\}$). Set $I_{\mathrm{sh}} = \{i \in I : \alpha_i \in \Delta_{\mathrm{sh}}\}$ and define $P_{\mathrm{sh}} = \bigoplus_{i \in I_{\mathrm{sh}}} \mathbb{Z}\omega_i$ and $P^+_{\mathrm{sh}} = P_{\mathrm{sh}} \cap P^+$. Consider also the subalgebras $\mathfrak{h}_{\mathrm{sh}} = \bigoplus_{i \in I_{\mathrm{sh}}} \mathbb{C}h_i$, $\mathfrak{b}^{\pm}_{\mathrm{sh}} = \mathfrak{h}_{\mathrm{sh}} \oplus \mathfrak{n}^{\pm}_{\mathrm{sh}}$, where $\mathfrak{n}^{\pm}_{\mathrm{sh}} = \bigoplus_{\pm\alpha \in R^+_{\mathrm{sh}}} \mathfrak{g}_{\alpha}$, and $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{n}^-_{\mathrm{sh}} \oplus \mathfrak{h}_{\mathrm{sh}} \oplus \mathfrak{n}^+_{\mathrm{sh}}$. Then if $\Delta_{\mathrm{sh}} \neq \varnothing$, $\mathfrak{g}_{\mathrm{sh}}$ is a simply laced Lie subalgebra of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_{\mathrm{sh}}$ and Δ_{sh} can be identified with the choice of simple roots associated to the given triangular decomposition. The subsets Q_{sh}, Q^+_{sh}, and the Weyl group $\mathcal{W}_{\mathrm{sh}}$ are defined in the obvious way. The restriction of $(\ ,\ )$ to $\mathfrak{g}_{\mathrm{sh}}$ is an invariant symmetric and nondegenerate bilinear form on $\mathfrak{g}_{\mathrm{sh}}$, but the normalization is not the same as the one we fixed for $\mathfrak{g}$. Indeed, $(\alpha, \alpha) = 2/r^{\vee}$ for all $\alpha \in R_{\mathrm{sh}}$. The set $\{x^{\pm}_{\alpha}, h_i : \alpha \in R^+_{\mathrm{sh}}, i \in I_{\mathrm{sh}}\}$ is a Chevalley basis for $\mathfrak{g}_{\mathrm{sh}}$.

Observe that $U_{\mathbb{Z}}(\mathfrak{g}) \cap U(\mathfrak{g}_{\mathrm{sh}})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ generated by $(x^{\pm}_{\alpha})^{(k)}$, $\alpha \in \Delta_{\mathrm{sh}}$; and, hence, Proposition 2.1.2 implies that $U_{\mathbb{F}}(\mathfrak{g}_{\mathrm{sh}})$ can be naturally identified with a subalgebra of $U_{\mathbb{F}}(\mathfrak{g})$. Similar observation apply to $U_{\mathbb{Z}}(\mathfrak{a})$ for $\mathfrak{a} = \mathfrak{n}^{\pm}_{\mathrm{sh}}, \mathfrak{h}_{\mathrm{sh}}$.

Consider the linear map $\mathfrak{h}^* \to \mathfrak{h}^*_{\mathrm{sh}}$, $\lambda \mapsto \bar{\lambda}$, given by restriction and let $i_{\mathrm{sh}} : \mathfrak{h}^*_{\mathrm{sh}} \to \mathfrak{h}^*$ be the linear map such that $i_{\mathrm{sh}}(\bar{\alpha}) = \alpha$ for all $\alpha \in \Delta_{\mathrm{sh}}$. In particular, $\overline{i_{\mathrm{sh}}(\mu)} = \mu$ for all $\mu \in \mathfrak{h}^*_{\mathrm{sh}}$. Given $\lambda \in P$, consider the function $\eta_{\lambda} : P_{\mathrm{sh}} \to P$ given by

$$\eta_{\lambda}(\mu) = i_{\mathrm{sh}}(\mu) + \lambda - i_{\mathrm{sh}}(\bar{\lambda}). \tag{2.4.1}$$

Lemma 2.4.1. *If $\lambda \in P^+$, $\mu \in P^+_{\mathrm{sh}}$, and $\mu \leq \bar{\lambda}$, then $\eta_{\lambda}(\mu) \in P^+$.*

Proof. For each $i \in I_{\mathrm{sh}}$, take $m_i \in \mathbb{Z}_{\geq 0}$ such that $\mu = \bar{\lambda} - \sum_{i \in I_{\mathrm{sh}}} m_i \bar{\alpha}_i$. In particular, $\eta_{\lambda}(\mu) = \lambda - \sum_{i \in I_{\mathrm{sh}}} m_i \alpha_i$. Then for $j \in I_{\mathrm{sh}}$ we have $\eta_{\lambda}(\mu)(h_j) = \mu(h_j) \geq 0$, while for $j \in I \setminus I_{\mathrm{sh}}$ we have $\eta_{\lambda}(\mu)(h_j) = \lambda(h_j) - \sum_{i \in I_{\mathrm{sh}}} m_i \alpha_i(h_j) \geq \lambda(h_j) \geq 0$. □

The affine Kac–Moody algebra associated to $\mathfrak{g}_{\mathrm{sh}}$ is naturally isomorphic to the subalgebra

$$\hat{\mathfrak{g}}_{\mathrm{sh}} := \mathfrak{g}_{\mathrm{sh}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

of $\hat{\mathfrak{g}}$, and under this isomorphism $\hat{\mathfrak{h}}_{\mathrm{sh}}$ is identified with $\mathfrak{h}_{\mathrm{sh}} \oplus \mathbb{C}c \oplus \mathbb{C}d$. The subalgebras $\mathfrak{g}_{\mathrm{sh}}[t]$ and $\hat{\mathfrak{n}}^{\pm}_{\mathrm{sh}}$, as well as $\hat{P}_{\mathrm{sh}}$, $\hat{Q}_{\mathrm{sh}}$, etc., are defined in the obvious way. Moreover, $U_{\mathbb{F}}(\tilde{\mathfrak{g}}_{\mathrm{sh}})$ and $U_{\mathbb{F}}(\mathfrak{g}_{\mathrm{sh}}[t])$ can be naturally identified with a subalgebra of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$.

3. Finite-dimensional modules

3.1. ***Modules for hyperalgebras.*** We now review the finite-dimensional representation theory of $U_{\mathbb{F}}(\mathfrak{g})$. If the characteristic of $\mathbb{F}$ is zero, then $U_{\mathbb{F}}(\mathfrak{g}) \cong U(\mathfrak{g}_{\mathbb{F}})$ and the results stated here can be found in [Humphreys 1978]. The literature for the positive characteristic setting is more often found in the context of algebraic groups, in which case $U_{\mathbb{F}}(\mathfrak{g})$ is known as the hyperalgebra or algebra of distributions of an algebraic group of the same Lie type as $\mathfrak{g}$ (see Part II of [Jantzen 2003]). A more detailed review in the present context can be found in [Jakelić and Moura 2007, Section 2].

Let V be a $U_{\mathbb{F}}(\mathfrak{g})$-module. A nonzero vector $v \in V$ is called a weight vector if there exists $\mu \in U_{\mathbb{F}}(\mathfrak{h})^*$ such that $hv = \mu(h)v$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$. The subspace consisting of weight vectors of weight μ is called a weight space of weight μ and it will be denoted by V_μ. If $V = \bigoplus_{\mu \in U_{\mathbb{F}}(\mathfrak{h})^*} V_\mu$, then V is said to be a weight module. If $V_\mu \neq 0$, μ is said to be a weight of V and $\mathrm{wt}(V) = \{\mu \in U_{\mathbb{F}}(\mathfrak{h})^* : V_\mu \neq 0\}$ is said to be the set of weights of V. Notice that the inclusion (2.2.2) induces an inclusion $P \hookrightarrow U_{\mathbb{F}}(\mathfrak{h})^*$. In particular, we can consider the partial order $\leq$ on $U_{\mathbb{F}}(\mathfrak{h})^*$ given by $\mu \leq \lambda$ if $\lambda - \mu \in Q^+$ and we have

$$(3.1.1) \qquad (x_\alpha^\pm)^{(k)} V_\mu \subseteq V_{\mu \pm k\alpha} \text{ for all } \alpha \in R^+, k > 0, \mu \in U_{\mathbb{F}}(\mathfrak{h})^*.$$

If V is a weight-module with finite-dimensional weight spaces, its character is the function $\mathrm{ch}(V) : U_{\mathbb{F}}(\mathfrak{h})^* \to \mathbb{Z}$ given by $\mathrm{ch}(V)(\mu) = \dim V_\mu$. As usual, if V is finite-dimensional, $\mathrm{ch}(V)$ can be regarded as an element of the group ring $\mathbb{Z}[U_{\mathbb{F}}(\mathfrak{h})^*]$ where we denote the element corresponding to $\mu \in U_{\mathbb{F}}(\mathfrak{h})^*$ by e^μ. By the inclusion (2.2.2) the group ring $\mathbb{Z}[P]$ can be regarded as a subring of $\mathbb{Z}[U_{\mathbb{F}}(\mathfrak{h})^*]$ and, moreover, the action of $\mathcal{W}$ on P induces an action of $\mathcal{W}$ on $\mathbb{Z}[P]$ by ring automorphisms where $w \cdot e^\mu = e^{w\mu}$.

If $v \in V$ is weight vector such that $(x_\alpha^+)^{(k)} v = 0$ for all $\alpha \in R^+$, $k > 0$, then v is said to be a highest-weight vector. If V is generated by a highest-weight vector, then it is said to be a highest-weight module. Similarly, one defines the notions of lowest-weight vectors and modules by replacing $(x_\alpha^+)^{(k)}$ by $(x_\alpha^-)^{(k)}$.

Theorem 3.1.1. *Let V be a $U_{\mathbb{F}}(\mathfrak{g})$-module.*

(a) *If V is finite-dimensional, then V is a weight-module,* $\mathrm{wt}(V) \subseteq P$, *and* $\dim V_\mu = \dim V_{\sigma\mu}$ *for all $\sigma \in \mathcal{W}$, $\mu \in U_{\mathbb{F}}(\mathfrak{h})^*$. In particular,* $\mathrm{ch}(V) \in \mathbb{Z}[P]^{\mathcal{W}}$.

(b) *If V is a highest-weight module of highest weight λ, then* $\dim(V_\lambda) = 1$ *and V_μ is nonzero only if $\mu \leq \lambda$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.*

(c) *For each $\lambda \in P^+$, the $U_{\mathbb{F}}(\mathfrak{g})$-module $W_{\mathbb{F}}(\lambda)$ given by the quotient of $U_{\mathbb{F}}(\mathfrak{g})$ by the left ideal $I_{\mathbb{F}}(\lambda)$ generated by*

$$U_{\mathbb{F}}(\mathfrak{n}^+)^0, \quad h-\lambda(h) \quad \textit{and} \quad (x_\alpha^-)^{(k)}, \ \textit{for all } h \in U_{\mathbb{F}}(\mathfrak{h}), \alpha \in R^+, k > \lambda(h_\alpha),$$

is nonzero and finite-dimensional. Moreover, every finite-dimensional highest-weight module of highest weight λ is a quotient of $W_{\mathbb{F}}(\lambda)$.

(d) *If V is finite-dimensional and irreducible, then there exists a unique $\lambda \in P^+$ such that V is isomorphic to the irreducible quotient $V_{\mathbb{F}}(\lambda)$ of $W_{\mathbb{F}}(\lambda)$. If the characteristic of $\mathbb{F}$ is zero, then $W_{\mathbb{F}}(\lambda)$ is irreducible.*

(e) *For each $\lambda \in P^+$, $\mathrm{ch}(W_{\mathbb{F}}(\lambda))$ is given by the Weyl character formula. In particular, $\mu \in \mathrm{wt}(W_{\mathbb{F}}(\lambda))$ if and only if $\sigma\mu \le \lambda$ for all $\sigma \in \mathcal{W}$. Moreover, $W_{\mathbb{F}}(\lambda)$ is a lowest-weight module with lowest weight $w_0\lambda$.* □

Remark 3.1.2. The module $W_{\mathbb{F}}(\lambda)$ defined in Theorem 3.1.1(c) is called the Weyl module (or costandard module) of highest weight λ. The known proofs of Theorem 3.1.1(e) make use of geometric results such as Kempf's Vanishing Theorem.

We shall need the following lemma in the proof of Lemma 5.2.5 below.

Lemma 3.1.3. *Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g})$-module, $\mu \in P$, and $\alpha \in R^+$. If $v \in V_\mu \setminus \{0\}$ is such that $(x_\alpha^-)^{(k)}v = 0$ for all $k > 0$, then $\mu(h_\alpha) \in \mathbb{Z}_{\le 0}$ and $(x_\alpha^+)^{(-\mu(h_\alpha))}v \ne 0$.* □

Remark 3.1.4. In characteristic zero, it is well known that the following stronger statement holds: if $v \in V_\mu \setminus \{0\}$ is such that $\mu(h_\alpha) \in \mathbb{Z}_{\le 0}$, then $(x_\alpha^+)^{(-\mu(h_\alpha))}v \ne 0$. In positive characteristic this stronger statement is not true for all finite-dimensional representations.

The next lemma can be proved exactly as in [Naoi 2012, Lemma 4.5].

Lemma 3.1.5. *Let $m_i \in \mathbb{Z}_{\ge 0}$, $i \in I$, V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{n}^-)$-module, and suppose $v \in V$ satisfies $(x_i^-)^{(k)}v = 0$ for all $i \in I$, $k > m_i$. Then, given $\alpha \in R^+$, we have $(x_\alpha^-)^{(k)}v = 0$ for all $k > \sum_{i\in I} n_i m_i$ where n_i are such that $h_\alpha = \sum_{i \in I} n_i h_i$.* □

3.2. ***Modules for hyper loop algebras.*** We now recall some basic results about the category of finite-dimensional $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-modules in the same spirit as Section 3.1. The results of this subsection can be found in [Jakelić and Moura 2007, Section 3] and references therein.

Given a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module V and $\xi \in U_{\mathbb{F}}(\tilde{\mathfrak{h}})^*$, let

$$V_\xi = \{v \in V : \text{ for all } x \in U_{\mathbb{F}}(\tilde{\mathfrak{h}}), \text{ there exists } k > 0 \text{ such that } (x - \xi(x))^k v = 0\}.$$

We say that V is an ℓ-weight module if $V = \bigoplus_{\boldsymbol{\omega}\in\mathcal{P}_{\mathbb{F}}} V_{\boldsymbol{\omega}}$. In this case, regarding V as a $U_{\mathbb{F}}(\mathfrak{g})$-module, we have

$$V_\mu = \bigoplus_{\substack{\boldsymbol{\omega}\in\mathcal{P}_{\mathbb{F}}:\\ \mathrm{wt}(\boldsymbol{\omega})=\mu}} V_{\boldsymbol{\omega}} \text{ for all } \mu \in P \quad\text{and}\quad V = \bigoplus_{\mu\in P} V_\mu.$$

A nonzero element of $V_{\boldsymbol{\omega}}$ is said to be an ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$. An ℓ-weight vector v is said to be a highest-ℓ-weight vector if $U_{\mathbb{F}}(\tilde{\mathfrak{h}})v = \mathbb{F}v$ and $(x^+_{\alpha,r})^{(k)}v = 0$ for all $\alpha \in R^+$ and all $r, k \in \mathbb{Z}$, $k > 0$. If V is generated by a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$, V is said to be a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$.

Theorem 3.2.1. *Let V be a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module.*

(a) *If V is finite-dimensional, then V is an ℓ-weight module. Moreover, if V is finite-dimensional and irreducible, then V is a highest-ℓ-weight module whose highest ℓ-weight lies in $\mathcal{P}^+_{\mathbb{F}}$.*

(b) *If V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}^+_{\mathbb{F}}$, then* $\dim V_{\boldsymbol{\omega}} = 1$ *and $V_\mu \neq 0$ only if $\mu \leq \mathrm{wt}(\boldsymbol{\omega})$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.*

(c) *For each $\boldsymbol{\omega} \in \mathcal{P}^+_{\mathbb{F}}$, the local Weyl module $W_{\mathbb{F}}(\boldsymbol{\omega})$ is nonzero and has finite dimension. Moreover, every finite-dimensional highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$ is a quotient of $W_{\mathbb{F}}(\boldsymbol{\omega})$.*

(d) *If V is finite-dimensional and irreducible, then there exists a unique $\boldsymbol{\omega} \in \mathcal{P}^+_{\mathbb{F}}$ such that V is isomorphic to the irreducible quotient $V_{\mathbb{F}}(\boldsymbol{\omega})$ of $W_{\mathbb{F}}(\boldsymbol{\omega})$.*

(e) *For $\mu \in P$ and $\boldsymbol{\omega} \in \mathcal{P}^+_{\mathbb{F}}$, μ is in $\mathrm{wt}(W_{\mathbb{F}}(\boldsymbol{\omega}))$ if and only if $\mu \in \mathrm{wt}(W_{\mathbb{F}}(\mathrm{wt}(\boldsymbol{\omega})))$, or equivalently if $w\mu \leq \mathrm{wt}(\boldsymbol{\omega})$ for all $w \in \mathcal{W}$.* □

3.3. ***Graded modules for hyper current algebras.*** Recall the following elementary fact.

Lemma 3.3.1. *Let A be a ring, $I \subset A$ a left ideal, $B = \mathbb{F} \otimes_{\mathbb{Z}} A$ an $\mathbb{F}$-algebra, and J the image of I in B, that is, J is the $\mathbb{F}$-span of $\{(1 \otimes a) \in B : a \in I\}$. Then $\mathbb{F} \otimes_{\mathbb{Z}} (A/I)$ is a left B-module, J is a left ideal of B, and we have an isomorphism of left B-modules $B/J \cong \mathbb{F} \otimes_{\mathbb{Z}} (A/I)$.* □

We shall use Lemma 3.3.1 with A being one of the integral forms so that B is the corresponding hyperalgebra.

Given $\lambda \in P^+$, let $I^c_{\mathbb{Z}}(\lambda) \subset U_{\mathbb{Z}}(\mathfrak{g}[t])$ be the left ideal generated by

$$U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0, \quad U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0, \quad h - \lambda(h), \quad (x^-_\alpha)^{(k)},$$

for all $h \in U_{\mathbb{Z}}(\mathfrak{h})$, $\alpha \in R^+$, $k > \lambda(h_\alpha)$, and set

$$W^c_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}(\mathfrak{g}[t])/I^c_{\mathbb{Z}}(\lambda).$$

Similarly, if $\ell \geq 0$ is also given, let $I_{\mathbb{Z}}(\ell, \lambda)$ be the left ideal of $U_{\mathbb{Z}}(\mathfrak{g}[t])$ generated by

$$U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0, \quad U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0, \quad h - \lambda(h), \quad (x^-_{\alpha,s})^{(k)},$$
$$\text{for all } h \in U_{\mathbb{Z}}(\mathfrak{h}), \alpha \in R^+, s, k \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda(h_\alpha) - r^\vee_\alpha \ell s\}.$$

Then set

$$D_{\mathbb{Z}}(\ell, \lambda) = U_{\mathbb{Z}}(\mathfrak{g}[t])/I_{\mathbb{Z}}(\ell, \lambda).$$

Notice that $W^c_{\mathbb{Z}}(\lambda)$ and $D_{\mathbb{Z}}(\ell, \lambda)$ are weight modules.

Since the ideals defining $W^c_{\mathbb{F}}(\lambda)$ and $D_{\mathbb{F}}(\ell, \lambda)$ (see Section 1.5) are the images of $I^c_{\mathbb{Z}}(\lambda)$ and $I_{\mathbb{Z}}(\ell, \lambda)$ in $U_{\mathbb{F}}(\mathfrak{g}[t])$, respectively, an application of Lemma 3.3.1 gives isomorphisms of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules

$$W^c_{\mathbb{F}}(\lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} W^c_{\mathbb{Z}}(\lambda) \quad \text{and} \quad D_{\mathbb{F}}(\ell, \lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} D_{\mathbb{Z}}(\ell, \lambda).$$

As before, $D_{\mathbb{Z}}(\ell, \lambda)$ is a quotient of $W^c_{\mathbb{Z}}(\lambda)$ for all $\lambda \in P^+$ and all $\ell > 0$. We shall see next (Proposition 3.3.2) that the latter is a finitely generated $\mathbb{Z}$-module and, hence, so is the former. Together with Corollary 1.5.3, this implies that

(3.3.1) $$D_{\mathbb{Z}}(\ell, \lambda) \text{ is a free } \mathbb{Z}\text{-module.}$$

The proof of the next proposition is an adaptation of that of [Jakelić and Moura 2007, Theorem 3.11]. The extra details can be found in [Macedo 2013].

Proposition 3.3.2. *For every $\lambda \in P^+$, the $U_{\mathbb{Z}}(\mathfrak{g}[t])$-module $W^c_{\mathbb{Z}}(\lambda)$ is a finitely generated $\mathbb{Z}$-module.*

We now prove an analogue of Theorem 1.5.4 for graded local Weyl modules.

Corollary 3.3.3. *Let $\lambda \in P^+$ and v be the image of 1 in $W^c_{\mathbb{C}}(\lambda)$. Then $U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is a free $\mathbb{Z}$-module of rank $\dim(W^c_{\mathbb{C}}(\lambda))$. Moreover, $U_{\mathbb{Z}}(\mathfrak{g}[t])v = \bigoplus_{\mu \in P}(U_{\mathbb{Z}}(\mathfrak{g}[t])v \cap W^c_{\mathbb{C}}(\lambda)_\mu)$. In particular, $U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is an integral form for $W^c_{\mathbb{C}}(\lambda)$.*

Proof. To simplify notation, set $L = U_{\mathbb{Z}}(\mathfrak{n}^-)v$. Let also ϑ be as in the proof of Proposition 3.3.2. Since v satisfies the relations satisfied by ϑ, it follows that there exists an epimorphism of $U_{\mathbb{Z}}(\mathfrak{g}[t])$-modules $W_{\mathbb{Z}}(\lambda) \to L$, $\vartheta \mapsto v$. Since $W_{\mathbb{Z}}(\lambda)$ is finitely generated, it follows that so is L. On the other hand, since $L \subseteq W^c_{\mathbb{C}}(\lambda)$, it is also torsion free and, hence, a free $\mathbb{Z}$-module of finite rank. Since $U_{\mathbb{Z}}(\mathfrak{n}^-)$ spans $U(\mathfrak{n}^-)$ and $W^c_{\mathbb{C}}(\lambda) = U(\mathfrak{n}^-)v$, it follows that L contains a basis of $W^c_{\mathbb{C}}(\lambda)$. This implies that the rank of L is at least $\dim(W^c_{\mathbb{C}}(\lambda))$. On the other hand, $\mathbb{C} \otimes_{\mathbb{Z}} L$ is a $\mathfrak{g}[t]$-module generated by the vector $1 \otimes v$ which satisfies the relations (1.5.1). Therefore, it is a quotient of $W^c_{\mathbb{C}}(\lambda)$. Since $\dim(\mathbb{C} \otimes_{\mathbb{Z}} L) = \operatorname{rank}(L)$, the first and the last statements follow. The second statement is clear since L is obviously a weight module. □

Consider the category $\mathscr{G}_{\mathbb{F}}$ of $\mathbb{Z}$-graded finite-dimensional representations of $U_{\mathbb{F}}(\mathfrak{g}[t])$. Recall the functors τ_m defined in the paragraph preceding Remark 1.5.1. For each $U_{\mathbb{F}}(\mathfrak{g})$-module V, let $\mathrm{ev}_0(V)$ be the module in $\mathscr{G}_{\mathbb{F}}$ obtained by extending the action of $U_{\mathbb{F}}(\mathfrak{g})$ to one of $U_{\mathbb{F}}(\mathfrak{g}[t])$ on V by setting $U_{\mathbb{F}}(\mathfrak{g}[t]_+)V = 0$. For $\lambda \in P^+$, $r \in \mathbb{Z}$, set $V_{\mathbb{F}}(\lambda, r) = \mathrm{ev}_r(V_{\mathbb{F}}(\lambda))$ where $\mathrm{ev}_r = \tau_r \circ \mathrm{ev}_0$.

Theorem 3.3.4. *Let $\lambda \in P^+$.*

(a) *If $V \in \mathscr{G}_{\mathbb{F}}$ is simple, then it is isomorphic to $V_{\mathbb{F}}(\lambda, r)$ for unique $(\lambda, r) \in P^+ \times \mathbb{Z}$.*

(b) *$W^c_{\mathbb{F}}(\lambda)$ is finite-dimensional.*

(c) *If V is a graded finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module generated by a weight vector v of weight λ satisfying $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$, then V is a quotient of $W^c_{\mathbb{F}}(\lambda)$.*

Proof. To prove part (a), suppose $V \in \mathscr{G}_{\mathbb{F}}$ is simple. If $V[r], V[s] \neq 0$ for $s < r \in \mathbb{Z}$, $(\bigoplus_{k \geq r} V[k])$ would be a proper submodule of V, contradicting the fact that it is simple. Thus there must exist a unique $r \in \mathbb{Z}$ such that $V[r] \neq 0$. Since $U_{\mathbb{F}}(\mathfrak{g}[t]_+)$ changes degrees, $V = V[r]$ must be a simple $U_{\mathbb{F}}(\mathfrak{g})$-module. This shows that $V \cong V_{\mathbb{F}}(\lambda, r)$ for some $\lambda \in P^+$, $r \in \mathbb{Z}$.

To prove part (b), observe that $W^c_{\mathbb{F}}(\lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} W^c_{\mathbb{Z}}(\lambda)$ (see Lemma 3.3.1). Thus the dimension of $W^c_{\mathbb{F}}(\lambda)$ must be at most the number of generators of $W^c_{\mathbb{Z}}(\lambda)$, which is proved to be finite in Proposition 3.3.2.

To prove part (c), observe that the $U_{\mathbb{F}}(\mathfrak{g})$-submodule $V' = U_{\mathbb{F}}(\mathfrak{g})v \subseteq V$ is a finite-dimensional highest-weight module of highest weight λ. Thus, by Theorem 3.1.1(c), V' is a quotient of $W_{\mathbb{F}}(\lambda)$. The statement follows by comparing the defining relations of V and $W^c_{\mathbb{F}}(\lambda)$. □

Remark 3.3.5. Denote by v the image of 1 in $W^c_{\mathbb{F}}(\lambda)$. From the defining relations (1.5.1) it follows that $\mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is a quotient of $W^c_{\mathbb{F}}(\lambda)$. It follows from Theorem 1.5.2(b) that $\mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}[t])v \cong W^c_{\mathbb{F}}(\lambda)$ for all $\lambda \in P^+$ (see Section 3.4 below). Moreover, since $\mathbb{F} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}(\lambda) \cong W^c_{\mathbb{F}}(\lambda)$, Theorem 1.5.2(b) also implies that $W_{\mathbb{Z}}(\lambda)$ is free.

3.4. ***Proof of* (1.5.4).** The argument of the proof will use Corollary 1.5.3, the characteristic-zero version of parts (c) and (d) of Theorem 1.5.2, and the following proposition.

Proposition 3.4.1 [Naoi 2012, Corollary A]. *Let $\lambda \in P^+$. Then* $\dim W^c_{\mathbb{C}}(\lambda) = \prod_{i \in I} (\dim W^c_{\mathbb{C}}(\omega_i))^{\lambda(h_i)}$. □

We shall also need the following general construction. Given a $\mathbb{Z}_{s \geq 0}$-filtered $U_{\mathbb{F}}(\mathfrak{g}[t])$-module W, we can consider the associated $U_{\mathbb{F}}(\mathfrak{g}[t])$-module $\mathrm{gr}(W) = \bigoplus_{s \geq 0} W_s / W_{s-1}$, which obviously has the same dimension as W. Suppose now that W is any cyclic $U_{\mathbb{F}}(\mathfrak{g}[t])$-module and fix a generator w. Then the $\mathbb{Z}$-grading on $U_{\mathbb{F}}(\mathfrak{g}[t])$ induces a filtration on W. Namely, set w to have degree zero and define the s-th filtered piece of W by $W_s = F^s U_{\mathbb{F}}(\mathfrak{g}[t])w$ where $F^s U_{\mathbb{F}}(\mathfrak{g}[t]) = \bigoplus_{r\leq s} U_{\mathbb{F}}(\mathfrak{g}[t])[r]$. Then $\mathrm{gr}(W)$ is cyclic since it is generated by the image of w in $\mathrm{gr}(W)$.

Recall the notation fixed for (1.5.4): $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{\times}$, $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, $\boldsymbol{\varpi}$ is the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. Also recall that, using (1.5.3), (1.5.4) will be proved if we show that

$$\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) \leq \dim W_{\mathbb{K}}(\boldsymbol{\omega}).$$

Fix $w \in W_{\mathbb{F}}(\boldsymbol{\varpi})_\lambda \setminus \{0\}$. Not only does w generate $W_{\mathbb{F}}(\boldsymbol{\varpi})$ as a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module, but it also follows from the proof of [Jakelić and Moura 2007, Theorem 3.11] (with a correction incorporated in the proof of [Jakelić and Moura 2010, Theorem 3.7]) that $U_{\mathbb{F}}(\mathfrak{n}^-[t])w = W_{\mathbb{F}}(\boldsymbol{\varpi})$. Hence, we can apply the general construction reviewed above to $W_{\mathbb{F}}(\boldsymbol{\varpi})$. Set $V = \mathrm{gr}(W_{\mathbb{F}}(\boldsymbol{\varpi}))$ and denote the image of w in V by v. The module V is finite-dimensional and v is a highest-weight vector of weight λ satisfying $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ (the latter follows since $\dim(V_\lambda) = 1$, V is graded, and $U_{\mathbb{F}}(\mathfrak{h}[t])$ is commutative). Hence, v satisfies the defining relations (1.5.1) of $W_{\mathbb{F}}^c(\lambda)$. In particular,

$$\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) \leq \dim W_{\mathbb{F}}^c(\lambda).$$

Since $\dim W_{\mathbb{F}}^c(\lambda) = \dim W_{\mathbb{K}}^c(\lambda)$ by Corollary 1.5.3, it now suffices to show that

$$\dim W_{\mathbb{K}}^c(\lambda) = \dim W_{\mathbb{K}}(\boldsymbol{\omega}).$$

For proving this, consider the decomposition of $\boldsymbol{\omega}$ of the form

$$\boldsymbol{\omega} = \prod_{j=1}^{m} \boldsymbol{\omega}_{\lambda_j, a_j}$$

for some $m \geq 0$, $a_j \in \mathbb{K}^\times$, $a_i \neq a_j$ for $i \neq j$, $\lambda_j \in P^+$ such that $\lambda = \sum_{j=1}^m \lambda_j$. By Theorem 1.5.2(d), in characteristic zero, $W_{\mathbb{K}}(\boldsymbol{\omega}) \cong \otimes_{j=1}^m W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda_j,a_j})$. In characteristic zero, Theorem 1.5.2(c) implies that $\dim W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda_j,a_j}) = \dim W_{\mathbb{K}}^c(\lambda_j)$. Hence,

$$\dim W_{\mathbb{K}}(\boldsymbol{\omega}) = \prod_{j=1}^{m} \dim W_{\mathbb{K}}^c(\lambda_j) = \prod_{j=1}^{m}\prod_{i\in I} \dim W_{\mathbb{K}}^c(\omega_i)^{\lambda_j(h_i)} = \prod_{i\in I} W_{\mathbb{K}}^c(\omega_i)^{\lambda(h_i)} = \dim W_{\mathbb{K}}^c(\lambda).$$

Here, the second and last equality follow from Proposition 3.4.1 and the others are clear. This completes the proof of (1.5.4).

Notice that all equalities of dimensions proved here actually imply the corresponding equalities of characters. In particular, it follows that

$$\mathrm{ch}(W_{\mathbb{F}}(\boldsymbol{\varpi})) = \prod_{i\in I} (\mathrm{ch}(W_{\mathbb{C}}^c(\omega_i)))^{\mathrm{wt}(\boldsymbol{\varpi})(h_i)} \quad \text{for all } \boldsymbol{\varpi} \in \mathcal{P}_{\mathbb{F}}^+. \tag{3.4.1}$$

3.5. ***Joseph–Mathieu–Polo relations for Demazure modules.*** We now explain the reason we call the module $D_{\mathbb{F}}(\ell,\lambda)$ a Demazure module. We begin with the following lemma. Let γ be the Dynkin diagram automorphism of $\mathfrak{g}$ induced by w_0 and recall from Section 2.2 that it induces an automorphism of $U_{\mathbb{F}}(\mathfrak{g}[t])$ also denoted by γ.

Lemma 3.5.1. *Let $\lambda\in P^+$, $\ell\geq 0$, and set $\lambda^*=-w_0\lambda$. Let W be the pull-back of $D_{\mathbb{F}}(\ell,\lambda^*)$ by γ. Then $D_{\mathbb{F}}(\ell,\lambda)\cong W$.*

Proof. Let $v\in D_{\mathbb{F}}(\ell,\lambda^*)_{\lambda^*}\setminus\{0\}$. By (1.5.1) and (1.5.2) we have

$$U_{\mathbb{F}}(\mathfrak{n}^+[t])^0v=U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0v=0,\quad hv=\lambda^*(h)v,\quad (x^-_{\alpha,s})^{(k)}v=0,$$

for all $h\in U_{\mathbb{F}}(\mathfrak{h})$, $\alpha\in R^+$, $s,k\in\mathbb{Z}_{\geq 0}$, $k>\max\{0,\lambda^*(h_\alpha)-s\ell r^\vee_\alpha\}$. Denote by w the vector v regarded as an element of W. Evidently, $W=U_{\mathbb{F}}(\mathfrak{g}[t])w$. Since γ restricts to automorphisms of $U_{\mathbb{F}}(\mathfrak{n}^+[t])$ and of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)$, it follows that $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0w=U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0w=0$, while (2.2.5) implies that $w\in W_\lambda$. Finally, (2.2.4) and (2.2.5) together imply that

$$(x^-_{\alpha,s})^{(k)}w=0\ \text{ for all } \alpha\in R^+,\ s,k\in\mathbb{Z}_{\geq 0},\ k>\max\{0,\lambda(h_\alpha)-s\ell r^\vee_\alpha\}.$$

This shows that w satisfies the defining relations of $D_{\mathbb{F}}(\ell,\lambda)$ and, hence, there exists an epimorphism from $D_{\mathbb{F}}(\ell,\lambda)$ onto W. Since $(\lambda^*)^*=\lambda$, reversing the roles of λ and λ^* we get an epimorphism on the other direction. Since these are finite-dimensional modules, we are done. □

In order to continue, we need the concepts of weight vectors, weight spaces, weight modules and integrable modules for $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$ which are similar to those for $U_{\mathbb{F}}(\mathfrak{g})$ (see Section 3.1) by replacing I with $\hat{I}$ and P with $\hat{P}'$. Also, using the obvious analogue of (2.2.2), we obtain an inclusion $\hat{P}'\hookrightarrow U_{\mathbb{F}}(\hat{\mathfrak{h}}')^*$. Let V be a $\mathbb{Z}$-graded $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module whose weights lie in $\hat{P}'$. As before, let $V[r]$ denote the r-th graded piece of V. For $\mu\in\hat{P}$, say $\mu=\mu'+m\delta$ with $\mu'\in\hat{P}'$, $m\in\mathbb{Z}$, set

$$V_\mu=\{v\in V[m]:hv=\mu'(h)v \text{ for all } h\in U_{\mathbb{F}}(\hat{\mathfrak{h}}')\}.$$

If $V_\mu\neq 0$ we shall say that μ is a weight of V and let $\operatorname{wt}(V)=\{\mu\in\hat{P}:V_\mu\neq 0\}$.

We record the following partial affine analogue of Theorem 3.1.1.

Theorem 3.5.2. *Let V be a graded $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module.*

(a) *If V is integrable, then V is a weight-module and $\operatorname{wt}(V)\subseteq\hat{P}$. Moreover, $\dim V_\mu=\dim V_{\sigma\mu}$ for all $\sigma\in\widehat{\mathcal{W}}$, $\mu\in\hat{P}$.*

(b) *If V is a highest-weight module of highest weight λ, $\dim(V_\lambda)=1$ and $V_\mu\neq 0$ only if $\mu\leq\lambda$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.*

(c) *Let $\Lambda \in \hat{P}^+$ and $m = \Lambda(d)$. Then the $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module $\hat{W}_{\mathbb{F}}(\Lambda)$ generated by a vector v of degree m satisfying the defining relations*

$$U_{\mathbb{F}}(\hat{\mathfrak{n}}^+)^0 v = 0, \quad hv = \Lambda(h)v \quad \textit{and} \quad (x_i^-)^{(k)} v = 0,$$

for all $h \in U_{\mathbb{F}}(\hat{\mathfrak{h}}')$, $i \in \hat{I}$, $k > \Lambda(h_i)$, is nonzero and integrable. Moreover, for every positive real root α, we have

$$(x_\alpha^-)^{(k)} v = 0 \ \textit{for all } k > \Lambda(h_\alpha). \tag{3.5.1}$$

Furthermore, every integrable highest-weight module of highest weight Λ is a quotient of $\hat{W}_{\mathbb{F}}(\Lambda)$. □

Given $\Lambda \in \hat{P}^+$, $\sigma \in \widehat{\mathcal{W}}$, the Demazure module $V_{\mathbb{F}}^\sigma(\Lambda)$ is defined as the $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-submodule generated by $\hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda}$ (see [Fourier and Littelmann 2007; Mathieu 1989; Naoi 2012]). In particular, $V_{\mathbb{F}}^\sigma(\Lambda) \cong V_{\mathbb{F}}^{\sigma'}(\Lambda)$ if $\sigma\Lambda = \sigma'\Lambda$ for some $\sigma' \in \widehat{\mathcal{W}}$. Our focus is on the Demazure modules which are stable under the action of $U_{\mathbb{F}}(\mathfrak{g})$. Since $V_{\mathbb{F}}^\sigma(\Lambda)$ is defined as a $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-module, it is stable under the action of $U_{\mathbb{F}}(\mathfrak{g})$ if, and only if,

$$U_{\mathbb{F}}(\mathfrak{n}^-)^0 \hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda} = 0. \tag{3.5.2}$$

In particular, since $V_{\mathbb{F}}^\sigma(\Lambda)$ is an integrable $U_{\mathbb{F}}(\mathfrak{sl}_\alpha)$-module for any $\alpha \in R^+$, it follows that $(\sigma\Lambda)(h_\alpha) \le 0$ for all $\alpha \in R^+$. Conversely, using the exchange condition for Coxeter groups (see [Humphreys 1990, Section 5.8]), one easily deduces that, for all $i \in \hat{I}$, we have

$$(x_i^\varepsilon)^{(k)} \hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda} = 0 \ \text{ for all } k > 0$$

where $\varepsilon = +$ if $\sigma\Lambda(h_i) \ge 0$ and $\varepsilon = -$ if $\sigma\Lambda(h_i) \le 0$. This implies that if $\sigma\Lambda(h_i) \le 0$ for all $i \in I$, then $V_{\mathbb{F}}^\sigma(\Lambda)$ is $U_{\mathbb{F}}(\mathfrak{g})$-stable. Thus, henceforth, assume $(\sigma\Lambda)(h_i) \le 0$ for all $i \in I$ and observe that this implies that $\sigma\Lambda$ must have the form

$$\sigma\Lambda = \ell\Lambda_0 + w_0\lambda + m\delta \ \text{ for some } \lambda \in P^+, m \in \mathbb{Z}, \text{ and } \ell = \Lambda(c). \tag{3.5.3}$$

Conversely, given $\ell \in \mathbb{Z}_{\ge 0}$, $\lambda \in P^+$, and $m \in \mathbb{Z}$, since $\widehat{\mathcal{W}}$ acts simply transitively on the set of alcoves of $\hat{\mathfrak{h}}^*$ (see [Humphreys 1990, Theorem 4.5.(c)]), there exists a unique $\Lambda \in \hat{P}^+$ such that $\ell\Lambda_0 + w_0\lambda + m\delta \in \widehat{\mathcal{W}}\Lambda$. Thus, if $\sigma \in \widehat{\mathcal{W}}$ and $\Lambda \in \hat{P}^+$ are such that

$$\sigma\Lambda = \ell\Lambda_0 + w_0\lambda + m\delta, \tag{3.5.4}$$

then $V_{\mathbb{F}}^\sigma(\Lambda)$ is $U_{\mathbb{F}}(\mathfrak{g})$-stable. Henceforth, we fix σ, Λ, w_0, λ, and m as in (3.5.4). Notice that if $\gamma = \pm\alpha + s\delta \in \hat{R}^+$ with $\alpha \in R^+$, then

$$\sigma\Lambda(h_\gamma) = \pm w_0\lambda(h_\alpha) + s\ell r_\alpha^\vee.$$

The following lemma is a rewriting of [Mathieu 1989, Lemme 26] using the above fixed notation.

Lemma 3.5.3. *The $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^{+})$-module $V_{\mathbb{F}}^{\sigma}(\Lambda)$ is isomorphic to the $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^{+})$-module generated by a vector v of degree m satisfying the following defining relations*: $hv = \sigma\Lambda(h)v$, $h \in U_{\mathbb{F}}(\hat{\mathfrak{h}}')$, $U_{\mathbb{F}}(\mathfrak{h}[t]_{+})^{0}v = U_{\mathbb{F}}(\mathfrak{n}^{-}[t]_{+})^{0}v = 0$, *and*

(3.5.5) $(x_{\alpha,s}^{+})^{(k)}v = 0$ *for all* $\alpha \in R^{+}, s \geq 0, k > \max\{0, -w_0\lambda(h_\alpha) - s\ell r_\alpha^\vee\}$. □

Remark 3.5.4. Mathieu [1989] attributes Lemma 3.5.3 to Joseph and Polo. This is the reason for the title of this subsection. The original version of this lemma in [Mathieu 1989] gives generator and relations for any Demazure module, not only for the $U_{\mathbb{F}}(\mathfrak{g})$-stable ones.

The following is the main result of this subsection.

Proposition 3.5.5. *The graded $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules $V_{\mathbb{F}}^{\sigma}(\Lambda)$ and $D_{\mathbb{F}}(\ell, \lambda, m)$ are isomorphic.*

Proof. It suffices to prove the statement for $m = 0$, so for simplicity we assume that this is the case. Proceeding as in [Fourier and Littelmann 2007, Corollary 1] (see also [Naoi 2012, Proposition 3.6]) we show that $V_{\mathbb{F}}^{\sigma}(\Lambda)$ is a quotient of $D_{\mathbb{F}}(\ell, \lambda)$. Namely, let v be a nonzero vector in $\hat{W}_{\mathbb{F}}(\Lambda)_\mu$ where $\mu = w_0\sigma\Lambda$. Quite clearly v generates $V_{\mathbb{F}}^{\sigma}(\Lambda)$. It follows that v is an extremal weight vector and, hence, satisfies the relations

$$(x_\gamma^{\pm})^{(k)}v = 0 \;\text{ for all } k > \max\{0, \mp\mu(h_\gamma)\} \tag{3.5.6}$$

and all positive real roots γ. In particular, taking $\gamma = \alpha + s\delta$ with $\alpha \in R^{+}$ and $s \geq 0$, it follows that

$$-\mu(h_\gamma) = -\lambda(h_\alpha) - \ell r_\alpha^\vee s \leq 0,$$

showing $(x_{\alpha,s}^{+})^{(k)}v = 0$ for all $k > 0$. Similarly, taking $\gamma = -\alpha + s\delta$, we get

$$-\mu(h_\gamma) = \lambda(h_\alpha) - \ell r_\alpha^\vee s,$$

which shows that v satisfies the relations determined by (1.5.2). It remains to be shown that $U_{\mathbb{F}}(\mathfrak{h}[t]_{+})^{0}v = 0$. This can be proved as in [Mathieu 1989, Lemme 26]. Alternatively, this can also be shown by proving that there exists a surjective map from $D(\ell, \lambda^*)$ to the pull-back of $V_{\mathbb{F}}^{\sigma}(\Lambda)$ by the automorphism ψ defined in Section 2.2 (similarly to what we do in the next paragraph), and then comparing weights (one uses a vector as in Lemma 3.5.3 to prove the existence of such a map). It now suffices to show that $\dim(D_{\mathbb{F}}(\ell, \lambda)) \leq \dim(V_{\mathbb{F}}^{\sigma}(\Lambda))$.

Now let v be in $D_{\mathbb{F}}(\ell, \lambda^*)_{\lambda^*} \setminus \{0\}$, W be the pull-back of $D_{\mathbb{F}}(\ell, \lambda^*)$ by ψ, and w denote v when regarded as an element of W. Since $U_{\mathbb{F}}(\mathfrak{n}^{+}[t])^{0}v = 0$, and since

(3.5.2) implies that $\psi(U_{\mathbb{F}}(\mathfrak{n}^-[t])^0) = U_{\mathbb{F}}(\mathfrak{n}^+[t])^0$, it follows that $U_{\mathbb{F}}(\mathfrak{n}^-[t])^0 w = 0$. Also, ψ restricts to an automorphism of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)$ and, hence, $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 w = 0$. Since $hv = \lambda^*(h)v$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$, (2.2.3) implies that $hw = w_0\lambda(h)w$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$. Finally, the defining relations of v and (2.2.1) imply that

$$(x^+_{\alpha,s})^{(k)} w = (x^-_{\alpha,s})^{(k)} v = 0 \text{ for all } \alpha \in R^+,\, s \geq 0,\, k > \max\{0, \lambda^*(h_\alpha) - s\ell r^\vee_\alpha\}.$$

Thus w satisfies all the defining relations of $V^\sigma_{\mathbb{F}}(\Lambda)$ in Lemma 3.5.3. Hence, W is a quotient of $V^\sigma_{\mathbb{F}}(\Lambda)$ and therefore $\dim(W) \leq \dim(V^\sigma_{\mathbb{F}}(\Lambda))$. Since $\dim(D_{\mathbb{F}}(\ell, \lambda^*)) = \dim(D_{\mathbb{F}}(\ell, \lambda))$ by Lemma 3.5.1, we are done. □

Corollary 3.5.6. *$D_{\mathbb{F}}(\ell, \lambda)$ is isomorphic to the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I^-_{\mathbb{F}}(\ell, \lambda)$ generated by $h - w_0\lambda(h)$, $h \in U_{\mathbb{F}}(\mathfrak{h})$, $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0$, $U_{\mathbb{F}}(\mathfrak{n}^-[t])^0$, and*

$$(x^+_{\alpha,s})^{(k)} \text{ for all } \alpha \in R^+,\, s \geq 0,\, k > \max\{0, -w_0\lambda(h_\alpha) - s\ell r^\vee_\alpha\}. \qquad \square$$

Remark 3.5.7. Observe that the difference between our first definition of $D_{\mathbb{F}}(\ell, \lambda)$ and the one given by Corollary 3.5.6 lies on exchanging a "highest-weight generator" by a "lowest-weight" one. More precisely, let v be as in Lemma 3.5.3. Then the isomorphism of Proposition 3.5.5 must send v to a nonzero element in $D_{\mathbb{F}}(\ell, \lambda)_{w_0\lambda}$. In particular, if w is in $D_{\mathbb{F}}(\ell, \lambda)_{w_0\lambda}$, it satisfies the relations listed in Lemma 3.5.3. The second part of our proof of Proposition 3.5.5 differs from the one given in [Fourier and Littelmann 2007, Corollary 1] in characteristic zero. It is claimed there that a vector in $D_{\mathbb{F}}(\ell, \lambda)_{w_0\lambda}$ must satisfy several relations, including (3.5.5), without further justification. Proposition 3.5.5 implies that this is true, but we do not see how to deduce it so directly (even in characteristic zero) since we cannot use extremal-weight vector theory to such vectors $D_{\mathbb{F}}(\ell, \lambda)$ a priori contained in an integrable module for the full affine hyperalgebra.

Corollary 3.5.8. *Let $\mathfrak{g} = \mathfrak{sl}_2$ and consider the subalgebra $\mathfrak{a} = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]_+ \subseteq \mathfrak{g}[t]$. For $\ell, \lambda \in \mathbb{Z}_{\geq 0}$, let $I'_{\mathbb{F}}(\ell, \lambda)$ be the left ideal of $U_{\mathbb{F}}(\mathfrak{a})$ generated by the generators of $I_{\mathbb{F}}(\ell, \lambda)$ which lie in $U_{\mathbb{F}}(\mathfrak{a})$. Then, given $k, l, s \in \mathbb{Z}_{\geq 0}$ with $k > \max\{0, \lambda - s\ell\}$, we have*

$$(x^+_i)^{(l)} (x^-_{i,s})^{(k)} \in U_{\mathbb{F}}(\mathfrak{a}) U_{\mathbb{F}}(\mathfrak{n}^+)^0 \oplus I'_{\mathbb{F}}(\ell, \lambda) \tag{3.5.7}$$

where i is the unique element of I.

Proof. The statement is a hyperalgebraic version of [Naoi 2012, Lemma 4.10] and the proof follows a similar outline. Namely, by using the automorphism of $\mathfrak{g}[t]$ determined by $x^\pm_{i,r} \mapsto x^\mp_{i,r}$, $i \in I$, $r \in \mathbb{Z}_{\geq 0}$, we observe that (3.5.7) is equivalent to

$$(x^-_i)^{(l)} (x^+_{i,s})^{(k)} \in U_{\mathbb{F}}(\mathfrak{a}^-) U_{\mathbb{F}}(\mathfrak{n}^-)^0 + I''_{\mathbb{F}}(\ell, \lambda) \tag{3.5.8}$$

for all $k, l, s \in \mathbb{Z}_{\geq 0}$, $k > \max\{0, \lambda - s\ell\}$, where $\mathfrak{a}^- = \mathfrak{n}^-[t]_+ \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]$ and $I''_{\mathbb{F}}(\ell, \lambda)$ is the left ideal of $U_{\mathbb{F}}(\mathfrak{a}^-)$ generated by the generators of $I^-_{\mathbb{F}}(\ell, \lambda)$ given in Corollary 3.5.6 which lie in $U_{\mathbb{F}}(\mathfrak{a}^-)$. Since $\mathfrak{g}[t]=\mathfrak{a}^-\oplus\mathfrak{n}^-$, the PBW theorem implies that

$$U_{\mathbb{F}}(\mathfrak{g}[t]) = U_{\mathbb{F}}(\mathfrak{a}^-)U_{\mathbb{F}}(\mathfrak{n}^-)^0 \oplus U_{\mathbb{F}}(\mathfrak{a}^-),$$

and, hence, $(x_i^-)^{(l)}(x_{i,s}^+)^{(k)} = u+u'$ with $u\in U_{\mathbb{F}}(\mathfrak{a}^-)U_{\mathbb{F}}(\mathfrak{n}^-)^0$ and $u'\in U_{\mathbb{F}}(\mathfrak{a}^-)$. Consider the Demazure module $D_{\mathbb{F}}(\ell,\lambda)$ and let $w\in D_{\mathbb{F}}(\ell,\lambda)_{-\lambda}\setminus\{0\}$. It follows from the proof of Proposition 3.5.5 that if $k>\max\{0,\lambda-s\ell\}$, then

$$u'w = \big((x_i^-)^{(l)}(x_{i,s}^+)^{(k)} - u\big)w = 0.$$

Since $\hat{\mathfrak{b}}'^{+} = \mathfrak{a}^-\oplus\mathbb{C}c$ and $\mathfrak{a}^-$ is an ideal of $\hat{\mathfrak{b}}'^{+}$, it follows from Lemma 3.5.3 that $I''_{\mathbb{F}}(\ell,\lambda)$ is the annihilating ideal of w inside $U_{\mathbb{F}}(\mathfrak{a})$, and, hence, $u'\in I''_{\mathbb{F}}(\ell,\lambda)$. □

4. Joseph's Demazure flags

4.1. ***Quantum groups.*** Let $\mathbb{C}(q)$ be the field of rational functions on an indeterminate q. Let also $C=(c_{ij})_{i,j\in\hat{I}}$ be the Cartan matrix of $\hat{\mathfrak{g}}$, and d_i, with $i\in\hat{I}$, be nonnegative relatively prime integers such that the matrix DC, with $D=\operatorname{diag}(d_i)_{i\in I}$, is symmetric. Set $q_i=q^{d_i}$ and for $m,n\in\mathbb{Z}$, $n\geq 0$, set

$$[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, \quad [n]_{q_i}! = [n]_{q_i}[n-1]_{q_i}\cdots[1]_{q_i},$$
$$\begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}[m-1]_{q_i}\dots[m-n+1]_{q_i}}{[n]_{q_i}!}.$$

The quantum group $U_q(\hat{\mathfrak{g}}')$ is a $\mathbb{C}(q)$-associative algebra (with 1) with generators $x_i^{\pm}, k_i^{\pm 1}$, $i\in\hat{I}$ subject to the following defining relations for all $i,j\in\hat{I}$:

$$k_ik_i^{-1}=1,\quad k_ik_j=k_jk_i,\quad k_ix_j^{\pm}k_i^{-1} = q_i^{\pm c_{ij}}x_j^{\pm},\quad [x_i^+,x_j^-] = \delta_{ij}\frac{k_i-k_i^{-1}}{q_i-q_i^{-1}},$$
$$\sum_{m=0}^{1-c_{ij}}(-1)^m\begin{bmatrix} 1-c_{ij} \\ m \end{bmatrix}_{q_i}(x_i^{\pm})^{1-c_{ij}-m}x_j^{\pm}(x_i^{\pm})^m = 0,\quad i\neq j.$$

Let $U_q(\hat{\mathfrak{n}}^{\pm})$ be the subalgebra generated by $x_i^{\pm}$, $i\in\hat{I}$, and $U_q(\hat{\mathfrak{b}}^{\pm})$ be the subalgebra generated by $U_q(\hat{\mathfrak{n}}^{\pm})$ together with $k_i^{\pm1}$, $i\in\hat{I}$.

We shall need an integral form of $U(\hat{\mathfrak{g}}')$. Let $\mathbb{Z}_q=\mathbb{Z}[q,q^{-1}]$, $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^{\pm})$ be the $\mathbb{Z}_q$-subalgebra of $U_q(\hat{\mathfrak{n}}^{\pm})$ generated by $(x_i^{\pm})^m/([m]_{q_i}!)$, $i\in\hat{I}$, $m\geq 0$, and $U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$ be the $\mathbb{Z}_q$-subalgebra of $U_q(\hat{\mathfrak{g}}')$ generated by $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^{\pm})$ and k_i, $i\in\hat{I}$. Let also $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^{\pm}) = U_q(\hat{\mathfrak{b}}^{\pm})\cap U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$. Then $U_{\mathbb{Z}_q}(\mathfrak{a})$, where $\mathfrak{a}=\hat{\mathfrak{g}}', \hat{\mathfrak{n}}^{\pm}, \hat{\mathfrak{b}}^{\pm}$, is a free $\mathbb{Z}_q$-module such that the natural map $\mathbb{C}(q)\otimes_{\mathbb{Z}_q}U_{\mathbb{Z}_q}(\mathfrak{a})\to U_q(\mathfrak{a})$ is a $\mathbb{C}(q)$-algebra isomorphism. In other words, $U_{\mathbb{Z}_q}(\mathfrak{a})$ is a $\mathbb{Z}_q$-form of $U_q(\mathfrak{a})$. Moreover, letting $\mathbb{Z}$ be a $\mathbb{Z}_q$-module where q acts as 1, there exists an epimorphism of $\mathbb{Z}$-algebras

$\mathbb{Z}\otimes_{\mathbb{Z}_q} U_{\mathbb{Z}_q}(\mathfrak{a}) \to U_{\mathbb{Z}}(\mathfrak{a})$, which is an isomorphism if $\mathfrak{a}=\hat{\mathfrak{n}}^{\pm}$, and whose kernel is the ideal generated by $k_i - 1$, $i \in \hat{I}$, for $\mathfrak{a} = \hat{\mathfrak{g}}'$, $\hat{\mathfrak{b}}^{\pm}$.

Given $\Lambda \in \hat{P}^+$, let $V_q(\Lambda)$ be the simple (type 1) $U_q(\hat{\mathfrak{g}}')$-module of highest weight Λ. Given a highest-weight vector $v \in V_q(\Lambda)$, set $V_{\mathbb{Z}_q}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^-)v$, which is a $\mathbb{Z}_q$-form of $V_q(\Lambda)$. Given $\sigma \in \widehat{\mathcal{W}}$ and a nonzero vector $v \in V_q(\Lambda)$ of weight $\sigma\Lambda$, set $V^{\sigma}_{\mathbb{Z}_q}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)v$, which is a free $\mathbb{Z}_q$-module as well as a $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-module, and $\mathbb{C}\otimes_{\mathbb{Z}_q} V^{\sigma}_{\mathbb{Z}_q}(\Lambda) \cong V^{\sigma}_{\mathbb{C}}(\Lambda)$. In particular,

$$V^{\sigma}_{\mathbb{Z}}(\Lambda) := \mathbb{Z}\otimes_{\mathbb{Z}_q} V^{\sigma}_{\mathbb{Z}_q}(\Lambda) \tag{4.1.1}$$

is an integral form of $V^{\sigma}_{\mathbb{C}}(\Lambda)$.

4.2. ***Crystals.*** A normal crystal associated to the root data of $\hat{\mathfrak{g}}$ defined as a set B equipped with maps $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z}$, for each $i \in \hat{I}$, and $\mathrm{wt} : B \to \hat{P}$ satisfying

(1) $\varepsilon_i(b) = \max\{n : \tilde{e}_i b \neq 0\}$, $\varphi_i(b) = \max\{n : \tilde{f}_i b \neq 0\}$, for all $i \in \hat{I}$, $b \in B$;

(2) $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)(h_i)$, for all $i \in \hat{I}$, $b \in B$;

(3) for $b, b' \in B$, $b' = \tilde{e}_i b$ if and only if $\tilde{f}_i b' = b$;

(4) if $b \in B$ and $i \in \hat{I}$ are such that $\tilde{e}_i b \neq 0$, then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$.

For convenience, we extend $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$, wt to $B \sqcup \{0\}$ by setting them to map 0 to 0. Denote by $\mathscr{E}$ the submonoid of the monoid of maps $B \sqcup \{0\} \to B \sqcup \{0\}$ generated by $\{\tilde{e}_i : i \in \hat{I}\}$, and similarly define $\mathscr{F}$. A normal crystal is said to be of highest weight $\Lambda \in \hat{P}^+$ if there exists $b_\Lambda \in B$ satisfying

$$\mathrm{wt}(b_\Lambda) = \Lambda, \quad \mathscr{E} b_\Lambda = \{0\}, \quad \text{and} \quad \mathscr{F} b_\Lambda = B.$$

Given $B' \subset B$ and $\mu \in \hat{P}$, define $B'_\mu = \{b \in B' : \mathrm{wt}(b) = \mu\}$ and define the character of B' as $\mathrm{ch}(B') = \sum_{\mu \in \hat{P}} \# B'_\mu e^\mu \in \mathbb{Z}[\hat{P}]$.

Given crystals B_1 and B_2, a morphism from B_1 to B_2 is a map $\psi : B_1 \to B_2 \sqcup \{0\}$ satisfying

(1) if $\psi(b) \neq 0$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$, for all $i \in \hat{I}$;

(2) if $\tilde{e}_i b \neq 0$, then $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$;

(3) if $\tilde{f}_i b \neq 0$, then $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

The set $B_1 \times B_2$ admits a structure of crystal denoted by $B_1 \otimes B_2$ (see [Joseph 2003, Section 2.4]). There is, up to isomorphism, exactly one family $\{B(\Lambda) : \Lambda \in \hat{P}^+\}$ of normal highest-weight crystals such that for all $\lambda, \mu \in \hat{P}^+$, the crystal structure of $B(\lambda) \otimes B(\mu)$ induces a crystal structure on its subset $\mathscr{F}(b_\lambda \otimes b_\mu)$, the inclusion is a homomorphism of crystals, and $\mathscr{F}(b_\lambda \otimes b_\mu) \cong B(\lambda + \mu)$.

Given a crystal B and $\sigma \in \widehat{\mathscr{W}}$ with a fixed reduced expression $\sigma = s_{i_1} \dots s_{i_n}$, define

$$\mathscr{E}^\sigma = \{\tilde{e}_{i_1}^{m_1} \dots \tilde{e}_{i_n}^{m_n} : m_j \in \mathbb{N}\} \subset \mathscr{E} \quad \text{and} \quad \mathscr{F}^\sigma = \{\tilde{f}_{i_1}^{m_1} \dots \tilde{f}_{i_n}^{m_n} : m_j \in \mathbb{N}\} \subset \mathscr{F}.$$

If $B = B(\Lambda)$, $\Lambda \in \hat{P}^+$ and $\sigma \in \widehat{\mathscr{W}}$, define the Demazure subset $B^\sigma(\Lambda) = \mathscr{F}^\sigma b_\Lambda \subseteq B(\Lambda)$. Then $B^\sigma(\Lambda)$ is $\mathscr{E}$-stable: $\mathscr{E} B^\sigma(\Lambda) \subset B^\sigma(\Lambda) \sqcup \{0\}$. It was proved in [Joseph 2003, Section 4.6] that $\mathrm{ch}(V_{\mathbb{C}}^\sigma(\Lambda)) = \mathrm{ch}(B^\sigma(\Lambda))$. This fact and the following theorem are the main results of [Joseph 2003] that we shall need.

Theorem 4.2.1. *Let $\Lambda, \mu \in \hat{P}^+$. For any $\sigma \in \widehat{\mathscr{W}}$, there exist a finite set J and elements $\sigma_j \in \widehat{\mathscr{W}}$, $b_j \in B^\sigma(\Lambda)$ for each $j \in J$, satisfying*

(1) $b_\mu \otimes B^\sigma(\Lambda) = \sqcup_{j \in J} B_j$ *where* $B_j := \mathscr{F}^{\sigma_j}(b_\mu \otimes b_j)$;

(2) $\mathscr{E}(b_\mu \otimes b_j) = \{0\}$;

(3) $\mathrm{ch}(B_j) = \mathrm{ch}(B^{\sigma_j}(\nu_j))$, *where* $\nu_j = \mu + \mathrm{wt}(b_j) \in \hat{P}^+$.

Remark 4.2.2. The proof of Theorem 4.2.1 establishes an algorithm to find the set J and the elements σ_j, b_j.

4.3. ***Globalizing.*** The theory of global basis of Kashiwara shows, in particular, that for each $\Lambda \in \hat{P}^+$, there is a map $G : B(\Lambda) \to V_q(\Lambda)$ such that

$$V_{\mathbb{Z}_q}(\Lambda) = \bigoplus_{b \in B(\Lambda)} \mathbb{Z}_q G(b), \tag{4.3.1}$$

the weight of $G(b)$ is $\mathrm{wt}(b)$ and $G(b_\Lambda)$ is a highest-weight vector of $V_q(\Lambda)$.

Fix $\Lambda, \mu \in \hat{P}^+, \sigma \in \widehat{\mathscr{W}}$ and let J, b_j, σ_j, ν_j, j be in J, be as in Theorem 4.2.1. Let b be in $B(\Lambda)_{\sigma\Lambda}$ and set $V_{\mathbb{Z}_q}^\sigma(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)G(b)$. Similarly, let b'_j be the unique element of B_j such that $\mathrm{wt}(b'_j) = \sigma_j \nu_j$. Choose a linear order on J such that $\mathrm{wt}(b_j) < \mathrm{wt}(b_k)$ only if $j > k$. For $j \in J$, let Y_j be the $\mathbb{Z}_q$-submodule of $V_q(\mu) \otimes V_q^\sigma(\Lambda)$ spanned by $G(b_\mu) \otimes G(b)$ with $b \in B_k$, $k \le j$, and set

$$y_j = G(b_\mu) \otimes G(b'_j). \tag{4.3.2}$$

Let also $Z_j = \sum_{k \le j} U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^-)(G(b_\mu) \otimes G(b_k))$. Since J is linearly ordered and finite, say $\#J = n$ and identify it with $\{1, \dots, n\}$. For convenience, set $Y_0 = \{0\}$. Observe that $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$ is a filtration of the $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-module $G(b_{\Lambda_0}) \otimes V_{\mathbb{Z}_q}^\sigma(\Lambda)$. The following result was proved in [Joseph 2006, Corollary 5.10].

Theorem 4.3.1. *Suppose $\mathfrak{g}$ is simply laced and $\mu(h_i) \le 1$ for all $i \in \hat{I}$. Then*:

(a) *The $\mathbb{Z}_q$-module Y_j is $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)$-stable for all $j \in J$.*

(b) *For all $j \in J$, Y_j / Y_{j-1} is isomorphic to $V_{\mathbb{Z}_q}^{\sigma_j}(\nu_j)$. In particular, Y_j/Y_{j-1} is a free $\mathbb{Z}_q$-module.*

(c) *For all* $j \in J$, *the image of* $\{G(b_\mu) \otimes G(b) : b \in B_j\}$ *in* Y_j/Y_{j-1} *is a* $\mathbb{Z}_q$*-basis of* Y_j/Y_{j-1}.

(d) *For each* $j \in J$, Z_j *is* $U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$*-stable and* $Y_j = Z_j \cap \big(G(b_\mu) \otimes V^{\sigma}_{\mathbb{Z}_q}(\Lambda)\big)$.

Remark 4.3.2. The above theorem was proved in [Joseph 2006] for any simply-laced symmetric Kac–Moody Lie algebra. However, as pointed out in [Naoi 2012, Remark 4.15], the proof also holds for $\hat{\mathfrak{sl}}_2$.

It follows from Theorem 4.3.1 and the fact that $G(b_\mu)$ is a highest-weight vector of $V_q(\Lambda)$ (4.3.1) that

$$Y_j = \sum_{k \le j} U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+) y_j. \tag{4.3.3}$$

4.4. ***Simply laced Demazure flags.*** Given $\ell \ge 0$, $\lambda \in P^+$, $m \in \mathbb{Z}$, let $D_{\mathbb{F}}(\ell, \lambda, m) = \tau_m(D_{\mathbb{F}}(\ell, \lambda))$ and $D_{\mathbb{Z}}(\ell, \lambda, m) = \tau_m(D_{\mathbb{Z}}(\ell, \lambda))$.

Theorem 4.4.1. *Suppose* $\mathfrak{g}$ *is simply laced, let* μ *be in* P^+ *and* $\ell' > \ell \ge 0$. *Then there exist* $k > 0$, $\mu_1, \dots, \mu_k \in P^+$, $m_1, \dots, m_k \in \mathbb{Z}_{\ge 0}$, *and a filtration of* $U_{\mathbb{Z}}(\mathfrak{g}[t])$*-modules* $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = D_{\mathbb{Z}}(\ell, \mu)$ *such that* D_j *and* D_j/D_{j-1} *are free* $\mathbb{Z}$*-modules for all* $j = 1, \dots, k$, *and* $D_j/D_{j-1} \cong D_{\mathbb{Z}}(\ell', \mu_j, m_j)$. *Moreover, for all* $j \in J$, *there exists* $\vartheta_j \in D_j$ *such that*

(i) *the image of* ϑ_j *in* D_j/D_{j-1} *satisfies the defining relations of* $D_{\mathbb{Z}}(\ell', \mu_j, m_j)$;

(ii) $D_j = \sum_{k \le j} U_{\mathbb{Z}}(\mathfrak{n}^-[t])\vartheta_k$.

Proof. The proof follows closely that of [Naoi 2012, Corollary 4.16]. First notice that it is enough to prove the theorem for $\ell' = \ell + 1$. Then let $\Lambda \in \widehat{P}^+$ and $w \in \widehat{\mathcal{W}}$ be such that $w\Lambda = \ell\Lambda_0 + w_0\mu$, and let $V^w_{\mathbb{Z}_q}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)G(b)$ where $b \in B(\Lambda)_{w\Lambda}$.

From Section 4.3, we know that the $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-submodule $G(b_{\Lambda_0}) \otimes V^w_{\mathbb{Z}_q}(\Lambda) \subseteq V_q(\Lambda_0) \otimes V_q(\Lambda)$ admits a filtration $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$. For each $j = 1, \dots, k$, let $D_j = \mathbb{Z} \otimes_{\mathbb{Z}_q} Y_j$, and observe that

$$D_k = \mathbb{Z} \otimes_{\mathbb{Z}_q} \big(G(b_{\Lambda_0}) \otimes_{\mathbb{Z}_q} V^w_{\mathbb{Z}_q}(\Lambda)\big) \cong \big(\mathbb{Z} \otimes_{\mathbb{Z}_q} G(b_{\Lambda_0})\big) \otimes_{\mathbb{Z}} \big(\mathbb{Z} \otimes_{\mathbb{Z}_q} V^w_{\mathbb{Z}_q}(\Lambda)\big) \cong \mathbb{Z}_{\Lambda_0} \otimes_{\mathbb{Z}} D_{\mathbb{Z}}(\ell, \mu),$$

where $\mathbb{Z}_{\Lambda_0}$ is a $U_{\mathbb{Z}}(\hat{\mathfrak{b}}^+)$-module on which $U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+)^0$ and $U_{\mathbb{Z}}(\mathfrak{g})^0$ act trivially and $U_{\mathbb{Z}}(\hat{\mathfrak{h}})$ acts by Λ_0. Moreover, as a $\mathbb{Z}$-module it is free of rank 1. Thus D_k is isomorphic to $D_{\mathbb{Z}}(\ell, \mu)$ as a $U_{\mathbb{Z}}(\mathfrak{g}[t])$-module. It follows from Theorem 4.3.1(d) that D_j is a $U_{\mathbb{Z}}(\mathfrak{g}[t])$-module for all $j = 1, \dots, k$ and, hence, so is D_j/D_{j-1}. So we have a filtration of $U_{\mathbb{Z}}(\mathfrak{g}[t])$-modules $0 = D_0 \subset D_1 \subset \cdots \subset D_k = D_{\mathbb{Z}}(\ell, \mu)$.

By Theorem 4.3.1(b), $Y_j/Y_{j-1} \cong V^{\sigma_j}_{\mathbb{Z}_q}(\nu_j)$ for some $\sigma_j \in \widehat{\mathcal{W}}$ and $\nu_j \in \widehat{P}^+$. By (4.1.1) $D_j/D_{j-1} \cong V^{\sigma_j}_{\mathbb{Z}}(\nu_j)$. Thus D_j/D_{j-1} is isomorphic to $D_{\mathbb{Z}}(\ell_j, \mu_j, m_j)$ for some $\mu_j \in P^+, m_j \in \mathbb{Z}$ and $\ell_j = \nu_j(c)$; see (3.5.3). Since all the weights of $V_q(\Lambda_0)\otimes V_q(\Lambda)$ are of the form $\Lambda+\Lambda_0-\eta$ for some $\eta\in\hat{Q}^+$, and $\alpha_i(c)=0$ for all $i\in\hat{I}$, it follows that $\ell_j=\ell+1$ for all j.

Keep denoting the image of y_j in D_j by y_j (see (4.3.2)). It follows that $D_j=\sum_{k\le j}U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+)y_j$ by (4.3.3). As in Remark 3.5.7, we now replace the "lowest-weight" generators y_j by "highest-weight generators". Thus, let b''_j be the unique element of B_j such that $\mathrm{wt}(b''_j)=w_0\sigma_j\nu_j=(\ell+1)\Lambda_0+\mu_j+m_j\delta$ and let ϑ_j be defined similarly to y_j by replacing b'_j by b''_j. □

The next corollary is now immediate.

Corollary 4.4.2. *Let $\mathfrak{g}$, μ, ℓ', ℓ, k, μ_j, $j=1,\dots,k$, be as in Theorem 4.4.1. Then there exists a filtration of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules $0=D_0\subseteq D_1\subseteq\cdots\subseteq D_k=D_{\mathbb{F}}(\ell,\mu)$, such that $D_j/D_{j-1}\cong D_{\mathbb{F}}(\ell',\mu_j)$ for all $j=1,\dots,k$.* □

5. Proof of Theorem 1.5.2

5.1. ***The isomorphism between Demazure and graded local Weyl modules.*** Recall that for $\mathfrak{g}=\mathfrak{sl}_2$, a characteristic-free proof of Theorem 1.5.2(a) was given in [Jakelić and Moura 2014]. Thus, assume $\mathfrak{g}$ is simply laced of rank higher than 1 and recall from Remark 1.5.1 that $D_{\mathbb{F}}(1,\lambda)$ is a quotient of $W^c_{\mathbb{F}}(\lambda)$. To prove the converse, let w be the image of 1 in $W^c_{\mathbb{F}}(\lambda)$. In order to show that $W^c_{\mathbb{F}}(\lambda)$ is a quotient of $D_{\mathbb{F}}(1,\lambda)$, it remains to prove that

$$(5.1.1)\qquad (x^-_{\alpha,s})^{(k)}w=0\ \text{ for all }\alpha\in R^+,\ s>0,\ k>\max\{0,\lambda(h_\alpha)-s\}.$$

Given $\alpha\in R^+$, consider the subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ (see Section 2.3) and let W_α be the $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$-submodule of $W^c_{\mathbb{F}}(\lambda)$ generated by w. Clearly, W_α is a quotient of the graded local Weyl module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ with highest weight $\lambda(h_\alpha)$, where we have identified the weight lattice of $\mathfrak{sl}_2$ with $\mathbb{Z}$ as usual. Since we already know that the theorem holds for $\mathfrak{sl}_2$, it follows that w must satisfy the same relations as the generator of the corresponding Demazure module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$. In particular, (5.1.1) holds and so does Theorem 1.5.2(a).

5.2. ***A smaller set of relations for nonsimply laced Demazure modules.*** In this subsection we assume $\mathfrak{g}$ is not simply laced and prove the following analogue of [Naoi 2012, Proposition 4.1].

Proposition 5.2.1. *For all $\lambda\in P^+$, $D_{\mathbb{F}}(1,\lambda)$ is isomorphic to the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}(\lambda)$ generated by*

$$(5.2.1)\qquad U_{\mathbb{F}}(\mathfrak{n}^+[t])^0,\quad U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0,\quad h-\lambda(h),\quad (x_i^-)^{(k)},\quad (x^-_{\alpha,s})^{(\ell)}$$

for all $h\in U_{\mathbb{F}}(\mathfrak{h})$, $i\in I\setminus I_{\mathrm{sh}}$, $\alpha\in R^+_{\mathrm{sh}}$, $s\ge 0$, $k>\lambda(h_i)$, $\ell>\max\{0,\lambda(h_\alpha)-sr^\vee\}$.

Let w be in $D_{\mathbb{F}}(1,\lambda)_\lambda\setminus\{0\}$ and V be the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module generated by a vector v with defining relations given by (5.2.1). In particular, there exists a unique epimorphism $V \to D_{\mathbb{F}}(1,\lambda)$ mapping v to w. To prove the converse, observe first that since $(x_i^-)^{(k)}v = 0$ for all $i \in I, k > \lambda(h_i)$, Lemma 3.1.5 implies that $(x_\alpha^-)^{(k)}v = 0$ for all $\alpha \in R^+, k > \lambda(h_\alpha)$. In particular, V is a quotient of $W_{\mathbb{F}}^c(\lambda)$ and, hence, it is finite-dimensional. It remains to show that

$$(x_{\alpha,s}^-)^{(k)}v = 0 \text{ for all } \alpha \in R^+ \setminus R_{\text{sh}}^+, s > 0, k > \max\{0, \lambda(h_\alpha) - sr_\alpha^\vee\}.$$

These relations will follow from the next few lemmas.

Lemma 5.2.2. *Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, λ be in P^+, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$. If $\alpha \in R^+$ is long, then $(x_{\alpha,s}^-)^{(k)}v = 0$ for all $s \geq 0, k > \max\{0, \lambda(h_\alpha) - s\}$.*

Proof. Consider the subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ (see Section 2.3). By Theorem 3.3.4 (c), the submodule $W = U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])v$ is a quotient of the local graded Weyl module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ with highest weight $\lambda(h_\alpha)$. Theorem 1.5.2 (a) implies that $W \cong D_{\mathbb{F}}^\alpha(1, \lambda(h_\alpha))$ where the latter is the corresponding Demazure module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$. In particular, v satisfies the relations (1.5.2). □

Lemma 5.2.3. *Assume $\mathfrak{g}$ is not of type G_2. Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, λ be in P^+, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ and $(x_{\alpha,s}^-)^{(k)}v = 0$ for all $\alpha \in R_{\text{sh}}^+, k > \max\{0, \lambda(h_\alpha) - 2s\}$. Then for every short root γ, we have $(x_{\gamma,s}^-)^{(k)}v = 0$ for all $s \geq 0, k > \max\{0, \lambda(h_\gamma) - 2s\}$.*

Proof. The proof will proceed by induction on $\text{ht}(\gamma)$. If $\text{ht}(\gamma) = 1$, then γ is simple and, hence, $\gamma \in R_{\text{sh}}^+$. Thus, suppose $\text{ht}(\gamma) > 1$ and that $\gamma \notin R_{\text{sh}}^+$. By [Naoi 2012, Lemma 4.6], there exist $\alpha, \beta \in R^+$ such that $\gamma = \alpha + \beta$ with α long and β short. Notice that $\{\alpha, \beta\}$ form a simple system of a rank-two root subsystem. In particular, $h_\gamma = 2h_\alpha + h_\beta$ and, hence, $\lambda(h_\gamma) = 2\lambda(h_\alpha) + \lambda(h_\beta)$.

Fix $s \geq 0$ and suppose first that $\lambda(h_\gamma) - 2s \geq 0$. In this case, we can choose $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$a + b = s, \quad \lambda(h_\alpha) - a \geq 0, \quad \text{and} \quad \lambda(h_\beta) - 2b \geq 0.$$

Indeed, $b = \max\{0, s - \lambda(h_\alpha)\}$ and $a = s - b$ satisfy these conditions. Then Lemma 5.2.2 implies that $(x_{\alpha,a}^-)^{(k)}v = 0$ for all $k > \lambda(h_\alpha) - a$, while the induction hypothesis implies that $(x_{\beta,b}^-)^{(k)}v = 0$ for all $k > \lambda(h_\beta) - 2b$. Applying Lemma 3.1.5 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}_{\alpha,\beta}^{a,b})$ (see Section 2.3), it follows that $(x_{\gamma,s}^-)^{(k)}v = 0$ for all $k > 2(\lambda(h_\alpha) - a) + (\lambda(h_\beta) - 2b) = \lambda(h_\gamma) - 2s$.

Now suppose $\lambda(h_\gamma) - 2s \leq 0$; this implies $s - \lambda(h_\alpha) = s - \frac{1}{2}(\lambda(h_\gamma) - \lambda(h_\beta)) \geq \lambda(h_\beta)/2 \geq 0$. We need to show that $(x_{\gamma,s}^-)^{(k)}v = 0$ for all $k > 0$. Letting $a = \lambda(h_\alpha)$ and $b = s - \lambda(h_\alpha)$, we have

$$a + b = s, \quad \lambda(h_\alpha) - a \leq 0, \quad \text{and} \quad \lambda(h_\beta) - 2b \leq 0.$$

Then Lemma 5.2.2 implies that $(x^-_{\alpha,a})^{(k)}v = 0$ for all $k > 0$, while the induction hypothesis implies that $(x^-_{\beta,b})^{(k)}v = 0$ for all $k > 0$. The result follows from an application of Lemma 3.1.5 as before. □

It remains to prove an analogue of Lemma 5.2.3 for $\mathfrak{g}$ of type G_2. This is much more technically complicated and will require that we assume that characteristic of $\mathbb{F}$ is at least 5. For the remainder of this subsection we assume $\mathfrak{g}$ is of type G_2 and set $I = \{1, 2\}$ so that α_1 is short. Given $\gamma = s\alpha_1 + l\alpha_2 \in R^+$, set $s_\gamma = s$. Set also

$$\mathfrak{n}^+[t]_> = \bigoplus_{\gamma\in R^+}\bigoplus_{s\geq s_\gamma} \mathbb{C}x^+_{\gamma,s}, \quad \mathfrak{n}^+[t]_< = \bigoplus_{\gamma\in R^+}\bigoplus_{s=0}^{s_\gamma-1} \mathbb{C}x^+_{\gamma,s}, \quad \mathfrak{a} = \mathfrak{n}^-[t]\oplus\mathfrak{h}[t]\oplus\mathfrak{n}^+[t]_>,$$

and observe that $\mathfrak{n}^+[t]_>$ and $\mathfrak{n}^+[t]_<$ are subalgebras of $\mathfrak{n}^+[t]$ such that $\mathfrak{n}^+[t] = \mathfrak{n}^+[t]_> \oplus \mathfrak{n}^+[t]_<$. The hyperalgebras $U_\mathbb{F}(\mathfrak{n}^+[t]_>)$, $U_\mathbb{F}(\mathfrak{n}^+[t]_<)$, and $U_\mathbb{F}(\mathfrak{a})$ are then defined in the usual way (see Section 1.3) and the PBW theorem implies that

$$U_\mathbb{F}(\mathfrak{n}^+[t]) = U_\mathbb{F}(\mathfrak{n}^+[t]_>) \oplus U_\mathbb{F}(\mathfrak{n}^+[t])U_\mathbb{F}(\mathfrak{n}^+[t]_<)^0. \tag{5.2.2}$$

We now prove a version of [Naoi 2012, Lemma 4.11] for hyperalgebras.

Lemma 5.2.4. *Given $\lambda \in P^+$, let $I'_\mathbb{F}(\lambda)$ be the left ideal of $U_\mathbb{F}(\mathfrak{a})$ generated by the generators of $I_\mathbb{F}(\lambda)$ described in (5.2.1) which lie in $U_\mathbb{F}(\mathfrak{a})$. Then*

$$I_\mathbb{F}(\lambda) \subseteq I'_\mathbb{F}(\lambda) \oplus U_\mathbb{F}(\mathfrak{a})U_\mathbb{F}(\mathfrak{n}^+[t]_<)^0.$$

Proof. Recall that $I_\mathbb{F}(\lambda)$ is the left ideal of $U_\mathbb{F}(\mathfrak{g}[t])$ generated by the set $\mathcal{I}$ whose elements are the elements in $U_\mathbb{F}(\mathfrak{n}^+[t])^0$, $U_\mathbb{F}(\mathfrak{h}[t]_+)^0$, together with the elements

$$\binom{h_i}{l} - \binom{\lambda(h_i)}{l}, \quad (x^-_2)^{(m)}, \quad (x^-_{1,s})^{(k)}$$

for $i \in I$, $k, l, m, s \in \mathbb{Z}_{\geq 0}$, $m > \lambda(h_2)$, $k > \max\{0, \lambda(h_1) - 3s\}$. To simplify notation, set $U_< = U_\mathbb{F}(\mathfrak{n}^+[t]_<)$ and $J = I'_\mathbb{F}(\lambda) \oplus U_\mathbb{F}(\mathfrak{a})U_\mathbb{F}(\mathfrak{n}^+[t]_<)^0$. Observe that $U_\mathbb{F}(\mathfrak{a})J \subseteq J$. Therefore, since $U_\mathbb{F}(\mathfrak{g}[t]) = U_\mathbb{F}(\mathfrak{a})U_<$ by (5.2.2) and we clearly have $\mathcal{I} \subseteq J$, it suffices to show that

$$U^0_<\mathcal{I} \subseteq J.$$

We will decompose the set $\mathcal{I}$ into parts, and prove the inclusion for each part. Namely, we first decompose $\mathcal{I}$ into $\big(\mathcal{I} \cap U_\mathbb{F}(\mathfrak{n}^+[t])U_\mathbb{F}(\mathfrak{h}[t])\big) \sqcup \big(\mathcal{I} \cap U_\mathbb{F}(\mathfrak{n}^-[t])\big)$, and then we further decompose $\mathcal{I} \cap U_\mathbb{F}(\mathfrak{n}^-[t])$ as

$$\{(x^-_2)^{(m)} : m > \lambda(h_2)\} \sqcup \{(x^-_{1,s})^{(k)} : s \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda(h_1) - 3s\}\}.$$

Since $\mathfrak{h}[t] \oplus \mathfrak{n}^+[t]$ is a subalgebra of $\mathfrak{g}[t]$, the PBW theorem tells us that $U_\mathbb{F}(\mathfrak{n}^+[t])U_\mathbb{F}(\mathfrak{h}[t]) = U_\mathbb{F}(\mathfrak{h}[t])U_\mathbb{F}(\mathfrak{n}^+[t])$, and therefore

$$U^0_<\big(\mathcal{I} \cap U_\mathbb{F}(\mathfrak{n}^+[t])U_\mathbb{F}(\mathfrak{h}[t])\big) \subseteq U_\mathbb{F}(\mathfrak{h}[t])U_\mathbb{F}(\mathfrak{n}^+[t]).$$

Now, by (5.2.2), $U_{\mathbb{F}}(\mathfrak{h}[t])U_{\mathbb{F}}(\mathfrak{n}^+[t]) \subseteq J$, so $U^0_<(\mathscr{I} \cap U_{\mathbb{F}}(\mathfrak{n}^+[t])U_{\mathbb{F}}(\mathfrak{h}[t])) \subseteq J$. In particular, we have shown that

$$U_{\mathbb{F}}(\mathfrak{g}[t])U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 \subseteq J. \tag{5.2.3}$$

It remains to show that

$$U^0_<\big(\mathscr{I} \cap U_{\mathbb{F}}(\mathfrak{n}^-[t])\big) \subseteq J.$$

We begin by proving that $U^0_< U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq J$, where $\mathfrak{n}_2^-$ is the subalgebra spanned by x_2^-. Consider the natural Q-grading on $U_{\mathbb{F}}(\mathfrak{g}[t])$, and for $\eta \in Q$ let $U_{\mathbb{F}}(\mathfrak{g}[t])_\eta$ denote the corresponding graded piece. Observe that $\mathfrak{m}_2 := \mathfrak{n}^+[t]_< \oplus \mathfrak{n}_2^-$ is a subalgebra of $\mathfrak{g}[t]$ and that

$$U^0_< U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq \bigoplus_\eta U_{\mathbb{F}}(\mathfrak{m}_2)_\eta,$$

where the sum runs over $\mathbb{Z}_{>0}\alpha_1 \oplus \mathbb{Z}\alpha_2$. Together with the PBW theorem, this implies that

$$U^0_< U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq U_{\mathbb{F}}(\mathfrak{n}_2^-)U^0_< \subseteq U_{\mathbb{F}}(\mathfrak{a})U^0_< \subseteq J.$$

Finally, we show that $U^0_<\mathscr{I}_1 \subseteq J$, where $\mathscr{I}_1 = \big(\mathscr{I} \cap U_{\mathbb{F}}(\mathfrak{n}_1^-[t])\big)$ and $\mathfrak{n}_1^-$ is the subalgebra spanned by x_1^-. Consider

$$\mathfrak{n}^+[t]^1_< = \bigoplus_{\gamma \in R^+\setminus\{\alpha_1\}} \bigoplus_{s=0}^{s_\gamma-1} \mathbb{C}x^+_{\gamma,s},$$

which is a subalgebra of $\mathfrak{n}^+[t]_<$ such that $\mathfrak{n}^+[t]_< = \mathfrak{n}_1^+ \oplus \mathfrak{n}^+[t]^1_<$, where $\mathfrak{n}_1^+ = \mathbb{C}x_1^+$. Moreover, $\mathfrak{m}_1 := \mathfrak{n}^+[t]^1_< \oplus \mathfrak{n}_1^-[t]$ is a subalgebra of $\mathfrak{g}[t]$ such that $U(\mathfrak{m}_1)_\eta \neq 0$ only if $\eta \in \mathbb{Z}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2$ and $U(\mathfrak{m}_1)_0 = \mathbb{C}$. This implies that

$$U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 U_{\mathbb{F}}(\mathfrak{n}_1^-[t]) = U_{\mathbb{F}}(\mathfrak{n}_1^-[t])U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0.$$

Since $U^0_< = U_{\mathbb{F}}(\mathfrak{n}_1^+)U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 \oplus U_{\mathbb{F}}(\mathfrak{n}_1^+)^0$, we get

$$\begin{aligned} U^0_<\mathscr{I}_1 &\subseteq \big(U_{\mathbb{F}}(\mathfrak{n}_1^+)U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 + U_{\mathbb{F}}(\mathfrak{n}_1^+)^0\big)\mathscr{I}_1 \\ &\subseteq U_{\mathbb{F}}(\mathfrak{n}_1^+)U_{\mathbb{F}}(\mathfrak{n}_1^-[t])U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 + U_{\mathbb{F}}(\mathfrak{n}_1^+)^0\mathscr{I}_1 \\ &\subseteq U_{\mathbb{F}}(\mathfrak{g}[t])U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 + U_{\mathbb{F}}(\mathfrak{n}_1^+)^0\mathscr{I}_1. \end{aligned}$$

The first summand in the last line is in J by (5.2.3) while the second one is in J by Corollary 3.5.8 (with $\lambda = \lambda(h_1)$ and $\ell = 3$) together with (5.2.3). □

Set $\mathfrak{h}_i = \mathbb{C}h_i$, $i \in I$, and $\mathfrak{b} = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t]_+ \oplus \mathfrak{h}_2 \oplus \mathfrak{n}^+[t]_>$. Observe that $\mathfrak{b}$ is an ideal of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{h}_1$. One easily checks that there exists a unique Lie algebra homomorphism $\phi : \mathfrak{b} \to \mathfrak{g}[t]$ such that

$$\phi(x^\pm_{\gamma,r}) = x^\pm_{\gamma,r\mp s_\gamma} \quad \text{for all } \gamma \in R^+.$$

Moreover, ϕ is the identity on $\mathfrak{h}[t]_+ + \mathfrak{sl}_{\alpha_2}$. Also, ϕ can be extended to a Lie algebra map $\mathfrak{a} \to U(\mathfrak{g}[t])$ by setting $\phi(h_1) = h_1 - 3$ (see [Naoi 2012, Section 4.2]). Proceeding as in Section 2.2, one sees that ϕ induces an algebra homomorphism $U_{\mathbb{F}}(\mathfrak{a}) \to U_{\mathbb{F}}(\mathfrak{g}[t])$ also denoted by ϕ.

We are ready to prove the analogue of Lemma 5.2.3 for type G_2.

Lemma 5.2.5. *Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, $\lambda \in P^+$, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ and $(x_{1,s}^-)^{(k)} v = 0$ for all $k >$* $\max\{0, \lambda(h_1) - 3s\}$. *Then for every short root γ, we have $(x_{\gamma,s}^-)^{(k)} v = 0$ for all* $s \geq 0$, $k > \max\{0, \lambda(h_\gamma) - 3s\}$.

Proof. Notice that the conclusion of the lemma is equivalent to

$$(x_{\gamma,s}^-)^{(k)} \in I_{\mathbb{F}}(\lambda) \ \text{ for all } s \geq 0, k > \max\{0, \lambda(h_\gamma) - 3s\}$$

for every short root γ. Recall that the short roots in R^+ are α_1, $\alpha := \alpha_1 + \alpha_2$ and $\vartheta := 2\alpha_1 + \alpha_2$ while the long roots are α_2, $\beta := 3\alpha_1 + \alpha_2$ and $\theta := 3\alpha_1 + 2\alpha_2$. For $\gamma = \alpha$, we have $h_\gamma = h_1 + 3h_2$ and the proof is similar to that of Lemma 5.2.3 (the details can be found in [Macedo 2013]). We shall use that the lemma holds for $\gamma = \alpha$ in the remainder of the proof. It remains to show that the lemma holds with $\gamma = \vartheta$. Notice that $h_\vartheta = 2h_1 + 3h_2$ and thus we want to prove that

$$(5.2.4) \qquad (x_{\vartheta,s}^-)^{(k)} \in I_{\mathbb{F}}(\lambda) \ \text{ for all } s \geq 0, k > \max\{0, 2\lambda(h_1) + 3\lambda(h_2) - 3s\}.$$

We prove (5.2.4) by induction on $\lambda(h_1)$. Following [Naoi 2012], we prove the cases $\lambda(h_1) \in \{0, 1, 2\}$ and then we show that (5.2.4) for $\lambda - 3\omega_1$ in place of λ implies it for λ. To shorten notation, set $a = \lambda(h_1)$, $b = \lambda(h_2)$.

(1) Assume $a = 0$. Since $\alpha_1 \in R^+_{\text{sh}}$, it follows that $(x_1^-)^{(k)} v = 0$ for all $k > 0$. By Lemma 5.2.2, we have $(x_{2,s}^-)^{(k)} v = 0$ for all $k > \max\{0, b - s\}$. Applying Lemma 3.1.5 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{\alpha_1,\alpha_2})$, it follows that $(x_{\vartheta,s}^-)^{(k)} v = 0$ for all $k > 3\max\{0, b - s\} = \max\{0, 2a + 3b - 3s\}$ as desired.

(2) Assume $a = 1$. This time we have $(x_1^-)^{(k)} v = 0$ for all $k > 1$. We split in 3 subcases.

(2.1) Suppose $b > s - 1$, and notice $2a + 3b - 3s > 0$. Lemma 5.2.2 implies $(x_{2,s}^-)^{(k)} v = 0$ for all $k > \max\{0, b - s\} = b - s$. Applying Lemma 3.1.5 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{\alpha_1,\alpha_2})$, it follows that $(x_{\vartheta,s}^-)^{(k)} v = 0$ for all $k > 2 + 3(b - s) = 2a + 3b - 3s$.

(2.2) Suppose $b = s - 1$, in which case $2a + 3b - 3s < 0$. Notice that $h_\beta = h_1 + h_2$ and, hence, $\lambda(h_\beta) = a + b = s$. Lemma 5.2.2 then implies that $(x_{\beta,s}^-)^{(k)} v = 0$ for all $k > 0$. Notice that $\{-\alpha_1, \beta\}$ form a basis for R. Since, $(x_1^+)^{(k)} v = 0$ for all $k > 0$, Lemma 3.1.5 applied to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{-\alpha_1,\beta})$ implies that $(x_{\vartheta,s}^-)^{(k)} v = 0$ for all $k > 0$.

(2.3) Suppose $b < s-1$, in which case $2a+3b-3s<0$. This time we apply Lemma 3.1.5 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{1,s-2}_{\alpha_1,\alpha_2})$. Indeed, we have $(x^-_{1,1})^{(k)}v=0$ for all $k>\max\{0,a-3\}=0$ and Lemma 5.2.2 implies that $(x^-_{2,s-2})^{(k)}v=0$ for all $k>\max\{0,b-(s-2)\}=0$. Thus, since$3(b-s)<-3$ and $a=1$, we have $\max\{0,2a+3b-3s\}=0$ and Lemma 3.1.5 implies that $(x^-_{\vartheta,s})^{(k)}v=0$ for all $k>0$.

(3) Assume $a=2$. We split in subcases as before.

(3.1) If $b>s-1$, the proof is similar to that of step (2.1).

(3.2) Suppose $b=s-1$, and notice that $2a+3b-3s=1$. Hence, we want to show that (5.2.4) holds for $k>1$. For $k>3$ we apply Lemma 3.1.5 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{1,s-2}_{\alpha_1,\alpha_2})$ in a similar fashion as we did in step (2.3) (the same can be conclude using the argument from step (2.2). For $k\in\{2,3\}$ we need our hypothesis on the characteristic of $\mathbb{F}$. Assume we have chosen the Chevalley basis so that $x^-_\vartheta=[x^+_1,x^-_\beta]$ and observe that (1.1.4) implies that $[x^+_1,x^-_\vartheta]=\pm 2x^-_\alpha$. Using this, one easily checks that

$$(x^-_{\vartheta,s})^{(2)}=(x^+_1)^{(2)}(x^-_{\beta,s})^{(2)}-\tfrac{1}{2}x^+_1(x^-_{\beta,s})^{(2)}x^+_1-\tfrac{1}{2}x^-_{\beta,s}x^-_{\vartheta,s}x^+_1\mp x^-_{\beta,s}x^-_{\alpha,s}.$$

Using the case $\gamma=\alpha$ and Lemma 5.2.2 we see that $x^-_{\alpha,s}v=(x^-_{\beta,s})^{(2)}v=0$. Hence, since $2\in\mathbb{F}^\times$, (5.2.4) holds for $k=2$. For $k=3$, we have $(x^-_{\vartheta,s})^{(3)}=\frac{1}{3}x^-_{\vartheta,s}(x^-_{\vartheta,s})^{(2)}$ and, since $3\in\mathbb{F}^\times$, (5.2.4) also holds for $k=3$.

(3.3) If $b<s-1$ the proof is similar to that of step (2.3).

(4) Assume $a\geq 3$ and that (5.2.4) holds for $\lambda-3\omega_1$.

(4.1) Suppose $s\geq 2$ and recall the definition of the map $\phi: U_{\mathbb{F}}(\mathfrak{a})\to U_{\mathbb{F}}(\mathfrak{g}[t])$. The induction hypothesis together with Lemma 5.2.4 implies that

$$(x^-_{\vartheta,s-2})^{(k)}\in I'_{\mathbb{F}}(\lambda-3\omega_1)\ \text{ for all } k>\max\{0,2a+3b-3s\},$$

and therefore

$$(x^-_{\vartheta,s})^{(k)}=\phi\big((x^-_{\vartheta,s-2})^{(k)}\big)\in\phi(I'_{\mathbb{F}}(\lambda-3\omega_1))\ \text{ for all } k>\max\{0,2a+3b-3s\}.$$

One easily checks that ϕ sends the generators of $I'_{\mathbb{F}}(\lambda-3\omega_1)$ to generators of $I_{\mathbb{F}}(\lambda)$, completing the proof of (5.2.4) for $s\geq 2$.

(4.2) For $s=0$, notice that $U_{\mathbb{F}}(\mathfrak{g})v$ is a quotient of $W_{\mathbb{F}}(\lambda)$, and (5.2.4) follows. Equivalently, apply Lemma 3.1.5 to $U_{\mathbb{F}}(\mathfrak{g}^{0,0}_{\alpha_1,\alpha_2})=U_{\mathbb{F}}(\mathfrak{g})$ and the proof is similar to that of step (2.1).

(4.3) If $s=1$ and $b\geq 1$, we have $2a+3b-3s>0$ and the usual application of Lemma 3.1.5 to $U_{\mathbb{F}}(\mathfrak{g}^{0,1}_{\alpha_1,\alpha_2})$ completes the proof of (5.2.4). If $s=1$ and $b=0$, we need to show that $(x^-_{\vartheta,1})^{(k)}v=0$ for $k>2a-3$.

Consider the subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\vartheta[t])\cong U_{\mathbb{F}}(\mathfrak{sl}_2[t])$ defined in Section 2.3. Since $\lambda(h_\vartheta)=2a$, it follows that $W:=U_{\mathbb{F}}(\mathfrak{sl}_\vartheta[t])v$ is a quotient of the $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-module $W^c_{\mathbb{F}}(2a)$, where we identified the weight lattice of $\mathfrak{sl}_2$ with $\mathbb{Z}$ as usual. Since

$W^c_{\mathbb{F}}(2a) \cong D_{\mathbb{F}}(1,2a)$ by Theorem 1.5.2(a), the defining relations of $D_{\mathbb{F}}(1,2a)$ imply $(x^-_{\vartheta,1})^{(k)}v = 0$ for $k > 2a-1$. It remains to check that $(x^-_{\vartheta,1})^{(k)}v = 0$ for $k \in \{2a-2, 2a-1\}$.

Suppose by contradiction that $(x^-_{\vartheta,1})^{(2a-1)}v \neq 0$, and notice that

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-1)}v = 0 \ \text{ for all } k > 0. \tag{5.2.5}$$

Indeed,

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-1)}v \in W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)}$$

is a vector of degree $2a-1 > 1$ for all $k \geq 0$. By the Weyl group invariance of the character of $W^c_{\mathbb{F}}(2a)$, we know that $W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)} = 0$ if $k > 1$, and that $W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)}$ is one-dimensional concentrated in degree zero if $k = 1$. This proves (5.2.5). Then Lemma 3.1.3 implies that

$$(x^+_{\vartheta})^{(2a-2)}(x^-_{\vartheta,1})^{(2a-1)}v \neq 0.$$

On the other hand, it follows from Lemma 2.1.1 that

$$(x^+_{\vartheta})^{(2a-2)}(x^-_{\vartheta,1})^{(2a-1)}v = x^-_{\vartheta,2a-1}v.$$

Since $2a-1 \geq 2$ and $2a-3(2a-1) = -4a+3 < 0$, it follows from step (4.1) that $x^-_{\vartheta,2a-1}v = 0$ yielding a contradiction as desired.

Similarly, assume by contradiction that $(x^-_{\vartheta,1})^{(2a-2)}v \neq 0$ and notice that

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-2)}v = 0 \ \text{ for all } k > 1.$$

Suppose first that $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v = 0$ as well. It then follows from Lemma 3.1.3 that

$$(x^+_{\vartheta})^{(2a-4)}(x^-_{\vartheta,1})^{(2a-2)}v \neq 0.$$

On the other hand, Lemma 2.1.1 implies that

$$(x^+_{\vartheta})^{(2a-4)}(x^-_{\vartheta,1})^{(2a-2)}v = (x^-_{\vartheta,a-1})^{(2)}v + \sum_{r=a}^{2a-2} x^-_{\vartheta,2a-2-r}x^-_{\vartheta,r}v.$$

Since $a-1 \geq 2$, step (4.1) implies that $(x^-_{\vartheta,r})^{(k)}v = 0$ for all $r \geq a-1$, $k > 0$, implying that the right-hand side is zero, which is a contradiction. It remains to check the possibility that $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v \neq 0$. In this case it follows that $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v$ is a lowest-weight vector for the algebra $U_{\mathbb{F}}(\mathfrak{sl}_{\vartheta})$ and, hence, Lemma 3.1.3 implies that

$$(x^+_{\vartheta})^{(2a-2)}x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v \neq 0.$$

Using (2.1.1) we get

$$(x^+_{\vartheta})^{(2a-2)}x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v = \left(x^-_{\vartheta}(x^+_{\vartheta})^{(2a-2)} + (x^+_{\vartheta})^{(2a-3)}\right)(x^-_{\vartheta,1})^{(2a-2)}v.$$

Lemma 2.1.1 together with step (4.1) will again imply that the right-hand side is zero. This completes the proof. □

5.3. ***Existence of Demazure flag.*** If $\mathfrak{g}$ is simply laced, Theorem 1.5.2(b) follows immediately from part (a) with $k = 1$. Thus, assume from now on that $\mathfrak{g}$ is not simply laced and recall the notation introduced in Section 2.4.

Given $\lambda \in P^+$, let $\mu = \bar{\lambda} \in P^+_{\mathrm{sh}}$ and v be the image of 1 in $W^c_{\mathbb{C}}(\lambda)$. Consider $W^{\mathrm{sh}}_{\mathbb{C}} := U(\mathfrak{g}_{\mathrm{sh}}[t])v$ and $W^{\mathrm{sh}}_{\mathbb{Z}} := U_{\mathbb{Z}}(\mathfrak{g}_{\mathrm{sh}}[t])v$. By [Naoi 2012, Lemma 4.17], there is an isomorphism of $U(\mathfrak{g}_{\mathrm{sh}}[t])$-modules $W^{\mathrm{sh}}_{\mathbb{C}} \cong D_{\mathbb{C}}(1, \mu)$. By Corollary 3.3.3, $W^{\mathrm{sh}}_{\mathbb{Z}}$ is an integral form of $W^c_{\mathbb{C}}(\mu) \cong D_{\mathbb{C}}(1, \mu)$. Hence, we have an isomorphism of $U_{\mathbb{Z}}(\mathfrak{g}_{\mathrm{sh}}[t])$-modules $W^{\mathrm{sh}}_{\mathbb{Z}} \cong D_{\mathbb{Z}}(1, \mu)$.

Since $\mathfrak{g}_{\mathrm{sh}}$ is of type A, Theorem 4.4.1 implies that there exist $k > 0$, $\mu_1, \dots, \mu_k \in P^+_{\mathrm{sh}}$, $m_1, \dots, m_k \in \mathbb{Z}_{\geq 0}$, and a filtration of $U_{\mathbb{Z}}(\mathfrak{g}_{\mathrm{sh}}[t])$-modules $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = W^{\mathrm{sh}}_{\mathbb{Z}}$, such that D_j and D_j/D_{j-1} are free $\mathbb{Z}$-modules, and $D_j/D_{j-1} \cong D_{\mathbb{Z}}(r^\vee, \mu_j, m_j)$ for all $j = 1, \dots, k$. In particular,

$$W^{\mathrm{sh}}_{\mathbb{Z}}/D_j \text{ is a free } \mathbb{Z}\text{-module for all } j = 0, \dots, k. \tag{5.3.1}$$

Set $\lambda_j = \eta_\lambda(\mu_j) \in P^+$ where η_λ is defined in (2.4.1), $W^j_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{g}[t])D_j$ and $W^j_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} W^j_{\mathbb{Z}}$. It is easy to see that we have $0 = W^0_{\mathbb{F}} \subseteq W^1_{\mathbb{F}} \subseteq \cdots \subseteq W^k_{\mathbb{F}}$, and $\lambda_k = \lambda$ since $\mu_k = \mu$. Hence, we are left to show that

$$W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong D_{\mathbb{F}}(1, \lambda_j, m_j) \text{ for all } j = 1, \dots, k, \quad \text{and} \quad W^k_{\mathbb{F}} \cong W^c_{\mathbb{F}}(\lambda).$$

Notice that $W^k_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{g}[t])v$. Then Corollary 3.3.3 implies that $W^k_{\mathbb{Z}}$ is an integral form of $W^c_{\mathbb{C}}(\lambda)$. Since $\mathbb{Z}$ is a PID and $W^k_{\mathbb{Z}}$ is a finitely generated, free $\mathbb{Z}$-module, it follows that $W^j_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank for all $j = 1, \dots, k$. Set $W^j_{\mathbb{C}} = U(\mathfrak{g}[t])D_j$. It follows from [Naoi 2012, Proposition 4.18] (which is Theorem 1.5.2(b) in characteristic zero) that $W^j_{\mathbb{C}}/W^{j-1}_{\mathbb{C}} \cong D_{\mathbb{C}}(1, \lambda_j, m_j)$ for all $j = 1, \dots, k$. Moreover, since $W^j_{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{Z}} W^j_{\mathbb{Z}}$, we have

$$\mathbb{C} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}}) \cong (W^j_{\mathbb{C}}/W^{j-1}_{\mathbb{C}}) \cong D_{\mathbb{C}}(1, \lambda_j, m_j).$$

Therefore, $W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module of rank $\dim(D_{\mathbb{C}}(1, \lambda_j, m_j))$ for all $j = 1, \dots, k$. Since $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong \mathbb{F} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}})$, it follows that

$$\dim(W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}) \geq \dim(D_{\mathbb{C}}(1, \lambda_j, m_j)) = \dim(D_{\mathbb{F}}(1, \lambda_j, m_j)).$$

Now, let $v_j \in D_j$ be as in Theorem 4.4.1, w be the image of v in $W^k_{\mathbb{F}}$, $u_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\mathrm{sh}}[t])$ be such that $v_j = u_j v$, and $w_j = u_j w$. It follows that

$$W^j_{\mathbb{Z}} = \sum_{n \leq j} U_{\mathbb{Z}}(\mathfrak{g}[t])v_n \quad \text{and} \quad W^j_{\mathbb{F}} = \sum_{n \leq j} U_{\mathbb{F}}(\mathfrak{g}[t])w_n.$$

We will show that the image $\overline{w}_j$ of w_j in $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}$ satisfies the relations described in Proposition 5.2.1, which implies that $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}$ is a quotient of $D_{\mathbb{F}}(1, \lambda_j, m_j)$ and, hence, $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong D_{\mathbb{F}}(1, \lambda_j, m_j)$ for all $j = 1, \dots, k$.

By construction, v_j is a weight vector of weight λ_j and degree m_j, and so is w_j. Since $D_j/D_{j-1} \cong D_{\mathbb{Z}}(r^\vee, \mu_j, m_j)$, it follows that

$$U_{\mathbb{F}}(\mathfrak{n}^+_{\mathrm{sh}}[t])^0 \overline{w}_j = U_{\mathbb{F}}(\mathfrak{h}_{\mathrm{sh}}[t]_+)^0 \overline{w}_j = 0 \quad \text{and} \quad (x^-_{\alpha,s})^{(k)} \overline{w}_j = 0$$

for all $\alpha \in R^+_{\mathrm{sh}}$, $s \geq 0$, $k > \max\{0, \lambda(h_\alpha) - sr^\vee\}$, $j = 1, \dots, k$. Thus, it remains to show that

$$(x^+_{\alpha,s})^{(m)} \overline{w}_j = \Lambda_{i,r} \overline{w}_j = (x^-_\alpha)^{(k)} \overline{w}_j = 0$$

for all $\alpha \in R^+ \setminus R^+_{\mathrm{sh}}$, $s \geq 0$, $r, m > 0$, $k > \lambda_j(h_i)$, $j = 1, \dots, k$. Since,

$$\lambda_j + m\alpha \notin \lambda - Q^+ \text{ for all } \alpha \in R^+ \setminus R^+_{\mathrm{sh}}, m > 0, \tag{5.3.2}$$

we get $(x^+_{\alpha,s})^{(m)} w_j = 0$ for all $m > 0$, $s \geq 0$. In particular, it follows that $\overline{w}_j$ is a highest-weight vector of weight λ_j and, hence, $(x^-_\alpha)^{(k)} \overline{w}_j = 0$ for all $\alpha \in R^+$, $k > \lambda(h_\alpha)$. Finally, we show that

$$\Lambda_{i,r} \overline{w}_j = 0 \text{ for all } i \in I \setminus I_{\mathrm{sh}}, r > 0, j = 1, \dots, k. \tag{5.3.3}$$

Observe that

$$\Lambda_{i,r} u_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\mathrm{sh}}) U_{\mathbb{Z}}(\mathfrak{h}[t]_+).$$

In particular, $\Lambda_{i,r} v_j \in W^{\mathrm{sh}}_{\mathbb{Z}} \cap W^j_{\mathbb{Z}}$. We will show that $\Lambda_{i,r} v_j \in D_{j-1}$ which implies (5.3.3). Let $y_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\mathrm{sh}})$ be such that $\Lambda_{i,r} u_j = y_j$ modulo $U_{\mathbb{Z}}(\mathfrak{n}^-_{\mathrm{sh}}) U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0$. Thus, we want to show that

$$y_j v \in D_{j-1}. \tag{5.3.4}$$

We prove this recursively on $j = 1, \dots, k$. Notice that since $\mathbb{C} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}} / W^{j-1}_{\mathbb{Z}}) \cong D_{\mathbb{C}}(1, \lambda_j, m_j)$, there exists $n_j \in \mathbb{Z}_{>0}$ such that $n_j y_j v \in W^{j-1}_{\mathbb{Z}}$, $j = 1, \dots, k$. In particular, since $W^0_{\mathbb{Z}} = 0$ and $W^1_{\mathbb{Z}}$ is a torsion-free $\mathbb{Z}$-module, (5.3.4) follows for $j = 1$. Next, we show that (5.3.4) implies

$$W^j_{\mathbb{Z}} \cap W^{\mathrm{sh}}_{\mathbb{Z}} = D_j. \tag{5.3.5}$$

Indeed, it follows from (5.3.2) and (5.3.4) that

$$W^j_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{n}^-[t]) U_{\mathbb{Z}}(\mathfrak{g}_{\mathrm{sh}}[t]) v_j + W^{j-1}_{\mathbb{Z}}.$$

Since $U_{\mathbb{Z}}(\mathfrak{h}_{\mathrm{sh}}[t]_+)^0 U_{\mathbb{Z}}(\mathfrak{n}^+_{\mathrm{sh}})^0 v_j \in D_{j-1}$ and, by the induction hypothesis, $W^{j-1}_{\mathbb{Z}} \cap W^{\mathrm{sh}}_{\mathbb{Z}} = D_{j-1}$, (5.3.5) follows by observing that

$$(U_{\mathbb{Z}}(\mathfrak{n}^-[t]) v_j) \cap W^{\mathrm{sh}}_{\mathbb{Z}} \subseteq D_j$$

(which is easily verified by weight considerations). Finally, observe that since $n_{j+1} y_{j+1} v$ is in $W^j_{\mathbb{Z}} \cap W^{\mathrm{sh}}_{\mathbb{Z}} = D_j$, (5.3.1) implies that $y_{j+1} v \in D_j$. Thus, (5.3.5) for j implies (5.3.4) for $j + 1$ and the recursive step is proved.

Remark 5.3.1. It follows from the above that $W_{\mathbb{F}}^{j}/W_{\mathbb{F}}^{j-1} \cong D_{\mathbb{F}}(1, \lambda_j, m_j)$ for any field $\mathbb{F}$. Hence, $W_{\mathbb{Z}}^{j}/W_{\mathbb{Z}}^{j-1}$ must be isomorphic to $D_{\mathbb{Z}}(1, \lambda_j, m_j)$ for all $j = 1, \ldots, k$.

It remains to show that $W_{\mathbb{F}}^{k} \cong W_{\mathbb{F}}^{c}(\lambda)$. Since Theorem 3.3.4(c) implies that we have a projection $W_{\mathbb{F}}^{c}(\lambda) \twoheadrightarrow W_{\mathbb{F}}^{k}$ of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules, it suffices to show that $\dim(W_{\mathbb{F}}^{c}(\lambda)) \leq \dim(W_{\mathbb{F}}^{k})$. This follows if we show that there exists a filtration $0 = \tilde{W}_{\mathbb{F}}^{0} \subseteq \tilde{W}_{\mathbb{F}}^{1} \subseteq \cdots \subseteq \tilde{W}_{\mathbb{F}}^{k} = W_{\mathbb{F}}^{c}(\lambda)$ such that $\tilde{W}_{\mathbb{F}}^{j}/\tilde{W}_{\mathbb{F}}^{j-1}$ is a quotient of $D_{\mathbb{F}}(1, \lambda_j, m_j)$ for all $j = 1, \ldots, k$. Let w' be the image of 1 in $W_{\mathbb{F}}^{c}(\lambda)$, $w'_j = u_j w' \in W_{\mathbb{F}}^{c}(\lambda)$, $\tilde{W}_{\mathbb{F}}^{j} := \sum_{n \leq j} U_{\mathbb{F}}(\mathfrak{g}[t]) w'_n \subseteq W_{\mathbb{F}}^{c}(\lambda)$, and $\overline{w}'_j$ be the image of w'_j in $\tilde{W}_{\mathbb{F}}^{j}/\tilde{W}_{\mathbb{F}}^{j-1}$. Observe that $\tilde{W}_{\mathbb{F}}^{k} = W_{\mathbb{F}}^{c}(\lambda)$. We need to show that $\overline{w}'_j$ satisfies the defining relations of $D_{\mathbb{F}}(1, \lambda_j)$ listed in Proposition 5.2.1. Let $\tilde{D}_j = \mathbb{F} \otimes_{\mathbb{Z}} D_j$ and $D'_j = \sum_{n \leq j} U_{\mathbb{F}}(\mathfrak{g}_{\mathrm{sh}}[t]) w'_n$. Notice that D'_k is a quotient of $W_{\mathbb{F}}^{c}(\mu) \cong \tilde{D}_k$ and let $\pi : \tilde{D}_k \to D'_k$ be a $U_{\mathbb{F}}(\mathfrak{g}_{\mathrm{sh}}[t])$-module epimorphism such that $v_k \mapsto w'_k$ (we keep denoting the image of v_j in $\tilde{D}_j$ by v_j). In particular, $w'_j = \pi(v_j)$ and π induces an epimorphism $\tilde{D}_j \to D'_j$ for all $j = 1, \ldots, k$. Hence,

$$x w'_j \in D'_{j-1} \quad \text{for all } x \in U_{\mathbb{Z}}(\mathfrak{g}_{\mathrm{sh}}[t]) \ \text{ such that } x v_j \in D_{j-1}.$$

This immediately implies that

$$U_{\mathbb{F}}(\mathfrak{n}_{\mathrm{sh}}^{+}[t])^0 \overline{w}'_j = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 \overline{w}'_j = 0 \quad \text{and} \quad (x_{\alpha,s}^{-})^{(k)} \overline{w}'_j = 0$$

for all $\alpha \in R_{\mathrm{sh}}^{+}$, $s \geq 0$, $k > \max\{0, \lambda(h_\alpha) - s r^\vee\}$, $j = 1, \ldots, k$. Note that (5.3.4) has been used here. The relations

$$(x_{\alpha,s}^{+})^{(m)} \overline{w}'_j = (x_i^{-})^{(k)} \overline{w}'_j = 0$$

for all $\alpha \in R^+ \setminus R_{\mathrm{sh}}^{+}$, $i \in I \setminus I_{\mathrm{sh}}$, $s \geq 0$, $m > 0$, $k > \lambda_j(h_i)$, $j = 1, \ldots, k$ follow from (5.3.2) as before.

5.4. ***The isomorphism between local Weyl modules and graded local Weyl modules.*** We now prove Theorem 1.5.2(c). Recall the definition of the automorphism φ_a of $U_{\mathbb{F}}(\mathfrak{g}[t])$ from Section 2.2. In particular, let $\tilde{a} \in \mathbb{A}^\times$ be such that its image in $\mathbb{F}$ is a. Denote by $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ the pull-back of $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$ (regarded as a $U_{\mathbb{F}}(\mathfrak{g}[t])$-module) by φ_a.

Notice that

$$\dim W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = \dim W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}) = \dim W_{\mathbb{K}}^{c}(\lambda) = \dim W_{\mathbb{F}}^{c}(\lambda).$$

Here, the first equality follows from (1.5.4), the second from (3.4.1) (with $\mathbb{F} = \mathbb{K}$) together with Proposition 3.4.1, and the third from Corollary 1.5.3. Since $\dim \varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})) = \dim W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$, Theorem 1.5.2(c) follows if we show that $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ is a quotient of $W_{\mathbb{F}}^{c}(\lambda)$.

Let $w \in W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})_\lambda \setminus \{0\}$ and use the symbol w_a to denote w when regarded as an element of $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$. Since $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = U_{\mathbb{F}}(\mathfrak{g}[t]) w$ and φ_a is an automorphism of $U_{\mathbb{F}}(\mathfrak{g}[t])$, it follows that $\varphi_a^* W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = U_{\mathbb{F}}(\mathfrak{g}[t])w_a$. Thus, we need to show that w_a satisfies the defining relations (1.5.1) of $W_{\mathbb{F}}^c(\lambda)$. Since φ_a fixes every element of $U_{\mathbb{F}}(\mathfrak{g})$, w_a is a vector of weight λ annihilated by $(x_\alpha^-)^{(k)}$ for all $\alpha \in R^+, k > \lambda(h_\alpha)$. Equation (2.2.6) implies that φ_a maps $U_{\mathbb{F}}(\mathfrak{n}^+[t])$ to itself and, hence, $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 w_a = 0$. Therefore, it remains to show that

$$U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 w_a = 0.$$

To show this, let v be in $W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}})_\lambda \setminus \{0\}$ and $L = U_{\mathbb{A}}(\mathfrak{g}[t])v$. By (1.5.4), $\mathbb{F} \otimes_{\mathbb{A}} L \cong W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$. In particular, the action of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0$ on $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ is obtained from the action of $U_{\mathbb{A}}(\mathfrak{h}[t]_+)^0$ on $\varphi_{\tilde{a}}^*(W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}))$ which, in turn, is obtained from the action of $U_{\mathbb{K}}(\mathfrak{h}[t]_+)^0$. Since $U_{\mathbb{K}}(\mathfrak{h}[t]_+)$ is generated by $h_{i,r}$, $i \in I, r > 0$, we are left to show that

$$h_{i,r} v_a = 0,$$

where v_a is the vector v regarded as an element of $\varphi_{\tilde{a}}^*(W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}))$. It is well known that the irreducible quotient of $W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}})$ is the evaluation module with evaluation parameter $\tilde{a}$ (see [Jakelić and Moura 2007, Section 3B]). Hence, $h_{i,s}v = \tilde{a}^s \lambda(h_i)v$ for all $i \in I$, $s \in \mathbb{Z}$. Using this, it follows that, for all $i \in I$, $r > 0$, we have

$$h_{i,r} v_a = (h_i \otimes (t-\tilde{a})^r)v = \sum_{s=0}^{r} \binom{r}{s} (-\tilde{a})^s h_{i,r-s} v = \lambda(h_i)\tilde{a}^r \sum_{s=0}^{r} \binom{r}{s} (-1)^s v = 0.$$

5.5. ***A tensor product theorem.*** We say that $\boldsymbol{\omega}, \boldsymbol{\pi} \in \mathcal{P}_{\mathbb{F}}^+$ are relatively prime if for all $i, j \in I$ the polynomials $\boldsymbol{\omega}_i(u)$ and $\boldsymbol{\pi}_j(u)$ are relatively prime in $\mathbb{F}[u]$. The goal of this subsection is to prove the following theorem from which we will deduce Theorem 1.5.2(d).

Theorem 5.5.1. *Suppose $\boldsymbol{\omega}, \boldsymbol{\pi} \in \mathcal{P}_{\mathbb{F}}^+$ are relatively prime and that V and W are quotients of $W_{\mathbb{F}}(\boldsymbol{\omega})$ and $W_{\mathbb{F}}(\boldsymbol{\pi})$, respectively. Then $V \otimes W$ is generated by its top weight space.*

Theorem 5.5.1 was proved in [Chari and Pressley 2001] in the case $\mathbb{F} = \mathbb{C}$. Although the proof we present here follows the same general lines, there are several extra technical issues to be taken care of arising from the fact that $U_{\mathbb{C}}(\tilde{\mathfrak{g}})$ is generated by $x_{\alpha,r}^{\pm}$, $\alpha \in R^+$, $r \in \mathbb{Z}$, while, in the case of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$, we also need arbitrarily large divided powers of these elements. We start the proof by establishing a few technical lemmas. Recall the definition of $X_{\alpha,m,s}^-(u)$ in Section 2.1 and set

$$X_{\alpha;s}^-(u) = X_{\alpha,1,s+1}^-(u)$$

To shorten notation, we shall often write $X_{\alpha;s}^-$ instead of $X_{\alpha;s}^-(u)$.

Fix $\boldsymbol{\omega} \in \mathcal{P}^+$ and let w be a highest-ℓ-weight vector of $W_{\mathbb{F}}(\boldsymbol{\omega})$. Given $\beta \in R^+$, define $\boldsymbol{\omega}_\beta(u) \in \mathbb{F}[u]$ by

$$\boldsymbol{\omega}_\beta(u) w = \Lambda_\beta(u) w.$$

One can easily check (see [Chari and Pressley 2001, Lemma 3.1]) that if ϑ is the highest short root of $\mathfrak{g}$ and $\beta \in R^+$, then there exists $\boldsymbol{\omega}_{\vartheta,\beta} \in \mathcal{P}^+$ such that

$$\boldsymbol{\omega}_\vartheta = \boldsymbol{\omega}_\beta \boldsymbol{\omega}_{\vartheta,\beta}.$$

Lemma 5.5.2. *For all $\beta \in R^+$, $k, l, s \in \mathbb{Z}$, $0 \le l \le k$, $k > \lambda(h_\beta)$, we have*

$$\big(\boldsymbol{\omega}_\vartheta X_{\beta;s}^{-\ (k-l)}\big)_{k+\deg(\boldsymbol{\omega}_{\vartheta,\beta})} w = 0.$$

Proof. We will need the following particular case of Lemma 2.1.1:

(5.5.1)
$$(x_{\beta,-s}^+)^{(l)} (x_{\beta,s+1}^-)^{(k)} = (-1)^l \big((X_{\beta;s}^-(u))^{(k-l)} \Lambda_\beta(u) \big)_k \mod U_{\mathbb{Z}}(\tilde{\mathfrak{g}}) U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+)^0$$

for all $k, l, s \in \mathbb{Z}$, $0 \le l \le k$. It follows from (5.5.1) and the definition of $\boldsymbol{\omega}_\beta$ that

(5.5.2) $$\big(\boldsymbol{\omega}_\beta X_{\beta;s}^{-\ (k-l)}\big)_k w = 0 \ \text{ for all } k, l, s \in \mathbb{Z}, 0 \le l \le k,\ k > \lambda(h_\beta).$$

Hence, for such k, l, s, we have

$$\begin{aligned}\big(\boldsymbol{\omega}_\vartheta X_{\beta;s}^{-\ (k-l)}\big)_{k+\deg(\boldsymbol{\omega}_{\vartheta,\beta})} w &= \big(\boldsymbol{\omega}_{\vartheta,\beta} \boldsymbol{\omega}_\beta X_{\beta;s}^{-\ (k-l)}\big)_{k+\deg(\boldsymbol{\omega}_{\vartheta,\beta})} w \\ &= \sum_{j=0}^{\deg(\boldsymbol{\omega}_{\vartheta,\beta})} (\boldsymbol{\omega}_{\vartheta,\beta})_j \big(\boldsymbol{\omega}_\beta X_{\beta;s}^{-\ (k-l)}\big)_{k+\deg(\boldsymbol{\omega}_{\vartheta,\beta})-j} w = 0,\end{aligned}$$

where the last equality follows from (5.5.2) since $k + \deg(\boldsymbol{\omega}_{\vartheta,\beta}) - j > \lambda(h_\beta)$. □

Let $\mathcal{R} = R^+ \times \mathbb{Z} \times \mathbb{Z}_{\ge 0}$ and Ξ be the set of functions $\xi : \mathbb{N} \to \mathcal{R}$ given by $j \mapsto \xi_j = (\beta_j, s_j, k_j)$, such that $k_j = 0$ for all j sufficiently large. Define the degree of ξ to be $d(\xi) = \sum_j k_j$. Let Ξ_d be the subset of functions of degree d and $\Xi_d^< = \bigcup_{d'<d} \Xi_{d'}$. Given $\xi \in \Xi$ such that $\xi_j = (\beta_j, s_j, k_j)$ for all $j \in \mathbb{N}$ and $k_j = 0$ for $j > m$, set

(5.5.3) $$x^\xi = (x_{\beta_1,s_1}^-)^{(k_1)} \cdots (x_{\beta_m,s_m}^-)^{(k_m)} \quad \text{and} \quad w^\xi = x^\xi w.$$

It will be convenient to write $\deg(w^\xi) = d(\xi) = \deg(x^\xi)$. The next lemma is an easy consequence of [Mitzman 1985, Lemma 4.2.13].

Lemma 5.5.3. *Let α be in R^+, $s \in \mathbb{Z}$, d, k be in $\mathbb{Z}_{\ge 0}$, and ξ be in Ξ_d. Then $x^\xi (x_{\alpha,s}^-)^{(k)}$ is in the span of*

$$\{(x_{\alpha,s}^-)^{(k)} x^\xi\} \cup \{x^\varsigma : \varsigma \in \Xi_{d+k}^<\}.$$ □

Lemma 5.5.4. *Let β be in R^+, k in $\mathbb{Z}$, d, r, s in $\mathbb{Z}_{\geq 0}$, $r \leq s$, $s > \lambda(h_\beta)$ and ξ in Ξ_d. Then $(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)})_s w^\xi$ is in the span of vectors of the form w^ς with $\varsigma \in \Xi^<_{r+d}$.*

Proof. If $d = 0$, it follows from (5.5.2) that $(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)})_s w^\xi = 0$, which proves the lemma in this case. We now proceed by induction on d. Thus, let $d > 0$ and write $w^\xi = (x^-_{\beta_1,s_1})^{(k_1)} \cdots (x^-_{\beta_l,s_l})^{(k_l)} w$ with $k_1 \neq 0$. Let also $\xi' \in \Xi$ be such that

$$\xi'_j = \begin{cases} \xi_j, & \text{if } j \neq 1, \\ (\beta_1, s_1, 0), & \text{if } j = 1. \end{cases}$$

Then, by Lemma 5.5.3, we have

$$\big(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)}\big)_s w^\xi = \big(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)}\big)_s (x^-_{\beta_1,s_1})^{(k_1)} w^{\xi'} = (x^-_{\beta_1,s_1})^{(k_1)} \big(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)}\big)_s w^{\xi'} + X w^{\xi'}$$

where X is in the span of $\{x^\varsigma : \varsigma \in \Xi^<_{r+k_1}\}$. In particular, $X w^{\xi'}$ is in the span of vectors of the desired form. Since $d(\xi') = d - k_1 < d$, the induction hypothesis implies that $(\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)})_s w^{\xi'}$ is in the span of vectors associated to elements of $\Xi^<_{r+d-k_1}$. Therefore, $(x^-_{\beta_1,s_1})^{(k_1)} (\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)})_s w^{\xi'}$ is in the span of vectors associated to elements of $\Xi^<_{r+d}$ as desired. □

Proof of Theorem 5.5.1. Let $w_{\boldsymbol{\omega}}$ and $w_{\boldsymbol{\pi}}$ be highest-ℓ-weight vectors for V and W, respectively. Let also

$$M = U_{\mathbb{F}}(\tilde{\mathfrak{g}})(w_{\boldsymbol{\omega}} \otimes w_{\boldsymbol{\pi}}) = U_{\mathbb{F}}(\tilde{\mathfrak{n}}^-)(w_{\boldsymbol{\omega}} \otimes w_{\boldsymbol{\pi}}).$$

Our goal is to show that $M = V \otimes W$. Since the vectors $w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}}$, $\xi, \xi' \in \Xi$ span $V \otimes W$, it suffices to show that these vectors are in M. We do this by induction on $d(\xi) + d(\xi')$ which obviously starts when $d(\xi) + d(\xi') = 0$ since, in this case, $w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}} = w_{\boldsymbol{\omega}} \otimes w_{\boldsymbol{\pi}}$.

Let $n \geq 0$, and suppose, by induction hypothesis, that

$$w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}} \in M \quad \text{for all } \xi, \xi' \in \Xi \text{ such that } d(\xi) + d(\xi') \leq n. \tag{5.5.4}$$

In order to complete the induction step, it suffices to show that

$$w^\xi_{\boldsymbol{\omega}} \otimes (x^-_{\beta,l})^{(r)} w^{\xi'}_{\boldsymbol{\pi}} \in M \quad \text{and} \quad ((x^-_{\beta,l})^{(r)} w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} \in M \tag{5.5.5}$$

for all $\beta \in R^+$, $r, l \in \mathbb{Z}$, $r \geq 1$, $\xi, \xi' \in \Xi$, such that $d(\xi) + d(\xi') + r = n + 1$. We prove (5.5.5) by a further induction on $r \geq 1$. Henceforth we fix $\beta \in R^+$.

Observe that the hypothesis on $\boldsymbol{\omega}$ and $\boldsymbol{\pi}$ implies that $\boldsymbol{\omega}_\vartheta$ and $\boldsymbol{\pi}_\vartheta$ are relatively prime. Therefore, we can choose $R, S \in \mathbb{F}[u]$ such that

$$R\boldsymbol{\omega}_\vartheta + S\boldsymbol{\pi}_\vartheta = 1.$$

Set

$$\delta = \deg(R\boldsymbol{\omega}_\vartheta) = \deg(S\boldsymbol{\pi}_\vartheta) \quad \text{and} \quad m = \max\{\mathrm{wt}(\boldsymbol{\omega})(h_\beta), \mathrm{wt}(\boldsymbol{\pi})(h_\beta)\}.$$

We claim that for all $\xi \in \Xi$ and $k \in \mathbb{Z}$,

$$(R\boldsymbol{\omega}_\vartheta X_{\beta;k}^{-\,(r)})_s w^\xi_{\boldsymbol{\omega}} \in \mathrm{span}(\{w^\varsigma_{\boldsymbol{\omega}} : \varsigma \in \Xi^<_{d(\xi)+r}\}) \quad \text{for all } s > m + \delta. \tag{5.5.6}$$

Indeed,

$$(R\boldsymbol{\omega}_\theta {X^-_{\beta;k}}^{(r)})_s w^\xi_{\boldsymbol{\omega}} = \sum_{j=0}^{\deg R} R_j (\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_{s-j} w^\xi_{\boldsymbol{\omega}}$$

and, since $s - j > m + \delta - j \geq m + \deg(\boldsymbol{\omega}_\vartheta) \geq \mathrm{wt}(\boldsymbol{\omega})(h_\beta)$, the claim follows from Lemma 5.5.4. Similarly one proves that

$$(5.5.7) \qquad (S\boldsymbol{\pi}_\vartheta {X^-_{\beta;k}}^{(r)})_s w^\xi_{\boldsymbol{\pi}} \in \mathrm{span}(\{w^\varsigma_{\boldsymbol{\pi}} : \varsigma \in \Xi^<_{d(\xi)+r}\}) \text{ for all } s > m + \delta.$$

We are ready to start the proof of (5.5.5). Suppose $d(\xi) + d(\xi') = n$ and let $\ell > m + \delta$. Then

$$\begin{aligned}
&(R\boldsymbol{\omega}_\vartheta X^-_{\beta;k})_\ell (w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}}) \\
&\qquad = ((R\boldsymbol{\omega}_\vartheta X^-_{\beta;k})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} + w^\xi_{\boldsymbol{\omega}} \otimes ((1 - S\boldsymbol{\pi}_\vartheta) X^-_{\beta;k})_\ell w^{\xi'}_{\boldsymbol{\pi}} \\
&\qquad = ((R\boldsymbol{\omega}_\vartheta X^-_{\beta;k})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} - w^\xi_{\boldsymbol{\omega}} \otimes (S\boldsymbol{\pi}_\vartheta X^-_{\beta;k})_\ell w^{\xi'}_{\boldsymbol{\pi}} + w^\xi_{\boldsymbol{\omega}} \otimes x^-_{\beta;\ell+k} w^{\xi'}_{\boldsymbol{\pi}}.
\end{aligned}$$

It follows from (5.5.6), (5.5.7) and (5.5.4) that $((R\boldsymbol{\omega}_\vartheta X^-_{\beta;k})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}}$ and $w^\xi_{\boldsymbol{\omega}} \otimes (S\boldsymbol{\pi}_\vartheta X^-_{\beta;k})_\ell w^{\xi'}_{\boldsymbol{\pi}}$ are in M. Since $(R\boldsymbol{\omega}_\vartheta X^-_{\beta;k})_\ell (w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}})$ is in M by definition, it follows that $w^\xi_{\boldsymbol{\omega}} \otimes x^-_{\beta;\ell+k} w^{\xi'}_{\boldsymbol{\pi}}$ is in M for all $k \in \mathbb{Z}$, which proves the first statement in (5.5.5) with $r = 1$. The second statement is proved similarly by looking at $(S\boldsymbol{\pi}_\vartheta X^-_{\beta;k})_\ell (w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}})$.

Let $r > 1$, $\xi, \xi' \in \Xi$ be such that $r + d(\xi) + d(\xi') = n + 1$ and set $\ell = r\ell'$ with ℓ' such that $\ell > m + \delta$. Then

$$\begin{aligned}
&(R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell (w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}}) \\
&\quad = ((R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} + w^{\xi'}_{\boldsymbol{\pi}} \otimes (R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^{\xi'}_{\boldsymbol{\pi}} + v \\
&\quad = ((R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} + w^\xi_{\boldsymbol{\omega}} \otimes ((1 - S\boldsymbol{\pi}_\vartheta) {X^-_{\beta;k}}^{(r)})_\ell w^{\xi'}_{\boldsymbol{\pi}} + v \\
&\quad = ((R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} - w^\xi_{\boldsymbol{\omega}} \otimes (S\boldsymbol{\pi}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^{\xi'}_{\boldsymbol{\pi}} \\
&\qquad\qquad + w^\xi_{\boldsymbol{\omega}} \otimes ({X^-_{\beta;k}}^{(r)})_\ell w^{\xi'}_{\boldsymbol{\pi}} + v \\
&\quad = ((R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} - w^\xi_{\boldsymbol{\omega}} \otimes (S\boldsymbol{\pi}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^{\xi'}_{\boldsymbol{\pi}} \\
&\qquad\qquad + w^\xi_{\boldsymbol{\omega}} \otimes (x^-_{\beta,\ell'+k})^{(r)} w^{\xi'}_{\boldsymbol{\pi}} + w^\xi_{\boldsymbol{\omega}} \otimes X w^{\xi'}_{\boldsymbol{\pi}} + v,
\end{aligned}$$

where v is in the span of vectors of the form

$$\left(\prod_i (x^-_{\beta,s_i})^{(a_i)} w^\xi_{\boldsymbol{\omega}}\right) \otimes \left(\prod_j (x^-_{\beta,s_j})^{(b_j)} w^{\xi'}_{\boldsymbol{\pi}}\right) \text{ with } 1 \leq a_i, b_j < r, \sum_i a_i + \sum_j b_j = r,$$

and X is in the span of elements of the form

$$(x^-_{\beta,s_1})^{(r_1)} (x^-_{\beta,s_2})^{(r_2)} \cdots (x^-_{\beta,s_n})^{(r_n)} \text{ with } r_1 + \cdots + r_n = r, \quad 0 < r_j < r.$$

Again, $(R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell(w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}})$ is in M by definition, while (5.5.6), (5.5.7), and (5.5.4), imply that

$$((R\boldsymbol{\omega}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell w^\xi_{\boldsymbol{\omega}}) \otimes w^{\xi'}_{\boldsymbol{\pi}} \in M \quad \text{and} \quad w^\xi_{\boldsymbol{\omega}} \otimes (S\boldsymbol{\pi}_\vartheta X^-_{\beta;k})_\ell w^{\xi'}_{\boldsymbol{\pi}} \in M.$$

By induction hypothesis on r, it follows that v and $w^\xi_{\boldsymbol{\omega}} \otimes X w^{\xi'}_{\boldsymbol{\pi}}$ are in M, which then implies that $w^\xi_{\boldsymbol{\omega}} \otimes (x^-_{\beta,\ell'+k})^{(r)} w^{\xi'}_{\boldsymbol{\pi}}$ is in M for all $k \in \mathbb{Z}$, completing the proof of the first statement of (5.5.5). The second statement is proved similarly by looking at $(S\boldsymbol{\pi}_\vartheta {X^-_{\beta;k}}^{(r)})_\ell(w^\xi_{\boldsymbol{\omega}} \otimes w^{\xi'}_{\boldsymbol{\pi}})$. □

5.6. ***The tensor product factorization of local Weyl modules.*** Theorem 1.5.2(d) clearly follows if we prove

$$W_{\mathbb{F}}(\boldsymbol{\varpi}_1) \otimes W_{\mathbb{F}}(\boldsymbol{\varpi}_2) \cong W_{\mathbb{F}}(\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2) \tag{5.6.1}$$

whenever $\boldsymbol{\varpi}_1, \boldsymbol{\varpi}_2 \in \mathcal{P}^+_{\mathbb{F}}$ are relatively prime.

In order to show (5.6.1), let $w_{\boldsymbol{\varpi}_1}$ and $w_{\boldsymbol{\varpi}_2}$ be highest-ℓ-weight vectors for $W_{\mathbb{F}}(\boldsymbol{\varpi}_1)$ and $W_{\mathbb{F}}(\boldsymbol{\varpi}_2)$, respectively. It is well known that $w_{\boldsymbol{\varpi}_1} \otimes v_{\boldsymbol{\varpi}_2}$ satisfies the defining relations of $W_{\mathbb{F}}(\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2)$, so there exists a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module map $\phi : W_{\mathbb{F}}(\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2) \to W_{\mathbb{F}}(\boldsymbol{\varpi}_1) \otimes W_{\mathbb{F}}(\boldsymbol{\varpi}_2)$ that sends $w_{\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2}$ to $w_{\boldsymbol{\varpi}_1} \otimes w_{\boldsymbol{\varpi}_2}$. Theorem 5.5.1 implies that ϕ is surjective. Hence, it suffices to show that

$$\dim(W_{\mathbb{F}}(\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2)) = \dim(W_{\mathbb{F}}(\boldsymbol{\varpi}_1) \otimes W_{\mathbb{F}}(\boldsymbol{\varpi}_2)). \tag{5.6.2}$$

In fact, recall from Remark 1.5.5 that there exist $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathcal{P}^\times_{\mathbb{A}}$ such that $\boldsymbol{\varpi}_1$ and $\boldsymbol{\varpi}_2$ are the images of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ in $\mathcal{P}^+_{\mathbb{F}}$, respectively. It then follows from (1.5.4) that

$$\begin{gathered} \dim(W_{\mathbb{F}}(\boldsymbol{\varpi}_1\boldsymbol{\varpi}_2)) = \dim(W_{\mathbb{K}}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2)) \quad \text{and} \\ \dim(W_{\mathbb{K}}(\boldsymbol{\omega}_i)) = \dim(W_{\mathbb{F}}(\boldsymbol{\varpi}_i)), \quad i = 1, 2. \end{gathered} \tag{5.6.3}$$

On the other hand, it follows from Theorem 1.5.2(d) in characteristic zero that

$$\dim(W_{\mathbb{K}}(\boldsymbol{\omega}_1\boldsymbol{\omega}_2)) = \dim(W_{\mathbb{K}}(\boldsymbol{\omega}_1)) \dim(W_{\mathbb{K}}(\boldsymbol{\omega}_2)). \tag{5.6.4}$$

Since (5.6.3) and (5.6.4) clearly imply (5.6.2), we are done.

5.7. ***Fusion products.*** We finish the paper with an application of Theorems 1.5.2 and 5.5.1 related to the concept of fusion products originally introduced in the characteristic-zero setting. Namely, we deduce the positive characteristic counterpart of [Naoi 2012, Corollary B] (compare [Fourier and Littelmann 2007, Corollary A] for simply-laced $\mathfrak{g}$).

Let V and W be as in Theorem 5.5.1, set $\lambda = \mathrm{wt}(\boldsymbol{\omega}) + \mathrm{wt}(\boldsymbol{\pi})$, and fix $v \in (V \otimes W)_\lambda \setminus \{0\}$. Then Theorem 5.5.1 implies that $V \otimes W = U_{\mathbb{F}}(\tilde{\mathfrak{g}})v$. In fact, as mentioned in Section 3.4, we actually have

$$V \otimes W = U_{\mathbb{F}}(\mathfrak{n}^-[t])v.$$

Define the fusion product of V and W, denoted $V * W$, as the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module $\mathrm{gr}(V \otimes W)$ with the module structure determined by v as described in the paragraph after Proposition 3.4.1. Evidently, if we have a collection $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_m$ of relatively prime elements of $\mathcal{P}_{\mathbb{F}}^+$ and, for each $j \in \{1, \dots, m\}$, V_j is a quotient of $W_{\mathbb{F}}(\boldsymbol{\omega}_j)$, we can define the fusion product $V_1 * \cdots * V_m$ in a similar way.

Proposition 5.7.1. *Let $\lambda \in P^+$, $m \in \mathbb{Z}_{>0}$ and $\boldsymbol{\omega}_j \in \mathcal{P}_{\mathbb{F}}^+$, $j = 1, \dots, m$, be relatively prime and such that $\lambda = \sum_{j=1}^m \mathrm{wt}(\boldsymbol{\omega}_j)$. Then*

$$W_{\mathbb{F}}^c(\lambda) \cong W_{\mathbb{F}}(\boldsymbol{\omega}_1) * \cdots * W_{\mathbb{F}}(\boldsymbol{\omega}_m).$$

Proof. One easily checks that a vector in $(W_{\mathbb{F}}(\boldsymbol{\omega}_1) * \cdots * W_{\mathbb{F}}(\boldsymbol{\omega}_m))_\lambda$ satisfies the defining relations of $W_{\mathbb{F}}^c(\lambda)$ (compare the proof of (1.5.4) in Section 3.4), showing that $W_{\mathbb{F}}(\boldsymbol{\omega}_1) * \cdots * W_{\mathbb{F}}(\boldsymbol{\omega}_m)$ is a quotient of $W_{\mathbb{F}}^c(\lambda)$. On the other hand, setting $\boldsymbol{\omega} = \prod_{j=1}^m \boldsymbol{\omega}_j$, we have

$$\begin{aligned}\dim(&W_{\mathbb{F}}(\boldsymbol{\omega}_1) * \cdots * W_{\mathbb{F}}(\boldsymbol{\omega}_m)) \\ &= \dim(W_{\mathbb{F}}(\boldsymbol{\omega}_1) \otimes \cdots \otimes W_{\mathbb{F}}(\boldsymbol{\omega}_m)) = \dim(W_{\mathbb{F}}(\boldsymbol{\omega})) = \dim(W_{\mathbb{F}}^c(\lambda)). \qquad \square\end{aligned}$$

The following corollary, which is the characteristic-free version of [Naoi 2012, Corollary B], is now easily deduced.

Corollary 5.7.2. *Let $m \in \mathbb{Z}_{>0}$, $\lambda_j \in P^+$ and $a_j \in \mathbb{F}^\times$, $j = 1, \dots, m$, be such that $a_i \neq a_j$ for $i \neq j$. Then, for $\lambda = \sum_{j=1}^m \lambda_j$, $W_{\mathbb{F}}^c(\lambda) \cong W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda_1,a_1}) * \cdots * W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda_m,a_m})$.* $\square$

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ON CURVES AND POLYGONS WITH THE EQUIANGULAR CHORD PROPERTY

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To the memory of Eugene Gutkin

Let C be a smooth, convex curve on either the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$ or the Euclidean plane $\mathbb{E}^2$ with the following property: there exists α and parametrizations $x(t)$ and $y(t)$ of C such that, for each t, the angle between the chord connecting $x(t)$ to $y(t)$ and C is α at both ends.

Assuming that C is not a circle, E. Gutkin completely characterized the angles α for which such a curve exists in the Euclidean case. We study the infinitesimal version of this problem in the context of the other two constant curvature geometries, and in particular, we provide a complete characterization of the angles α for which there exists a nontrivial infinitesimal deformation of a circle through such curves with corresponding angle α. We also consider a discrete version of this property for Euclidean polygons, and in this case, we give a complete description of all nontrivial solutions.

1. Introduction

Given a smooth, convex oriented closed curve C in the Euclidean plane $\mathbb{E}^2$ and $x, y \in C$, $x \neq y$, let $|xy|$ denote the oriented chord connecting x to y. Motivated by his study of mathematical billiards, E. Gutkin [1993] asked the following:

Question 1. Assume the existence of parametrizations $x(t)$ and $y(t)$ of C such that, for each t,

(1) $x'(t), y'(t) \neq 0$;

(2) $x(t) \neq y(t)$;

(3) there exists $\alpha \in (0, \pi]$ such that both angles between C and $|x(t)y(t)|$ equal α.

Then if C is not a circle, what are all possible values of α?

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Keywords: mathematical billiards, capillary floating problem, geometric rigidity.

Gutkin provides a complete answer to Question 1 by establishing the following necessary and sufficient condition for α: there exists an integer $k \geq 2$ such that

$$k \tan \alpha = \tan(k\alpha); \tag{1-1}$$

see [Gutkin 1993; 2012; Tabachnikov 1995]. In particular, only a countable number of values of the angle α are possible.

In terms of billiards, the billiard ball map on the interior of C has a horizontal invariant circle given by the condition that the angle made by the trajectories with the boundary of the table is equal to α. This statement can also be interpreted in terms of capillary floating with zero gravity in neutral equilibrium; see [Finn 2009; Finn and Sloss 2009].

We call a curve satisfying this equiangular chord property a *Gutkin curve*; we will refer to the corresponding angle α as the *contact angle*.

We generalize Gutkin's theorem in two directions: to curves in the standard 2-sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$ and to polygons in $\mathbb{E}^2$ via a discretized version of Question 1. For $\mathbb{S}^2$ and $\mathbb{H}^2$, we consider the following infinitesimal version of Gutkin's question:

Question 2. In either $\mathbb{H}^2$ or $\mathbb{S}^2$, for which angles α are there nontrivial infinitesimal deformations of a radius-R circle through Gutkin curves with contact angle α?

Here, a *nontrivial deformation* of a circle is a deformation that does not correspond to a circle solution (of a different radius).

Our first result yields an answer to Question 2:

Theorem 1.1. *Assume that a circle of radius R in $\mathbb{S}^2$ or in $\mathbb{H}^2$ admits a nontrivial infinitesimal deformation through Gutkin curves with contact angle α. Define angles c via*

$$\cot c = \cos R \cot \alpha$$

in the spherical case and

$$\cot c = \cosh R \cot \alpha$$

in the hyperbolic case. Then there exists $k \in \mathbb{N}$, $k \geq 2$, such that

$$k \tan c = \tan kc.$$

Thus, as in the Euclidean case, only a countable number of values of the contact angle α are possible for a given radius R.

Note that, in the Euclidean plane, Gutkin curves with contact angle $\alpha = \pi/2$ are precisely the curves of constant width; the same holds in the spherical and hyperbolic settings; see [Leichtweiss 2005] for curves of constant width in non-Euclidean geometries.

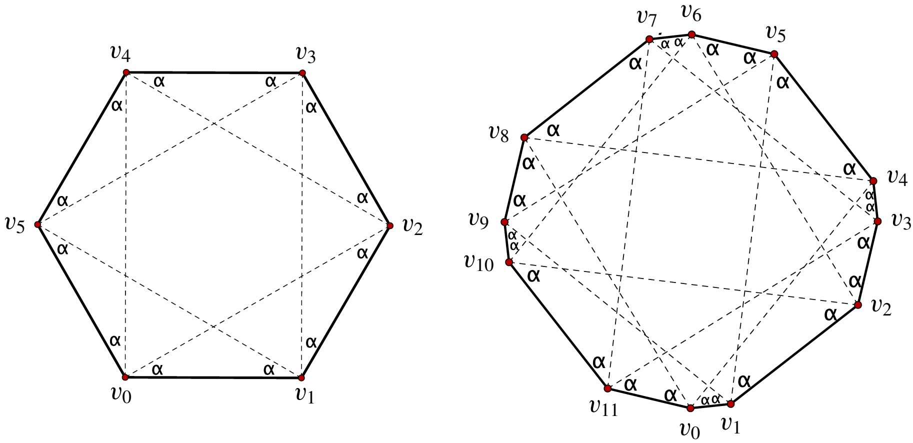

Figure 1. Gutkin (6, 2)-gon and (12, 4)-gon.

In Section 4, we consider the following analog of Gutkin's theorem for polygons in $\mathbb{E}^2$. Let P be a convex n-gon with vertices $\{v_0, \dots, v_{n-1}\}$ in their cyclic order. For $k \in \mathbb{N}$, $2 \le k \le n/2$, a k-diagonal is a straight line segment connecting vertices of P whose indices differ by k modulo n. Then P is a *nontrivial Gutkin* (n, k)*-gon* if P is not regular and there exists α such that, for any k-diagonal D, both contact angles between D and P equal α (see Figure 1 for examples). That is, for each i,

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \alpha,$$

where $\angle v_{i+1} v_i v_{i+k}$ denotes the angle between the edge $|v_{i+1} v_i|$ and the k-diagonal $|v_i v_{i+k}|$.

Our second result is a complete characterization of the pairs (n, k) for which a nontrivial Gutkin (n, k)-gon exists:

Theorem 1.2. *A nontrivial Gutkin* (n, k)*-gon in the Euclidean plane exists if and only if* n *and* $k-1$ *are not coprime.*

Interestingly, the main ingredient of our proof is the Diophantine equation

$$\tan \frac{kr\pi}{n} \tan \frac{\pi}{n} = \tan \frac{k\pi}{n} \tan \frac{r\pi}{n},$$

which is a discrete version of (1-1). This equation also appeared in [Tabachnikov 2006], and it was solved in [Connelly and Csikós 2009].

2. A proof of Gutkin's theorem in E^2

Although the existing proofs of Gutkin's theorem in $\mathbb{E}^2$ [Gutkin 1993; 2012; Tabachnikov 1995] are very clear and simple, our goal in this paper is to study the situations in $\mathbb{S}^2$ and $\mathbb{H}^2$. Therefore, in this section, we reprove (the necessary part of) Gutkin's

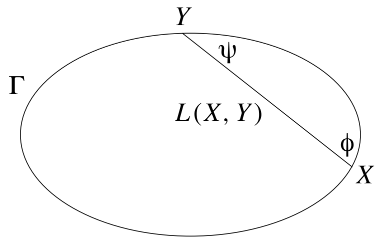

Figure 2. Curve Γ with chord xy.

theorem using methods that can be applied to the other constant-curvature settings. This proof is motivated by the study of integrable billiards by M. Bialy [1993; 2013].

Let $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be a periodic unit-speed parametrization of a smooth strictly convex curve Γ. For $x, y \in \mathbb{R}$, let X and Y be the points $\tilde{\gamma}(x)$ and $\tilde{\gamma}(y)$, ϕ and ψ the angles made by the chord XY with Γ, and $L = |XY|$ the length of the chord, the generating function of the billiard ball map. See Figure 2.

We have

$$L_x = -\cos\phi, \quad L_y = \cos\psi,$$

$$L_{xy} = \frac{\sin\phi\sin\psi}{L}, \; L_{xx} = \frac{\sin^2\phi}{L} - \kappa(x)\sin\phi, \; L_{yy} = \frac{\sin^2\psi}{L} - \kappa(y)\sin\psi, \tag{2-1}$$

where κ is the curvature of the curve and subscripts denote partial differentiation; see, e.g., [Bialy 1993].

We interpret $L(x, y)$ as a function on the torus $\Gamma\times\Gamma$. If Γ is a Gutkin curve with contact angle α, then there exists a curve s on this torus where both angles, ϕ and ψ, have the same constant value α.

We seek a reparametrization $\gamma(t(x)) = \tilde{\gamma}(x)$ so that the values $t(x)$ and $t(y)$ of the new parameter at the points X and Y differ by a constant: $2c = t(y) - t(x)$. Denote d/dt by a prime.

Proposition 2.1. *The parameter t is determined by the condition $x' = a/\kappa(x)$, where a is a constant.*

Proof. Since α is constant as a function of t,

$$0 = L_{xt} = L_{xx}x' + L_{xy}y' \quad\text{and}\quad 0 = L_{yt} = L_{xy}x' + L_{yy}y'. \tag{2-2}$$

This implies that $L_{xx}L_{yy} = L_{xy}^2$ along our curve, and substituting from (2-1), we have

$$\frac{\sin\alpha}{\kappa(x)} + \frac{\sin\alpha}{\kappa(y)} = L. \tag{2-3}$$

We compute y'/x' from (2-1)–(2-3),

$$\frac{y'}{x'} = -\frac{L_{xy}}{L_{yy}} = \frac{\sin\alpha}{\kappa(y)L - \sin\alpha} = \frac{\kappa(x)}{\kappa(y)},$$

which implies the claim. □

Since the curvature is the rate of turning of the direction of the curve, Proposition 2.1 defines (up to a multiplicative coefficient) the angular parameter along the curve. Note that $0 \le x \le L(\gamma)$ and $0 \le t \le T$, where T is the upper bound of t and $L(\gamma)$ is the length of γ. It follows that

$$T = \int_0^T dt = \frac{1}{a}\int_0^{L(\gamma)} \kappa(x)\,dx.$$

Choose $a = 1$ to make $T = 2\pi$, which agrees with the angle. Then $c = \alpha$.

In view of Proposition 2.1, we set

$$f_1 := f(t-\alpha) = \frac{\sin\alpha}{\kappa(x)} \quad \text{and} \quad f_2 := f(t+\alpha) = \frac{\sin\alpha}{\kappa(y)}.$$

From (2-3), we have

$$L = \frac{\kappa(x)\sin\alpha + \kappa(y)\sin\alpha}{\kappa(x)\kappa(y)} = f_1 + f_2.$$

It follows that $L' = f_1' + f_2'$. By the chain rule, we have

$$L_x x' + L_y y' = \cot\alpha(f_2 - f_1) = f_1' + f_2',$$

and therefore,

$$f'(t+\alpha) + f'(t-\alpha) = \cot\alpha(f(t+\alpha) - f(t-\alpha)). \tag{2-4}$$

Since $f(t)$ is a function with period 2π, using the Fourier expansion, we obtain $f(t) = \sum b_k e^{ikt}$, where $b_k \in \mathbb{C}$ and $b_{-k} = \overline{b_k}$. Thus,

$$f(t \pm \alpha) = \sum b_k e^{\pm ik\alpha} e^{ikt} \quad \text{and} \quad f'(t\pm\alpha) = \sum b_k ike^{\pm ik\alpha} e^{ikt}.$$

Let LHS be the left-hand side of (2-4) and RHS the right-hand side. It follows that

$$\text{LHS} = \sum b_k ik(e^{ik\alpha} + e^{-ik\alpha})e^{ikt} \quad \text{and} \quad \text{RHS} = \cot\alpha \sum b_k (e^{ik\alpha} - e^{-ik\alpha})e^{ikt}.$$

Equating both sides, we have

$$b_k(k\cos k\alpha - \cot\alpha \sin k\alpha) = 0.$$

For $k = 1$, this automatically holds, and if $b_k \ne 0$ for some $k \ge 2$, then

$$k\tan\alpha = \tan k\alpha.$$

If the curve is a circle, then $f(t)$ is constant and all $b_k = 0$, and if the curve is not a circle, then $b_k \neq 0$ for some $k \geq 1$. It remains to show that $b_1 = 0$.

Recall that x is arc length and t is the angular parameter on the curve γ. Then $\gamma_x = (\cos t, \sin t)$ and $dt/dx = \kappa$. Therefore,

$$\gamma_t = \frac{1}{\kappa}(\cos t, \sin t) \quad \text{and} \quad \int_0^{2\pi} \gamma_t \, dt = 0.$$

Hence,

$$\int_0^{2\pi} \frac{\cos t}{\kappa} \, dt = \int_0^{2\pi} \frac{\sin t}{\kappa} \, dt = 0;$$

that is, the function f is L^2-orthogonal to the first harmonics. Hence, f has no first harmonics in the Fourier expansion; that is, $b_1 = 0$.

3. Infinitesimal analogs of Gutkin's theorem in $\mathbb{S}^2$ and $\mathbb{H}^2$

We prove Theorem 1.1 in detail for $\mathbb{S}^2$. The hyperbolic case being analogous, we only indicate the necessary changes.

Let γ be a Gutkin curve, and as before, let x and y be arc length parameters. Then ϕ and ψ should have constant value, namely, the contact angle α. By [Bialy 2013], we have the following formulas for the first and second partials of L (valid along the curve $s \subset \Gamma \times \Gamma$):

$$L_x = -\cos\alpha, \quad L_y = \cos\alpha,$$

$$L_{xy} = \frac{\sin^2\alpha}{\sin L}, \quad L_{xx} = \frac{\sin^2\alpha}{\tan L} - \kappa(x)\sin\alpha, \quad L_{yy} = \frac{\sin^2\alpha}{\tan L} - \kappa(y)\sin\alpha. \tag{3-1}$$

(The function κ is the geodesic curvature of the curve.) Once again, we seek a parametrization on the curve such that the values of the parameter at points x and y differ by a constant: $t(y) = t(x) + 2c$.

Proposition 3.1. *The desired parametrization $\gamma(t)$ is given by the equation*

$$x' = \frac{a}{\sqrt{\kappa^2(x) + \sin^2\alpha}},$$

where a is a constant.

Proof. Equation (2-2) holds along our curve as before, so $L_{xx}L_{yy} = L_{xy}^2$. Substitute from (3-1) to obtain the equation

$$\left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)\left(\kappa(y) - \frac{\sin\alpha}{\tan L}\right) = \frac{\sin^2\alpha}{\sin^2 L}. \tag{3-2}$$

Then we can compute y'/x' from (2-2),

$$\frac{y'}{x'} = -\frac{L_{xx}}{L_{xy}} = \left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)\frac{\sin L}{\sin\alpha} = \frac{\sqrt{\kappa(x) - \frac{\sin\alpha}{\tan L}}}{\sqrt{\kappa(y) - \frac{\sin\alpha}{\tan L}}}, \tag{3-3}$$

with the last equality due to (3-2). Next, we claim that

$$\frac{\sqrt{\kappa(x) - \frac{\sin\alpha}{\tan L}}}{\sqrt{\kappa(y) - \frac{\sin\alpha}{\tan L}}} = \frac{\sqrt{\kappa^2(x) + \sin^2\alpha}}{\sqrt{\kappa^2(y) + \sin^2\alpha}}, \tag{3-4}$$

which, along with (3-3), implies the statement of the proposition.

It remains to prove (3-4). Rewrite (3-2) as

$$\kappa(x)\kappa(y) - \frac{\sin\alpha}{\tan L}(\kappa(x) + \kappa(y)) - \sin^2\alpha = 0,$$

and multiply by $\kappa(y) - \kappa(x)$ to obtain

$$\kappa(x)\kappa^2(y) - \frac{\sin\alpha}{\tan L}\kappa^2(y) + \kappa(x)\sin^2\alpha = \kappa^2(x)\kappa(y) - \frac{\sin\alpha}{\tan L}\kappa^2(x) + \kappa(y)\sin^2\alpha,$$

or

$$\left(\kappa(x) - \frac{\sin\alpha}{\tan L}\right)(\kappa^2(y) + \sin^2\alpha) = \left(\kappa(y) - \frac{\sin\alpha}{\tan L}\right)(\kappa^2(x) + \sin^2\alpha).$$

This implies (3-4). □

We choose a in such a way that

$$T = \frac{1}{a}\int_0^{L(\gamma)} \sqrt{\kappa^2(x) + \sin^2\alpha}\, dx = 2\pi \tag{3-5}$$

in order to make Fourier expansion more convenient.

Define a function f on the curve by

$$\cot f = \frac{\kappa}{\sin\alpha}. \tag{3-6}$$

Remark 3.2. The meaning of the function f is illustrated in Figure 3. Let O be the center of the osculating circle at point $x \in \gamma$, and let R be its radius. Then $\cot R = \kappa(x)$. Drop the perpendicular from O to the segment xy. Then we have a right triangle PxO with an angle $\pi/2 - \alpha$. Solving a right spherical triangle yields $\cot|Px|\sin\alpha = \cot R$. Hence, $f = |Px|$.

Denote by f_1 and f_2 the values of this function at points y and x.

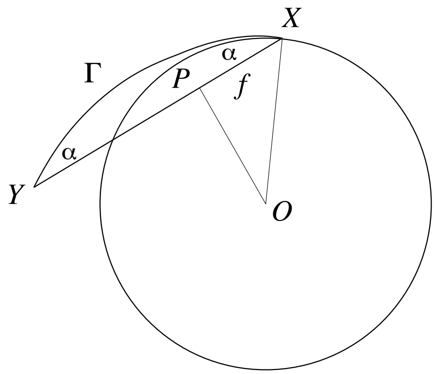

Figure 3. Geometric interpretation of the function f.

Proposition 3.3. *One has*

$$a \cot\alpha(\sin f_1 - \sin f_2) = f_1' + f_2'. \tag{3-7}$$

Proof. First, note that Proposition 3.1 and (3-6) imply that

$$x' = \frac{a \sin f}{\sin\alpha}. \tag{3-8}$$

Next, as before, $L_{xx}L_{yy} = L_{xy}^2$, and substituting from (3-1), we obtain

$$\cot L = \frac{\kappa(x)\kappa(y) - \sin^2\alpha}{\kappa(x)\sin\alpha + \kappa(y)\sin\alpha}.$$

Substituting $\kappa(x)$ and $\kappa(y)$ from (3-6) yields

$$\cot L = \frac{\cot f_1 \cot f_2 - 1}{\cot f_1 + \cot f_2} = \cot(f_1 + f_2).$$

Thus, $L = f_1 + f_2$, and hence, $L' = f_1' + f_2'$. By the chain rule,

$$L' = L_x x' L_y y' = \frac{a\cos\alpha}{\sin\alpha}(\sin f_1 - \sin f_2),$$

where the last equality is due to (3-1) and (3-8). This implies the statement. □

Remark 3.4. Equation (3-7) appeared in [Tabachnikov 2006] in a study of a different rigidity problem also related to a flotation problem (Ulam's problem on bodies that float in equilibrium in all positions) and to a problem of bicycle kinematics.

Equation (3-7) is an analog of (2-4), but unlike the Euclidean case, it is nonlinear, and we do not know how to solve it. Thus, we resort to linearization of the problem, that is, start from a circle γ_0 of radius R and then deform it to find infinitesimal solutions.

Write $f_1(t) = f(t+c)$ and $f_2(t) = f(t-c)$, where the constant c depends on the Gutkin curve and the contact angle (in the Euclidean case, $c = \alpha$). For a circle on S^2, we compute the relation between R, α and c and the value of a.

Lemma 3.5. *One has*

$$\cos\alpha = \frac{\cos c}{\sqrt{\sin^2 R \cos^2 c + \cos^2 R}} \quad \textit{or equivalently} \quad \cot c = \cos R \cot\alpha,$$

and

$$a = \sqrt{\cos^2 R + \sin^2\alpha \sin^2 R}.$$

Proof. The circle of radius R is parametrized as

$$\gamma_0(t) = (\sin R \cos t, \sin R \sin t, \cos R),$$

where $t \in [0, 2\pi]$. We need to find the angle α made by the geodesic segment $[\gamma_0(-c), \gamma_0(c)]$ with this circle.

The great circle through points $\gamma_0(-c)$ and $\gamma_0(c)$ is the parametric curve

$$\Gamma(s) = \frac{\cos c}{\sqrt{\sin^2 R \cos^2 c + \cos^2 R}}(\sin R \cos c, 0, \cos R) + \sin s(0, 1, 0),$$

and $\Gamma(s_0) = \gamma_0(c)$ for $\sin s_0 = \sin R \sin c$. It remains to compute the velocity vectors $d\Gamma(s)/ds$ and $d\gamma_0(t)/dt$, evaluate them at $s = s_0$ and $t = c$, respectively, and compute the angle between these vectors. This straightforward computation yields the first formula of the lemma. A calculation using trigonometric identities yields the simpler, equivalent, formula.

To obtain the formula for a, note that the length and the geodesic curvature of the circle γ_0 are equal to $2\pi \sin R$ and $\cot R$, respectively. Then (3-5) yields the result. □

Remark 3.6. A referee pointed out that this lemma can be proved, in a simpler way, by applying formulas of spherical trigonometry to the spherical triangle XPO in Figure 3.

Now we are ready for the proof of Theorem 1.1 in the spherical case. Let γ_0 be a circle of radius R. Then the function f is a constant satisfying $\cot f = \cot R/\sin\alpha$ (see (3-6)), and the constants c and a are as in Lemma 3.5. Consider an infinitesimal deformation of the curve in the class of Gutkin curves with the contact angle α. Then f, c and a deform as

$$f \mapsto f + \varepsilon g(t), \quad c \mapsto c + \varepsilon\delta, \quad a \mapsto a + \varepsilon\beta,$$

where $g(t)$ is a 2π-periodic function and all the previous relations hold. Substitute into (3-7):

$$(a+\varepsilon\beta)\cot\alpha\big(\sin(f+\varepsilon g(t+c+\varepsilon\delta))-\sin(f+\varepsilon g(t-c-\varepsilon\delta))\big) \\ =\varepsilon(g'(t+c+\varepsilon\delta)+g'(t-c-\varepsilon\delta)).$$

Computing modulo ε^2 yields

$$a\cot\alpha\cos f(g(t+c)-g(t-c))=g'(t+c)+g'(t-c).$$

As before, this implies that, if $g(t)$ is not a constant (which would correspond to a trivial deformation to a circle of possibly different radius), then

$$k\cos kc=a\cot\alpha\cos f\sin kc$$

for each k for which the Fourier coefficient $b_k\neq 0$. Substituting the values of the constants f and a and eliminating α using Lemma 3.5 yields, after a straightforward, albeit tedious, computation,

$$k\cos kc=\cot c\sin kc \quad \text{or} \quad k\tan c=\tan kc.$$

For $k=1$, this formula holds for all c, and it remains to explain the condition $k\geq 2$ in the formulation of the theorem. The next proposition shows that the first Fourier coefficient b_1 vanishes.

Proposition 3.7. *The function $g(t)$ is L^2-orthogonal to the first harmonics; that is, its Fourier expansion does not contain* $\cos t$ *and* $\sin t$.

Proof. Let φ and θ be the spherical coordinates. Recall that the spherical metric is $\sin^2\theta\, d\varphi^2+d\theta^2$. The unperturbed curve $\gamma_0(t)$, the circle of latitude of radius R, has the coordinates (t, R). Consider its infinitesimal deformation

$$\gamma_\varepsilon(t)=(t+\varepsilon\bar{f}(t), R+\varepsilon\bar{g}(t)),$$

where $\bar{f}$ and $\bar{g}$ are 2π-periodic functions. The curvature of γ_0 is $\cot R$. Let $\cot R+\varepsilon k(t)$ be the curvature of γ_ε. Here and below, all computations are modulo ε^2.

Due to (3-6),

$$\sin\alpha\cot(f+\varepsilon g(t))=\cot R+\varepsilon k(t);$$

hence, up to a constant multiplier, $g=k$. We shall compute $k(t)$ and show that it is free from first harmonics.

We shall use Liouville's formula for curvature of a curve in an orthogonal coordinate system (u, v); see, e.g., [do Carmo 1976]. Recall this formula. Let ψ be the angle made by the curve with the curves $v=\text{const}$, let K_u and K_v be the geodesic curvatures of the coordinate curves $v = \text{const}$ and $u = \text{const}$ and let x be the arc length parameter on the curve. Then the curvature of the curve is

$$\frac{d\psi}{dx} + K_u \cos\psi + K_v \sin\psi. \tag{3-9}$$

Here u and v are the longitude and latitude, so $K_v = 0$ and $K_u(\varphi, \theta) = \cot\theta$.

Since

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta,$$

one has

$$\gamma_\varepsilon = \big(\sin R\cos t + \varepsilon(\bar g(t)\cos R\cos t - \bar f(t)\sin R\sin t), \\ \sin R\sin t + \varepsilon(\bar g(t)\cos R\sin t + \bar f(t)\sin R\cos t), \cos R - \varepsilon\bar g(t)\sin R\big).$$

Then

$$\gamma'_\varepsilon = \big(-\sin R\sin t + \varepsilon(-\bar g\cos R\sin t + \bar g'\cos R\cos t - \bar f\sin R\cos t - \bar f'\sin R\sin t), \\ \sin R\cos t + \varepsilon(\bar g\cos R\cos t + \bar g'\cos R\sin t - \bar f\sin R\sin t + \bar f'\sin R\cos t), \\ -\varepsilon\bar g'\sin R\big).$$

It follows that

$$|\gamma'_\varepsilon| = \sin R + \varepsilon(\bar g\cos R + \bar f'\sin R).$$

The angle ψ between γ'_ε and the circles of latitude is infinitesimal. Therefore, $\cos\psi = 1$ (modulo ε^2). Using the formula for γ'_ε, one computes this angle:

$$\psi = -\varepsilon\frac{\bar g'(t)}{\sin R}.$$

(The minus sign is due to the fact that increasing $\bar g$ pushes the curve down to the equator.) Hence,

$$\frac{d\psi}{dx} = \frac{\psi'}{x'} = \frac{\psi'}{|\gamma'_\varepsilon|} = -\varepsilon\frac{\bar g''(t)}{\sin^2 R}.$$

Finally,

$$\cot\theta = \cot(R + \varepsilon\bar g(t)) = \cot R - \varepsilon\frac{\bar g(t)}{\sin^2 R}.$$

Now (3-9) implies that, up to a constant factor, $k(t) = \bar g(t) + \bar g''(t)$. Since the differential operator $d^2/dx^2 + 1$ "kills" the first harmonics, the result follows. □

This concludes the proof in the spherical case.

For the case of $\mathbb{H}^2$, we apply a similar method, so we briefly describe the differences. The formulas for the partials of L read [Bialy 2013]

$$L_x = -\cos\alpha, \quad L_y = \cos\alpha,$$
$$L_{xy} = \frac{\sin^2\alpha}{\sinh L}, \quad L_{xx} = \frac{\sin^2\alpha}{\tanh L} - \kappa(x)\sin\alpha, \quad L_{yy} = \frac{\sin^2\alpha}{\tanh L} - \kappa(y)\sin\alpha.$$

The parametrization of a Gutkin curve is given by $x_t = a/\sqrt{\kappa(x)^2 - \sin^2\alpha}$, where the constant a is normalized so that the parameter t takes values in $[0, 2\pi]$. One defines the function $f(t)$ by $\coth f = \kappa/\sin\alpha$, and as before, one obtains a difference-differential equation

$$a\cot\alpha(\sinh f_1 - \sinh f_2) = f_1' + f_2'.$$

Analogs of Lemma 3.5 hold:

$$\cos\alpha = \frac{\cos c}{\sqrt{\cosh^2 R - \sinh^2 R\cos^2 c}} \quad \text{or equivalently} \quad \cot c = \cosh R\cot\alpha$$

and

$$a = \sqrt{\cosh^2 R - \sin^2\alpha\sinh^2 R}.$$

The computations in Euclidean space $\mathbb{R}^3$ involving the unit sphere are replaced by similar computations in the Minkowski space $\mathbb{R}^{1,2}$ involving a hyperboloid of two sheets, used as a model of $\mathbb{H}^2$.

4. Gutkin polygons

Refer to the introduction for the definition of a Gutkin (n, k)-gon. Let $G(n, k)$ denote the set of all Gutkin (n, k)-gons. Given $P \in G(n, k)$, it will be convenient to think of P as being embedded in the complex plane $\mathbb{C}$. Let l_i denote the side length, $|v_{i+1} - v_i|$.

Notice that if $n = 2k$, for every index i, one has $i - k = i + k$. Therefore, in this case, each vertex is the end point of exactly one diagonal. If $n \neq 2k$, then $i - k \neq i + k$, so each vertex is the endpoint of two diagonals. In this case, for each v_i, we call the angle between the two diagonals β_i; i.e., $\beta_i = \angle v_{i-k}v_iv_{i+k}$.

The first two propositions in this section will establish basic geometric properties of a Gutkin (n, k)-gon.

Proposition 4.1. *Given n and k, the associated contact angle is equal to $\pi(k-1)/n$ for any Gutkin (n, k)-gon.*

Proof. Let $P \in G(2k, k)$ for some $k \geq 2$. For each i, $\angle v_{i+k}v_iv_{i+1} = \angle v_{i+k}v_iv_{i-1} = \alpha$. Then all interior angles of P are equal to 2α. Since the sum of the interior angles

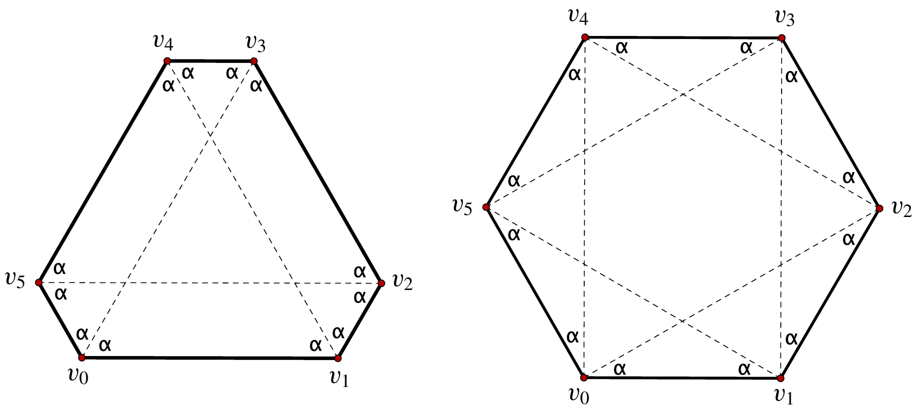

Figure 4. Two Gutkin polygons with angles labeled. Left: Gutkin (6, 3)-gon. Right: Gutkin (6, 2)-gon.

of any n-gon is equal to $\pi(n-2)$, we have $\alpha = \pi(n-2)/(2n)$, which is equal to $\pi(k-1)/n$.

Now assume that $n \neq 2k$. First, note that the sum of the interior angles of the Gutkin polygon equals $(n-2)\pi$ and also equals

$$\sum_{i=0}^{n-1} \beta_i + 2n\alpha;$$

see Figure 4. Therefore,

$$\alpha = \frac{\pi(n-2) - \sum_{i=0}^{n-1} \beta_i}{2n}. \tag{4-1}$$

For fixed n and k, let $P \in G(n,k)$. For $1 \le j \le \gcd(n,k)$, define the polygon

$$Q_j = \overline{v_j v_{j+k} v_{j+2k} \cdots v_{j+(nk/\gcd(n,k))-1}}.$$

Two examples of the Q_j are shown in Figure 5. Note that the sides of Q_j are the diagonals of P. The vertices of all Q_j form a disjoint partition of $\{v_0, v_1, \ldots, v_{n-1}\}$ into $\gcd(n,k)$ subsets of equal size. Thus, the sum of the interior angles of all Q_j is $\sum_{i=0}^{n-1} \beta_i$.

Each Q_j is a star polygon with the number of vertices $N = n/\gcd(n,k)$ and the turning number $W = k/\gcd(n,k)$. The sum of the interior angles of such a polygon equals $\pi(N-2W)$, that is, $\pi(n-2k)/\gcd(n,k)$. One has $\gcd(n,k)$ polygons Q_j; hence, the total sum of their exterior angles is $\pi(n-2k)$. Substituting into (4-1) yields the result. □

Proposition 4.2. *In a Gutkin (n,k)-gon, the interior angles associated to vertices v_i and v_{i+k-1} are equal for all i.*

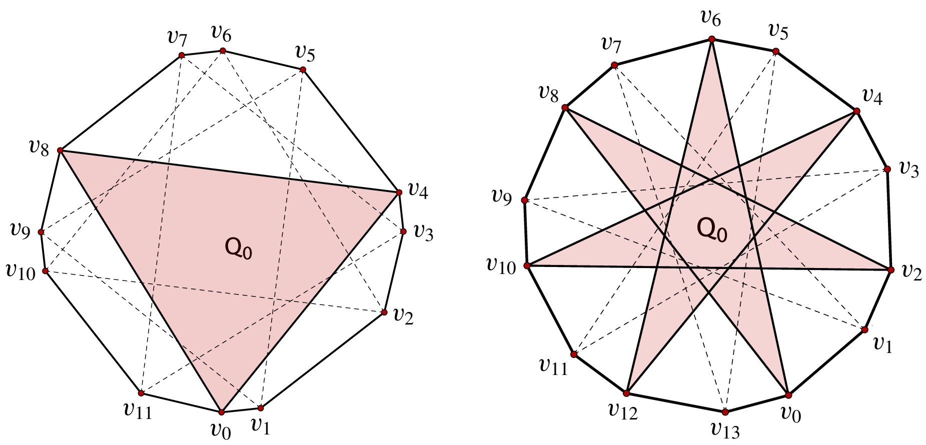

Figure 5. Polygons Q_0 on two Gutkin polygons. Left: Q_0 for a Gutkin $(12, 4)$-gon. Right: Q_0 for a Gutkin $(14, 6)$-gon.

Proof. Consider the self-intersecting quadrilateral $B_i = v_i v_{i+k} v_{i+k+1} v_{i+1}$; see Figure 6. Let w_i denote the intersection point of the two diagonals, $\overline{v_i v_{i+k}}$ and $\overline{v_{i+1} v_{i+k+1}}$. Notice that B_i is comprised of two triangles meeting at w_i. The opposite angles at w_i are equal, and the angle at v_i and v_{i+k+1} is equal to α. Therefore, the angles at v_{i+1} and v_{i+k} are equal, which are also equal to $\alpha + \beta_{i+1}$ and $\alpha + \beta_{i+k}$, respectively. Then $\beta_{i+1} = \beta_{i+k}$. Since the interior angle associated to any v_j is equal to $2\alpha + \beta_j$, the desired result follows. □

Corollary 4.3. *If n and $k - 1$ are coprime, then any $P \in G(n, k)$ is equiangular.*

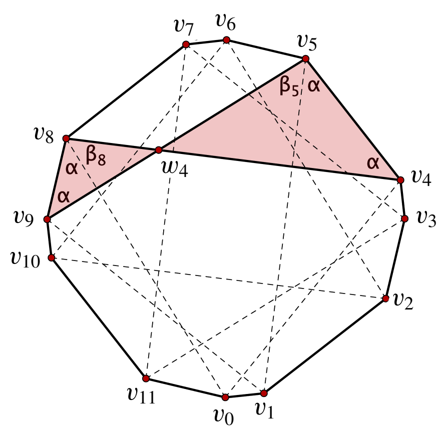

Figure 6. A Gutkin $(12, 4)$-gon. The shaded region is B_4.

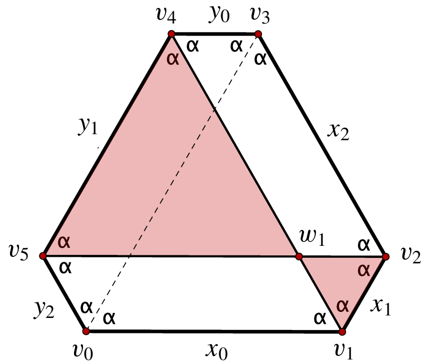

Figure 7. A Gutkin $(6, 3)$-gon with side lengths labeled. The shaded region is B_1.

The case $n = 2k$ is special in that Gutkin polygons abound (in the continuous case, this corresponds to the contact angle $\pi/2$, that is, when Gutkin curves are curves of constant width). Let $\mathbb{R}^n_+$ be the positive orthant.

Proposition 4.4. *The dimension of the space of Gutkin* $(2k, k)$*-gons, considered modulo similarities, equals* $k - 2$*. This quotient space is the intersection of a* $(k-2)$*-dimensional affine subspace with an open cube in* $\mathbb{R}^k$*.*

Proof. Let P be a Gutkin $(2k, k)$-gon. Consider the diagonals $\overline{v_i v_{i+k}}$ and $\overline{v_{i+1} v_{i+k+1}}$ of $G(2k, k)$; see Figure 7. Let w_i denote the intersection of these two diagonals, and let B_i be the bow-tie-shaped polygon $\overline{v_i v_{i+1} v_{i+k+1} v_{i+k}}$. Notice that $\triangle v_i v_{i+1} w_i$ and $\triangle v_{i+k+1} v_{i+k} w_i$ are both isosceles triangles and are similar.

Thus, $v_i w_i = v_{i+1} w_i$ and $v_{i+k} w_i = v_{i+k+1} w_i$. Hence, the diagonals $v_i v_{i+k}$ and $v_{i+1} v_{i+k+1}$ have equal length. Since i is arbitrary and the indices are circular, all diagonals have the same length, say, h. Since h is just a scaling factor, we set $h = 1$ for the remainder of the proof.

Notice that P is comprised of k polygons B_i. Let x_i denote the length of $\overline{v_i v_{i+1}}$ for $0 \le i \le k-1$, and let y_i denote the length of $\overline{v_{i+k} v_{i+k+1}}$, where $0 \le i \le k-1$. Note that x_i and y_i denote the lengths of the nonintersecting sides of B_i.

Assume that v_0 is at the origin and v_1 lies on the positive x axis, and recall that the vertices are labeled in counterclockwise order. This factors out the action of the isometry group of the plane. We shall show that $x_0, \ldots, x_{k-1}$ uniquely determine $y_0, \ldots, y_{k-1}$ and study the condition that these sides form a closed polygon.

Since the diagonals have fixed length equal to 1, one has $y_i = 2\cos\alpha - x_i$. Also, v_k is at the point $(\cos\alpha, \sin\alpha)$. Viewing the sides of $G(2k, k)$ as vectors, the i-th side is $x_i(\cos i\theta, \sin i\theta)$, where $\theta = \pi - 2\alpha = \pi/k$, and the sum of these vectors must be equal to v_k. Thus,

$$\sum_{i=0}^{k-1} x_i(\cos i\theta, \sin i\theta) = (\cos\alpha, \sin\alpha). \tag{4-2}$$

If the side lengths $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}$ form a closed polygon, then the sides with lengths y_i must start at v_k and end at v_0. In other words, the side lengths satisfy

$$v_k + \sum_{i=0}^{k-1} y_i(\cos(\pi + i\theta), \sin(\pi + i\theta)) = v_0. \tag{4-3}$$

Simplifying the left-hand side yields

$$\begin{aligned}(\cos\alpha, \sin\alpha) &+ \sum_{i=0}^{k-1} y_i(-\cos i\theta, -\sin i\theta)\\ &= (\cos\alpha, \sin\alpha) - \sum_{i=0}^{k-1}(2\cos\alpha - x_i)(\cos i\theta, \sin i\theta)\\ &= (\cos\alpha, \sin\alpha) - 2\cos\alpha \sum_{i=0}^{k-1}(\cos i\theta, \sin i\theta) + \sum_{i=0}^{k-1} x_i(\cos i\theta, \sin i\theta)\\ &= (\cos\alpha, \sin\alpha) - 2\cos\alpha(1, \tan\alpha) + (\cos\alpha, \sin\alpha) = (0, 0) = v_0.\end{aligned}$$

Thus, (4-2) implies (4-3).

Hence, $G(2k, k)$ is determined by the k-tuple $x_0, \ldots, x_{k-1}$ satisfying the two linear equations (4-2). In addition, $0 < x_i < 2\cos\alpha$ for all i. This implies the result. □

Next we consider other equiangular cases.

Proposition 4.5. *The quotient space of the space of equiangular Gutkin (n, k)-gons by the group of similarities is identified with the intersection of an M-dimensional affine subspace with $\mathbb{R}^n_+$, where M is equal to the number of positive integers $2 \le r \le n-2$ satisfying the equation*

$$\tan\frac{kr\pi}{n}\tan\frac{\pi}{n} = \tan\frac{k\pi}{n}\tan\frac{r\pi}{n}. \tag{4-4}$$

Proof. Let $P \in G(n, k)$ be embedded in the complex plane with $v_0 = 0$ and v_1 on the positive real axis. Let $x_i = |v_{i+1} - v_i|$ for $0 \le i \le k-1$ be the side lengths of P. Let $\omega = \exp(2\pi/n)$. Notice that $v_{i+1} - v_i = x_i\omega^i$, and a diagonal can be represented as

$$v_{i+k} - v_i = a_i\omega^{i+m}, \tag{4-5}$$

where $a_i \in \mathbb{R}$, $a_i > 0$ and $m = (k-1)/2$. Notice that in this representation,

$$\arg(v_{i+1} - v_i) = (2\pi i)/n,$$
$$\arg(v_{i+k} - v_{i+k-1}) = 2\pi(i+k-1)/n,$$
$$\arg(v_{i+k} - v_i) = \pi(2i+k-1)/n.$$

Then

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \pi(k-1)/n = \alpha.$$

Moreover,

$$\begin{aligned} v_{i+k} - v_i &= (v_{i+k} - v_{i+k-1}) + (v_{i+k-1} - v_{i+k-2}) + \cdots + (v_{i+1} - v_i) \\ &= \omega^{i+k-1} x_{i+k-1} + \omega^{i+k-2} x_{i+k-2} + \cdots + \omega^i x_i \\ &= \omega^i x_i + \omega^{i+1} x_{i+1} + \cdots + \omega^{i+k-1} x_{i+k-1}. \end{aligned}$$

From (4-5), $v_{i+k} - v_i$ is also equal to $a_i \omega^{i+m}$. Thus,

$$\begin{aligned} a_i \omega^{i+m} &= \omega^i x_i + \omega^{i+1} x_{i+1} + \cdots + \omega^{i+k-1} x_{i+k-1} \\ a_i &= \omega^{-m} x_i + \omega^{1-m} x_{i+1} + \cdots + \omega^{k-1-m} x_{k-1}. \end{aligned}$$

Using $a_i - \overline{a_i} = 0$, one has

$$(\omega^{-m} - \omega^m) x_i + (\omega^{1-m} - \omega^{m-1}) x_{i+1} + \cdots + (\omega^{k-1-m} - \omega^{m-k+1}) x_{k-1} = 0.$$

This gives a system of n linear equations on variables x_i. The coefficient matrix, A, is a circulant matrix where the first row is equal to

$$\begin{pmatrix} \omega^{-m} - \omega^m & \omega^{1-m} - \omega^{m-1} & \cdots & \omega^{k-1-m} - \omega^{m-k+1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then the eigenvalues of A are

$$\lambda_r = \sum_{\nu=0}^{k-1} (\omega^{\nu-m} - \omega^{m-\nu}) \omega^{\nu r}; \tag{4-6}$$

see [Davis 1979].

We expect one of the eigenvalues to be equal to zero because we have not factorized by scaling yet. If no other eigenvalue equals zero, then only trivial solutions exist. Now, we compute λ_r in three cases: $r = 0$, $r = 1$ or $r = n-1$, and $2 \le r \le n-2$.

For $r = 0$, we have

$$\lambda_0 = \omega^{-m} \sum_{\nu=0}^{k-1} \omega^\nu - \omega^m.$$

Let h be equal to $|\omega^i + \cdots + \omega^{i+k}|$. By rotational symmetry, h does not vary with i. Now evaluating the above equation,

$$\lambda_0 = h\omega^{-m}\omega^m - h\omega^m\omega^{-m} = 0.$$

Thus, for $r = 0$, A has eigenvalue λ_0 equal to zero.

Assume that λ_r is equal to zero for some other r. Set (4-6) to zero and simplify:

$$\sum_{\nu=0}^{k+1} \omega^{(r+1)\nu} = \omega^{k-1} \sum_{\nu=0}^{k-1} \omega^{(r-1)\nu}. \tag{4-7}$$

For $r = 1$, (4-7) can be written as $k\omega^{k-1} = \sum_{\nu=0}^{k-1} \omega^{2\nu}$. Then $k = \left|\sum_{\nu=0}^{k-1} \omega^{2\nu}\right|$. This is true only if the $\omega^{2\nu}$ are collinear, which is clearly not the case. Thus, $\lambda_1 \neq 0$ and likewise for $r = n-1$.

For $2 \le r \le n-2$, using geometric series, we can rewrite (4-7) as

$$\frac{\omega^{k(r+1)} - 1}{\omega^{r+1} - 1} = \omega^{k-1} \frac{\omega^{k(r-1)} - 1}{\omega^{r-1} - 1}. \tag{4-8}$$

After expanding this equation in terms of sines and cosines and using trigonometric identities, one rewrites it as (4-4). For any solution r, one obtains $\lambda_r = 0$. This implies the claim. □

We are ready to prove Theorem 1.2.

If n and $k-1$ are coprime, then a Gutkin polygon is equiangular by Corollary 4.3. Connelly and Csikós [2009] show that a solution to (4-4) for integer values $1 < k$ and $r < n/2$ must satisfy $k + r = n/2$ and $n \mid (k-1)(r-1)$. Since n and $k-1$ are coprime, there are no solutions. Note also that, if r is a solution, so is $n-r$. Thus, by Proposition 4.5, the matrix A has corank 1 and the Gutkin polygon must be regular.

It remains to construct a nontrivial Gutkin polygon for noncoprime n and $k-1$. Let $p = \gcd(n, k-1)$ and $q = n/p$. Choose angles $\theta_1, \ldots, \theta_p$ such that $\theta_1 + \cdots + \theta_p = 2\pi/q$. Divide a unit circle into q equal parts, and divide each of these equal arcs into p arcs of lengths $\theta_1, \ldots, \theta_p$ in this order. One obtains an inscribed n-gon. See Figure 8 for $n = 8$ and $k = 3$.

Lemma 4.6. *The constructed n-gon is a Gutkin polygon.*

Proof. The angular measure of an inscribed angle is half that of the subtended arc. It follows that

$$\angle v_{i+1} v_i v_{i+k} = \angle v_{i+k-1} v_{i+k} v_i = \frac{\theta_1 + \cdots + \theta_p}{2} = \frac{\pi}{q}.$$ □

Since the choice of the angles $\theta_1, \ldots, \theta_p$ was arbitrary, we obtain a $(p-1)$-parameter family of pairwise nonsimilar Gutkin polygons.

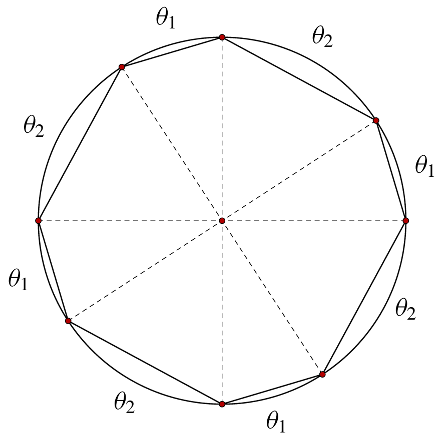

Figure 8. Constructing a nontrivial Gutkin polygon.

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THE WELL-POSEDNESS OF NONLINEAR SCHRÖDINGER EQUATIONS IN TRIEBEL-TYPE SPACES

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Sufficient and necessary conditions for embeddings between $F^s_{p,q}$ (Triebel spaces) and $N^s_{p,q}$ (new spaces constructed by combining frequency-uniform decomposition with $L^p(\ell^q)$) are obtained. Moreover, we study the Cauchy problem for generalized nonlinear Schrödinger equations in $L^r(0, T, N^s_{p,q})$.

1. Introduction and notation

We study NLS (nonlinear Schrödinger equations) by using frequency-uniform decomposition techniques. Suppose that Q_k is the unit cube centered at k and $\{Q_k\}_{k\in\mathbb{Z}^n}$ is a decomposition of $\mathbb{R}^n$. Such decompositions go back to the work of N. Wiener [1932], and we call them Wiener decompositions of $\mathbb{R}^n$. We can write

$$\Box_k \sim F^{-1}\chi_{Q_k}F \quad \text{for } k\in\mathbb{Z}^n, \tag{1}$$

where χ_E denotes the characteristic function on the set E. Since Q_k is just a translation of Q_0, the $\Box_k$ have the same localized structures in the frequency space, and are called the frequency-uniform decomposition operators.

Compared with the dyadic decomposition, the frequency-uniform decomposition has many advantages for the Schrödinger semigroup. It is known that

$$S(t) = e^{it\Delta} : L^p \to L^p$$

if and only if $p = 2$. This is one of the main reasons that we can not solve NLS in $L^p (p \neq 2)$. However, if we consider the frequency-uniform decomposition, we have

$$\|\Box_k S(t) f\|_{L^p} \lesssim (1+|t|)^{n|1/2-1/p|}\|\Box_k f\|_{L^p},$$

which enables us to solve NLS in frequency-uniform decomposition spaces.

Roughly speaking, combining dyadic decomposition operators with function spaces $L^p(\ell^q)$, we can introduce Triebel spaces [Triebel 1992]. Combining frequency-uniform decomposition operators with function spaces $L^p(\ell^q)$, we can introduce

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Triebel-type spaces. From a PDE point of view, the combination of frequency-uniform decomposition operators and Banach function spaces $L^p(\ell^q)$ seems to be important in making nonlinear estimates, which contains an automatic decomposition on high-low frequencies.

We now give an exact definition on frequency-uniform decomposition operators. Since χ_{Q_k} is not differentiable, one needs to replace χ_{Q_k} in (1) by a smooth cut-off function. We first denote $|\xi|_\infty := \max_{i=1,\dots,n} |\xi_i|$. Let $\rho \in S(\mathbb{R}^n) : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to $\{\xi : |\xi_i|_\infty \leqslant 1\}$ with $\rho(\xi) = 1$ if $|\xi|_\infty \leqslant \frac{1}{2}$, and $\rho(\xi) = 0$ if $|\xi|_\infty \geqslant 1$. Let ρ_k be a translation of ρ:

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n.$$

Since $\rho_k(\xi) = 1$ in the unit closed cube $Q_k : \{\xi \in \mathbb{R}^n : |\xi - k|_\infty \leqslant \frac{1}{2}\}$ and $\{Q_k\}_{k \in \mathbb{Z}^n}$ is a covering of $\mathbb{R}^n$, one has that for all $\xi \in \mathbb{R}^n$, $\sum_{k\in\mathbb{Z}^n} \rho_k(\xi) \geqslant 1$. Set

$$\sigma_k(\xi) = \rho_k(\xi) \Big(\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \Big)^{-1}, \quad k \in \mathbb{Z}^n.$$

Then we have

$$(\sharp) \qquad \begin{cases} |\sigma_k(\xi)| \geqslant c & \xi \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, & \\ \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1 & \text{for all } \xi \in \mathbb{R}^n \\ |D^\alpha \sigma_k(\xi)| \leqslant C_{|\alpha|}, & \text{for all } \xi \in \mathbb{R}^n,\ \alpha \in (N \cup \{0\})^n. \end{cases}$$

Hence, the set

$$\Upsilon_n = \{\{\sigma_k\}_{k\in\mathbb{Z}^n} : \{\sigma_k\}_{k\in\mathbb{Z}^n} \text{ satisfies } (\sharp)\}$$

is nonempty. Let $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon_n$ be a function sequence. We call $\{\Box_k\}_{k\in\mathbb{Z}^n}$ the frequency-uniform decomposition operators, where $\Box_k := F^{-1}\sigma_k F$, $k \in \mathbb{Z}^n$. If we combine these decompositions with $L^p(\ell^q)$, we can introduce a new type of function spaces as follows. For any $k \in \mathbb{Z}^n$, we set $|k| = |k_1| + \cdots + |k_n|$, $\langle k \rangle = 1 + |k|$. If $0 < p < \infty$, $0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we denote

$$N^s_{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \|f\|_{N^s_{p,q}} = \Big\| \Big(\sum_{k\in\mathbb{Z}^n} \langle k \rangle^{sq} |\Box_k f|^q \Big)^{1/q} \Big\|_p < \infty \Big\}.$$

If $p = \infty$, $0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we set

$$(3) \quad N^s_{\infty,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \exists \{f_k(x)\}_{k=0}^\infty \subset L^\infty(\mathbb{R}^n) \text{ such that } f = \sum_{k=0}^\infty F^{-1}\sigma_k F f_k \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n,\ell^q)} < \infty \Big\},$$

and

$$\|f\|_{N^s_{\infty,q}(\mathbb{R}^n)} = \inf \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n,\ell^q)},$$

where the infimum is taken over all admissible representations of f in the sense of (3).

One purpose of this paper is to study the semilinear estimates, dual estimates, Strichartz estimates and time-space estimates in the Triebel-type spaces. Furthermore, from the definitions, we see that Triebel spaces and Triebel-type spaces are rather similar; both of them are the combinations of frequency decomposition operators and function spaces $L^p(\ell^q)$. In fact, we have the following inclusion results:

Theorem 1.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$.

(a) *If* $s_1 > s_2 + \sigma(p,q)$ (*when* $\sigma(p,q) = 0$, $s_1 \geqslant s_2$), *then* $N^{s_1}_{p,q} \subset F^{s_2}_{p,q}$, *where*

$$\sigma(p,q) = \max\Big\{0, n\Big(\frac{1}{p} - \frac{1}{q}\Big), n\Big(1 - \frac{1}{q} - \frac{1}{p}\Big)\Big\}.$$

(b) *If* $s_1 > s_2 + \tau(p,q)$ (*when* $\tau(p,q) = 0$, $s_1 \geqslant s_2$), *then* $F^{s_1}_{p,q} \subset N^{s_2}_{p,q}$, *where*

$$\tau(p,q) = \max\Big\{0, n\Big(\frac{1}{q} - \frac{1}{p}\Big), n\Big(\frac{1}{q} + \frac{1}{p} - 1\Big)\Big\}.$$

Theorem 1.2. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$.

(a) *If* $F^{s_1}_{p,q} \subset N^{s_2}_{p,q}$, *then* $s_1 \geqslant s_2 + \tau(p,q)$, *where*

$$\tau(p,q) = \max\Big\{0, n\Big(\frac{1}{q} - \frac{1}{p}\Big), n\Big(\frac{1}{q} + \frac{1}{p} - 1\Big)\Big\}.$$

(b) *If* $N^{s_1}_{p,q} \subset F^{s_2}_{p,q}$, *then* $s_1 \geqslant s_2 + \sigma(p,q)$, *where*

$$\sigma(p,q) = \max\Big\{0, n\Big(\frac{1}{p} - \frac{1}{q}\Big), n\Big(1 - \frac{1}{q} - \frac{1}{p}\Big)\Big\}.$$

In recent decades, a large amount of work has been devoted to the study of well-posedness in Besov and modulation spaces (for example, see [Kato 1987; 1989; 1995; Cazenave and Weissler 1989; 1990; Wang 1993; Kenig et al. 1995; Pecher 1997; Nakamura and Ozawa 1998; Cazenave 2003; Wang et al. 2006; 2009; 2011; Wang and Huang 2007; Wang and Hudzik 2007; Chen and Fan 2011]). Our second goal is to explore solutions of NLS in the Triebel-type spaces. We will use the smooth effect estimates, together with the frequency-uniform decomposition techniques, to study NLS, and we show that it is locally well-posed in a class of Triebel-type spaces.

Theorem 1.3. *Assume* $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, $2 \leqslant p \leqslant k+2$, $r \geqslant k+2$; *then for any initial data* $u_0 \in H^s$, $s > \frac{1}{2}n$, *there exists* $T^* := T^*(\|u_0\|_{H^s}) > 0$ *such that the initial value problem*

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau$$

has a unique solution

$$u \in C(0, T^*, H^s) \cap L^r_{\text{loc}}(0, T^*; N^s_{p,2}).$$

Moreover, if $T^* < \infty$, *then*

$$\|u\|_{C(0,T^*,H^s)\cap L^r(0,T^*;N^s_{p,2})} = \infty,$$

where $S(t) = F^{-1}e^{-it|\xi|^2}F$.

Theorem 1.4. *Assume* $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, $2 \leqslant p \leqslant k+2$, $r \geqslant k+2$, $p' \leqslant q \leqslant p$, *then for any initial data* $u_0 \in N^s_{p',q}$, $s > n(1-\frac{1}{q})$, *there exists* T *such that the initial value problem*

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau$$

has a unique solution

$$u \in L^r_{\text{loc}}(0, T; N^s_{p,q}).$$

The rest of this paper is divided into five sections and an appendix. In Section 2 we will state some properties of $N^s_{p,q}$, which are useful to establish the embedding inclusions between Triebel spaces and $N^s_{p,q}$. In Section 3 we will prove Theorems 1.1 and 1.2. Section 4 is devoted to considering the multiplication algebra of $N^s_{p,q}$. Some dispersive smooth effects for the Schrödinger semigroup will be given in Section 5 and Theorems 1.3 and 1.4 will be proved in Section 6. The Appendix derives a complex interpolation in Triebel-type spaces (Theorem A.14) and shows some properties of the modulation spaces (Theorems A.1–A.8) that are used in Sections 2–5.

Notation. Throughout the paper, we set

$$L^r(\mathbb{R}, N^s_{p,q}) := \left\{ f \in \mathcal{S}' : \left(\int_{\mathbb{R}} \|f\|^r_{N^s_{p,q}}\, dt \right)^{1/r} < \infty \right\}.$$

We shall sometimes write $X \lesssim Y$ to denote the estimate $X \leqslant CY$ for some C. For any $s \in \mathbb{R}$, $0 < p, q \leq \infty$, we set

$$M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k f\|^q_{L^p} \right)^{1/q} < \infty \right\};$$

$M^s_{p,q} := M^s_{p,q}(\mathbb{R}^n)$ is called a modulation space, first introduced by Feichtinger [2003] in the case $1 \leqslant p, q \leqslant \infty$.

2. Basic properties of $N^s_{p,q}$

In order to study the Cauchy problem in $N^s_{p,q}$, we first give some properties of $N^s_{p,q}$:

Proposition 2.1. *Let* $-\infty < s < \infty$, $0 < p < \infty$, $0 < q \leqslant \infty$. *The following inclusions hold*:

(1) *Let* $q_1 \le q_2$. *Then*

$$N^s_{p,q_1} \subset N^s_{p,q_2}.$$

(2) *Let* $\varepsilon q_2 > n$. *Then*

$$N^{s+\varepsilon}_{p,q_1} \subset N^s_{p,q_2}.$$

(3) *Let* $p < \infty$. *Then*

$$M^s_{p,p\wedge q} \subset N^s_{p,q} \subset M^s_{p,p\vee q}.$$

Proof. Since $\ell^p \subset \ell^{p+a}, a \ge 0$, we can get (1) directly. Let us observe that

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq_2}|a_k|^{q_2}\Big)^{1/q_2} &= \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s+\varepsilon)q_2}\langle k\rangle^{-\varepsilon q_2}|a_k|^{q_2}\Big)^{1/q_2}\\ &\lesssim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{s+\varepsilon}|a_k| \qquad\qquad (\varepsilon q_2 > n).\end{aligned}$$

Taking $a_k = \Box_k f$, we can show that (2) holds with the help of (1).

Finally we prove (3). Let $b_k = \langle k\rangle^s \Box_k f$. There are two cases:

Case 1. $q \le p$. In this case, we have

$$\|b_k\|_{\ell^p(L^p)} \le \|b_k\|_{L^p(\ell^q)} \le \|b_k\|_{\ell^q(L^p)}.$$

Actually, noticing that $\ell^q \subset \ell^p$, we have $\|b_k\|_{\ell^p(L^p)} = \|b_k\|_{L^p(\ell^p)} \le \|b_k\|_{L^p(\ell^q)}$. So, the first part of the above inequality holds. Moreover, by Minkowski's inequality, we have

$$\|b_k\|_{L^p(\ell^q)} = \Big\|\sum_{k=0}^{\infty} |b_k|^q\Big\|^{1/q}_{L^{p/q}} \leqslant \Big(\sum_{k=0}^{\infty} \||b_k|^q\|_{L^{p/q}}\Big)^{1/q} = \|b_k\|_{\ell^q(L^p)}.$$

This proves the second part.

Case 2. $p \le q$. By Minkowski's inequality and $\ell^p \subset \ell^q$, we have

$$\begin{aligned}\|b_k\|_{\ell^q(L^p)} &= \Big(\Big\|\int_{\mathbb{R}^n} |b_k|^p\,dx\Big\|_{\ell^{q/p}}\Big)^{1/p} \leqslant \Big(\int_{\mathbb{R}^n} \||b_k|^p\|_{\ell^{q/p}}\,dx\Big)^{1/p}\\ &= \|b_k\|_{L^p(\ell^q)} \leqslant \|b_k\|_{\ell^p(L^p)},\end{aligned}$$

□

Proposition 2.2 (completeness). *For any $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$, we have*:

(1) $N^s_{p,q}$ *is a quasi-Banach space. Moreover, if* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, *then* $N^s_{p,q}$ *is a Banach space.*

(2) $\mathcal{S}(\mathbb{R}^n) \subset N^s_{p,q} \subset \mathcal{S}'(\mathbb{R}^n)$.

(3) *If* $0 < p, q < \infty$, *then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $N^s_{p,q}$.

Proof.

Step 1. Thanks to [Wang and Hudzik 2007], we obtain that $\mathcal{S} \subset M^s_{p,\infty} \subset \mathcal{S}'$.

Step 2. We prove that $\mathcal{S} \subset N^s_{p,q}$. From Step 1, we have that $\mathcal{S} \subset M^{s+\varepsilon}_{p,\infty}$. By Proposition 2.1 and Theorem A.5, we know that $M^{s+\varepsilon}_{p,\infty} \subset M^s_{p,p\wedge q} \subset M^s_{p,q} \cap N^s_{p,q}$ $(\varepsilon > n/p \wedge q)$. Thus, we have the desired result.

Step 3. Similarly to Step 2, we can also prove that $N^s_{p,q} \subset \mathcal{S}'$ (by $M^s_{p,p\vee q} \subset \mathcal{S}'$). We omit the details of the proof.

Step 4. $N^s_{p,q}$ is a quasinormed space. Now we prove completeness. Let $\{f_\ell\}^\infty_{\ell=1}$ be a Cauchy sequence in $N^s_{p,q}$ (with respect to a fixed quasinorm in $N^s_{p,q}$). Part (2) of the theorem shows that $\{f_\ell\}^\infty_{\ell=1}$ is also a Cauchy sequence in $\mathcal{S}'$. Because $\mathcal{S}'$ is a complete locally convex topological linear space, we can find a limit element $f \in \mathcal{S}'$. Then $F^{-1}\sigma_k F f_\ell$ converges to $F^{-1}\sigma_k F f$ in $\mathcal{S}'(\mathbb{R}^n)$ if $\ell \to \infty$. On the other hand, $\{F^{-1}\sigma_k F f_\ell\}^\infty_{\ell=1}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ $(N^s_{p,q} \subset M^s_{p,q\vee p})$. By Theorem A.2, it is also a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. This shows that the limiting element of $\{F^{-1}\sigma_k F f_\ell\}^\infty_{\ell=1}$ in $L^p(\mathbb{R}^n)$ (which is the same as in $L^\infty(\mathbb{R}^n)$) coincides with $\{F^{-1}\sigma_k F f\}$. Now it follows by standard arguments that f belongs to $N^s_{p,q}$ and that f_ℓ converges in $N^s_{p,q}$ to f. Hence, $N^s_{p,q}$ is complete.

Step 5. We prove that if $-\infty < s < \infty$ and $0 < p, q < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $N^s_{p,q}(\mathbb{R}^n)$. Let $f \in N^s_{p,q}$; then we put

$$f_N(x) = \sum_{k=0}^{N} F^{-1}\sigma_k F f.$$

Note $f_N \in N^s_{p,q}(\mathbb{R}^n)$. Consequently (by Theorem A.6),

$$\begin{aligned}\|f - f_N\|_{N^s_{p,q}(\mathbb{R}^n)} &\leqslant c \left\| \left(\sum_{k=N}^{\infty} \sum_{r=-1}^{1} \langle k\rangle^{sq} |F^{-1}\sigma_k \sigma_{k+r} F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\leqslant c \left\| \left(\sum_{k=N}^{\infty} \langle k\rangle^{sq} |F^{-1}\sigma_k F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Lebesgue's bounded convergence theorem proves that the right-hand side of above inequality tends to zero if $N \to \infty$. Hence, f_N approximates f in $N^s_{p,q}(\mathbb{R}^n)$. Next, we let $\varphi \in \mathcal{S}$ with $\varphi(0)=1$ and supp $F\varphi \subset \{y : |y| \leqslant 1\}$. Let $f_\delta(x)=\varphi(\delta x) f(x)$ with $0<\delta<1$. Then $(f_N)_\delta \in \mathcal{S}(\mathbb{R}^n)$ approximates f_N in $L_p^\Omega := \{f \in L^p : \text{supp } \hat{f} \subset \Omega\}$ with $\Omega = \{y : |y| \leqslant 2^{N+2}\}$ if $\delta \to 0$. But this is also an approximation of f_N in $N^s_{p,q}(\mathbb{R}^n)$. This proves that $\mathcal{S}(\mathbb{R}^n)$ is dense in $N^s_{p,q}(\mathbb{R}^n)$. □

Proposition 2.3 (dual space). *Assume $-\infty < s < \infty$. The following inclusions hold:*

(a) *Let $1 \leqslant p < \infty$ and $1 < q < \infty$. Then*

$$(N^s_{p,q})^* = N^{-s}_{p',q'}.$$

(b) *Let $0 < p < 1$ and $0 < q \leqslant 1$. Then*

$$(N^s_{p,q})^* = M^{-s}_{\infty,\infty}.$$

Proof.

Step 1. For $1 \leqslant p < \infty$, $0 < q < \infty$, [Triebel 1983] showed similar results for Triebel spaces. We can prove the result similarly as for Triebel spaces, and omit the details.

Step 2. For $0 < p < 1$ and $0 < q \leqslant 1$, by the property of $N^s_{p,q}$, we have

$$M^s_{p,p\wedge q}(\mathbb{R}^n) \subset N^s_{p,q}(\mathbb{R}^n) \subset M^s_{p,p\vee q}(\mathbb{R}^n) \subset M^s_{1,1}(\mathbb{R}^n).$$

Then by Theorem A.3, we have

$$M^{-s}_{\infty,\infty}(\mathbb{R}^n) \supset (N^s_{p,q}(\mathbb{R}^n))^* \supset M^{-s}_{\infty,\infty}(\mathbb{R}^n).$$

This proves the second part of this proposition. □

Proposition 2.4 (equivalent norm). *Assume $\{\sigma_k\}_{k\in\mathbb{Z}^n}, \{\phi_k\}_{k\in\mathbb{Z}^n} \in \Upsilon_n$, $0 < p < \infty$, $0 < q \leqslant \infty$. Then $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ and $\{\phi_k\}_{k\in\mathbb{Z}^n}$ generate equivalent norms on $N^s_{p,q}$.*

Proof. Recall that [Feichtinger 2003; Wang and Hudzik 2007] showed the equivalence of $\|\cdot\|^{\sigma_k}_{M^s_{p,q}}$ and $\|\cdot\|^{\phi_k}_{M^s_{p,q}}$. By a similar argument as for modulation spaces and by Theorem A.6, we can obtain the claimed equivalence of $\|\cdot\|^{\sigma_k}_{N^s_{p,q}}$ and $\|\cdot\|^{\phi_k}_{N^s_{p,q}}$. □

Lemma 2.5. *Assume $(I-\Delta)^{s/2} f = F^{-1}(1+|\xi|^2)^{s/2} Ff$, $0 < p < \infty$, $0 < q \leqslant \infty$. Then we have*

$$\|(I-\Delta)^{s/2} f\|_{N_{p,q}} \sim \|f\|_{N^s_{p,q}}.$$

Proof. Analogously to the case of modulation spaces, by Theorem A.6 we can prove the consequence, and the details are omitted. □

Theorem 2.6. *Assume $1 \leqslant p_2 \leqslant p_1 < \infty$, $1 \leqslant q \leqslant \infty$. Then we have*

$$\|u\|_{N^s_{p_1,q}} \lesssim \|u\|_{N^s_{p_2,q}}.$$

Proof. By the definition of $N^s_{p,q}$, we have

$$\begin{aligned}
\|u\|_{N^s_{p_1,q}} &= \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} |\Box_i u|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&= \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big|\Box_i \sum_{|\ell|_\infty \leqslant 1} \Box_{i+\ell} u\Big|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&\leqslant \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big| |\sigma_i^\vee| * \Big| \Big(\sum_{|\ell|_\infty \leqslant 1} \Box_{i+\ell} u \Big) \Big| \Big|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&\lesssim \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big| \Big(\sum_{|\ell|_\infty \leqslant 1} \Box_{i+\ell} u \Big) \Big|^q \Big)^{1/q} \Big\|_{L^{p_2}} \\
&\lesssim \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} |\Box_i u|^q \Big)^{1/q} \Big\|_{L^{p_2}},
\end{aligned}$$

which implies the result. In the above equation, we used Theorem A.7 and $\||\sigma_0^\vee| * f\|_{L^{p_1}} \leqslant \|f\|_{L^{p_2}}$ ($p_1 \geqslant p_2$, Young's inequality). □

Theorem 2.7. *Assume $f \in L^2$. Then for any $s \in \mathbb{R}$, we have*

$$\|f\|_{H^s} \sim \|f\|_{N^s_{2,2}}.$$

Proof. First, we prove that $\|f\|_{L^2} \lesssim \|f\|_{N_{2,2}}$. By Plancherel's inequality, we have

$$\begin{aligned}
\|f\|_{L^2} &= \Big(\int_{\mathbb{R}^n} \Big| \sum_{i\in\mathbb{Z}^n} \Box_i f \Big|^2 dx \Big)^{1/2} = \Big(\int_{\mathbb{R}^n} \Big| \sum_{i\in\mathbb{Z}^n} \sigma_i F f \Big|^2 dx \Big)^{1/2} \\
&\lesssim \Big(\int_{\mathbb{R}^n} \sum_{i\in\mathbb{Z}^n} |\sigma_i F f|^2 dx \Big)^{1/2} \\
&= \Big(\int_{\mathbb{R}^n} \Big(\Big(\sum_{i\in\mathbb{Z}^n} |F^{-1}\sigma_i F f|^2 \Big)^{1/2} \Big)^2 dx \Big)^{1/2} = \|f\|_{N_{2,2}}.
\end{aligned}$$

The inverse inequality can be proved similarly $\Big(\sum_{i\in\mathbb{Z}^n} |\sigma_i F f|^2 \lesssim \Big| \sum_{i\in\mathbb{Z}^n} \sigma_i F f \Big|^2 \Big)$. □

Theorem 2.8. *Assume $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \leqslant \infty$, $0 < p < \infty$. Then, for $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, we have*

$$N^{s_1}_{p,q_1}(\mathbb{R}^n) \subset N^{s_2}_{p,q_2}(\mathbb{R}^n).$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}\|f\|_{N^{s_2}_{p,q_2}} &= \Big\|\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s_2q_2}|\square_k f|^{q_2}\Big)^{1/q_2}\Big\|_{L^p}\\ &= \Big\|\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{(s_2-s_1)q_2}\langle k\rangle^{s_1q_2}|\square_k f|^{q_2}\Big)^{1/q_2}\Big\|_{L^p}\\ &\leqslant \|f\|_{N^{s_1}_{p,q_1}}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)}\Big)^{(q_1-q_2)/(q_1q_2)}.\end{aligned}$$

Then by $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$ and

$$\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)} \lesssim \sum_{i=0}^{\infty}\langle i\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)},$$

we obtain the results. □

3. Embedding between $N^s_{p,q}$ and Triebel spaces

The embedding theorem is of importance for the study of nonlinear PDEs and we give the details of the proof. We start with the embedding for the same indices p, q.

Proof of Theorem 1.1.

Step 1. For $0 < p < \infty$, $0 < q \leqslant \infty$, $\varepsilon > 0$ (a small positive number), we have

$$N_{p,q} \subset F_{p,q} \quad \text{for all } q \leqslant 1 \wedge p, \tag{4}$$

$$N^{n(1/(p\wedge 1)-1/q)+\varepsilon}_{p,q} \subset F_{p,q} \quad \text{for all } 0 < p \leqslant q. \tag{5}$$

Let $a_k = \max(0, 2^{k-1} - \sqrt{n})$, $b_k = 2^{k+1} + \sqrt{n}$. We can easily find that $\Delta_k \square_i f = 0$ for $|i| \,\overline{\in}\, [a_k, b_k]$. First, we show the inclusion (4).

Case 1. $p \geqslant 1$, $q \leqslant 1$. By Theorem A.6, we have

$$\begin{aligned}\|\Delta_k f\|_{L^p(\ell^q)} &= \Big\|\Big(\sum_k |\Delta_k f|^q\Big)^{1/q}\Big\|_{L^p} \leqslant \Big\|\Big(\sum_k \Big|\sum_{\substack{i\in\mathbb{Z}^n\\ |i|\in[a_k,b_k]}}\Delta_k\square_i\square_i f\Big|^q\Big)^{1/q}\Big\|_{L^p}\\ &\leqslant C\Big\|\Big(\sum_k \sum_{\substack{i\in\mathbb{Z}^n\\ |i|\in[a_k,b_k]}}(|F^{-1}(\varphi_k\sigma_i)F\square_i f|)^q\Big)^{1/q}\Big\|_{L^p}\\ &\lesssim C\Big\|\Big(\sum_k \sum_{\substack{i\in\mathbb{Z}^n\\ |i|\in[a_k,b_k]}}|\square_i f|^q\Big)^{1/q}\Big\|_{L^p} \lesssim C\Big\|\Big(\sum_{i\in\mathbb{Z}^n}|\square_i f|^q\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

Case 2. $p<1,\ p=q$. By Theorem A.1, we have

$$N_{p,p}\subset M_{p,p}\subset B_{p,p}\subset F_{p,p}.$$

Then by combining Cases 1 and 2, we obtain the result (4).

Next, we show the second inclusion. By Theorem A.1, we obtain that

$$N_{p,q}^{n(1/(p\wedge 1)-1/q)+\varepsilon}\subset M_{p,q}^{n(1/p\wedge 1-1/q)+\varepsilon}\subset B_{p,q}^{\varepsilon}\subset B_{p,p}\subset F_{p,q}.$$

This proves (5). In the above discussion, we used Theorem A.15.

Step 2. We prove the following inclusions:

(a) If $0<q\leqslant 2$, $\varepsilon>0$ (a small positive number), then we have

$$F_{2,q}^{n(1/q-1/2)+\varepsilon}\subset N_{2,q}. \tag{6}$$

(b) If $p\geqslant 2$, $\varepsilon>0$ (a small positive number), then we have

$$N_{p,\infty}^{n(1-1/p)+\varepsilon}\subset F_{p,\infty}. \tag{7}$$

By Theorem A.1, one has that for $0<q\leqslant 2$,

$$B_{2,q}^{n(1/q-1/2)}\subset M_{2,q}.$$

Then by the embedding estimates in Proposition 2.1 and Theorem A.15, we have

$$F_{2,q}^{n(1/q-1/2)+2\varepsilon}\subset F_{2,\infty}^{n(1/q-1/2)+\varepsilon}\subset B_{2,\infty}^{n(1/q-1/2)+\varepsilon}\subset B_{2,q}^{n(1/q-1/2)}\subset M_{2,q}\subset N_{2,q},$$

which implies the result (6).

For the case $p\geqslant 2$, by Theorem A.1 and Proposition 2.1, we have that

$$N_{p,\infty}^{n(1-1/p)+\varepsilon}\subset M_{p,\infty}^{n(1-1/p)+\varepsilon}\subset B_{p,\infty}^{\varepsilon}\subset B_{p,p\wedge\infty}\subset F_{p,\infty}.$$

This proves (7).

Step 3. Assume $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$. Then for

$$\sigma(p,q)=\max\left(0,n\left(\frac{1}{p\wedge p'}-\frac{1}{q}\right)\right)$$

and $s_1>s_2+\sigma(p,q)$ (if $\sigma(p,q)=0$, $s_1\geqslant s_2$), we have

$$N_{p,q}^{s_1}\subset F_{p,q}^{s_2}. \tag{8}$$

Actually, thanks to (5), (7) and the dual versions of (6), we have

$$\begin{aligned} N_{p,\infty}^{n+\varepsilon}&\subset F_{p,\infty}, & 1\leqslant p<\infty,\\ N_{p,\infty}^{n(1-1/p)+\varepsilon}&\subset F_{p,\infty}, & p\geqslant 2,\\ N_{2,q}^{n(1/2-1/q)+\varepsilon}&\subset F_{2,q}, & 2\leqslant q\leqslant\infty. \end{aligned}$$

Taking $p = 1, \infty^-$ (i.e., a sufficient large number) and $q = \infty$, we have

$$N_{1,\infty}^{n+\varepsilon} \subset F_{1,\infty}, \quad N_{2,\infty}^{n/2+\varepsilon} \subset F_{2,\infty}, \quad N_{\infty^-,\infty}^{n(1-1/\infty^-)+\varepsilon} \subset F_{\infty^-,\infty}.$$

Applying the complex interpolation theorem to these three estimates, we obtain

$$N_{p,\infty}^{\max(n/p,n/p')+\varepsilon} \subset F_{p,\infty}, \quad 1 \leqslant p < \infty.$$

Moreover, by (4), we have

$$N_{p,1} \subset F_{p,1}, \quad 1 \leqslant p < \infty. \tag{9}$$

Recall that

$$N_{2,2} = F_{2,2}. \tag{10}$$

Applying the complex interpolation theorem separately to the above three estimates again, we obtain

$$N_{p,q}^{\sigma(p,q)+\varepsilon} \subset F_{p,q}.$$

(When $\sigma(p,q) = 0$, we apply the complex interpolation theorem to (9) and (10).)

Step 4. We show the sufficiency of $N_{p,q}^{s_1} \subset F_{p,q}^{s_2}$. By Step 3, we see that the conclusion holds if $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$. By Step 1, we have the result if $0 < p \leqslant 1$ or $0 < q < 1$.

Next, we prove the sufficiency of $F_{p,q}^{s_1} \subset N_{p,q}^{s_2}$. Set

$$\begin{aligned}
\mathbb{R}^2_{++} &= \left\{\left(\frac{1}{p}, \frac{1}{q}\right) : \frac{1}{p} > 0,\ \frac{1}{q} \geqslant 0\right\},\\
S_1 &= \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_{++} : \frac{1}{q} \geqslant \frac{1}{p},\ \frac{1}{p} \leqslant \frac{1}{2}\right\},\\
S_2 &= \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_{++} : \frac{1}{q} < \frac{1}{p},\ \frac{1}{p} + \frac{1}{q} \leqslant 1\right\},\\
S_3 &= \mathbb{R}^2_{++} - (S_1 \cup S_2).
\end{aligned}$$

Step 1$'$. For $1 \leqslant p < \infty$, $q = \infty$, we have

$$F_{p,\infty} \subset N_{p,\infty}. \tag{11}$$

Actually, by the definition of $N_{p,\infty}$, $F_{p,\infty}$ and Theorem A.7, we have (δ_j is the Littlewood–Paley decomposition)

$$\begin{aligned}
\|f\|_{N_{p,\infty}} &= \Big\| \sup_{i\in\mathbb{Z}^n} |\Box_i f| \Big\|_{L^p} = \Big\| \sup_{i\in\mathbb{Z}^n} \Big| F^{-1}\sigma_i \sum_{\ell=-4}^{4} \delta_{j+\ell} Ff \Big| \Big\|_{L^p}\\
&\leqslant \Big\| \sup_{j\in\mathbb{Z}^n} |F^{-1}\delta_j Ff| \Big\|_{L^p} = \|f\|_{F_{p,\infty}}.
\end{aligned}$$

Step 2′. For $0 < p \leqslant 1$, we have

$$F^{n(1/p-1)}_{p,\infty} \subset N_{p,\infty}. \tag{12}$$

By Theorem A.6, we have that for $|i| \in [2^{j-1}, 2^j)$,

$$\begin{aligned}
\|f\|_{N_{p,\infty}} &= \Big\| \sup_{i\in\mathbb{Z}^n} |\Box_i f| \Big\|_{L^p} \\
&= \Big\| \sup_{i\in\mathbb{Z}^n} \Big| F^{-1}\sigma_i \sum_{\ell=-4}^{4} \delta_{j+\ell} F f \Big| \Big\|_{L^p} \\
&\lesssim \|2^{-js}\sigma_i(2^j\cdot)\|_{H^x} \|f\|_{F^s_{p,\infty}} \qquad (s \geqslant n(1/p-1)) \\
&\lesssim \|f\|_{F^s_{p,\infty}} \qquad (s \geqslant n(1/p-1),\ \|2^{-js}\sigma_i(2^j\cdot)\|_{H^x} \lesssim C),
\end{aligned}$$

which implies the result (12).

Step 3′. $(1/p, 1/q) \in S_3$, $\tau(p,q) = n(1/p + 1/q - 1)$. Let

$$\frac{1}{p_0} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_0} = 0, \quad \frac{1}{p_1} = \frac{1}{2}, \quad \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{2}.$$

Assume $\theta = 1/q(1/p + 1/q - \frac{1}{2})^{-1}$, we have

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$
$$\frac{1}{p} + \frac{1}{q} - 1 = (1-\theta)\Big(\frac{1}{p_0} - 1\Big) + \Big(\frac{1}{q_1} - \frac{1}{2}\Big)\theta.$$

By (6) and (12), we have

$$F^{n(1/q_1-1/2)+\varepsilon}_{2,q_1} \subset N_{2,q_1}, \quad F^{n(1/p_0-1)}_{p_0,\infty} \subset N_{p_0,\infty}.$$

A complex interpolation yields

$$F^{n(1/p+1/q-1)+\varepsilon}_{p,q} \subset N_{p,q}.$$

Step 4′. $(1/p, 1/q) \in S_1$, $\tau(p,q) = n(1/q - 1/p)$. Let $(1/p, 1/q) \in \dot{S}_1$ (inner point of s). Then $(1/p, 1/q) \in \dot{S}_1$ is a point at the line segment connecting $(1/\infty^-, 0)$ and some point $(1/p_1, 1/q_1) \in \{(1/p, 1/q) : p = 2,\ q < 2\}$. By (11), Step 3′ and complex interpolation, we have

$$F^{n(1/q-1/p)+\varepsilon}_{p,q} \subset N_{p,q}.$$

Step 5′. $(1/p, 1/q) \in S_2$, $\tau(p,q) = 0$. We can obtain the results by $F_{2,2} \subset N_{2,2}$ and the dual version of (4), i.e., for $1 \leqslant p < \infty$, $F_{p,\infty} \subset N_{p,\infty}$. □

Proof of Theorem 1.2.

Step 1. For the first part, we need to show that for any $0<\eta\ll 1$, $F^{\tau(p,q)-\eta}_{p,q}\not\subset N_{p,q}$.

Case 1. $(1/p,1/q)\in S_3$, $\tau(p,q)=1/p+1/q-1$. Let $f=F^{-1}\delta_j$, $j\gg 1$. We have

$$\|f\|_{F^{\tau(p,q)-\eta}_{p,q}}=\left\|\left(\sum_j 2^{j(\tau(p,q)-\eta)q}|\Delta_j f|^q\right)^{1/q}\right\|_{L^p}$$
$$\lesssim \sum_{\ell=-1}^{1} 2^{j(\tau(p,q)-\eta)(j+\ell)}\|F^{-1}\delta_{j+\ell}\delta_j\|_{L^p}\lesssim 2^{jn/q-j\eta}.$$

Denote

$$\Lambda_0=\{k\in\mathbb{Z}^n: B(k,\sqrt{n})\cap\{\xi:|\xi|\in[0,2)\}\neq\varnothing\},$$
$$\Lambda_j=\{k\in\mathbb{Z}^n: B(k,\sqrt{n})\cap\{\xi:|\xi|\in[2^{j-1},2^{j+1})\}\neq\varnothing\}.$$

If $k\in\Lambda_j$, then $|k|\sim 2^j$. Noticing that at most $O(2^{nj})$ unit cubes intersect with Λ_j, we have

$$\|f\|_{N_{p,q}}=\left\|\left(\sum_k|\Box_k f|^q\right)^{1/q}\right\|_{L^p}\geqslant\left\|\left(\sum_{k\in\Lambda_j}|\Box_k f|^q\right)^{1/q}\right\|_{L^p}$$
$$\gtrsim\left\|\left(\sum_{k\in\Lambda_j}|F^{-1}\sigma_k\delta_j|^q\right)^{1/q}\right\|_{L^p}\gtrsim 2^{jn/q}.$$

Based on the above observation, we have

$$\|f\|_{N_{p,q}}\gtrsim 2^{\eta j}\|f\|_{F^{\tau(p,q)-\eta}_{p,q}},$$

which implies that $F^{\tau(p,q)-\eta}_{p,q}\not\subset N_{p,q}$.

Case 2. $(1/p,1/q)\in S_2$, $\tau(p,q)=0$. We consider the case $q=\infty$. Taking $k(j)=(2^j,0,\dots,0)$ and $f=F^{-1}\sigma_{k(j)}$, we have that

$$\|f\|_{N_{p,\infty}}\gtrsim 1\gtrsim 2^{\eta j}\|f\|_{F^{-\eta}_{p,\infty}}.$$

If $q<\infty$, we need to show that

$$N_{p,q}\not\subset F^{\varepsilon}_{p,r},\quad 1\leqslant p<\infty. \tag{13}$$

Assume $f\in\mathcal{S}$, $\operatorname{supp}\hat{f}\subset\{\xi:|\xi_i|<\frac{1}{2},\ i=1,\dots,n\}$. Let $N\gg 1$, $0<\varepsilon\ll 1$,

$$k(j)=(2^{Nj},0,\dots,0)\in\mathbb{Z}^n,$$
$$\widehat{F}(\xi)=\sum_{j=1}^{\infty}2^{-\varepsilon Nj}\hat{f}(\xi-k(j)).$$

We see that

$$\|F\|_{N_{p,q}} = \Big\| \Big(\sum_i |\Box_i F|^q \Big)^{1/q} \Big\|_{L^p}$$
$$\lesssim \Big\| \Big(\sum_j \sum_{|\ell|_\infty \leqslant 1} 2^{-\varepsilon N q j} |F^{-1} \sigma_{k(j)+\ell} \hat{f}(\cdot - k(j))| \Big)^{1/q} \Big\|_{L^p} \lesssim 1$$

On the other hand, letting $s > \varepsilon$, we have

$$\|F\|_{F^s_{p,r}} = \Big\| \Big(\sum_j 2^{jrs} |\Delta_j F|^r \Big)^{1/r} \Big\|_{L^p}$$
$$= \Big\| \Big(\sum_j 2^{Njrs} |F^{-1} \delta_{Nj} 2^{-\varepsilon N j} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p}$$
$$\gtrsim \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \delta_{Nj} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p}$$
$$\gtrsim \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \delta_{Nj} \chi_{Q_{k(j)}} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p}$$
$$= \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \chi_{Q_{k(j)}} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p}$$
$$\gtrsim \infty \qquad (\text{let } \delta(\xi) = 1, \text{if } |\xi| \in [\tfrac{3}{4}, \tfrac{5}{4}]),$$

which implies (13). Then by its dual version, we obtain the claimed results.

Case 3. $(1/p, 1/q) \in S_1$, $\tau(p,q) = n(1/q - 1/p)$. Let $\sigma_k(\xi) = 0$ if

$$\xi \,\overline{\in}\, \tilde{Q}_k := \{\xi : |\xi_i - k_i| \leqslant \tfrac{5}{8}, 1 \leqslant i \leqslant n\}$$

and $\delta_j(\xi) = 1$ if $\xi \in D_j := \{\xi : \frac{5}{4} 2^{j-1} \leqslant |\xi| \leqslant \frac{3}{4} 2^{j+1}\}$. Assume

$$A_j = \{k \in \mathbb{Z}^n : \widetilde{Q}_k \subset D_j\}, \quad j \gg 1.$$

Let $f \in \mathcal{S}$, supp $\hat{f} \subset B(0, \frac{1}{8})$ and

$$g(x) = \sum_{k \in A_j} e^{ixk} (\tau_k f)(x), \quad \tau_k f = f(\cdot - k).$$

It is easy to see that $\operatorname{supp} \widehat{\tau_k f} \subset B(0, \frac{1}{8})$ and $\operatorname{supp} \tau_k(\widehat{\tau_k f}) \cap \operatorname{supp} \sigma_\ell = \varnothing$, if $k \neq \ell$. Then, we have

$$\begin{aligned}
\|g\|_{N_{p,q}} &= \left\| \left(\sum_i |\Box_i g|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_i \left| \Box_i \sum_{k \in A_j} e^{ixk} (\tau_k f)(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{i \in A_j} |F^{-1} \sigma_i F g|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_{i \in A_j} |F^{-1} \sigma_0 (\widehat{\tau_k f})|^q \right)^{1/q} \right\|_{L^p} \\
&\gtrsim 2^{jn/q}.
\end{aligned}$$

On the other hand, by $\operatorname{supp} \widehat{g} \subset \{\xi : 2^{j-1} \leqslant |\xi| \leqslant 2^j\}$, we have

$$\begin{aligned}
\|g\|_{F^{n(1/q-1/p)-\eta}_{p,q}} &= \left\| \left(\sum_j 2^{jq(n(1/q-1/p)-\eta)} |\Delta_j g|^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|\Delta_j g\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|g\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|g\|_{L^\infty}^{1-2/p} \|g\|_{L^2}^{2/p}.
\end{aligned}$$

By Plancherel's identity, we have

$$\|g\|_{L^2} = \|\widehat{g}\|_{L^2} = \left(\int_{\mathbb{R}^n} \sum_{k \in A_j} |\tau_k (e^{-ik\xi} \hat{f}(\xi))|^2 \, d\xi \right)^{1/2} \lesssim 2^{nj/2}.$$

We can further assume that $f(x) = f(|x|)$ is a decreasing function on $|x|$. Then $|f(x-k)| \lesssim (1+|x-k|)^{-N}$, $N \gg 1$ and a straightforward computation will lead to $|g(x)| \lesssim 1$.

Based on the above observation, we have

$$\|g\|_{F^{n(1/q-1/p)-\eta}_{p,q}} \lesssim 2^{nj/q - \eta j}.$$

This proves the results.

Step 2. We study the second embedding estimate. Set

$$\begin{aligned}
R_1 &= \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \mathbb{R}^2_{++} : \frac{1}{q} \geqslant \frac{1}{p},\ \frac{1}{p} + \frac{1}{q} \geqslant 1 \right\}, \\
R_2 &= \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \mathbb{R}^2_{++} : \frac{1}{q} \leqslant \frac{1}{p},\ \frac{1}{p} \geqslant \frac{1}{2} \right\}, \\
R_3 &= \mathbb{R}^2_{++} - (R_1 \cup R_2).
\end{aligned}$$

We consider three cases:

Case 1′. $(1/p, 1/q) \in R_1$. We have discussed it in Case 2 (dual version).

Case 2′. $(1/p, 1/q) \in R_3$. For $1 \leqslant p, q < \infty$, we obtain the claimed results by Case 1. For the case $q = \infty$, letting $f = F^{-1}\delta_j$, we have

$$\|f\|_{F_{p,q}} \gtrsim 2^{jn(1-1/p)} \quad \text{and} \quad \|f\|_{N^{n(1-1/p)}_{p,\infty}} \lesssim 2^{jn(1-1/p)},$$

which implies the result.

Case 3′. $(1/p, 1/q) \in R_2$. Assume $f \in \mathcal{S}$, $f(0) = 1$ and

$$\operatorname{supp} \hat{f} \subset Q_0 := \{\xi : |\xi_i| \leqslant \tfrac{1}{2},\ 1 \leqslant i \leqslant n\}.$$

Choose $0 < a \ll 1$. Denote $f_a(x) = f(x/a)$. Then

$$\operatorname{supp} \hat{f_a} \subset Q_{0,a} := \Big\{\xi : |\xi_i| \leqslant \frac{1}{2a},\ 1 \leqslant i \leqslant n\Big\}.$$

Let

$$D_j = \{\xi : \tfrac{5}{4}2^{j-1} \leqslant |\xi| \leqslant \tfrac{3}{4}2^{j+1}\} \quad \text{and} \quad Q_{k(i),a} := k(i) + Q_{0,a}.$$

One has that $Q_{k(i),a}$ and D_j overlap at most $O(a^n 2^{jn})$ cubes. Moreover, there is a $\beta > 0$ such that $f_a(x) > \frac{1}{2}$, $x \in B(0, \beta)$. Let $A_j = \{k(i) : i = 1, \dots, O(a^n 2^{jn})\}$ and $g(x) = \sum_{k \in A_j} e^{ixk}(\tau_k f_a)(x)$. By a straightforward computation, we obtain

$$\|g\|_{F_{p,q}} \gtrsim (a\beta)^{n/p} 2^{jn/p} \quad \text{and} \quad \|g\|_{N^{n(1/p-1/q)}_{p,q}} \lesssim 2^{jn/p}.$$

This finishes the proof. □

4. Multiplication algebra

It is well known that $B^s_{p,q}$ is a multiplication algebra if $s > n/p$; see [Cazenave and Weissler 1990]. The regularity indices, for which $N^s_{p,q}$ constitutes a multiplication algebra, are quite different from those of Besov space. Set

$$\mathbb{R}^2_{++} = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) : \frac{1}{p} > 0,\ \frac{1}{q} \geqslant 0\Big\}, \qquad D_1 = (0, 1] \times [0, 1],$$
$$D_2 = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) \in \mathbb{R}^2_{++} : \frac{1}{q} - 1 < \frac{1}{p} \leqslant \frac{1}{q},\ \frac{1}{q} > 1\Big\},$$
$$D_3 = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) \in \mathbb{R}^2_{++} : \frac{1}{p} \geqslant \frac{1}{q},\ \frac{1}{p} > 1\Big\}.$$

Theorem 4.1. *Assume that* $p > 0$, $1 - 1/(p+1) < q \leqslant \infty$ *and*

$$s > \begin{cases} n\Big(1 - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_1, \\ 0 & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_2, \\ np\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_3. \end{cases}$$

Then $N^s_{p,q}$ *is a multiplication algebra, i.e.,*

$$\|fg\|_{N^s_{p,q}} \lesssim \|f\|_{N^s_{p,q}} \|g\|_{N^s_{p,q}}$$

holds for all $f, g \in N^s_{p,q}$.

Proof.

Step 1. $0 < p < \infty$, $q = \infty$.

Case 1. $1 \leqslant p < \infty$, $q = \infty$, $s > n$. Let $\Lambda_{i,j} = \{k \in \mathbb{Z}^n : |i - j - k| < C(n)\}$. Then it is easy to see that $\Box_i fg = \Box_i \sum_j \Box_j f \sum_{k(i,j)\in\Lambda_{i,j}} \Box_{k(i,j)} g$. Suppose $f, g \in N^s_{p,q}$, by Theorems A.6, 2.6, 2.8 and $\|a_i * b_i\|_{\ell^\infty} \leqslant \|a_i\|_{\ell^1} \|b_i\|_{\ell^\infty}$, we have

$$\begin{aligned}
\|fg\|_{N^s_{p,\infty}} &= \Big\| \sup_i \langle i\rangle^s |\Box_i fg| \Big\|_{L^p} \\
&= \Big\| \sup_i \langle i\rangle^s \Big| \Box_i \sum_j \Box_j f \sum_{k(i,j)\in\Lambda_{i,j}} \Box_{k(i,j)} g \Big| \Big\|_{L^p} \\
&\lesssim \Big\| \sup_i \langle i\rangle^s \sum_j |\Box_j f| |\Box_{k(i,j)} g| \Big\|_{L^p} \qquad (|i - j - k(i,j)| < C(n)) \\
&\lesssim \Big\| \sup_i \sum_j (\langle j\rangle^s + \langle k(i,j)\rangle^s) |\Box_j f| |\Box_{k(i,j)} g| \Big\|_{L^p} \\
&\lesssim \Big\| \sup_i \langle i\rangle^s |\Box_i g| \Big\|_{L^{p_1}} \Big\| \sum_j |\Box_j f| \Big\|_{L^{p_2}} \\
&\qquad + \Big\| \sup_i \langle i\rangle^s |\Box_i f| \Big\|_{L^{p_1}} \Big\| \sum_j |\Box_j g| \Big\|_{L^{p_2}} \\
&\lesssim \|g\|_{N^s_{p_1,\infty}} \|f\|_{N_{p_2,1}} + \|f\|_{N^s_{p_1,\infty}} \|g\|_{N_{p_2,1}} \lesssim \|f\|_{N^s_{p,\infty}} \|g\|_{N^s_{p,\infty}},
\end{aligned}$$

where $1/p = 1/p_1 + 1/p_2$.

Case 2. $0 < p < 1$, $q = \infty$, $s > n$. By the similar argument as in Case 1, we have

$$\|fg\|_{N^s_{p,\infty}} \lesssim \Big\| \sup_i \sum_j (\langle j\rangle^s + \langle k(i,j)\rangle^s)|\Box_j f||\Box_{k(i,j)} g| \Big\|_{L^p}$$
$$\lesssim \Big\| \sup_i \langle i\rangle^s |\Box_i g| \Big\|_{L^p} \Big\| \sum_j |\Box_j f| \Big\|_{L^\infty} + \Big\| \sup_i \langle i\rangle^s |\Box_i f| \Big\|_{L^p} \Big\| \sum_j |\Box_j g| \Big\|_{L^\infty}.$$

On the other hand, by Theorem A.7, we have

$$\Big\| \sum_j |\Box_j f| \Big\|_{L^\infty} = \Big\| \sum_j \Big| \Box_j \sum_{\ell=-1}^{1} \Box_{j+\ell} f \Big| \Big\|_{L^\infty}$$
$$\lesssim \Big\| \sum_j |\sigma_j^\vee| * \Big| \sum_{\ell=-1}^{1} \Box_{j+\ell} f \Big| \Big\|_{L^\infty} \lesssim \Big\| \sum_j |\Box_j f| \Big\|_{L^1} = \|f\|_{N_{1,1}}.$$

Then by Proposition 2.1 and Theorem A.5, we have

$$\|f\|_{N_{1,1}} = \|f\|_{M_{1,1}} \leqslant \|f\|_{M^s_{p,\infty}} \leqslant \|f\|_{N^s_{p,\infty}} \quad \text{for } s > n.$$

This finishes the proof of Case 2.

Combining Cases 1 and 2, we have $\|fg\|_{N^s_{p,\infty}} \lesssim \|f\|_{N^s_{p,\infty}} \|g\|_{N^s_{p,\infty}}$ for $s > n$.

Step 2. $0 < p = q \leqslant 1$. Suppose $f, g \in N_{p,p}$; from Proposition 2.1 and Theorem A.4, we have

$$\|fg\|_{N_{p,p}} = \|fg\|_{M_{p,p}} \leqslant \|f\|_{M_{p,p}} \|g\|_{M_{p,p}} = \|f\|_{N_{p,p}} \|g\|_{N_{p,p}}.$$

Step 3. $1 \leqslant p < \infty$, $q = 1$, $s \geqslant 0$. Suppose $f, g \in N^s_{p,q}$. By Theorems A.4, A.7, 2.6 and $\|a_i * b_i\|_{\ell^1} \lesssim \|a_i\|_{\ell^1} \|b_i\|_{\ell^1}$, we have

$$\|fg\|_{N^s_{p,1}} = \Big\| \sum_i \langle i\rangle^s |\Box_i fg| \Big\|_{L^p}$$
$$\leqslant \Big\| \sum_i \langle i\rangle^s \Big| |F^{-1}(\sigma_i)| * \Big(\sum_j \Box_j f \sum_{k(i,j)\in\Lambda_{i,j}} \Box_{k(i,j)} g \Big) \Big| \Big\|_{L^p}$$
$$\lesssim \Big\| \sum_i \sum_j \Big(\langle j\rangle^s |\Box_j f| \sum_{k(i,j)\in\Lambda_{i,j}} \langle k(i,j)\rangle^s |\Box_{k(i,j)} g| \Big) \Big\|_{L^p}$$
$$\lesssim \Big\| \Big(\sum_i \langle i\rangle^s |\Box_i f| \Big) \Big(\sum_i \langle i\rangle^s |\Box_i g| \Big) \Big\|_{L^p} \lesssim \|f\|_{N^s_{p,1}} \|g\|_{N^s_{p,1}},$$

where we used the fact that $|i - j - k(i,j)| < c(n)$.

Step 4. Let $(1/p, 1/q) \in D_1$. It is easy to see that $(1/p, 1/q)$ is a point of the line segment connecting $(1/p, 0)$ and $(1/p, 1)$. At the point $(1/p, 0)$, in Step 1, we have shown that $N^s_{p,\infty}$ is a multiplication algebra if $s > n$. For $(1/p, 1)$, in Step 3, we have shown that $N^s_{p,1}$ is a multiplication algebra if $s \geqslant 0$. Using complex interpolation (Theorem A.14), we obtain that for $(1/p, 1/q) \in D_1$, $N^s_{p,q}$ is a multiplication algebra if $s > n(1-1/q)$.

If $(1/p, 1/q) \in D_2$, then it belongs to the segment by connecting $(1/p_0, 1)$ and $(1/\bar{p}, 1/\bar{p})$, where $1/p_0 < 1/p - 1/q + 1$ and $\bar{p} = 1 - p(1-q)(1-p_0)/(p - p_0 q)$. In Step 3, we see that for $s \geqslant 0$, $N^s_{P_0,1}$ is a multiplication algebra; in Step 2, we see that $N_{\bar{p},\bar{p}}$ is a multiplication algebra, if $s \geqslant 0$. Then complex interpolation between them gives that for $(1/p, 1/q) \in D_2$ and $s \geqslant 0$, $N^s_{p,q}$ is a multiplication algebra.

If $(1/p, 1/q) \in D_3$, then one can make a line segment connecting $(1/p, 1/p)$ and $(1/p, 0)$. For $(1/p, 1/p)$, we see that once $s \geqslant 0$, $N^s_{p,p}$ is a multiplication algebra. For $(1/p, 0)$, we see that once $s \geqslant n$, $N^s_{p,\infty}$ is a multiplication algebra. Then we use complex interpolation to obtain that $N^s_{p,q}$ is a multiplication algebra if $s > np(1/p - 1/q)$. □

Remark. Assume that $k \in \mathbb{Z}^+$, $p > 0$, $1 - 1/(p+1) < q \leqslant \infty$ and

$$s > \begin{cases} n\Big(1 - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_1, \\ 0 & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_2, \\ np\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_3. \end{cases}$$

Then $N^s_{p,q}$ is a multiplication algebra, i.e.,

$$\|u^k\|_{N^s_{p,q}} \lesssim \|u\|^k_{N^s_{p,q}}$$

holds for all $u \in N^s_{p,q}$ (for $p \geqslant 1$, we obtain $\|u^k\|_{N^s_{p,q}} \lesssim \|u\|^k_{N^s_{kp,q}}$).

Proof. We obtain the result by the similar argument as for the above theorem. □

5. Smooth effects of the Schrödinger semigroup

In this section we will discuss a kind of Strichartz estimates. This kind of estimate was first introduced by R. S. Strichartz [1977], then developed by Pecher [1984], Ginibre and Velo [1995] and Wang, Han, and Huang [2011]. Set

$$S(t) = F^{-1} e^{-it|\xi|^2} F.$$

Our aim is to derive the estimates of $S(t)$ in the spaces $N^s_{p,q}$.

Theorem 5.1. *Assume* $2 \leqslant p < \infty$, $p' \leqslant q \leqslant p$, *and* $1/p + 1/p' = 1$. *Then, for any* $s \in \mathbb{R}$, *we have*

$$\|S(t)f\|_{N^s_{p,q}} \lesssim (1+|t|)^{-n(1/2-1/p)}\|f\|_{N^s_{p',q}}.$$

Proof. By Proposition 2.1 and Theorem A.8, we have

$$\begin{aligned}\|S(t)f\|_{N^s_{p,q}} \leqslant \|S(t)f\|_{M^s_{p,q}} &\leqslant (1+t)^{-n(1/2-1/p)}\|f\|_{M^s_{p',q}} \\ &\leqslant (1+t)^{-n(1/2-1/p)}\|f\|_{N^s_{p',q}}.\end{aligned}$$

In view of the estimates above, the theorem is proved. □

Theorem 5.2. *Assume* $r \geqslant 1$, $p' \leqslant q \leqslant p$, $2 \leqslant p < \infty$, *and* $Af = \int_0^t S(t-\tau)f(\tau)\,d\tau$. *Then for any* $s \in \mathbb{R}$, *we have*

$$\|Af\|_{L^r(-T,T;N^s_{p,q})} \lesssim T^{2/r}\|f\|_{L^{r'}(-T,T;N^s_{p',q})}.$$

Proof. By Theorem 5.1, we have

$$\begin{aligned}\|Af\|_{L^r(-T,T;N^s_{p,q})} &\leqslant \left\|\int_0^t \|S(t-\tau)f(\tau)\|_{N^s_{p,q}}\,d\tau\right\|_{L^r(-T,T)} \\ &\leqslant \left\|\int_0^t (1+|t-\tau|)^{-n(1/2-1/p)}\|f(\tau)\|_{N^s_{p',q}}\,d\tau\right\|_{L^r(-T,T)} \\ &\lesssim \left\|\int_0^\infty \chi_{\tau\in[0,t]}\|f(\tau)\|_{N^s_{p',q}}\right\|_{L^r(-T,T)} \\ &\lesssim T^{1/r}\left(\int_{-T}^T \|f(\tau)\|^r_{L^{r'}(-T,T,N^s_{p',q})}\,dt\right)^{1/r} \\ &\lesssim T^{2/r}\|f\|_{L^{r'}(-T,T;N^s_{p',q})}.\end{aligned}$$
□

Theorem 5.3. *Assume* $r \geqslant 1$ *and* $2 \leqslant p < \infty$. *Then we have*

$$\|S(t)f\|_{L^r(-T,T,N^s_{p,2})} \lesssim T^{1/r}\|f\|_{H^s}.$$

Proof. We show that for any $T > 0$, $I = (-T,T)$, $\varphi \in \mathcal{S}$ and $\psi \in C_c([0,T), C_c^\infty(\mathbb{R}^n))$,

$$\left|\int_{-T}^T (S(t)\varphi, \psi(t))\,dt\right| \lesssim T^{1/r}\|\varphi\|_2\|\psi\|_{L^{r'}(I,N_{p',2})}.$$

Actually, we have

$$\left|\int_{-T}^T (S(t)\varphi, \psi(t))\,dt\right| \lesssim \|\varphi\|_2\left\|\int_{-T}^T S(-t)\psi(t)\,dt\right\|_2.$$

Thanks to Theorem 5.2, we have

$$\begin{aligned}\left\|\int_{-T}^{T} S(-t)\psi(t)\,dt\right\|_2^2 &= \left|\int_{-T}^{T}\left(\psi(t), \int_{-T}^{T} S(t-\tau)\psi(\tau)\,d\tau\right)dt\right| \\ &\lesssim \|\psi\|_{L^{r'}(I,N_{p',2})}\left\|\int_{-T}^{T} S(t-\tau)\psi(\tau)\,d\tau\right\|_{L^r(I,N_{p,2})} \\ &\lesssim T^{2/r}\|\psi\|_{L^{r'}(I,N_{p',2})}^2.\end{aligned}$$

Then by $\|(I-\Delta)^{s/2} f\|_{N_{p,2}} \sim \|f\|_{N_{p,2}^s}$ (Lemma 2.5) and density, we are done. □

Lemma 5.4. *Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$. Assume also $r \geqslant k+2$, $2 \leqslant p \leqslant k+2$, $p' \leqslant q \leqslant p$ and $A(f) = \int_0^t S(t-\tau) f(u(\tau))\,d\tau$. Then for any $s > n(1-1/q)$, we obtain*

$$\|A(f)\|_{L^r(0,T;N_{p,q}^s)} \lesssim T^{1-k/r}\|u\|_{L^r(0,T;N_{p,q}^s)}^{k+1}.$$

Proof. By Theorems 2.6, 5.2 and 4.1, we have

$$\begin{aligned}\|A(f)\|_{L^r(0,T;N_{p,q}^s)} &= \left\|\int_0^t S(t-\tau) f(u(\tau))\,d\tau\right\|_{L^r(0,T;N_{p,q}^s)} \\ &\lesssim T^{2/r}\|f\|_{L^{r'}(0,T;N_{p',q}^s)} \\ &= T^{2/r}\left(\int_0^T \|f(u(\tau))\|_{N_{p',q}^s}^{r'}\,d\tau\right)^{1/r'} \\ &\lesssim T^{2/r}\left(\int_0^T \|u(\tau)\|_{N_{(k+1)p',q}^s}^{(k+1)r'}\,d\tau\right)^{1/r'} \\ &\lesssim T^{2/r}\left(\int_0^T \|u(\tau)\|_{N_{p,q}^s}^{r}\,d\tau\right)^{(k+1)/r} T^{(r-k-2)/r} \\ &\lesssim T^{1-k/r}\|u\|_{L^r(0,T;N_{p,q}^s)}^{k+1}.\end{aligned}$$

□

Lemma 5.5. *Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$. Assume also $r \geqslant k+1$, $1 \leqslant p \leqslant 2(k+1)$ and $A(f) = \int_0^t S(t-\tau) f(u(\tau))\,d\tau$. Then for any $s > \frac{1}{2}n$, we obtain*

$$\|A(f)\|_{C(0,T;H_2^s)} \lesssim T^{1-(k+1)/r}\|u\|_{L^r(0,T;N_{p,2}^s)}^{k+1}.$$

Proof. By Theorems 2.6, 2.7, 4.1 and 5.1, we have

$$\begin{aligned}\|A(f)\|_{C(0,T;H_2^s)} &\leqslant \int_0^T \|f(u(\tau))\|_{H_2^s}\,d\tau \leqslant \int_0^T \|u\|_{N_{2(k+1),2}^s}^{k+1}\,d\tau \\ &\leqslant \int_0^T \|u\|_{N_{p,2}^s}^{k+1}\,d\tau \leqslant T^{1-(k+1)/r}\|u\|_{L^r(0,T;N_{p,2}^s)}^{k+1}.\end{aligned}$$

□

6. Well-posedness of nonlinear Schrödinger equations

In this section we study the well-posedness of the Schrödinger equations

$$iu_t + \Delta u = f(u), \quad u(0,x) = u_0(x).$$

The solution, $u(x,t)$, of the above Cauchy problem is given by

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau, \tag{14}$$

where $S(t) = F^{-1}e^{it|\xi|^2}F$.

Proof of Theorem 1.3. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$$\mathcal{D} = \{u \in C(0,T;H^s) \cap L^r(0,T;N^s_{p,2}) : \|u\|_{L^r(0,T;N^s_{p,2})} < \delta,\ \|u\|_{C(0,T;H^s)} < \delta\}$$

be equipped with the metric

$$d(u,\upsilon) = \|u-\upsilon\|_{L^r(0,T;N^s_{p,2})\cap C(0,T;H^s)}.$$

It is easy to see that $(\mathcal{D}, d)$ is a complete metric space. Now we consider the map

$$J : u(t) \to S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

We shall prove that there exists $T, \delta > 0$ such that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map.

By the nonlinear term estimate (Theorems 4.1, 5.4 and 5.5) and Theorem 5.3, we have

$$\|Ju\|_{L^r(0,T;N^s_{p,2})} \lesssim T^{1/r}\|u_0\|_{H^s} + T^{1-k/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}$$

and

$$\|Ju\|_{C(0,T;H^s)} \lesssim \|u_0\|_{H^s} + T^{1-(k+1)/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}.$$

Then we let

$$\delta = 2C\|u_0\|_{H^s}, \quad T < 1$$

and take T such that

$$2CT^{(1-(k+1)/r)}\delta^k \leqslant \tfrac{1}{2}.$$

Then we have

$$\|Ju\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})} < \delta$$

and

$$\|Ju - J\upsilon\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})} \leqslant \tfrac{1}{2}\|u-\upsilon\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})}.$$

In this way, we obtain that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map. So, there exists a $u \in \mathcal{D}$ satisfying (14). Using standard argument, we can extend the solution (considering the mapping

$$J : u(t) \to S(t-T)u_T - i\int_T^t S(t-\tau) f(u(\tau))\, d\tau, \tag{15}$$

and noticing that $u(T) \in H^s$ [Li and Chen 1989], we can use the same way as in the above to solve (15)). And we can find a maximum $T^* > 0$ which satisfies the conditions in the theorem. □

Proof of Theorem 1.4. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$$\mathcal{D} = \{u \in L^r(0, T; N^s_{p,q}) : \|u\|_{L^r(0,T;N^s_{p,q})} < \delta\},$$

which is equipped with the metric

$$d(u, v) = \|u - v\|_{L^r(0,T;N^s_{p,q})}.$$

It is easy to see that $(\mathcal{D}, d)$ is a complete metric space. Now we consider the map

$$J : u(t) \to S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

By the nonlinear term estimate (Theorems 4.1 and 5.4) and Theorem 5.1, we have

$$\|Ju\|_{L^r(0,T;N^s_{p,q})} \lesssim T^{1/r}\|u_0\|_{N^s_{p',q}} + T^{1-k/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,q})}.$$

Then we let

$$\delta = 2C\|u_0\|_{N^s_{p',q}}, \quad T < 1$$

and take T such that

$$2CT^{(1-k/r)}\delta^k \leqslant \tfrac{1}{2}.$$

Then we have

$$\|Ju\|_{L^r(0,T;N^s_{p,q})} < \delta$$

and

$$\|Ju - Jv\|_{L^r(0,T;N^s_{p,q})} \leqslant \tfrac{1}{2}\|u - v\|_{L^r(0,T;N^s_{p,q})}.$$

In this way, we proved that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map. So, there exists a $u \in \mathcal{D}$ satisfying (14). □

Appendix

We list some properties of modulation spaces and some inequalities used in this paper; most of them are well-known to those familiar with PDEs. Moreover, we sketch a proof of complex interpolation on $N^s_{p,q}$.

Theorem A.1. *Assume $0 < p, q \leqslant \infty$ and $s_1, s_2 \in \mathbb{R}$. Then we have*:

(1) $B^{s_1}_{p,q} \subset M^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \tau(p,q)$, *where*

$$\tau(p,q) = \max\left\{0, n\left(\frac{1}{q} - \frac{1}{p}\right), n\left(\frac{1}{q} + \frac{1}{p} - 1\right)\right\}.$$

(2) $M^{s_1}_{p,q} \subset B^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \sigma(p,q)$, *where*

$$\sigma(p,q) = \max\left\{0, n\left(\frac{1}{p} - \frac{1}{q}\right), n\left(1 - \frac{1}{q} - \frac{1}{p}\right)\right\}.$$

For details of the proof, refer to [Toft 2004; Wang et al. 2006; Sugimoto and Tomita 2007].

Theorem A.2. *Assume $0 < p \leqslant q \leqslant \infty$. Let $\Omega \subset \mathbb{R}^n$ be a compact set*, $\operatorname{diam} \Omega < 2R$. *Then there exists $C(p,q,R) > 0$ such that*

$$\|f\|_{L^q} \leqslant C\|f\|_{L^p} \quad \text{for all } f \in L^p_\Omega,$$

where $L^p_\Omega = \{f \in L^p : \operatorname{supp} \hat{f} \subset \Omega\}$.

For details of the proof, refer to [Wang et al. 2009; 2011].

Theorem A.3. *Suppose $0 < p, q < \infty$. Then we have*

$$(M^s_{p,q})^* = M^{-s}_{(1\vee p)',(1\vee q)'}.$$

For details of the proof, refer to [Han and Wang 2012].

Theorem A.4. *Assume that*

$$s > \begin{cases} n\left(1 - 1 \wedge \frac{1}{q}\right), & \left(\frac{1}{p}, \frac{1}{q}\right) \in D_1, \\ n\left(1 \vee \frac{1}{p} \vee \frac{1}{q} - \frac{1}{q}\right), & \left(\frac{1}{p}, \frac{1}{q}\right) \in D_2. \end{cases}$$

Then $M^s_{p,q}$ is a multiplication algebra, i.e.,

$$\|fg\|_{M^s_{p,q}} \lesssim \|f\|_{M^s_{p,q}} \|g\|_{M^s_{p,q}}$$

holds for all $f, g \in M^s_{p,q}$.

For details of the proof, refer to [Han and Wang 2012]. Note that

$$D_1 = \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{q} \geqslant \frac{2}{p}, \frac{1}{p} \leqslant \frac{1}{2}\right\}, \quad D_2 = \mathbb{R}^2_+ \setminus D_1.$$

Theorem A.5 (embedding). *Assume* $s_1, s_2 \in \mathbb{R}$ *and* $0 < p_1, p_2, q_1, q_2 \leqslant \infty$. *We have*:

(a) *If* $s_2 \leqslant s_1$, $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, *then* $M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}$.

(b) *If* $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, *then* $M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2}$.

For details of the proof, refer to [Wang et al. 2011; Han and Wang 2012].

Theorem A.6 (further multiplier assertions). *Let* $0 < p < \infty$ *and* $0 < q \leqslant \infty$. *Let* $\Omega = \{\Omega_k\}_{k=0}^{\infty}$ *be a sequence of compact subsets of* $\mathbb{R}^n$. *Let* $d_k > 0$ *be the diameter of* Ω_k. *If* $x > n(n/\min(1, p, q) - \frac{1}{2})$, *then there exists a constant* C *such that*

$$\|F^{-1} M_k F f_k\|_{L^p(\ell^q)} \leqslant C \sup_i \|M_i(d_i \cdot)\|_{H^x} \|f_k\|_{L^p(\ell^q)}$$

holds for all systems $\{f_k\}_{k=0}^{\infty} \in L^p_\Omega(\ell^q)$ *and all sequences* $\{M_k(x)\}_{k=0}^{\infty} \subset H^x$.

For details of the proof, refer to [Triebel 1983]. Note that $L^p_\Omega(\ell^q)$ is defined as

$\{f : f = \{f_k\}_{k=0}^{\infty} \subset S', \operatorname{supp} F f_k \subset \Omega_k$ if $k = 0, 1, 2, \ldots$ and $\|f_k\|_{L^p(\ell^q)} < \infty\}$.

Theorem A.7. *Let* $0 < p, q \leqslant \infty$ *and* (X, μ), (Y, υ) *be two measure spaces. Let* T *be a positive linear operator mapping* $L^p(X)$ *into* $L^q(Y)$ (*resp.* $L^{q,\infty}(Y)$) *with norm* A. *Let* B *be a Banach space. Then* T *has a* B*-valued extension* $\vec{T}$ *that maps* $L^p(X, B)$ *into* $L^q(Y, B)$ (*resp.* $L^{q,\infty}(Y, B)$) *with the same norm.*

For details of the proof, refer to [Grafakos 2004].

Theorem A.8. *Assume* $s \in \mathbb{R}$, $2 \leqslant p < \infty$, $0 < q < \infty$ *and* $1/p + 1/p' = 1$. *Then we have*

$$\|\Box_k S(t) f\|_{L^p} \lesssim (1 + |t|)^{-n(1/2 - 1/p)} \|\Box_k f\|_{L^{p'}},$$
$$\|S(t) f\|_{M^s_{p,q}} \lesssim (1 + |t|)^{-n(1/2 - 1/p)} \|f\|_{M^s_{p',q}}.$$

For details of the proof, refer to [Wang et al. 2011].

We start with some abstract theory about complex interpolation on quasi-Banach spaces. Let $\mathcal{S} = \{z : 0 < \operatorname{Re} z < 1\}$ be a strip in the complex plane. Its closure $\{z : 0 \leqslant \operatorname{Re} z \leqslant 1\}$ is denoted by $\bar{\mathcal{S}}$. We say that $f(z)$ is an $\mathcal{S}'$-analytic function in $\mathcal{S}$ if the following properties are satisfied:

(a) For every fixed $z \in \bar{\mathcal{S}}$, $f(z) \in \mathcal{S}'(\mathbb{R}^n)$.

(b) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $F^{-1}\varphi F f(x, z)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \bar{\mathcal{S}}$.

(c) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $F^{-1}\varphi F f(x, z)$ is an analytic function in $\mathcal{S}$ for every fixed $x \in \mathbb{R}^n$.

We denote the set of all $\mathcal{S}'$-analytic functions in $\mathcal{S}$ by $A(\mathcal{S}'(\mathbb{R}^n))$. The idea we used here is due to [Triebel 1983; Han and Wang 2012].

Definition A.9. Let A_0 and A_1 be quasi-Banach spaces, and $0 < \theta < 1$. We define

$$F(A_0, A_1) = \Big\{\varphi(z) \in A(\mathcal{S}'(\mathbb{R}^n)) : \varphi(\ell + it) \in A_\ell,\ \ell = 0, 1,\ \text{for all } t \in \mathbb{R}, \\ \|\varphi(z)\|_{F(A_0,A_1)} = \max_{\ell=0,1} \sup_{t\in\mathbb{R}} \|\varphi(\ell + it)\|_{A_\ell}\Big\}$$

and

$$(A_0, A_1)_\theta = \Big\{f \in \mathcal{S}' : \exists \varphi(z) \in F(A_0, A_1), \\ \text{such that } f = \varphi(\theta),\ \|f\|_{(A_0,A_1)_\theta} = \inf_\varphi \|\varphi(z)\|_{F(A_0,A_1)}\Big\},$$

where the infimum is taken over all $\varphi(z) \in F(A_0, A_1)$ such that $\varphi(\theta) = f$.

The following three propositions are essentially known in [Triebel 1983; Han and Wang 2012].

Proposition A.10. *Adopt the notation in Definition A.9; then*

$$((A_0, A_1)_\theta, \|\cdot\|_{(A_0,A_1)_\theta})$$

is a quasi-Banach space.

Proposition A.11. *Adopt the notation in Definition A.9; then we have*

$$\|f\|_{(A_0,A_1)_\theta} = \inf_\varphi \Big(\sup_{t\in\mathbb{R}} \|\varphi(it)\|_{A_0}^{1-\theta} \sup_{t\in\mathbb{R}} \|\varphi(1 + it)\|_{A_1}^{\theta}\Big),$$

where the infimum is taken over all $\varphi(z) \in F(A_0, A_1)$ such that $\varphi(\theta) = f$.

Proposition A.12. *Let T be a continuous multilinear operator from*

$$A_0^{(1)} \times A_0^{(2)} \times \cdots \times A_0^{(m)}$$

to B_0 and from $A_1^{(1)} \times A_1^{(2)} \times \cdots \times A_1^{(m)}$ to B_1, satisfying

$$\|T(f^{(1)}, f^{(2)}, \dots, f^{(m)})\|_{B_0} \leqslant C_0 \prod_{j=1}^m \|f^{(j)}\|_{A_0^{(j)}},$$

$$\|T(f^{(1)}, f^{(2)}, \dots, f^{(m)})\|_{B_1} \leqslant C_1 \prod_{j=1}^m \|f^{(j)}\|_{A_1^{(j)}},$$

$$f^{(j)} \in A_0^{(j)} \cap A_1^{(j)}.$$

Then T is continuous from $(A_0^{(1)}, A_1^{(1)})_\theta \times (A_0^{(2)}, A_1^{(2)})_\theta \times \cdots \times (A_0^{(m)}, A_1^{(m)})_\theta$ to $(B_0, B_1)_\theta$ with norm at most $C_0^{1-\theta} C_0^{\theta}$, provided $0 \leqslant \theta \leqslant 1$.

Theorem A.13 (complex interpolation). *Suppose* $0<\theta<1$, $0<p_0,p_1<\infty$, $0<q_0,q_1\leqslant\infty$ *and*

$$s=(1-\theta)s_0+\theta s_1,\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},$$

then we have

$$(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})_\theta=N^s_{p,q}.$$

Sketch of proof. Let $g\in N^s_{p,q}(\mathbb{R}^n)$ and $g_k(x)=F^{-1}\sigma_k Fg$. Let also $\psi_k(x)=\sum_{\ell=-1}^{1}\sigma_{k+\ell}(x)$ for $k=0,1,2,\dots$ (with $\sigma_{-1}=0$). In particular, $\psi_k(x)=1$ if $x\in\operatorname{supp}\sigma_k$.

Set

$$g_k^*(x)=\sup_{x\in\mathbb{R}^n}\frac{|g_k(x-y)|}{1+|\langle k\rangle y|^a},\quad x\in\mathbb{R}^n,\ a>\frac{n}{\min(p,q)}.$$

For $z\in\bar{\mathcal{S}}$, we write

$$\begin{cases}a_1(z)=sq\Big(\dfrac{1-z}{q_0}+\dfrac{z}{q_1}\Big)-(1-z)s_0-zs_1,\\ a_2(z)=p\Big(\dfrac{1-z}{p_0}+\dfrac{z}{p_1}\Big)-q\Big(\dfrac{1-z}{q_0}+\dfrac{z}{q_1}\Big),\\ a_3(z)=1-p\Big(\dfrac{1-z}{p_0}+\dfrac{z}{p_1}\Big),\\ a_4(z)=q\Big(\dfrac{1-z}{q_0}+\dfrac{z}{q_1}\Big).\end{cases}$$

We put

$$f(z)=\sum_{k=0}^{\infty}F^{-1}\psi_kF\Bigg[\langle k\rangle^{a_1(z)}\Bigg(\sum_{\ell=0}^{k}|\langle\ell\rangle^s g_\ell^*|^q\Bigg)^{\frac{a_2(z)}{q}}\|\langle\ell\rangle^s g_\ell^*\|^{a_3^z}_{L^p(\mathbb{R}^n,\ell^q)}g_k^{a_4(z)}(x)\Bigg].$$

Obviously, $f(z)\in A(\mathcal{S}')$ and $f(\theta)=g$. Direct calculation shows

$$\|f(\ell+it)\|_{N^{s_\ell}_{p_\ell,q_\ell}}\lesssim\|g\|_{N^s_{p,q}},\quad \ell=0,1.$$

This proves that $N^s_{p,q}\subset(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})_\theta$.

Conversely, let $f\in F(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})$. If $\varphi\in A(\mathcal{S}')$ such that $\varphi(\theta)=f$, for some $\theta\in(0,1)$, we can find two positive functions $\mu_0(\theta,t)$ and $\mu_1(\theta,t)$ in $(0,1)\times\mathbb{R}$ satisfying

$$|f(z)|^r\leqslant\left(\frac{1}{1-\theta}\int_{\mathbb{R}}|f(it)|^r\mu_0(\theta,t)\,dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}}|f(1+it)|^r\mu_1(\theta,t)\,dt\right)^{\theta},$$

with $1/(1-\theta)\int_{\mathbb{R}}\mu_0(\theta,t)\,dt = 1/\theta\int_{\mathbb{R}}\mu_1(\theta,t)\,dt = 1$. Taking the $N^s_{p,q}$ norm of both sides and then applying Minkowski's inequality imply that

$$\begin{aligned}\|f\|_{N^s_{p,q}} &\leqslant \sup_{t\in\mathbb{R}}\|\langle k\rangle^{s_0}F^{-1}\sigma_k F\varphi(it)\|^{1-\theta}_{L^{p_0}(\mathbb{R},q_0)} \\ &\qquad\qquad\qquad\qquad \times \sup_{t\in\mathbb{R}}\|\langle k\rangle^{s_1}F^{-1}\sigma_k F\varphi(1+it)\|^{1-\theta}_{L^{p_1}(\mathbb{R},q_1)} \\ &\leqslant \|f\|_{F(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})}.\end{aligned}$$

This proves that $(N^{s_0}_{p_0,q_0}, N^{s_1}_{p_1,q_1})_\theta \subset N^s_{p,q}$. □

Theorem A.14 (complex interpolation). *Let $-\infty < s_0, s_1 < \infty$, $0 < p_0^{(j)}, p_1^{(j)} < \infty$, $0 < q_0^{(j)}, q_1^{(j)} \leqslant \infty$, $j = 1,\dots,m$. If T is a continuous multilinear mapping from*

$$N^{s_0^{(1)}}_{p_0^{(1)},q_0^{(1)}} \times \cdots \times N^{s_0^{(m)}}_{p_0^{(m)},q_0^{(m)}}$$

to $N^{s_0}_{p_0,q_0}$ with norm M_0, and also continuous multilinear from

$$N^{s_1^{(1)}}_{p_1^{(1)},q_1^{(1)}} \times \cdots \times N^{s_1^{(m)}}_{p_1^{(m)},q_1^{(m)}}$$

to $N^{s_1}_{p_1,q_1}$ with norm M_1, then T is continuous and multilinear from

$$N^{s^{(1)}}_{p^{(1)},q^{(1)}} \times \cdots \times N^{s^{(m)}}_{p^{(m)},q^{(m)}}$$

to $N^s_{p,q}$ with norm at most $M_0^{1-\theta}M_1^\theta$, provided $0 \leqslant \theta \leqslant 1$, and

$$s^{(j)} = (1-\theta)s_0^{(j)} + \theta s_1^j,$$

$$\frac{1}{p^{(j)}} = \frac{1-\theta}{p_0^{(j)}} + \frac{\theta}{p_1^{(j)}}, \qquad \frac{1}{q^{(j)}} = \frac{1-\theta}{q_0^{(j)}} + \frac{\theta}{q_1^{(j)}}, \qquad j = 1,\dots,m$$

This theorem is a natural consequence of Proposition A.12 and Theorem A.13.

Theorem A.15. *Let $s \in R$, $0 < p, q \leqslant \infty$. We have*:

(1) *If $\varepsilon > 0$, then*

$$F^{s+\varepsilon}_{p,q_1} \subset F^s_{p,q_2}, \qquad B^{s+\varepsilon}_{p,q_1} \subset B^s_{p,q_2}.$$

(2) *If $p < \infty$, then*

$$B^s_{p,p\wedge q} \subset F^s_{p,q} \subset B^s_{p,p\vee q}.$$

For details of the proof, refer to [Triebel 1983].

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HYPERSURFACES WITH CONSTANT CURVATURE QUOTIENTS IN WARPED PRODUCT MANIFOLDS

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We study rigidity problems for hypersurfaces with constant curvature quotients $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ in the warped product manifolds. Here $\mathcal{H}_{2k}$ is the k-th Gauss–Bonnet curvature and $\mathcal{H}_{2k+1}$ arises from the first variation of the total integration of $\mathcal{H}_{2k}$. Hence the quotients considered here are in general different from $\sigma_{2k+1}/\sigma_{2k}$, where σ_k are the usual mean curvatures. We prove several rigidity and Bernstein-type results for compact or noncompact hypersurfaces corresponding to such quotients.

1. Introduction

Let Σ^{n-1} be a closed smooth hypersurface isometrically immersed in an n-dimensional Riemannian manifold (M^n, g). Assume that Σ_t is a variation of Σ with the unit outward normal vector field ν_t as the variational vector field. It is well known that the first variation of the area functional $\mathrm{Area}(\Sigma_t)$ is given by

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma H\, d\mu,$$

where H is the mean curvature of Σ with respect to the inner normal and $d\mu$ is the area element of Σ. On the other hand, it is well known that the first variation of the total scalar curvature functional $\int_\Sigma R\, d\mu$ is given by

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Sigma_t} R\, d\mu_t = \int_\Sigma -2 \sum_{i,j=1}^{n-1} E^{ij} h_{ij}\, d\mu,$$

where $E^{ij} = R^{ij} - \frac{1}{2} R g^{ij}$ and h_{ij} are respectively the Einstein tensor and the second fundamental form of Σ with respect to the inner normal in the local coordinates.

There is a natural generalization of scalar curvature, called Gauss–Bonnet curvatures L_k for an integer $1 \le k \le \frac{1}{2}(n-1)$ for $(n-1)$-dimensional Riemannian

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manifolds. L_k are intrinsic curvature functions. When $n-1$ is even, the highest order Gauss–Bonnet curvature $L_{(n-1)/2}$ is exactly the Pfaffian in the Gauss–Bonnet–Chern formula. L_2 appeared first in [Lanczos 1938] and has been intensively studied in the theory of Gauss–Bonnet gravity, which is a generalization of Einstein gravity.

The first variation of the total Gauss–Bonnet curvature functional $\int_\Sigma L_k \, d\mu$ has been considered long time ago by Lovelock [1971]. Li [1985] also computed the first variation of these functionals as well as the second variation for submanifolds in the general ambient Riemannian manifolds. Recently an alternative computation was given by Labbi [2008b]:

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Sigma_t} L_k \, d\mu_t = \int_\Sigma -2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij} \, d\mu,$$

where $E_{(k)}^{ij}$ is the generalized Einstein tensor defined by (2-1). Labbi [2008a] referred to the critical point of $\int_\Sigma L_k \, d\mu$ as $2k$-minimal submanifolds. In this sense, the ordinary minimal submanifolds are referred as 0-minimal submanifolds.

For the ambient space $M^n = \mathbb{R}^n$, by the Gauss equation, one can verify that $L_k = (2k)!\sigma_{2k}$ and $-2\sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij} = (2k+1)!\sigma_{2k+1}$, where σ_k are the usual mean curvatures defined by the elementary symmetric functions of the principal curvatures of associated hypersurfaces. Hence the Gauss–Bonnet curvatures L_k as well as the integrand $-2\sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij}$ appear like higher order mean curvatures.

Throughout this paper, we use the notation

$$\mathcal{H}_{2k} := L_k, \quad \mathcal{H}_{2k+1} := -2 \sum_{i,j=1}^{n-1} E_{(k)}^{ij} h_{ij},$$

and call them $2k$-mean curvature and $(2k+1)$-mean curvature. By convention, we use $L_0 = 1$. We emphasize here that in general these mean curvatures are different from the usual ones defined by σ_k except $\mathcal{H}_0$ and $\mathcal{H}_1$. The 0-mean curvature $\mathcal{H}_0$ is equal to 1 and the 1-mean curvature $\mathcal{H}_1$ is equal to the usual mean curvature H.

We will consider some rigidity problems related to $\mathcal{H}_{2k}$ and $\mathcal{H}_{2k+1}$ in a class of Riemannian manifolds: warped product manifolds. A warped product manifold $(M, \bar{g})$ is the product manifold of an interval and an $(n-1)$-dimensional Riemannian manifold with some smooth positive warping function. Precisely,

$$M = [0, \bar{r}) \times_\lambda N^{n-1} \quad (0 < \bar{r} \le \infty)$$

is equipped with

$$\bar{g} = dr^2 + \lambda(r)^2 g_N,$$

where $\lambda : [0, \bar{r}) \to \mathbb{R}_+$ is a smooth positive function and (N^{n-1}, g_N) is an $(n-1)$-dimensional Riemannian manifold.

The rigidity problems for hypersurfaces in Riemannian manifolds with constant curvature functions are one of the central problems in the classical differential geometry. Historically, the rigidity problems for hypersurfaces in the Euclidean space was studied by Liebmann [1899], Hsiung [1954], Süss [1952], Alexandrov [1956; 1957; 1958a; 1958b], Alexandrov and Volkov [1958], Reilly [1977], Ros [1988], Korevaar [1988], etc. Recently, many works concerning the rigidity for hypersurfaces in warped product manifolds have appeared, see, for example, [Montiel 1999; Alías et al. 2013; Brendle 2013; Brendle and Eichmair 2013; Wu and Xia 2014] and the references therein.

In all above works, the curvature functions are related to the elementary symmetric functions σ_k of the principal curvatures of hypersurfaces. Our concern in this paper is the curvature functions $\mathcal{H}_{2k}$ and $\mathcal{H}_{2k+1}$. In view of the Gauss equation, for hypersurfaces in general ambient Riemannian manifolds, $\mathcal{H}_{2k}$ and $\mathcal{H}_{2k+1}$ depend not only on σ_k but also on the Riemannian curvature tensor of the ambient manifolds. Therefore, except for the case that the ambient spaces are the space forms, for which $\mathcal{H}_{2k}$ and $\mathcal{H}_{2k+1}$ can be written as linear combinations of σ_k, one cannot express them as pure functions on the principal curvatures of hypersurfaces.

The first attempt in which we succeed is the rigidity on the curvature quotients $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ in a class of warped product manifolds. These quotients can be viewed as a generalization of the usual mean curvature H since the case $k=0$ corresponds to H. We remark that the rigidity on the quotients of σ_k in a class of warped product manifolds has been considered in [Wu and Xia 2014]. However, as mentioned before, these two kinds of quotients have large differences in general. Many techniques seem to be difficult to apply for the quotients $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ considered here.

The first main result of this paper is stated as:

Theorem 1.1. *Define $(M^n, \bar{g})$ to be an n-dimensional warped product manifold $[0, \bar{r}) \times_\lambda N^{n-1}$ whose warped product function satisfies*

$$\lambda\lambda'' - (\lambda')^2 \geq 0 \quad (\textit{i.e., } \log\lambda \textit{ is convex}). \tag{1-1}$$

Let Σ^{n-1} be a closed star-shaped hypersurface in M such that the generalized Einstein tensor $E_{(k)}$ is semidefinite on Σ. For any integer k with $0 \leq k < \frac{1}{2}(n-1)$ and $\mathcal{H}_{2k}$ not vanishing on Σ, if the curvature quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ is a constant, then Σ is a slice $\{r_0\} \times N$ for some $r_0 \in [0, \bar{r})$ and the constant is $(n-1-2k)\log\lambda(r_0)$.

The star-shapedness means that Σ can be written as a graph over N, alternatively, $\langle \partial/\partial r, \nu\rangle \geq 0$, where ν is the outer normal of Σ. The method to prove Theorem 1.1 is to apply the maximum principle to an elliptic equation. This method was previously indicated by Montiel [1999] and was used widely in [Alías and Colares 2007; Alías et al. 2012].

The condition (1-1) imposed on M only depends on the warped product function λ, not the fiber manifold N. We notice that the condition excludes the usual space forms $\mathbb{R}^n$, $\mathbb{S}^n_+$ (hemisphere) and $\mathbb{H}^n$ (hyperbolic space) in which cases $\lambda\lambda'' - (\lambda')^2 = -1$. For $\mathbb{R}^n$, since the quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ is equal to $\sigma_{2k+1}/\sigma_{2k}$, the result still holds; see [Korevaar 1988; Koh 2000]. We will consider the case $\mathbb{S}^n_+$ and $\mathbb{H}^n$ elsewhere since the proof has a different flavor. We also notice that the condition (1-1) is satisfied by some local space forms such as $[0,\infty)\times_{e^r}\mathbb{R}^{n-1}$ or $[0,\infty)\times_{\cosh r}\mathbb{R}^{n-1}$. There are also nonconstant curvature manifolds which satisfy (1-1). A typical example for which the condition (1-1) is satisfied is the so-called Kottler–Schwarzschild spaces $[0,\infty)\times_\lambda N(\kappa)$, whose warped product fact λ satisfies $\lambda'(r)=\sqrt{\kappa+\lambda(r)^2-2m\lambda(r)^{2-n}}$ and $N(\kappa)$ is a closed space form of the constant sectional curvature $\kappa = 0$ or -1. See Appendix for a detailed explanation.

Note that $E^{ij}_{(1)} = R^{ij} - \frac{1}{2}Rg^{ij}$ is the Einstein tensor, so that in $k=1$ case, the semidefinite condition of $E_{(1)}$ is just the semidefiniteness of the Einstein tensor. In particular, if $M=\mathbb{R}^n$, one readily sees that $-E_{(k)} = \frac{1}{2}(2k)!T_{2k}$, where T_{2k} is the $2k$-Newton tensor associated to the hypersurface Σ, and the seminegative definite condition of $E_{(k)}$ relates to $2k$-convexity.

In order to extend the above result to noncompact hypersurfaces, we need a generalization of the Omori–Yau maximum principle for the trace-type semi-elliptic operators. The classical Omori–Yau maximum principle is initially stated for the Laplacian Δ. A Riemannian manifold Σ is said to satisfy the Omori–Yau maximum principle if for any function $u\in C^2(\Sigma)$ with $\sup_\Sigma u < +\infty$, there exists a sequence $\{p_i\}_{i\in\mathbb{N}}\subset\Sigma$ such that for each i, the following inequalities hold:

$$u(p_i) > \sup_\Sigma u - \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad \Delta u(p_i) < \frac{1}{i}.$$

This principle was first proved by Omori [1967] and later generalized by Yau [1975] under the condition that the Ricci curvature is bounded from below. It has proved to be very useful in the framework of noncompact manifolds and attracted considerable extending works. For example, it was improved by Chen and Xin [1992] and Ratto, Rigoli and Setti [Ratto et al. 1995] by assuming that the radial curvature decays slower than a certain decreasing function. Recently, the essence of the Omori–Yau maximum principle was captured by Pigola, Rigoli and Setti (see [Pigola et al. 2005, Theorem 1.9]) that the validity of the Omori–Yau maximum principle is assured by the existence of some nonnegative C^2 function satisfying some appropriate requirements, and thus may not necessarily depend on the curvature bounds. Also, they discussed the generalizations for the trace-type differential operators (see Definition 3.1) which will be used in this paper. For a detailed discussion of the sufficient condition to guarantee the Omori–Yau maximum principle for the trace-type differential operators to hold in the warped product manifolds, see [Alías et al. 2013] or Section 3 below.

We have a rigidity result for noncompact hypersurfaces:

Theorem 1.2. *Define $(M^n, \bar{g})$ to be an n-dimensional warped product manifold $[0, \bar{r}) \times_\lambda N^{n-1}$ whose warped product function satisfies $\lambda\lambda'' - (\lambda')^2 \geq 0$ with equality only at isolated points. Let (Σ^{n-1}, g) be a complete noncompact star-shaped hypersurface in M, which is contained in a slab $[r_1, r_2] \times N$, $0 \leq r_1 < r_2 < \bar{r}$, such that the generalized Einstein tensor $E_{(k)}$ being semidefinite on Σ. Assume the Omori–Yau maximum principle holds for the trace-type operator $\mathrm{tr}_g(-2E_{(k)}\nabla_g^2)$ on Σ. For an integer k with $0 \leq k < \frac{1}{2}(n-1)$ and $\mathcal{H}_{2k}$ not vanishing on Σ, if the curvature quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ is a constant, then Σ is a slice $\{r_0\} \times N$ for some $r_0 \in [r_1, r_2]$ and the constant is $(n-1-2k)\log\lambda(r_0)$.*

Motivated by the analogous Bernstein type result on the quotient of the usual mean curvatures [Aquino and de Lima 2014], we can establish a corresponding result in our case. More precisely, instead of assuming the curvature quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ being constant, we can establish the rigidity result via assuming a natural comparison inequality between $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ and its value on the slices.

Theorem 1.3. *Define $(M^n, \bar{g})$ to be an n-dimensional warped product manifold $[0, \bar{r}) \times_\lambda N$. Let (Σ^{n-1}, g) be a complete, star-shaped hypersurface in M, which is contained in a slab $[r_1, r_2] \times N$, $0 \leq r_1 < r_2 < \bar{r}$, such that the generalized Einstein tensor $E_{(k)}$ is semidefinite on Σ. Assume that the Omori–Yau maximum principle holds for the trace-type operator $\mathrm{tr}_g(-2E_{(k)}\nabla_g^2)$ on Σ and that the Gauss–Bonnet curvature $\mathcal{H}_{2k}$ is bounded by two positive constants, i.e., $0 < C_1 \leq \mathcal{H}_{2k} \leq C_2$. If*

$$\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \leq (n-1-2k)\frac{\lambda'(r)}{\lambda(r)} \quad \text{and} \quad |\nabla_g r|_g \leq \inf_\Sigma\left((n-1-2k)\frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right),$$

then the hypersurface Σ is a slice $\{r_0\} \times M$ for some $r_0 \in [r_1, r_2]$.

We remark that we do not assume the log-convexity of the warped product function for Theorem 1.3.

2. Preliminaries

In this section, we first recall the work of [Lovelock 1971] on the generalized Einstein tensors and Gauss–Bonnet curvatures. Throughout this paper, we use the notation R_{ijkl}, R_{ij} and R to indicate the Riemannian 4-tensor, the Ricci tensor in local coordinates and the scalar curvature respectively. We use the metric g to lower or raise an index and adopt the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

For an $(n-1)$-dimensional Riemannian manifold (Σ^{n-1}, g), the Einstein tensor $E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is very important in theoretical physics. It is a conversed quantity, i.e.,

$$\nabla_j E_i^j = 0,$$

where ∇ is the covariant derivative with respect to the metric g.

Lovelock [1971] studied the classification of tensors A satisfying

(i) $A^{ij} = A^{ji}$, i.e, A is symmetric.

(ii) $A^{ij} = A^{ij}(g, \partial g, \partial^2 g)$.

(iii) $\nabla_j A^{ij} = 0$, i.e., A is divergence-free.

(iv) A^{ij} is linear in the second derivatives of g.

It is clear that the Einstein tensor E_{ij} satisfies all above conditions. Lovelock classified all 2-tensors satisfying (i)–(iii). For an integer $0 \le k \le \frac{1}{2}(n-1)$, let us define a 2-tensor $E_{(k)}$ locally by

$$E^{ij}_{(k)} := -\frac{1}{2^{k+1}} g^{lj} \delta^{i i_1 i_2 \cdots i_{2k-1} i_{2k}}_{l j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}. \tag{2-1}$$

Here the generalized Kronecker delta is defined by

$$\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} = \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}.$$

One can check that $E_{(k)}$ satisfies (i)–(iii). Lovelock proved that any 2-tensor satisfying (i)–(iii) has the form

$$\sum_k \alpha^k E_{(k)},$$

with certain constants α^k, $k \ge 0$. The $E_{(k)}$ are called the generalized Einstein tensors.

For an integer $0 \le k \le \frac{1}{2}(n-1)$, the Gauss–Bonnet curvatures L_k are defined by

$$L_k := \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}. \tag{2-2}$$

When $2k = n-1$, L_k is the Euler density. When $k < \frac{1}{2}(n-1)$, L_k is called the dimensional continued Euler density in physics. We set $E_{(0)} = -\frac{1}{2}g$ and $L_0 = 1$. It is clear from the definitions (2-1) and (2-2) that

$$\operatorname{tr}_g(E_{(k)}) := E^{ij}_{(k)} g_{ij} = -\frac{n-1-2k}{2} L_k. \tag{2-3}$$

It is easy to see that $(E_{(1)})_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor and $L_1 = R$ is the scalar curvature. One can also check that

$$E^{ij}_{(2)} = 2RR^{ij} - 4R^{is}R_s{}^j - 4R_{sl}R^{silj} + 2R^i{}_{klm}R^{jklm} - \tfrac{1}{2}g^{ij}L_2,$$

and

$$L_2 = \tfrac{1}{4}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}R^{j_1j_2}{}_{i_1i_2}R^{j_3j_4}{}_{i_3i_4} = R_{ijsl}R^{ijsl} - 4R_{ij}R^{ij} + R^2.$$

In [Lovelock 1971], the author proved that the first variational formula for the total Gauss–Bonnet curvature functional is given in terms of the generalized Einstein tensor. It was also presented in [Li 1985; Labbi 2008b], although with different notation and formalism. For the convenience of readers, we include a proof here.

Proposition 2.1 [Lovelock 1971]. *Let (Σ^{n-1}, g) be a smooth closed manifold. Assume that g_t is a variation of g with $\frac{\partial}{\partial t}\big|_{t=0} g_{ij} = v_{ij}$ for a symmetric 2-tensor v, then*

$$\frac{d}{dt}\bigg|_{t=0}\int_{\Sigma_t} L_k\, d\mu_t = \int_\Sigma -E^{ij}_{(k)}v_{ij}\, d\mu. \tag{2-4}$$

In particular, if (Σ^{n-1}, g) is a closed, smooth hypersurface immersed in an n-dimensional Riemannian manifold $(M^n, \bar{g})$ and the variational vector field is given by the outward unit normal ν, then

$$\frac{d}{dt}\bigg|_{t=0}\int_{\Sigma_t} L_k\, d\mu_t = \int_\Sigma -2E^{ij}_{(k)}h_{ij}\, d\mu. \tag{2-5}$$

where h_{ij} denotes the second fundamental form of Σ with respect to $-\nu$.

Proof. By the simple fact that $\frac{d}{dt}\big|_{t=0} d\mu_t = \frac{1}{2}\operatorname{tr}_g v\, d\mu$ and the definition of L_k, we compute

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int_{\Sigma_t} L_k\, d\mu_t &= \int_\Sigma \frac{d}{dt}\bigg|_{t=0} L_k d\mu + \int_\Sigma \tfrac{1}{2}L_k \operatorname{tr}_g v\, d\mu \\ &= \int_\Sigma kP^{ij}_{(k)sl}\frac{d}{dt}\bigg|_{t=0} R_{ij}{}^{sl}\, d\mu + \int_\Sigma \tfrac{1}{2}L_k\operatorname{tr}_g v\, d\mu,\end{aligned} \tag{2-6}$$

where the 4-tensor $P_{(k)}$ is given by

$$P^{stlj}_{(k)} := \frac{1}{2^k}\delta^{i_1i_2\cdots i_{2k-3}i_{2k-2}st}_{j_1j_2\cdots j_{2k-3}j_{2k-2}j_{2k-1}j_{2k}}R_{i_1i_2}{}^{j_1j_2}\cdots R_{i_{2k-3}i_{2k-2}}{}^{j_{2k-3}j_{2k-2}}g^{j_{2k-1}l}g^{j_{2k}j}, \tag{2-7}$$

and

$$P^{ij}_{(k)sl} = P^{ijpq}_{(k)}g_{sp}g_{lq}.$$

We remark that $P_{(k)}$ shares the same symmetry as the Riemann curvature tensor, that is,

$$P^{stjl}_{(k)} = -P^{tsjl}_{(k)} = -P^{stlj}_{(k)} = P^{jlst}_{(k)}. \tag{2-8}$$

Furthermore, by applying the second Bianchi identity of the curvature tensor, one can check that $P_{(k)}$ has the crucial property of being divergence-free (see [Ge et al. 2014, Lemma 2.2] for a proof)

$$\nabla_s P_{(k)}^{stjl} = 0. \tag{2-9}$$

To calculate the first term in (2-6), we recall that if $(\partial/\partial t)g = v$, then the evolution equation of the curvature tensor is given by (see [Chow et al. 2006, Equation (2.66)])

$$\frac{d}{dt}R_{ijsl} = -\tfrac{1}{2}(\nabla_i\nabla_j v_{sl} - \nabla_i\nabla_l v_{sj} - \nabla_s\nabla_j v_{il} + \nabla_s\nabla_l v_{ij} - R_{ijsm}v^m{}_l - R_{ijml}v^m{}_s).$$

Then we use (2-8) and (2-9) to compute that

$$\begin{aligned}
&\int_\Sigma kP_{(k)sl}^{ij}\left(\frac{d}{dt}\bigg|_{t=0} R_{ij}{}^{sl}\right)d\mu \\
&= \int_\Sigma kP_{(k)sl}^{ij}\big(\tfrac{1}{2}(-\nabla_i\nabla_j v^{sl} + \nabla_i\nabla^l v_j^s + \nabla^s\nabla_j v_i^l - \nabla^s\nabla^l v_{ij}) \\
&\qquad\qquad + \tfrac{1}{2}(R_{ijm}{}^l v^{ms} - R_{ijm}{}^s v^{ml}) + (-R_{ijp}{}^l v^{sp} - R_{ijq}{}^s v^{lq})\big)\,d\mu \\
&= -\int_\Sigma kP_{(k)sl}^{ij} R_{ijm}{}^l v^{sm}\,d\mu,
\end{aligned} \tag{2-10}$$

where in the last equality we used (2-9), (2-8) and the simple observation that $(R_{ijm}{}^l v^{ms} - R_{ijm}{}^s v^{ml})$ and $(-R_{ijp}{}^l v^{sp} - R_{ij}{}^s{}_q v^{lq})$ are both antisymmetric with respect to the pair (s, l).

Going back to (2-6), we obtain that

$$\frac{d}{dt}\bigg|_{t=0}\int_{\Sigma_t} L_k\,d\mu_t = \int_\Sigma (-kP_{(k)}^{ijs}{}_l R_{ij}{}^{ml} + \tfrac{1}{2}L_k g^{ms})v_{ms}\,d\mu.$$

On the other hand, from definitions (2-1), (2-2) and (2-7), it is direct to check that

$$\begin{aligned}
E_{(k)}^{ms} &= -\frac{1}{2^{k+1}} g^{ls}\delta^{mi_1i_2\cdots i_{2k-1}i_{2k}}_{lj_1j_2\cdots j_{2k-1}j_{2k}} R_{i_1i_2}{}^{j_1j_2}\cdots R_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}} \\
&= -\frac{1}{2^{k+1}} g^{ms}\delta^{mi_1i_2\cdots i_{2k-1}i_{2k}}_{mj_1j_2\cdots j_{2k-1}j_{2k}} R_{i_1i_2}{}^{j_1j_2}\cdots R_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}} \\
&\qquad - \frac{2k}{2^{k+1}} g^{i_1s}\delta^{mi_1i_2\cdots i_{2k-1}i_{2k}}_{i_1j_1j_2\cdots j_{2k-1}j_{2k}} R_{i_1i_2}{}^{j_1j_2}\cdots R_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}} \\
&= -\tfrac{1}{2}L_k g^{ms} + kP_{(k)}^{ijs}{}_l R_{ij}{}^{ml}.
\end{aligned}$$

Hence we complete the proof of (2-4).

In the case that Σ is a hypersurface, one only needs to note that $\frac{\partial}{\partial t}g_{ij} = 2h_{ij}$ for the evolving hypersurfaces. □

The second aim of this section is to give several simple facts on the warped product manifolds. Let $M^n = [0, \bar{r}) \times_\lambda N^{n-1}$ $(0 < \bar{r} \le \infty)$ be a warped product manifold equipped with a Riemannian metric

$$\bar{g} = dr^2 + \lambda(r)^2 g_N.$$

where $\lambda : [0, \bar{r}) \to \mathbb{R}$ is a smooth positive function. Let Σ be a smooth hypersurface in $(M, \bar{g})$ with induced metric g. We denote by $\overline{\nabla}$ and ∇ the covariant derivatives with respect to $\bar{g}$ and g respectively. We define a vector field X on M by

$$X(r) = \lambda(r)\frac{\partial}{\partial r}.$$

Let $\{e_1, \dots, e_{n-1}\}$ be a local frame on Σ, it is well known that X is a conformal Killing vector field satisfying

$$\overline{\nabla}_{e_i} X(r) = \lambda'(r) e_i. \tag{2-11}$$

We denote by r the height function which is obtained by the projection of Σ in M onto the first factor $[0, \bar{r})$. Let $\phi(r)$ be a primitive of $\lambda(r)$.

Proposition 2.2. *The restriction of ϕ on Σ, still denoted by ϕ, satisfies*

$$\nabla_i \nabla_j \phi(r) = \lambda'(r) g_{ij} - \langle X, \nu\rangle h_{ij}. \tag{2-12}$$

The height function r on Σ satisfies

$$\nabla_i \nabla_j r = \frac{\lambda'(r)}{\lambda(r)} g_{ij} - \frac{\lambda'(r)}{\lambda(r)} \nabla_i r \nabla_j r - \langle \partial_r, \nu\rangle h_{ij}. \tag{2-13}$$

Consequently, we have

$$-2E^{ij}_{(k)} \nabla_i \nabla_j \phi(r) = (n-1-2k)\lambda'(r)\mathcal{H}_{2k} - \langle X, \nu\rangle \mathcal{H}_{2k+1}. \tag{2-14}$$

$$-2E^{ij}_{(k)} \nabla_i \nabla_j r = (n-1-2k)\frac{\lambda'(r)}{\lambda(r)}\mathcal{H}_{2k} + \frac{2\lambda'(r)}{\lambda(r)} E^{ij}_{(k)} \nabla_i r \nabla_j r - \langle \partial_r, \nu\rangle \mathcal{H}_{2k+1}. \tag{2-15}$$

Proof. Using (2-11), we have

$$\begin{aligned}\nabla_i \nabla_j \phi(r) &= \overline{\nabla}_i \overline{\nabla}_j \phi - \langle \overline{\nabla}\phi(r), \nu\rangle h_{ij} = \overline{\nabla}_i X_j - \langle X, \nu\rangle h_{ij} \\ &= \lambda'(r) g_{ij} - \langle X, \nu\rangle h_{ij}.\end{aligned}$$

Equation (2-13) follows from (2-12) and

$$\nabla_i \nabla_j r = \nabla_i \left(\frac{1}{\lambda(r)} \nabla_j \phi(r)\right) = \frac{1}{\lambda(r)} \nabla_i \nabla_j \phi(r) - \frac{\lambda'(r)}{\lambda(r)} \nabla_i r \nabla_j r.$$

For equations (2-14) and (2-15), we only need to notice that

$$-2E^{ij}_{(k)} g_{ij} = (n-1-2k) L_k = (n-1-2k)\mathcal{H}_{2k}$$

and

$$-2E^{ij}_{(k)} h_{ij} = \mathcal{H}_{2k+1}.$$

□

3. rigidity for the quotient $\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}$

In this section, we prove our main theorems.

Proof of Theorem 1.1. Since Σ is compact, there exist points $p_{\min}, p_{\max} \in \Sigma$ such that the height function r attains its maximum and minimum values, i.e.,

$$\min_{\Sigma} r = r(p_{\min}), \quad \max_{\Sigma} r = r(p_{\max}).$$

At these points,

$$\nabla r(p_{\min}) = \nabla r(p_{\max}) = 0, \tag{3-1}$$

$$\nabla^2 r(p_{\min}) \geq 0, \quad \nabla^2 r(p_{\max}) \leq 0. \tag{3-2}$$

It follows from (3-1) and the star-shapedness of Σ that

$$\langle \partial_r, \nu\rangle(p_{\min}) = \langle \partial_r, \nu\rangle(p_{\max}) = 1. \tag{3-3}$$

By using (3-1) and (3-3) in (2-15), we obtain

$$-2E_{(k)}^{ij}\nabla_i\nabla_j r(p_{\min}) = (n-1-2k)(\log\lambda)'(\min_{\Sigma} r)\mathcal{H}_{2k}(p_{\min}) - \mathcal{H}_{2k+1}(p_{\min}), \tag{3-4}$$

$$\begin{aligned} -2E_{(k)}^{ij}\nabla_i\nabla_j r(p_{\max}) \\ = (n-1-2k)(\log\lambda)'(\max_{\Sigma} r)\mathcal{H}_{2k}(p_{\max}) - \mathcal{H}_{2k+1}(p_{\max}). \end{aligned} \tag{3-5}$$

We claim that the quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ satisfies

$$\begin{aligned} \min_{\Sigma}\left(\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right) &\leq (n-1-2k)(\log\lambda)'(\min_{\Sigma} r), \\ (n-1-2k)(\log\lambda)'(\max_{\Sigma} r) &\leq \max_{\Sigma}\left(\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right). \end{aligned} \tag{3-6}$$

Consider first the case that $-2E_{(k)}^{ij}$ is positive semidefinite. It follows from (3-2), (3-4) and (3-5) that

$$(n-1-2k)(\log\lambda)'(\min_{\Sigma} r)\mathcal{H}_{2k}(p_{\min}) - \mathcal{H}_{2k+1}(p_{\min}) \geq 0, \tag{3-7}$$

$$(n-1-2k)(\log\lambda)'(\max_{\Sigma} r)\mathcal{H}_{2k}(p_{\max}) - \mathcal{H}_{2k+1}(p_{\max}) \leq 0. \tag{3-8}$$

From the fact that

$$-2E_{(k)}^{ij}g_{ij} = (n-1-2k)\mathcal{H}_{2k},$$

together with the assumption that $\mathcal{H}_{2k}$ is nonvanishing on Σ, we know that $\mathcal{H}_{2k} > 0$. Hence the claim in this case follows from (3-7) and (3-8) immediately. For the second case that $-2E_{(k)}^{ij}$ is negative semidefinite, similar argument applies by taking $\mathcal{H}_{2k} < 0$ into account. We finish the proof of the claim.

Now using the assumption that $\log\lambda$ is convex, we obtain from (3-6) that

$$\begin{aligned}\min_{\Sigma}\left(\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right) &\le (n-1-2k)(\log\lambda)'(\min_{\Sigma} r)\\ &\le (n-1-2k)(\log\lambda)'(\max_{\Sigma} r) \le \max_{\Sigma}\left(\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right).\end{aligned}$$

Since the quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ is constant, we have from above that

$$\frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} = (n-1-2k)(\log\lambda)'(\min_{\Sigma} r) = (n-1-2k)(\log\lambda)'(\max_{\Sigma} r), \tag{3-9}$$

which yields that $(\log\lambda)'(r)$ is a constant function on Σ. Substituting (3-9) into (2-14), we have

$$-2E^{ij}_{(k)}\nabla_i\nabla_j\phi(r) = \lambda(1-\langle\partial_r,\nu\rangle)\mathcal{H}_{2k+1}. \tag{3-10}$$

Notice that $\langle\partial_r,\nu\rangle \le 1$ and $\mathcal{H}_{2k+1} = c\mathcal{H}_{2k}$ does not change sign on Σ. Applying the classical maximum principle to the elliptic equation (3-10), we conclude that $\phi(r)$ is a constant function on Σ. Since ϕ is an increasing function with respect to r due to the fact $\phi' = \lambda > 0$, we conclude that the height function r is a constant function on Σ, i.e., Σ is a slice $\{r_0\}\times N$. □

To extend the previous result to noncompact hypersurfaces, we will apply a generalization of the Omori–Yau maximum principle for trace-type differential operators. Consider a Riemannian manifold (Σ, g) and a semi-elliptic operator $L = \operatorname{tr}_g(T\circ\nabla_g^2)$, where $T : T\Sigma \to T\Sigma$ is a positive semidefinite symmetric tensor. For simplicity we will omit the subscription g.

Definition 3.1. We say that the Omori–Yau maximum principle holds on Σ for L, if for any function $u \in C^2(\Sigma)$ with $\sup_\Sigma u < +\infty$, there exists a sequence $\{p_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that for each i, the following holds:

$$u(p_i) > \sup_{\Sigma} u - \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad Lu(p_i) < \frac{1}{i}.$$

Since $\inf_\Sigma u = -\sup_\Sigma(-u)$, the above is equivalent to that for any function $u \in C^2(\Sigma)$ with $\inf_\Sigma u > -\infty$, there exists a sequence $\{p_i\}_{i\in\mathbb{N}} \subset \Sigma$ such that for each i, the following holds:

$$u(p_i) < \inf_{\Sigma} u + \frac{1}{i}, \quad |\nabla u|(p_i) < \frac{1}{i}, \quad Lu(p_i) > -\frac{1}{i}.$$

Assume the generalized Omori–Yau maximum principle holds for the trace-type operator $L = \operatorname{tr}(-2E_{(k)}\nabla^2)$, one can prove the analogous result for noncompact hypersurfaces.

Proof of Theorem 1.2. Due to the same argument as in the proof of Theorem 1.1, we only need to prove the theorem in the case that $-2E_{(k)}^{ij}$ is positive semidefinite. By the generalized Omori–Yau maximum principle, we have two sequences $\{p_i\}$ and $\{q_i\}$ in Σ with properties

(i) $\lim_{i\to+\infty} \phi(r(p_i)) = \sup_{\Sigma} \phi(r)$, $\lim_{i\to+\infty} \phi(r(q_i)) = \inf_{\Sigma} \phi(r)$;

(ii) $|\nabla\phi(r)|(p_i) = \lambda(r(p_i))|\nabla r|(p_i) < \frac{1}{i}$, $|\nabla\phi(r)|(q_i) = \lambda(r(p_i))|\nabla r|(q_i) < \frac{1}{i}$;

(iii) $\operatorname{tr}(-2E_{(k)}\nabla^2\phi(r))(p_i) < \frac{1}{i}$, $\operatorname{tr}(-2E_{(k)}\nabla^2\phi(r))(q_i) > -\frac{1}{i}$.

Since $\phi(r)$ is strictly increasing due to $\phi'(r) = f(r) > 0$, we have

$$\lim_{i\to+\infty} r(p_i) = \sup_{\Sigma} r, \quad \lim_{i\to\infty} r(q_i) = \inf_{\Sigma} r,$$

and thus

$$\lim_{i\to+\infty} \langle \partial_r, \nu\rangle(p_i) = \lim_{i\to+\infty} \langle \partial_r, \nu\rangle(q_i) = 1.$$

Using the above facts in (2-14) and letting $i \to +\infty$, we get

$$(n-1-2k)(\log\lambda)'(\sup_{\Sigma} r) \le \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}} \le (n-1-2k)(\log\lambda)'(\inf_{\Sigma} r). \tag{3-11}$$

By the assumption that $(\log\lambda)'' \ge 0$ with equality only at isolated points, we obtain the desired result that r is constant. That is, Σ is a slice $\{r_0\} \times M$. □

In the following, we discuss some sufficient condition to guarantee the generalized Omori–Yau maximum principle to hold for Σ. Inspired by [Pigola et al. 2005], Alías et al. [2013, Theorem 1 and Corollary 3] proved that the Omori–Yau maximum principle holds for a trace-type elliptic operator $L = \operatorname{tr}(T \circ \nabla^2)$ with positive semidefinite T satisfying $\sup_{\Sigma} \operatorname{tr} T < \infty$ on a Riemannian manifold Σ, provided that the radial sectional curvature (the sectional curvature of the 2-planes containing $\nabla\rho$, where ρ is the distance function on Σ from a fixed point in Σ) of Σ satisfies the condition

$$K_{\Sigma}^{\mathrm{rad}}(\nabla\rho, \nabla\rho) > -G(\rho), \tag{3-12}$$

where $G : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying

$$G(0) > 0,\ G'(t) \ge 0,\ \int_0^{+\infty} \frac{1}{\sqrt{G(t)}} = +\infty,\ \limsup_{t\to+\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty. \tag{3-13}$$

A special case for which (3-12) holds is that the sectional curvature of Σ is bounded from below (one can choose $G(\rho) = C(1+\rho^2)$, where C is a constant).

In the case of warped product manifolds, Alías et al. gave a detailed discussion of (3-12). More precisely, they proved [ibid., Corollary 4] that for a hypersurface Σ in a slab of a warped product manifold $[r_1, r_2] \times N$, (3-12) holds for L with positive semidefinite T satisfying $\sup_\Sigma \operatorname{tr} T < \infty$, provided that the radial sectional curvature of the fiber manifold N satisfies

$$K_N^{\mathrm{rad}}(\nabla^N \hat{\rho}, \nabla^N \hat{\rho}) > -G(\hat{\rho}), \tag{3-14}$$

where $\hat{\rho}$ is the distance function on the fiber N from a fixed point in N and $G : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying the conditions listed in (3-13), together with $\sup_\Sigma \|h\|^2 < +\infty$ on Σ. Geometrically, the condition (3-14) means that the radial sectional curvature of the fiber manifold N has a strong quadratic decay at infinity, that is, one can choose $G(\rho) = C(1+\rho^2 \log^2(2+\rho))$ as shown in [Chen and Xin 1992]. In particular, when N has sectional curvature bounded from below or N is compact, (3-14) holds. As a direct result of Theorem 1.2, we have:

Corollary 3.2. *Let $(M^n, \bar{g})$ be as in Theorem 1.2. Assume that the radial sectional curvature of N satisfies (3-14). Let Σ^{n-1} be a complete, noncompact star-shaped hypersurface in M which is contained in a slab $[r_1, r_2] \times N$ with $\sup_\Sigma \|h\|^2 < +\infty$. Assume $-2E_{(k)}$ is semidefinite on Σ and $\sup_\Sigma \mathcal{H}_{2k} < \infty$ on Σ. If the quotient $\mathcal{H}_{2k+1}/\mathcal{H}_{2k}$ is constant, then the hypersurface is a slice $\{r_0\} \times N$.*

Following the argument close to the proof of Theorem 1.2, one may prove the Bernstein-type result in this case.

Proof of Theorem 1.3. By the generalized Omori–Yau maximum principle to the height function r, there exists a sequence $\{p_i\} \subset \Sigma$ such that

$$\lim_{i\to\infty} r(p_i) = \sup_\Sigma r, \quad \lim_{i\to\infty} |\nabla r|(p_i) = 0, \quad \lim_{i\to\infty} \sup \operatorname{tr}(-2E_{(k)}\nabla^2 r)(p_i) \le 0.$$

It follows from the semidefiniteness of $-2E_{(k)}$ and the positivity of $\mathcal{H}_{2k}$ that

$$0 \le \langle -2E_{(k)}\nabla r, \nabla r\rangle \le \operatorname{tr}(-2E_{(k)})|\nabla r|^2 \le (n-1-2k)C_2|\nabla r|^2.$$

From the fact $\langle \partial_r, \nu\rangle^2 = 1 - |\nabla r|^2$, we have

$$\lim_{i\to\infty} \langle \partial_r, \nu\rangle(p_i) = 1,$$

and thus

$$\lim_{i\to\infty} \langle -2E_{(k)}\nabla r, \nabla r\rangle(p_i) = 0.$$

Combining all the above facts together into (2-15), we have

$$0 \ge \lim_{i\to\infty} \sup \operatorname{tr}(-2E_{(k)}\nabla^2 r)(p_i) \ge C_1 \lim_{i\to\infty} \left((n-1-2k)\frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right)(p_i) \ge 0,$$

so that

$$\lim_{i\to\infty} \left((n-1-2k)\frac{\lambda'(r)}{\lambda(r)} - \frac{\mathcal{H}_{2k+1}}{\mathcal{H}_{2k}}\right)(p_i) = 0.$$

From the hypothesis, we have $\inf_\Sigma((n-1-2k)\lambda'(r)/\lambda(r)-\mathcal{H}_{2k+1}/\mathcal{H}_{2k})=0$, and thus $|\nabla r|\equiv 0$ on Σ, which yields that Σ is a slice $\{r_0\}\times M$ for some $r_0\in[0,\bar{r})$. □

Appendix: Kottler–Schwarzschild manifolds

The Kottler manifolds, or Kottler–Schwarzschild manifolds, are analogues of the Schwarzschild space in the setting of asymptotically locally hyperbolic manifolds. For $\kappa=1,0$ or -1, let $(N(\kappa),\hat{g})$ be a closed space form of constant sectional curvature κ. An n-dimensional Kottler–Schwarzschild manifold

$$P_{\kappa,m}=[\rho_{\kappa,m},\infty)\times N(\kappa)$$

is equipped with the metric

$$g_{\kappa,m}=\frac{d\rho^2}{V^2_{\kappa,m}(\rho)}+\rho^2\hat{g},\qquad V_{\kappa,m}=\sqrt{\rho^2+\kappa-\frac{2m}{\rho^{n-2}}}. \tag{A-1}$$

Let $\rho_0:=\rho_{\kappa,m}$ be the largest positive root of

$$\phi(\rho):=\rho^2+\kappa-\frac{2m}{\rho^{n-2}}=0.$$

Remark that in (A-1), in order to have a positive root ρ_0, if $\kappa=0$ or 1, the parameter m should be always positive; if $\kappa=-1$, the parameter m can be negative. In fact, in this case, $m\in[m_c,+\infty)$ and

$$m_c=-\frac{(n-2)^{(n-2)/2}}{n^{n/2}}.$$

Here the certain critical value m_c comes from the following. If $m\le 0$, one can solve the equation

$$\phi'(\rho)=2\rho+(n-2)\frac{2m}{\rho^{n-1}}=0,$$

to get the root $\rho_1=(-(n-2)m)^{1/n}$. Note the fact that $\phi(\rho_1)\le 0$, which yields

$$m\ge-\frac{(n-2)^{(n-2)/2}}{n^{n/2}}.$$

By a change of variable $r=r(\rho)$ with

$$r'(\rho)=\frac{1}{V_{\kappa,m}(\rho)},\qquad r(\rho_{\kappa,m})=0,$$

we can rewrite $P_{\kappa,m}$ as a warped product manifold $P_{\kappa,m}=[0,\infty)\times_{\lambda_\kappa}N(\kappa)$ equipped with the metric

$$g_{\kappa,m}:=\bar{g}:=dr^2+\lambda_\kappa(r)^2\hat{g},$$

where $\lambda_\kappa:[0,\infty)\to[\rho_{\kappa,m},\infty)$ is the inverse of $r(\rho)$, i.e., $\lambda_\kappa(r(\rho))=\rho$.

It is easy to check

$$\lambda_\kappa'(r) = V_{\kappa,m}(\rho) = \sqrt{\kappa + \lambda_\kappa(r)^2 - 2m\lambda_\kappa(r)^{2-n}},$$
$$\lambda_\kappa''(r) = \lambda_\kappa(r) + (n-2)m\lambda_\kappa(r)^{1-n}.$$

Hence

$$\lambda_\kappa\lambda_\kappa'' - (\lambda_\kappa')^2 = -\kappa + nm\lambda_\kappa^{2-n}.$$

For the case $\kappa = 0$, $m \geq 0$ and hence $\lambda_\kappa\lambda_\kappa'' - (\lambda_\kappa')^2 = nm\lambda_\kappa^{2-n} \geq 0$. For the case $\kappa = -1$, if $m \geq 0$, then $\lambda_\kappa\lambda_\kappa'' - (\lambda_\kappa')^2 = 1 + nm\lambda_\kappa^{2-n} > 0$. If

$$m \in \left[-\frac{(n-2)^{(n-2)/2}}{n^{n/2}}, 0\right),$$

then

$$\begin{aligned}\lambda_\kappa\lambda_\kappa'' - (\lambda_\kappa')^2 &= 1 + nm\lambda_\kappa^{2-n} \geq 1 + nm\rho_0^{2-n} \geq 1 + nm\rho_1^{2-n} \\ &= 1 + nm(-(n-2)m)^{(2-n)/n} = 1 - n(n-2)^{(2-n)/n}(-m)^{2/n} \\ &\geq 1 - n(n-2)^{(2-n)/n}\left(\frac{(n-2)^{(n-2)/2}}{n^{n/2}}\right)^{2/n} = 0.\end{aligned}$$

As a conclusion, the condition on the log convexity of λ holds for the Kottler–Schwarzschild manifolds with $\kappa = 0$ and -1. We remark that the log convexity of λ does not hold for the Kottler–Schwarzschild manifolds when $\kappa = 1$.

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THE FIRST TERMS IN THE EXPANSION OF THE BERGMAN KERNEL IN HIGHER DEGREES

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We establish the cancellation of the first $2j$ terms in the diagonal asymptotic expansion of the restriction to the $(0, 2j)$-forms of the Bergman kernel associated to the spin^c Dirac operator on high tensor powers of a positive line bundle twisted by a (not necessarily holomorphic) complex vector bundle, over a compact Kähler manifold. Moreover, we give a local formula for the first and the second (nonzero) leading coefficients, as well as for the third assuming that the first two vanish.

Introduction

The Bergman kernel of a Kähler manifold endowed with a positive line bundle L is the smooth kernel of the orthogonal projection on the kernel of the Kodaira Laplacian $\square^L = \bar{\partial}^L \bar{\partial}^{L,*} + \bar{\partial}^{L,*} \bar{\partial}^L$. The existence of a diagonal asymptotic expansion of the Bergman kernel associated with the p-th tensor power of L when $p \to +\infty$ and the form of the leading term were proved in [Tian 1990; Zelditch 1998; Catlin 1999]. Moreover, the coefficients in this expansion encode geometric information about the underlying manifold, and therefore they have been studied closely: the second and third terms were computed by Lu [2000], X. Wang [2005], L. Wang [2003] and Ma and Marinescu [2012] in different degrees of generality (see also the recent paper [Xu 2012]). This asymptotic analysis plays an important role in various problems of Kähler geometry; see, for instance, [Donaldson 2001; Fine 2012]. We refer the reader to [Ma and Marinescu 2007] (henceforth abbreviated [MM]) for a comprehensive study of the Bergman kernel and its applications. See also the survey [Ma 2011].

In fact, Dai, Liu and Ma [Dai et al. 2006] established the asymptotic development of the Bergman kernel in the symplectic case, using the heat kernel (see also [Ma and Marinescu 2006]). In [Charbonneau and Stern 2011], these asymptotics in the symplectic case have found an application in the study of the variation of Hodge structures of vector bundles. In that setting, the Bergman kernel is the kernel of a Kodaira-like Laplacian on a twisted bundle $L \otimes E$, where E is a (not

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necessarily holomorphic) complex vector bundle. Because of that, the Bergman kernel is no longer supported in degree 0 (unlike it did in the Kähler case), and the asymptotic development of its restriction to the $(0,2j)$-forms is related to the degree of 'nonholomorphicity' of E.

In this paper, we will show that the leading term in the asymptotics of the restriction to the $(0,2j)$-forms of the Bergman kernel is of order $p^{\dim X-2j}$ and we will compute it. That will lead to a local version of [Charbonneau and Stern 2011, Equation (1.3)], which is the main technical result of their paper; see Remark 0.6. After that, we will also compute the second term in the asymptotics, as well as the third term in the case where the first two vanish.

We now give more detail about our results. Let (X,ω,J) be a compact Kähler manifold of complex dimension n. Let (L,h^L) be a holomorphic Hermitian line bundle on X, and (E,h^E) a Hermitian complex vector bundle. We endow (L,h^L) with its Chern (i.e., holomorphic and Hermitian) connection ∇^L, and (E,h^E) with a Hermitian connection ∇^E, whose curvatures are $R^L=(\nabla^L)^2$ and $R^E=(\nabla^E)^2$.

Except in the beginning of Section 1A, we will always assume that (L,h^L,∇^L) satisfies the *prequantization condition*

$$\omega=\frac{\sqrt{-1}}{2\pi}R^L. \tag{0-1}$$

Let $g^{TX}(\cdot,\cdot)=\omega(\cdot,J\cdot)$ be the Riemannian metric on TX induced by ω and J. It induces a metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X):=\Lambda^\bullet(T^{*(0,1)}X)$; see Section 1A.

Let $L^p=L^{\otimes p}$ be the p-th tensor power of L. Let

$$\Omega^{0,\bullet}(X,L^p\otimes E)=\mathscr{C}^\infty(X,\Lambda^{0,\bullet}(T^*X)\otimes L^p\otimes E) \tag{0-2}$$

and $\bar\partial^{L^p\otimes E}:\Omega^{0,\bullet}(X,L^p\otimes E)\to\Omega^{0,\bullet+1}(X,L^p\otimes E)$ be the Dolbeault operator induced by the $(0,1)$-part of ∇^E (see (1-3)). Let $\bar\partial^{L^p\otimes E,*}$ be its dual with respect to the L^2-product. We set (see (1-6))

$$D_p=\sqrt{2}(\bar\partial^{L^p\otimes E}+\bar\partial^{L^p\otimes E,*}), \tag{0-3}$$

which exchanges odd and even forms.

Definition 0.1. Let

$$P_p\colon\Omega^{0,\bullet}(X,L^p\otimes E)\to\ker(D_p) \tag{0-4}$$

be the orthogonal projection onto the kernel of D_p. The operator P_p is called the *Bergman projection*. It has a smooth kernel with respect to $dv_X(y)$, denoted by $P_p(x,y)$, which is called the *Bergman kernel*.

Remark 0.2. If E is holomorphic, then by Hodge theory and the Kodaira vanishing theorem (see respectively [MM, Theorems 1.4.1 and 1.5.6]), we know that, for p

large enough, P_p is the orthogonal projection $\mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$. Here, by [Ma and Marinescu 2002, Theorem 1.1], we just know that

$$\ker(D_p|_{\Omega^{0,\mathrm{odd}}(X,L^p\otimes E)}) = 0 \tag{0-5}$$

for p large, so that $P_p \colon \Omega^{0,\mathrm{even}}(X, L^p \otimes E) \to \ker(D_p)$. In particular, $P_p(x, x) \in \mathscr{C}^\infty(X, \mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E))$.

By Theorem 1.3, D_p is a Dirac operator, which enables us to apply this result:

Theorem 0.3 [Dai et al. 2006, Theorem 1.1]. *There exist*

$$\boldsymbol{b}_r \in \mathscr{C}^\infty(X, \mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E)) \tag{0-6}$$

such that for any $k \in \mathbb{N}$ and $p \to +\infty$,

$$p^{-n} P_p(x, x) = \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} + O(p^{-k-1}), \tag{0-7}$$

that is, for every $k, l \in \mathbb{N}$, there exists a constant $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,

$$\left| p^{-n} P_p(x, x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} \right|_{\mathscr{C}^l(X)} \leq C_{k,l}\, p^{-k-1}. \tag{0-8}$$

Here $|\cdot|_{\mathscr{C}^l(X)}$ is the $\mathscr{C}^l$-norm for the variable $x \in X$.

To simplify the formulas, we denote by

$$\mathscr{R} = (R^E)^{0,2} \in \Omega^{0,2}(X, \mathrm{End}(E)) \tag{0-9}$$

the $(0, 2)$-part of R^E (which is zero if E is holomorphic). For $j \in [\![1, n]\!]$, let

$$I_j \colon \Lambda^{0,\bullet}(T^*X) \otimes E \to \Lambda^{0,j}(T^*X) \otimes E \tag{0-10}$$

be the natural orthogonal projection. The first main result in this paper is:

Theorem 0.4. *For any $k \in \mathbb{N}$, $k \geq 2j$, we have when $p \to +\infty$,*

$$p^{-n} I_{2j} P_p(x, x) I_{2j} = \sum_{r=2j}^{k} I_{2j} \boldsymbol{b}_r(x) I_{2j}\, p^{-r} + O(p^{-k-1}), \tag{0-11}$$

and moreover,

$$I_{2j} \boldsymbol{b}_{2j}(x) I_{2j} = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} I_{2j} (\mathscr{R}_x^j)(\mathscr{R}_x^j)^* I_{2j}, \tag{0-12}$$

where $(\mathscr{R}_x^j)^*$ is the dual of $\mathscr{R}_x^j$ acting on $(\Lambda^{0,\bullet}(T^*X) \otimes E)_x$.

Theorem 0.4 leads immediately to:

Corollary 0.5. *Uniformly in $x \in X$, when $p \to +\infty$, we have*

$$\mathrm{Tr}((I_{2j} P_p I_{2j})(x,x)) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \|\mathscr{R}_x^j\|^2 p^{n-2j} + O(p^{n-2j-1}). \tag{0-13}$$

Remark 0.6. By integrating (0-13) over X, we get

$$\mathrm{Tr}(I_{2j} P_p I_{2j}) = \frac{1}{(4\pi)^{2j}} \frac{1}{2^{2j}(j!)^2} \|\mathscr{R}^j\|_{L^2}^2 p^{n-2j} + O(p^{n-2j-1}), \tag{0-14}$$

which is the main technical result of [Charbonneau and Stern 2011, Equation (1.3)]; thus Corollary 0.5 can be viewed as a local version of it. The constant in (0-14) differs from the one in [ibid.] because our conventions are not the same as theirs (e.g., they chose $\omega = \sqrt{-1}R^L$).

Let $R_\Lambda^E := -\sqrt{-1}\sum_i R^E(w_i, \bar{w}_i)$ for $(\bar{w}_1, \dots, \bar{w}_n)$ an orthonormal frame of $T^{(0,1)}X$. Let R^{TX} be the curvature of the Levi-Civita connection ∇^{TX} of (X, g^{TX}), and for $(e_1, \dots, e_{2n})$ an orthonormal frame of TX, let $r^X = -\sum_{i,j} \langle R^{TX}(e_i, e_j)e_i, e_j\rangle$ be the scalar curvature of X.

For $j, k \in \mathbb{N}$ and $j \geq k$, we also define $\mathrm{C}_j(k)$ by

$$\mathrm{C}_j(k) := \frac{1}{(4\pi)^j} \frac{1}{2^k k!} \frac{1}{\prod_{s=k+1}^j (2s+1)}, \tag{0-15}$$

with the convention that $\prod_{s\in\varnothing} = 1$.

Let $\nabla^{\Lambda^{0,\bullet}}$ be the connection on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX}. Let $\nabla^{\Lambda^{0,\bullet}\otimes E}$ be the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by ∇^E and $\nabla^{\Lambda^{0,\bullet}}$, and let $\Delta^{\Lambda^{0,\bullet}\otimes E}$ be the associated Laplacian. For precise definitions, see Section 1A.

For every operator A acting on a Hermitian space, we define the positive (not necessarily definite) operator and the symmetric operator associated to A as

$$\mathrm{Pos}[A] = AA^* \quad \text{and} \quad \mathrm{Sym}[A] = A + A^*. \tag{0-16}$$

Finally, to simplify the notation, we define $\mathcal{T}_0(j)$, $\mathcal{T}_1(j)$, $\mathcal{T}_2(j)$ and $\mathcal{T}_3(j)$ as:

- $\mathcal{T}_0(0) = 0$, and for $j \geq 1$,

$$\mathcal{T}_0(j) = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{n} \sum_{k=0}^{j-1} I_{2j}(\mathrm{C}_j(j) - \mathrm{C}_j(k)) \mathscr{R}_x^{j-k-1} (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes E} \mathscr{R}_\cdot)(x) \mathscr{R}_x^k I_0. \tag{0-17}$$

- $\mathcal{T}_1(0)=\mathcal{T}_1(1)=0$, and for $j\geq 2$,

$$\mathcal{T}_1(j)=\frac{I_{2j}}{2\pi}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\Big\{(\mathrm{C}_j(j)-\mathrm{C}_j(q+1)) \times \mathscr{R}_x^{j-(q+2)}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{q-m}(\nabla_{w_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{m} + \mathrm{C}_j(m)\Big[\prod_{s=q+2}^{j}\Big(1+\frac{1}{2s}\Big)-1\Big] \times \mathscr{R}_x^{j-(q+2)}(\nabla_{w_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{q-m}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{m}\Big\}I_0, \tag{0-18}$$

- $\mathcal{T}_2(0)=0$, and for $j\geq 1$,

$$\mathcal{T}_2(j)=\frac{1}{4\pi}I_{2j}\sum_{k=0}^{j-1}\big\{(\mathrm{C}_j(k)-\mathrm{C}_j(j))\mathscr{R}_x^{j-(k+1)}(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^{k}\big\}I_0, \tag{0-19}$$

- for $j\geq 0$,

$$\mathcal{T}_3(j)=I_{2j}\sum_{k=0}^{j}\mathscr{R}_x^{j-k}\Big[\frac{1}{6}\Big(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\Big)r_x^X - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x}\Big]\mathscr{R}_x^{k}I_0. \tag{0-20}$$

Our second goal is to compute the second term in the expansion (0-11).

Theorem 0.7. *We can decompose $I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j}$ as the sum of four terms*:

$$I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j} = \mathrm{Pos}[\mathcal{T}_0(j)]+\mathrm{C}_j(j)\,\mathrm{Sym}[(\mathcal{T}_1(j)+\mathcal{T}_2(j)+\mathcal{T}_3(j))(\mathscr{R}_x^{j})^*I_{2j}]. \tag{0-21}$$

For instance, for $j=1$, using the fact that $(R^E_\Lambda)^*=R^E_\Lambda$, we find

$$128\pi^3 I_2\boldsymbol{b}_3(x)I_2 = \frac{1}{9}\,\mathrm{Pos}\Big[I_2\sum_{i=0}^{n}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)I_0\Big]-\frac{1}{6}\,\mathrm{Sym}[I_2(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x)\mathscr{R}_x^*I_2] -\frac{\sqrt{-1}}{6}I_2(R^E_\Lambda\mathscr{R}_x\mathscr{R}_x^*+\mathscr{R}_x\mathscr{R}_x^*R^E_\Lambda)I_2-\frac{2\sqrt{-1}}{3}I_2\mathscr{R}_xR^E_\Lambda\mathscr{R}_x^*I_2-\frac{r_x^X}{4}I_2\mathscr{R}_x\mathscr{R}_x^*I_2. \tag{0-22}$$

The last goal of this paper is to compute the third term in the expansion (0-11), assuming that the first two vanish.

Theorem 0.8. *Let $j\in[\![1,n]\!]$. If*

$$I_{2j}\boldsymbol{b}_{2j}(x)I_{2j}=I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j}=0, \tag{0-23}$$

then $\mathcal{T}_3$ equals

$$\mathcal{T}_3'(j) := -\sqrt{-1} I_{2j} \sum_{k=0}^{j} \frac{\mathrm{C}_j(k)}{4\pi(2k+1)} \mathcal{R}_x^{j-k} R^E_{\Lambda,x} \mathcal{R}_x^k I_0, \tag{0-24}$$

and

$$I_{2j} \boldsymbol{b}_{2j+2}(x) I_{2j} = \mathrm{Pos}[\mathcal{T}_1(j) + \mathcal{T}_2(j) + \mathcal{T}_3'(j)]. \tag{0-25}$$

Theorems 0.4, 0.7 and 0.8 yield to:

Corollary 0.9. *We have*

$$I_{2j} P_p(x,x) I_{2j} = O(p^{n-2j-3}) \iff \begin{cases} \mathcal{R}_x^j = 0, \\ \mathcal{T}_0(j) = 0, \\ \mathcal{T}_1(j) + \mathcal{T}_2(j) + \mathcal{T}_3'(j) = 0. \end{cases} \tag{0-26}$$

This paper is organized as follows. In Section 1 we compute the square of D_p and use a local trivialization to rescale it, and then give the Taylor expansion of the rescaled operator. In Section 2, we use this expansion to give a formula for the coefficients $\boldsymbol{b}_r$ appearing in (0-7), which will lead to a proof of Theorem 0.4. In Section 3, we prove Theorem 0.7 using the formula for $\boldsymbol{b}_r$. Finally, in Section 4, we prove Theorem 0.8 using the techniques and results of the preceding sections.

In this whole paper, when an index variable appears twice in a single term, it means that we are summing over all its possible values.

1. Rescaling D_p^2 and Taylor expansion

In this section, we follow the method of [MM, Chapter 4] that enables to prove the existence of $\boldsymbol{b}_r$ in (0-7) in the case of a holomorphic vector bundle E, and that still applies here (as pointed out in [MM, Section 8.1.1]). Then, in Sections 2 and 3, we will use this approach to understand $I_{2j} \boldsymbol{b}_r I_{2j}$ and prove Theorems 0.4 and 0.7.

In Section 1A, we will first prove Theorem 1.3, and then give a formula for the square of D_p, which will be the starting point of our approach.

In Section 1B, we will rescale the operator D_p^2 to get an operator $\mathscr{L}_t$, and then give the Taylor expansion of the rescaled operator.

In Section 1C, we will study more precisely the limit operator $\mathscr{L}_0$.

1A. ***The square of D_p.*** For further details on the material of this subsection, the lector can read [MM]. First of all let us give some notation.

The Riemannian volume form of (X, g^{TX}) is given by $dv_X = \omega^n/n!$. We will denote by $\langle \cdot, \cdot \rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes \mathbb{C}$ induced by g^{TX}.

For the rest of Section 1A, we will fix $(w_1, \dots, w_n)$ a local orthonormal frame of $T^{(1,0)}X$ with dual frame $(w^1, \dots, w^n)$. Then $(\bar{w}_1, \dots, \bar{w}_n)$ is a local orthonormal frame of $T^{(0,1)}X$ whose dual frame is denoted by $(\bar{w}^1, \dots, \bar{w}^n)$, and the vectors

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \quad \text{and} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j) \tag{1-1}$$

form a local orthonormal frame of TX.

We choose the Hermitian metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$ such that $\{\bar{w}^{j_1} \wedge \cdots \wedge \bar{w}^{j_k} : 1 \leq j_1 < \cdots < j_k \leq n\}$ is an orthonormal frame of $\Lambda^{0,\bullet}(T^*X)$.

For any Hermitian bundle (F, h^F) over X, let $\mathscr{C}^\infty(X, F)$ be the space of smooth sections of F. It is endowed with the L^2-Hermitian metric

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{h^F} \, dv_x(x). \tag{1-2}$$

The corresponding norm will be denoted by $\|\cdot\|_{L^2}$, and the completion of $\mathscr{C}^\infty(X, F)$ with respect to this norm by $L^2(X, F)$.

Let $\bar\partial^E$ be the Dolbeault operator of E. It is the (0, 1)-part of the connection ∇^E

$$\bar\partial^E := (\nabla^E)^{0,1} \colon \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, T^{*(0,1)}X \otimes E). \tag{1-3}$$

We extend it to get an operator

$$\bar\partial^E \colon \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet+1}(X, E) \tag{1-4}$$

by the Leibniz formula: for $s \in \mathscr{C}^\infty(X, E)$ and $\alpha \in \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X))$ homogeneous,

$$\bar\partial^E(\alpha \otimes s) = (\bar\partial\alpha) \otimes s + (-1)^{\deg\alpha} \alpha \otimes \bar\partial^E s. \tag{1-5}$$

We can now define the operator

$$D^E = \sqrt{2}(\bar\partial^E + \bar\partial^{E,*}) \colon \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet}(X, E), \tag{1-6}$$

where the dual is taken with respect to the L^2-norm associated with the Hermitian metrics $h^{\Lambda^{0,\bullet}}$ and h^E.

Let $\nabla^{\Lambda(T^*X)}$ be the connection on $\Lambda(T^*X)$ induced by the Levi-Civita connection ∇^{TX} of X. Since X is Kähler, ∇^{TX} preserves $T^{(0,1)}X$ and $T^{(1,0)}X$. Thus, it induces a connection $\nabla^{T^{*(0,1)}X}$ on $T^{*(0,1)}X$, and then a Hermitian connection $\nabla^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X)$. We then have that for any $\alpha \in \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X))$,

$$\nabla^{\Lambda^{0,\bullet}}\alpha = \nabla^{\Lambda(T^*X)}\alpha. \tag{1-7}$$

Note the important fact that $\nabla^{\Lambda(T^*X)}$ preserves the bi-grading on $\Lambda^{\bullet,\bullet}(T^*X)$.

Let $\nabla^{\Lambda^{0,\bullet}\otimes E} := \nabla^{\Lambda^{0,\bullet}} \otimes 1 + 1 \otimes \nabla^E$ be the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^E.

Proposition 1.1. *On* $\Omega^{0,\bullet}(X, E)$, *we have*

$$\bar\partial^E = \bar w^j \wedge \nabla^{\Lambda^{0,\bullet}\otimes E}_{\bar w_j}, \qquad \bar\partial^{E,*} = -i_{\bar w_j}\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_j}. \tag{1-8}$$

Proof. We still denote by ∇^E the extension of the connection ∇^E to $\Omega^{\bullet,\bullet}(X, E)$ by the usual formula $\nabla^E(\alpha\otimes s) = d\alpha\otimes s + (-1)^{\deg\alpha}\alpha\wedge\nabla^E s$ for $s \in \mathscr{C}^\infty(X, E)$ and $\alpha \in \mathscr{C}^\infty(X, \Lambda(T^*X))$ homogeneous. We know that $d = \varepsilon\circ\nabla^{\Lambda(T^*X)}$ where ε is the exterior multiplication (see [MM, Equation (1.2.44)]), so we get that $\nabla^E = \varepsilon\circ\nabla^{\Lambda(T^*X)\otimes E}$. Using (1-7), it follows that

$$\bar\partial^E = (\nabla^E)^{0,1} = \bar w^j \wedge \nabla^{\Lambda^{0,\bullet}\otimes E}_{\bar w_j},$$

which is the first part of (1-8).

The second part of our proposition follows classically from the first by exactly the same computation as in [MM, Lemma 1.4.4]. □

Definition 1.2. Let $v = v^{1,0}+v^{0,1} \in TX = T^{(1,0)}X\oplus T^{(0,1)}X$, and $\bar v^{(0,1),*} \in T^{*(0,1)}X$ the dual of $v^{1,0}$ for $\langle\cdot,\cdot\rangle$. We define the *Clifford action of* TX *on* $\Lambda^{0,\bullet}(T^*X)$ by

$$c(v) = \sqrt{2}(\bar v^{(0,1),*}\wedge -i_{v^{0,1}}). \tag{1-9}$$

We verify easily that for $u, v \in TX$,

$$c(u)c(v)+c(v)c(u) = -2\langle u, v\rangle, \tag{1-10}$$

and that for any skew-adjoint endomorphism A of TX,

$$\tfrac14\langle Ae_i, e_j\rangle c(e_i)c(e_j) = -\tfrac12\langle Aw_j, \bar w_j\rangle + \langle Aw_\ell, \bar w_m\rangle \bar w^m\wedge i_{\bar w_\ell} + \tfrac12\langle Aw_\ell, w_m\rangle i_{\bar w_\ell} i_{\bar w_m} + \tfrac12\langle A\bar w_\ell, \bar w_m\rangle \bar w^\ell\wedge\bar w^m\wedge. \tag{1-11}$$

Let $\nabla^{\det}$ be the Chern connection of $\det(T^{(1,0)}X) := \Lambda^n(T^{(1,0)}X)$, and ∇^{Cl} the Clifford connection on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX} and $\nabla^{\det}$ (see [MM, Equation (1.3.5)]). We also denote by ∇^{Cl} the connection on $\Lambda^{0,\bullet}(T^*X)\otimes E$ induced by ∇^{Cl} and ∇^E. By [loc. cit.], (1-11) and the fact that $\nabla^{\det}$ is holomorphic, we get

$$\nabla^{\mathrm{Cl}} = \nabla^{\Lambda^{0,\bullet}}. \tag{1-12}$$

Let $D^{c,E}$ be the associated spin^c Dirac operator

$$D^{c,E} = \sum_{j=1}^{2n} c(e_j)\nabla^{\mathrm{Cl}}_{e_j}\colon \Omega^{0,\bullet}(X, E)\to\Omega^{0,\bullet}(X, E). \tag{1-13}$$

By (1-8) and (1-12), we have:

Theorem 1.3. D^E *is equal to the spin*^{c} *Dirac operator* $D^{c,E}$ *acting on* $\Omega^{0,\bullet}(X, E)$.

Remark 1.4. Note that all the results proved in the beginning of this subsection hold without assuming the prequantization condition (0-1), but from now on we will use it.

Let (F, h^F) be a Hermitian vector bundle on X and let ∇^F be a Hermitian connection on F. Then the *Bochner Laplacian* Δ^F acting on $\mathscr{C}^\infty(X, F)$ is defined by

$$\Delta^F = -\sum_{j=1}^{2n}((\nabla^F_{e_j})^2 - \nabla^F_{\nabla^{TX}_{e_j} e_j}). \tag{1-14}$$

On $\Omega^{0,\bullet}(X)$, we define the *number operator* $\mathcal{N}$ by

$$\mathcal{N}|_{\Omega^{0,j}(X)} = j, \tag{1-15}$$

and we also denote by $\mathcal{N}$ the operator $\mathcal{N} \otimes 1$ acting on $\Omega^{0,\bullet}(X, F)$.

The bundle L^p is endowed with the connection ∇^{L^p} induced by ∇^L (which is also its Chern connection). Let $\nabla^{L^p\otimes E} := \nabla^{L^p} \otimes 1 + 1 \otimes \nabla^E$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. We will denote

$$D_p = D^{L^p\otimes E}. \tag{1-16}$$

Theorem 1.5. *The square of D_p is given by*

$$\begin{aligned} D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} &- R^E(w_j, \overline{w}_j) - 2\pi pn + 4\pi p\mathcal{N} \\ &+ 2(R^E + \tfrac{1}{2}R^{\det})(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell} + R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} \\ &+ R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m. \end{aligned} \tag{1-17}$$

Proof. By Theorem 1.3, we can use [MM, Theorem 1.3.5]

$$D_p^2 = \Delta^{\mathrm{Cl}} + \tfrac{1}{4}r^X + \tfrac{1}{2}(R^{L^p\otimes E} + \tfrac{1}{2}R^{\det})(e_i, e_j)c(e_i)c(e_j), \tag{1-18}$$

where r^X is the scalar curvature of X. From (1-12), we see that $\Delta^{\mathrm{Cl}} = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E}$. Moreover, $r^X = 2R^{\det}(w_j, \overline{w}_j)$ and $R^{L^p\otimes E} = R^E + pR^L$. Using the equivalent of (1-11) for 2-forms (substituting $A(\cdot\,,\cdot)$ for $\langle A\cdot\,,\cdot\rangle$) and the fact that R^L and $R^{\det}$ are (1, 1)-forms, (1-18) reads

$$\begin{aligned} D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} &+ \tfrac{1}{2}R^{\det}(w_j, \overline{w}_j) - (R^E(w_j, \overline{w}_j) + pR^L(w_j, \overline{w}_j) + \tfrac{1}{2}R^{\det}(w_j, \overline{w}_j)) \\ &+ 2(R^E + pR^L + \tfrac{1}{2}R^{\det})(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell} + R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} \\ &+ R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m. \end{aligned}$$

Thanks to (0-1), we have $R^L(w_\ell, \overline{w}_m) = 2\pi\delta_{\ell m}$. Moreover, $\mathcal{N} = \sum_\ell \overline{w}^\ell \wedge i_{\overline{w}_\ell}$, thus we get Theorem 1.5. □

1B. ***Rescaling D_p^2.*** In this subsection, we rescale D_p^2, but to do this we must define it on a vector space. Therefore, we will use normal coordinates to transfer the problem on the tangent space to X at a fixed point. Then we give a Taylor expansion of the rescaled operator, but the problem is that each operator acts on a different space, namely,

$$\mathrm{E}_p := \Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes E,$$

so we must first handle this issue.

Fix $x_0 \in X$. For the rest of this paper, we fix $\{w_j\}$ an orthonormal basis of $T_{x_0}^{(1,0)}X$, with dual basis $\{w^j\}$, and we construct an orthonormal basis $\{e_i\}$ of $T_{x_0}X$ from $\{w_j\}$ as in (1-1).

For $\varepsilon > 0$, we denote by $B^X(x_0, \varepsilon)$ and $B^{T_{x_0}X}(0, \varepsilon)$ the open balls in X and $T_{x_0}X$ with center x_0 and 0 and radius ε. If $\exp^X_{x_0}$ is the Riemannian exponential of X, then for ε small enough, $Z \in B^{T_{x_0}X}(0, \varepsilon) \mapsto \exp^X_{x_0}(Z) \in B^X(x_0, \varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}X$ with $\mathbb{R}^{2n}$ via the orthonormal basis $\{e_i\}$:

$$(Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_i Z_i e_i \in T_{x_0}X. \tag{1-19}$$

From now on, we will always identify $B^{T_{x_0}X}(0, \varepsilon)$ and $B^X(x_0, \varepsilon)$. Note that in this identification, the radial vector field $\mathcal{R} = \sum_i Z_i e_i$ becomes $\mathcal{R} = Z$, so Z can be viewed as a point or as a tangent vector.

For $Z \in B^{T_{x_0}X}(0, \varepsilon)$, we identify (L_Z, h^L_Z), (E_Z, h^E_Z) and $(\Lambda^{0,\bullet}_Z(T^*X), h^{\Lambda^{0,\bullet}}_Z)$ with $(L_{x_0}, h^L_{x_0})$, $(E_{x_0}, h^E_{x_0})$ and $(\Lambda^{0,\bullet}(T^*_{x_0}X), h^{\Lambda^{0,\bullet}}_{x_0})$ by parallel transport with respect to the connection ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$ along the geodesic ray $t \in [0,1] \mapsto tZ$. We denote by Γ^L, Γ^E and $\Gamma^{\Lambda^{0,\bullet}}$ the corresponding connection forms of ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$.

Remark 1.6. As $\nabla^{\Lambda^{0,\bullet}}$ preserves the degree, the identification between $\Lambda^{0,\bullet}(T^*X)$ and $\Lambda^{0,\bullet}(T^*_{x_0}X)$ is compatible with the degree. Thus, $\Gamma^{\Lambda^{0,\bullet}}_Z \in \bigoplus_j \operatorname{End}(\Lambda^{0,j}(T^*X))$.

Let S_L be a unit vector of L_{x_0}. It gives an isometry $L^p_{x_0} \simeq \mathbb{C}$, which induces an isometry

$$\mathrm{E}_{p,x_0} \simeq (\Lambda^{0,\bullet}(T^*X) \otimes E)_{x_0} =: \mathbb{E}_{x_0}. \tag{1-20}$$

Thus, in our trivialization, D_p^2 acts on $\mathbb{E}_{x_0}$, but this action may *a priori* depend on the choice of S_L. In fact, since the operator D_p^2 takes values in $\operatorname{End}(\mathrm{E}_{p,x_0})$ which is canonically isomorphic to $\operatorname{End}(\mathbb{E})_{x_0}$ (by the natural identification $\operatorname{End}(L^p) \simeq \mathbb{C}$), all our formulas do not depend on this choice.

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$, and let $\kappa(Z)$ be the smooth positive function defined for $|Z| \le \varepsilon$ by

$$dv_X(Z) = \kappa(Z)\, dv_{TX}(Z), \tag{1-21}$$

with $\kappa(0)=1$.

Definition 1.7. We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_0}X$. For $s\in\mathscr{C}^\infty(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ and $t=1/\sqrt{p}$, set

$$\begin{aligned}(S_t s)(Z)&=s(Z/t),\\ \nabla_t&=tS_t^{-1}\kappa^{1/2}\nabla^{\mathrm{Cl}_0}\kappa^{-1/2}S_t,\\ \nabla_0&=\nabla+\tfrac{1}{2}R^L_{x_0}(Z,\cdot),\\ \mathscr{L}_t&=t^2S_t^{-1}\kappa^{1/2}D_p^2\kappa^{-1/2}S_t,\\ \mathscr{L}_0&=-\sum_i(\nabla_{0,e_i})^2+4\pi\mathcal{N}-2\pi n.\end{aligned}\tag{1-22}$$

Let $\|\cdot\|_{L^2}$ be the L^2-norm induced by $h^{\mathbb{E}_{x_0}}$ and dv_{TX}. We can now state the key result in our approach to Theorems 0.4 and 0.7:

Theorem 1.8. *There exist second-order formally self-adjoint* (*with respect to* $\|\cdot\|_{L^2}$) *differential operators* $\mathcal{O}_r$ *with polynomial coefficients such that for all* $m\in\mathbb{N}$,

$$\mathscr{L}_t=\mathscr{L}_0+\sum_{r=1}^m t^r\mathcal{O}_r+O(t^{m+1}).\tag{1-23}$$

Furthermore, *each* $\mathcal{O}_r$ *can be decomposed as*

$$\mathcal{O}_r=\mathcal{O}_r^0+\mathcal{O}_r^{+2}+\mathcal{O}_r^{-2},\tag{1-24}$$

where $\mathcal{O}_r^k$ *changes the degree of the form it acts on by* k.

Proof. The first part of the theorem (i.e., (1-23)) is contained in [Ma and Marinescu 2008, Theorem 1.4]. We will briefly recall how they obtained this result.

Let Φ_E be the smooth self-adjoint section of $\mathrm{End}(\mathbb{E}_{x_0})$ on $B^{T_{x_0}X}(0,\varepsilon)$:

$$\begin{aligned}\Phi_E=-R^E(w_j,\overline{w}_j)+2(R^E+\tfrac{1}{2}R^{\det})(w_\ell,\overline{w}_m)\overline{w}^m\wedge i_{\overline{w}_\ell}\\ +R^E(w_\ell,w_m)i_{\overline{w}_\ell}i_{\overline{w}_m}+R^E(\overline{w}_\ell,\overline{w}_m)\overline{w}^\ell\wedge\overline{w}^m.\end{aligned}\tag{1-25}$$

We can see that we can decompose Φ_E as $\Phi_E^0+\Phi_E^{+2}+\Phi_E^{-2}$, where

(1-26)

$\Phi_E^0=R^E(w_j,\overline{w}_j)+2(R^E+\frac{1}{2}R^{\det})(w_\ell,\overline{w}_m)\overline{w}^m\wedge i_{\overline{w}_\ell}$ preserves the degree,

$\Phi_E^{+2}=R^E(\overline{w}_\ell,\overline{w}_m)\overline{w}^\ell\wedge\overline{w}^m$ rises the degree by 2,

$\Phi_E^{-2}=R^E(w_\ell,w_m)i_{\overline{w}_\ell}i_{\overline{w}_m}$ lowers the degree by 2.

Using Theorem 1.5, we find that

$$D_p^2=\Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E}+p(-2\pi n+4\pi\mathcal{N})+\Phi_E.\tag{1-27}$$

Let $g_{ij}(Z)=g^{TX}(e_i,e_j)(Z)$ and $(g^{ij}(Z))_{ij}$ be the inverse of the matrix $(g_{ij}(Z))_{ij}$. Let $(\nabla^{TX}_{e_i}e_j)(Z)=\Gamma^k_{ij}(Z)e_k$. As in [MM, Equation (4.1.34)], by (1-22) and (1-27), we get

$$\begin{aligned}\nabla_{t,\cdot} &= \kappa^{1/2}(tZ)\Big(\nabla_\cdot + t\Gamma^{\Lambda^{0,\bullet}}_{tZ} + \frac{1}{t}\Gamma^L_{tZ} + t\Gamma^E_{tZ}\Big)\kappa^{-1/2}(tZ),\\ \mathscr{L}_t &= -g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}) - 2\pi n + 4\pi\mathcal{N} + t^2\Phi_E(tZ).\end{aligned} \tag{1-28}$$

Moreover, $\kappa=(\det(g_{ij}))^{1/2}$, thus we can prove equation (1-23) as in [MM, Theorem 4.1.7] by taking the Taylor expansion of each term appearing in (1-28). Note that in [MM], every data has to be extended to $T_{x_0}X$ to make the analysis work, but as we admit the result, we do not have to worry about it and simply restrict ourselves to a neighborhood of x_0.

Now, it is clear that in the formula for $\mathscr{L}_t$ in (1-28), the term

$$\mathscr{L}^0_t := -g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}) - 2\pi n + 4\pi\mathcal{N} + t^2\Phi^0_E(tZ) \tag{1-29}$$

preserves the degree, because $\Gamma^{\Lambda^{0,\bullet}}$ does (as explained in Remark 1.6). Thus, using (1-26) and taking Taylor expansion of $\mathscr{L}_t$ in (1-28), we can write

$$\mathscr{L}^0_t = \mathscr{L}_0 + \sum_{r=1}^{\infty} t^r\mathcal{O}^0_r, \qquad t^2\Phi^{\pm 2}_E(tZ) = \sum_{r=2}^{\infty} t^r\mathcal{O}^{\pm 2}_r. \tag{1-30}$$

From (1-30), we get (1-24).

Finally, due to the presence of the conjugation by $\kappa^{1/2}$ in (1-22), $\mathscr{L}_t$ is a formally self-adjoint operator on $\mathscr{C}^\infty(\mathbb{R}^{2n},\mathrm{E}_{x_0})$ with respect to $\|\cdot\|_{L^2}$. So are $\mathscr{L}_0$ and $\mathcal{O}_r$. □

Recall that $\mathscr{R}=(R^E)^{0,2}\in\Omega^{0,2}(X,\mathrm{End}(E))$.

Proposition 1.9. *We have*

$$\mathcal{O}_1 = 0. \tag{1-31}$$

For $\mathcal{O}_2$, we have the formulas

$$\mathcal{O}^{+2}_2 = \mathscr{R}_{x_0}, \qquad \mathcal{O}^{-2}_2 = (\mathscr{R}_{x_0})^*, \tag{1-32}$$

and

$$\begin{aligned}\mathcal{O}^0_2 = \tfrac{1}{3}\langle R^{TX}_{x_0}(Z,e_i)Z,e_j\rangle\nabla_{0,e_i}\nabla_{0,e_j} - R^E_{x_0}(w_j,\bar{w}_j) - \tfrac{1}{6}r^X_{x_0}\\ +(\langle\tfrac{1}{3}R^{TX}_{x_0}(Z,e_k)e_k + \tfrac{\pi}{3}R^{TX}_{x_0}(z,\bar{z})Z,e_j\rangle - R^E_{x_0}(Z,e_j))\nabla_{0,e_j}.\end{aligned} \tag{1-33}$$

Proof. For $F=L$, E or $\Lambda^{0,\bullet}(T^*X)$, it is known that (see, for instance, [MM, Lemma 1.2.4])

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^F)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^F)_{x_0}(Z,e_j)\frac{Z^\alpha}{\alpha!}, \tag{1-34}$$

and in particular,

$$\Gamma^F_Z(e_j) = \tfrac{1}{2} R^F_{x_0}(Z, e_j) + O(|Z|^2). \tag{1-35}$$

Furthermore, we know from the Gauss lemma (see, e.g., [MM, Equation (1.2.19)]) that

$$g_{ij}(Z) = \delta_{ij} + O(|Z|^2) \tag{1-36}$$

This implies that

$$\kappa(Z) = |\det(g_{ij}(Z))|^{1/2} = 1 + O(|Z|^2). \tag{1-37}$$

Moreover, the second line of [MM, Equation (4.1.103)] entails

$$\frac{\sqrt{-1}}{2\pi} R^L_Z(Z, e_j) = \langle JZ, e_j\rangle + O(|Z|^3), \tag{1-38}$$

and thus by (1-34) and (1-38),

$$\Gamma^L_Z = \tfrac{1}{2} R^L_{x_0}(Z, e_j) + O(|Z|^3). \tag{1-39}$$

Using (1-28), (1-35), (1-37) and (1-39), we see that

$$\nabla_t = \nabla_0 + O(t^2). \tag{1-40}$$

Finally, using again (1-28), (1-36) and (1-40), we get $\mathcal{O}_1 = 0$.

Concerning $\mathcal{O}_2^{\pm 2}$, from (1-30), we see that

$$\begin{aligned} \mathcal{O}_2^{+2} &= \Phi_E^{+2}(0) = R^E_{x_0}(\bar{w}_\ell, \bar{w}_m)\bar{w}^\ell \wedge \bar{w}^m = (R^E_{x_0})^{0,2} = \mathcal{R}_{x_0}, \\ \mathcal{O}_2^{-2} &= \Phi_E^{-2}(0) = R^E_{x_0}(w_\ell, w_m) i_{\bar{w}_\ell} i_{\bar{w}_m} = ((R^E_{x_0})^{0,2})^* = (\mathcal{R}_{x_0})^*. \end{aligned} \tag{1-41}$$

Finally, by (1-29) and [MM, Equation (4.1.34)], we see that our $\mathscr{L}^0_t$ corresponds to $\mathscr{L}_t$ in [MM]. Thus, by (1-30) and [MM, Equation (4.1.31)], our $\mathcal{O}_2^0$ is equal to their $\mathcal{O}_2$ (this is because in their case, E is holomorphic, so R^E is a $(1,1)$-form and there is no term changing the degree in $(\bar{\partial}^{L^p\otimes E} + \bar{\partial}^{L^p\otimes E,*})^2$; but the terms preserving the degree are the same as ours). Hence (1-33) follows from [MM, Theorem 4.1.25]. □

1C. ***Bergman kernel of the limit operator*** $\mathscr{L}_0$**.** In this subsection, we study more precisely the operator $\mathscr{L}_0$.

We introduce the complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus, we get $Z = z + \bar{z}$, $w_j = \sqrt{2}\partial/\partial z_j$ and $\bar{w}_j = \sqrt{2}\partial/\partial \bar{z}_j$. We will identify z to $\sum_j z_j \partial/\partial z_j$ and $\bar{z}$ to $\sum_j \bar{z}_j \partial/\partial \bar{z}_j$ when we consider z and $\bar{z}$ as vector fields.

Set

$$\begin{aligned} &b_j = -2\nabla_{0,\partial/\partial z_j}, \quad b_j^+ = 2\nabla_{0,\partial/\partial \bar{z}_j}, \\ &b = (b_1, \dots, b_n), \quad \mathscr{L} = -\sum_i (\nabla_{0,e_i})^2 - 2\pi n. \end{aligned} \tag{1-42}$$

By definition, $\nabla_0 = \nabla + \frac{1}{2} R^L_{x_0}(Z, \cdot)$ so we get

$$b_i = -2\frac{\partial}{\partial z_i} + \pi \bar{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, \tag{1-43}$$

and for any polynomial $g(z, \bar{z})$ in z and $\bar{z}$,

$$\begin{gathered}[b_i, b_j^+] = -4\pi\delta_{ij}, \quad [b_i, b_j] = [b_i^+, b_j^+] = 0, \\ [g(z,\bar{z}), b_j] = 2\frac{\partial}{\partial z_j} g(z,\bar{z}), \quad [g(z,\bar{z}), b_j^+] = -2\frac{\partial}{\partial \bar{z}_j} g(z,\bar{z}).\end{gathered} \tag{1-44}$$

Finally, a simple calculation shows

$$\mathscr{L} = \sum_i b_i b_i^+ \quad \text{and} \quad \mathscr{L}_0 = \mathscr{L} + 4\pi\mathcal{N}. \tag{1-45}$$

Recall that we denoted by $\|\cdot\|_{L^2}$ the L^2-norm associated with $h^{\mathbb{E}_{x_0}}$ and dv_{TX}. For this form we have $b_i^+ = (b_i)^*$, therefore $\mathscr{L}$ and $\mathscr{L}_0$ are self-adjoint.

The next theorem is proved in [MM, Theorem 4.1.20]:

Theorem 1.10. *The spectrum of the restriction of $\mathscr{L}$ to $L^2(\mathbb{R}^{2n})$ is $\mathrm{Sp}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = 4\pi\mathbb{N}$ and an orthogonal basis of the eigenspace for the eigenvalue $4\pi k$ is*

$$b^\alpha\left(z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right)\right) \quad \textit{with } \alpha, \beta \in \mathbb{N}^n \textit{ and } \sum_i \alpha_i = k. \tag{1-46}$$

In particular, an orthonormal basis of $\ker(\mathscr{L}|_{L^2(\mathbb{R}^{2n})})$ is

$$\left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right), \tag{1-47}$$

and thus if $\mathscr{P}(Z, Z')$ is the smooth kernel of $\mathscr{P}$, the orthogonal projection from $(L^2(\mathbb{R}^{2n}), \|\cdot\|_0)$ onto $\ker(\mathscr{L})$ (where $\|\cdot\|_0$ is the L^2-norm associated to $g^{TX}_{x_0}$) with respect to $dv_{TX}(Z')$, we have

$$\mathscr{P}(Z, Z') = \exp\left(-\frac{\pi}{2}(|z|^2 + |z'|^2 - 2z\cdot\bar{z}')\right). \tag{1-48}$$

Now let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0}), \|\cdot\|_{L^2})$ onto $N := \ker(\mathscr{L}_0)$, and $P^N(Z, Z')$ be its smooth kernel with respect to $dv_{TX}(Z')$. From (1-45), we have

$$P^N(Z, Z') = \mathscr{P}(Z, Z') I_0. \tag{1-49}$$

2. The first coefficient in the asymptotic expansion

In this section we prove Theorem 0.4. We will proceed as follows. In Section 2A, following [MM, Section 4.1.7], we will give a formula for $\boldsymbol{b}_r$ involving the $\mathscr{O}_k$ and $\mathscr{L}_0$. In Section 2B, we will see how this formula entails Theorem 0.4.

2A. ***A formula for $\boldsymbol{b}_r$.*** By Theorem 1.10 and (1-45), we know that for every $\lambda \in \delta$ the unit circle in $\mathbb{C}$, $(\lambda - \mathscr{L}_0)^{-1}$ exists.

Let $f(\lambda, t)$ be a formal power series on t with values in $\mathrm{End}(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0}))$:

$$f(\lambda, t) = \sum_{r=0}^{+\infty} t^r f_r(\lambda) \quad \text{with } f_r(\lambda) \in \mathrm{End}(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})). \tag{2-1}$$

Consider the equation of formal power series on t for $\lambda \in \delta$,

$$\Bigl(\lambda - \mathscr{L}_0 - \sum_{r=1}^{+\infty} t^r \mathscr{O}_r\Bigr) f(\lambda, t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})}. \tag{2-2}$$

We then find that

$$\begin{aligned} f_0(\lambda) &= (\lambda - \mathscr{L}_0)^{-1}, \\ f_r(\lambda) &= (\lambda - \mathscr{L}_0)^{-1} \sum_{j=1}^{r} \mathscr{O}_j f_{r-j}(\lambda). \end{aligned} \tag{2-3}$$

Thus by (1-31) and by induction,

$$f_r(\lambda) = \Biggl(\sum_{\substack{r_1+\cdots+r_k=r \\ r_j \geq 2}} (\lambda - \mathscr{L}_0)^{-1} \mathscr{O}_{r_1} \cdots (\lambda - \mathscr{L}_0)^{-1} \mathscr{O}_{r_k}\Biggr) (\lambda - \mathscr{L}_0)^{-1}. \tag{2-4}$$

Definition 2.1. Following [MM, Equation (4.1.91)], we define $\mathscr{F}_r$ by

$$\mathscr{F}_r = \frac{1}{2\pi\sqrt{-1}} \int_\delta f_r(\lambda)\, d\lambda, \tag{2-5}$$

and we denote by $\mathscr{F}_r(Z, Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

Theorem 2.2. *The following equation holds*:

$$\boldsymbol{b}_r(x_0) = \mathscr{F}_{2r}(0, 0). \tag{2-6}$$

Proof. This formula follows from [MM, Theorem 8.1.4] as [MM, Equation (4.1.97)] follows from [MM, Theorem 4.1.24], remembering that in our situation the Bergman kernel P_p is not supported in degree 0. □

2B. ***Proof of*** Theorem 0.4. Let $T_{\boldsymbol{r}}(\lambda)=(\lambda-\mathscr{L}_0)^{-1}\mathcal{O}_{r_1}\cdots(\lambda-\mathscr{L}_0)^{-1}\mathcal{O}_{r_k}(\lambda-\mathscr{L}_0)^{-1}$ be the term in the sum (2-4) corresponding to $\boldsymbol{r}=(r_1,\dots,r_k)$. Let $N^{\perp}$ be the orthogonal of N in $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$, and $P^{N^{\perp}}$ the associated orthogonal projector. In $T_{\boldsymbol{r}}(\lambda)$, each term $(\lambda-\mathscr{L}_0)^{-1}$ can be decomposed as

$$(\lambda-\mathscr{L}_0)^{-1}=(\lambda-\mathscr{L}_0)^{-1}P^{N^{\perp}}+\frac{1}{\lambda}P^{N}. \tag{2-7}$$

Set

$$L^{N^{\perp}}(\lambda)=(\lambda-\mathscr{L}_0)^{-1}P^{N^{\perp}},\qquad L^{N}(\lambda)=\frac{1}{\lambda}P^{N}. \tag{2-8}$$

By (1-45), $\mathscr{L}_0$ preserves the degree, and thus so do $(\lambda-\mathscr{L}_0)^{-1}$, $L^{N^{\perp}}$ and L^{N}.

For $\eta=(\eta_1,\dots,\eta_{k+1})\in\{N,N^{\perp}\}^{k+1}$, let

$$T_{\boldsymbol{r}}^{\eta}(\lambda)=L^{\eta_1}(\lambda)\mathcal{O}_{r_1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda). \tag{2-9}$$

We can decompose

$$T_{\boldsymbol{r}}(\lambda)=\sum_{\eta=(\eta_1,\dots,\eta_{k+1})}T_{\boldsymbol{r}}^{\eta}(\lambda), \tag{2-10}$$

and by (2-4) and (2-5),

$$\mathscr{F}_{2r}=\frac{1}{2\pi\sqrt{-1}}\sum_{\substack{r_1+\cdots+r_k=2r\\(\eta_1,\dots,\eta_{k+1})}}\int_{\delta}T_{\boldsymbol{r}}^{\eta}(\lambda)\,d\lambda. \tag{2-11}$$

Note that $L^{N^{\perp}}(\lambda)$ is an holomorphic function of λ, so

$$\int_{\delta}L^{N^{\perp}}(\lambda)\mathcal{O}_{r_1}\cdots L^{N^{\perp}}(\lambda)\mathcal{O}_{r_k}L^{N^{\perp}}(\lambda)\,d\lambda=0. \tag{2-12}$$

Thus, in (2-11), every nonzero term that appears contains at least one $L^{N}(\lambda)$,

$$\int_{\delta}T_{\boldsymbol{r}}^{\eta}(\lambda)\,d\lambda\neq 0\quad\Rightarrow\quad\text{there exists an } i_0 \text{ such that } \eta_{i_0}=N. \tag{2-13}$$

Now fix k and j in $\mathbb{N}$. Let $s\in L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ be a form of degree $2j$, $\boldsymbol{r}\in(\mathbb{N}\backslash\{0,1\})^k$ such that $\sum_i r_i=2r$ and $\eta=(\eta_1,\dots,\eta_{k+1})\in\{N,N^{\perp}\}^{k+1}$ such that there is an i_0 satisfying $\eta_{i_0}=N$. We want to find a necessary condition for $I_{2j}T_{\boldsymbol{r}}^{\eta}(\lambda)I_{2j}s$ to be nonzero.

Suppose then that $I_{2j}T_{\boldsymbol{r}}^{\eta}(\lambda)I_{2j}s\neq 0$. Since $L^{\eta_{i_0}}=\frac{1}{\lambda}P^{N}$, and N is concentrated in degree 0, we must have

$$\deg(\mathcal{O}_{r_{i_0}}L^{\eta_{i_0+1}}(\lambda)\mathcal{O}_{r_{i_0+1}}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s)=0;$$

but each $L^{\eta_i}(\lambda)$ preserves the degree, and by Theorem 1.8 each $\mathscr{O}_{r_i}$ lowers the degree at most by 2, so

$$0 = \deg(\mathscr{O}_{r_{i_0}} L^{\eta_{i_0+1}}(\lambda)\mathscr{O}_{r_{i_0+1}} \cdots L^{\eta_k}(\lambda)\mathscr{O}_{r_k} L^{\eta_{k+1}}(\lambda) I_{2j} s) \geq 2j - 2(k - i_0 + 1),$$

and thus

$$2j \leq 2(k - i_0 + 1). \tag{2-14}$$

Similarly, $L^{\eta_1}(\lambda)\mathscr{O}_{r_1} \cdots L^{\eta_k}(\lambda)\mathscr{O}_{r_k} L^{\eta_{k+1}}(\lambda) I_{2j} s$ must have a nonzero component in degree $2j$ and by Theorem 1.8 each $\mathscr{O}_{r_i}$ rises the degree at most by 2, so $2j$ must be less than or equal to the number of $\mathscr{O}_{r_i}$ appearing before $\mathscr{O}_{r_{i_0}}$, that is,

$$2j \leq 2(i_0 - 1). \tag{2-15}$$

With (2-14) and (2-15), we find

$$4j \leq 2k. \tag{2-16}$$

Finally, since for every i, $r_i \geq 2$ and $\sum_{i=1}^{k} r_i = 2r$, we have $2k \leq 2r$, and thus

$$4j \leq 2k \leq 2r. \tag{2-17}$$

Consequently, if $r < 2j$ we have $I_{2j} T^{\eta}_{\boldsymbol{r}}(\lambda) I_{2j} = 0$, and by (2-11), we find $I_{2j} \mathscr{F}_{2r} I_{2j} = 0$. Using Theorem 2.2, we find

$$I_{2j} \boldsymbol{b}_r I_{2j} = 0,$$

which, combined with Theorem 0.3, entails the first part of Theorem 0.4.

For the second part of this theorem, let us focus on the case $r = 2j$. We also suppose that $j \geq 1$, because in the case $j = 0$, [MM, Equation (8.1.5)] implies that $\boldsymbol{b}_0(x_0) = \mathscr{F}_0(0,0) = I_0 \mathscr{P}(0,0) = I_0$, so Theorem 0.4 is true for $j = 0$.

In $I_{2j} \mathscr{F}_{4j} I_{2j}$, there is only one term satisfying equations (2-14), (2-15) and (2-17). First we see that (2-17) implies that $r = k = 2j$ and for all i, $r_i = 2$, while (2-14) and (2-15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j. Moreover, only $\mathscr{O}_2^{+2}$ and $\mathscr{O}_2^{-2}$ appear in $I_{2j} \mathscr{F}_{4j} I_{2j}$, not $\mathscr{O}_2^{0}$, because the degree must decrease by $2j$ and then increase by $2j$ with $k = 2j$ $\mathscr{O}_{r_i}$ available. To summarize,

$$\begin{aligned} I_{2j} \mathscr{F}_{4j} I_{2j} &= \frac{1}{2\pi\sqrt{-1}} \int_{\delta} \Big(I_{2j} ((\lambda - \mathscr{L}_0)^{-1} P^{N^{\perp}} \mathscr{O}_2^{+2})^j \\ &\qquad\qquad \times \frac{1}{\lambda} P^N (\mathscr{O}_2^{-2} (\lambda - \mathscr{L}_0)^{-1} P^{N^{\perp}})^j I_{2j} \Big)\, d\lambda \\ &= I_{2j} ({\mathscr{L}_0}^{-1} P^{N^{\perp}} \mathscr{O}_2^{+2})^j P^N (\mathscr{O}_2^{-2} {\mathscr{L}_0}^{-1} P^{N^{\perp}})^j I_{2j} \\ &= I_{2j} ({\mathscr{L}_0}^{-1} \mathscr{O}_2^{+2})^j P^N (\mathscr{O}_2^{-2} {\mathscr{L}_0}^{-1})^j I_{2j}. \end{aligned} \tag{2-18}$$

Because by (1-45), $L^2(\mathbb{R}^{2n}, (\Lambda^{0,>0}(T^*X) \otimes E)_{x_0}) \subset N^{\perp}$, we removed the $P^{N^{\perp}}$ in the last line.

Let $A = I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^j P^N$. Since $(\mathcal{O}_2^{+2})^* = \mathcal{O}_2^{-2}$ (see Proposition 1.9) and $\mathscr{L}_0$ is self-adjoint, the adjoint of A is $A^* = P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^j I_{2j}$, and thus

$$I_{2j}\mathscr{F}_{4j}I_{2j} = AA^*. \tag{2-19}$$

Recall that $P^N = \mathscr{P} I_0$ (see (1-49)). Let $s \in L^2(\mathbb{R}^{2n}, E_{x_0})$; since $\mathscr{L}_0 = \mathscr{L} + 4\pi\mathscr{N}$ and $\mathscr{L}\mathscr{P}s = 0$, $(\mathscr{P}s)\mathscr{R}_{x_0}$ is an eigenfunction of $\mathscr{L}_0$ for the eigenvalue $2 \times 4\pi$. Thus,

$$\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2}P^N s = \mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2}\mathscr{P}s = \mathscr{L}_0{}^{-1}((\mathscr{P}s)\mathscr{R}_{x_0}) = \frac{1}{4\pi}\frac{1}{2}\mathscr{R}_{x_0}\mathscr{P}s. \tag{2-20}$$

Now, an easy induction shows that

$$A = \frac{1}{(4\pi)^j}\frac{1}{2 \times 4 \times \cdots \times 2j} I_{2j}\mathscr{R}_{x_0}^j\mathscr{P} = \frac{1}{(4\pi)^j}\frac{1}{2^j j!} I_{2j}\mathscr{R}_{x_0}^j\mathscr{P}. \tag{2-21}$$

Let $A(Z, Z')$ and $A^*(Z, Z')$ be the smooth kernels of A and A^* with respect to $dv_{TX}(Z')$. By (2-19), $I_{2j}\mathscr{F}_{4j}I_{2j}(0,0) = \int_{\mathbb{R}^{2n}} A(0, Z)A^*(Z, 0)\, dZ$. Thanks to

$$\int_{\mathbb{R}^{2n}} \mathscr{P}(0, Z)\mathscr{P}(Z, 0)\, dZ = (\mathscr{P} \circ \mathscr{P})(0, 0) = \mathscr{P}(0, 0) = 1 \tag{2-22}$$

and (2-21), we find (0-12).

3. The second coefficient in the asymptotic expansion

In this section, we prove Theorem 0.7. Using (2-6), we know that

$$I_{2j}\boldsymbol{b}_{2j+1}I_{2j}(0,0) = I_{2j}\mathscr{F}_{4j+2}I_{2j}(0,0). \tag{3-1}$$

In Section 3A, we decompose this into three terms, and then in Sections 3B and 3C we handle them separately.

Fix $j \in [\![0, n]\!]$. For every smoothing operator F acting on $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ in this section, we denote by $F(Z, Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

3A. ***Decomposition of the problem.*** Applying inequality (2-17) with $r = 2j+1$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$, the nonzero terms $\int_\delta T_{\boldsymbol{r}}^\eta(\lambda)\, d\lambda$ in the decomposition (2-11) satisfy $k = 2j$ or $k = 2j+1$. Since $\sum_i r_i = 4j+2$ and $r_i \geq 2$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$ there are three types of terms $T_{\mathrm{r}}^\eta(\lambda)$ with nonzero integral, in which

- for $k = 2j$,
  - there are $2j-2$ $\mathcal{O}_{r_i}$ equal to $\mathcal{O}_2$ and 2 equal to $\mathcal{O}_3$ and we denote by I the sum of these terms,
  - there are $2j-1$ $\mathcal{O}_{r_i}$ equal to $\mathcal{O}_2$ and 1 equal to $\mathcal{O}_4$ and we denote by II the sum of these terms,
- for $k = 2j+1$,
  - all the $\mathcal{O}_{r_i}$ are equal to $\mathcal{O}_2$ and we denote by III the sum of these terms.

We thus have a decomposition

$$I_{2j}\mathscr{F}_{4j+2}I_{2j} = \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{3-2}$$

Remark 3.1. Note that for I and II to be nonzero, we must have $j \geq 1$. Moreover, in the first two cases, as $k = 2j$, by the same reasoning as in Section 2B, (2-14) and (2-15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j, and that only $\mathcal{O}_2^{\pm 2}$, $\mathcal{O}_3^{\pm 2}$ and $\mathcal{O}_4^{\pm 2}$ appear in I and II, not some $\mathcal{O}_{r_i}^0$.

3B. *The term involving only $\mathcal{O}_2$.*

Lemma 3.2. *In any term $T_{\mathrm{r}}^{\eta}(\lambda)$ appearing in the sum* III (*with nonvanishing integral*), *the i_0 such that $\eta_{i_0} = N$ is unique and equal to j or $j+1$. If we denote by* III_a *and* III_b *the sums corresponding to these two cases*, *we have*

$$\begin{aligned}\mathrm{III}_a &= \sum_{k=0}^{j} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^{j}I_{2j},\\ \mathrm{III}_b &= (\mathrm{III}_a)^*,\\ \mathrm{III} &= \mathrm{III}_a+\mathrm{III}_b.\end{aligned} \tag{3-3}$$

Remark 3.3. For the same reason as for (2-18), we have removed the $P^{N^\perp}$ in (3-3) without getting any problem concerning the existence of $\mathscr{L}_0{}^{-1}$.

Proof. Fix a term $T_{\mathrm{r}}^{\eta}(\lambda)$ appearing in the sum III with nonvanishing integral. Using again the same reasoning as in Section 2B, we see that there exists at most two indices i_0 such that $\eta_{i_0} = N$, and that they are in $\{j, j+1\}$. Indeed, with only $2j+1$ $\mathcal{O}_{r_i}$ at our disposal, we need j of them before the first P^N, and j after the last one.

Now, the only possible term with $\eta_j = \eta_{j+1} = N$ is

$$(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j}P^N\mathcal{O}_2^{0}P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^{j}. \tag{3-4}$$

To prove that this term is vanishing, using (1-33), [Ma and Marinescu 2012, Equations (3.13), (3.16b) and 4.1a], we see that $\mathscr{P}\mathcal{O}_2^0\mathscr{P} = 0$, and so

$$P^N\mathcal{O}_2^0P^N = \mathscr{P}\mathcal{O}_2^0\mathscr{P}I_0 = 0. \tag{3-5}$$

We have proved the first part of the lemma.

The second part follows from the reasoning made at the beginning of this proof, and the facts that i_0 is unique, $\mathcal{O}_2^0$ is self-adjoint and $(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^* = \mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1}$. □

Let us compute the term that appears in (3-3),

$$\mathrm{III}_{a,k} := I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^{j}I_{2j}. \tag{3-6}$$

With (2-21), we know that

$$P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^j I_{2j} = \frac{1}{(4\pi)^j}\frac{1}{2^j j!}\mathscr{P}(\mathscr{R}_{x_0}^j)^* I_{2j}, \tag{3-7}$$

and

$$\begin{aligned} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k P^N \\ = \frac{1}{(4\pi)^k}\frac{1}{2^k k!} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}(\mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P})I_0. \end{aligned} \tag{3-8}$$

Let

$$R_{k\bar{m}\ell\bar{q}} = \left\langle R^{TX}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_m}\right)\frac{\partial}{\partial z_\ell}, \frac{\partial}{\partial \bar{z}_q}\right\rangle_{x_0} \quad \text{and} \quad R^E_{k\bar{\ell}} = R^E_{x_0}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_\ell}\right). \tag{3-9}$$

By [ibid., Lemma 3.1], we know that

$$R_{k\bar{m}\ell\bar{q}} = R_{\ell\bar{m}k\bar{q}} = R_{k\bar{q}\ell\bar{m}} = R_{\ell\bar{q}k\bar{m}} \quad \text{and} \quad r^X_{x_0} = 8R_{m\bar{m}q\bar{q}}. \tag{3-10}$$

Once again, our $\mathcal{O}_2^0$ correspond to the $\mathcal{O}_2$ of [Ma and Marinescu 2012] (see (1-33) and [ibid., Equations (3.13), (3.16b)]), so we can use [ibid., Equation (4.6)] to get

$$\begin{aligned} \mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P} = (\tfrac{1}{6}b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell + \tfrac{4}{3} b_q R_{\ell\bar{k}k\bar{q}} z_\ell \\ - \tfrac{1}{3}\pi b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell \bar{z}'_m + b_q R^E_{\ell\bar{q}} z_\ell)\mathscr{R}_{x_0}^k\mathscr{P}, \end{aligned} \tag{3-11}$$

Set

$$\begin{aligned} a &= \tfrac{1}{6} b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell, \quad & b &= \tfrac{4}{3} b_q R_{\ell\bar{k}k\bar{q}} z_\ell, \\ c &= -\tfrac{1}{3}\pi b_q R_{k\bar{m}\ell\bar{q}} z_k z_\ell \bar{z}'_m, \quad & d &= b_q R^E_{\ell\bar{q}} z_\ell. \end{aligned} \tag{3-12}$$

Thanks to (1-45), (1-46) and (3-11), we find

$$\mathscr{L}_0{}^{-1}\mathcal{O}_2^0\mathscr{R}_{x_0}^k\mathscr{P}I_0 = \left(\frac{a}{4\pi(2+2k)} + \frac{b+c+d}{4\pi(1+2k)}\right)\mathscr{R}_{x_0}^k\mathscr{P}I_0, \tag{3-13}$$

and by induction, (3-8) becomes

$$\begin{aligned} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k P^N \\ = \frac{1}{(4\pi)^{j+1}}\frac{1}{2^k k!} I_{2j}\mathscr{R}_{x_0}^{j-k}\left(\frac{a}{(2+2k)\cdots(2+2j)} + \frac{b+c+d}{(1+2k)\cdots(1+2j)}\right)\mathscr{R}_{x_0}^k\mathscr{P}I_0. \end{aligned} \tag{3-14}$$

Lemma 3.4. *We have*

$$\begin{aligned} (a\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) = \tfrac{1}{6} r^X_{x_0}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z), \quad (b\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) = -\tfrac{1}{3} r^X_{x_0}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z), \\ (c\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) = 0, \quad (d\mathscr{R}_{x_0}^k\mathscr{P})(0,Z) = -2R^E_{q\bar{q}}\mathscr{R}_{x_0}^k\mathscr{P}(0,Z). \end{aligned} \tag{3-15}$$

Proof. This lemma is a consequence of the relations (1-44) and (3-10). For demonstration, we will compute $(b\mathcal{R}^k_{x_0}\mathcal{P})(0,Z)$; the other terms are similar.

$$\begin{aligned}(b\mathcal{R}^k_{x_0}\mathcal{P})(0,Z) &= (\tfrac{4}{3}b_q R_{\ell\bar{k}k\bar{q}} z_\ell \mathcal{R}^k_{x_0}\mathcal{P})(0,Z)\\ &= \tfrac{4}{3}R_{\ell\bar{k}k\bar{q}}\mathcal{R}^k_{x_0}((z_\ell b_q - 2\delta_{\ell q})\mathcal{P})(0,Z)\\ &= -\tfrac{8}{3}R_{\ell\bar{k}k\bar{\ell}}\mathcal{R}^k_{x_0}\mathcal{P}(0,Z) = -\tfrac{1}{3}r^X_{x_0}\mathcal{R}^k_{x_0}\mathcal{P}(0,Z). \qquad\square\end{aligned}$$

Using (2-22), (3-6), (3-7) and (3-13), we find

$$\text{(3-16)}\quad \mathrm{III}_{a,k}(0,0) = I_{2j}\mathrm{C}_j(j)\mathcal{R}^{j-k}_{x_0}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right) r^X_{x_0} - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)} R^E_{q\bar{q}}\right]\mathcal{R}^k_{x_0}(\mathcal{R}^j_{x_0})^* I_{2j}.$$

Notice that $2R^E_{q\bar{q}} = R^E_{x_0}\big(\sqrt{2}\frac{\partial}{\partial z_q}, \sqrt{2}\frac{\partial}{\partial \bar{z}_q}\big) = R^E_{x_0}(w_q, \bar{w}_q) = \sqrt{-1}R^E_{\Lambda,x_0}$ by definition. Consequently,

$$\text{(3-17)}\quad \mathrm{III}_a(0,0) = I_{2j}\mathrm{C}_j(j)\sum_{k=0}^{j}\mathcal{R}^{j-k}_{x_0}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right) r^X_{x_0} - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x_0}\right]\mathcal{R}^k_{x_0}(\mathcal{R}^j_{x_0})^* I_{2j}.$$

3C. ***The two other terms.*** In this subsection, we suppose that $j \geq 1$ (see Remark 3.1). Moreover, the existence of any $\mathcal{L}_0{}^{-1}$ in this section follows from the reasoning done in Remark 3.3, and this operator will be used without further precision.

Due to (1-30), we have

$$\text{(3-18)}\qquad \mathcal{O}^{+2}_3 = \frac{d}{dt}\Phi^{+2}_{E_0}(tZ)\Big|_{t=0} = z_i\frac{\partial\mathcal{R}_\cdot}{\partial z_i}(0) + \bar{z}_i\frac{\partial\mathcal{R}_\cdot}{\partial\bar{z}_i}(0),$$

$$\text{(3-19)}\qquad \mathcal{O}^{+2}_4 = \frac{z_i z_j}{2}\frac{\partial^2\mathcal{R}_\cdot}{\partial z_i\partial z_j}(0) + z_i\bar{z}_j\frac{\partial^2\mathcal{R}_\cdot}{\partial z_i\partial\bar{z}_j}(0) + \frac{\bar{z}_i\bar{z}_j}{2}\frac{\partial^2\mathcal{R}_\cdot}{\partial\bar{z}_i\partial\bar{z}_j}(0).$$

The sum I can be decomposed into three subsums: I_a, I_b and I_c, in which the two $\mathcal{O}_3$ appear respectively both at the left, on either side or both at the right of P^N (see Remark 3.1). As usual, we have $\mathrm{I}_c = (\mathrm{I}_a)^*$.

In the same way, we can decompose II as $\mathrm{II}_a + \mathrm{II}_b$: in II_a the $\mathcal{O}_4$ appears at the left of P^N, and in II_b at the right. Once again, $\mathrm{II}_b = (\mathrm{II}_a)^*$.

Computation of $\mathrm{I}_b(0,0)$. To compute I_b, we first compute the kernel of

$$\text{(3-20)}\qquad A_k := I_{2j}(\mathcal{L}_0{}^{-1}\mathcal{O}^{+2}_2)^{j-k-1}(\mathcal{L}_0{}^{-1}\mathcal{O}^{+2}_3)(\mathcal{L}_0{}^{-1}\mathcal{O}^{+2}_2)^k\mathcal{P}I_0$$

at $(0,Z)$.

By (2-21) and (3-18),

$$A_k = I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\mathscr{R}_{x_0}^k \mathscr{P} I_0$$
$$= \frac{1}{(4\pi)^k}\frac{1}{2^k k!} I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\Big[z_i\frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)+\bar{z}_i\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\Big]\mathscr{R}_{x_0}^k\mathscr{P}I_0. \tag{3-21}$$

By Theorem 1.10, if $s \in N$, then $z_i s \in N$, so by the same calculation as in (2-21),

$$\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\Big[z_i\frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\Big]\mathscr{R}_{x_0}^k\mathscr{P}I_0\Big)(0,Z)$$
$$= \frac{1}{(4\pi)^j}\frac{1}{2^j j!}\Big(I_{2j}\Big[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\Big]z_i\mathscr{P}I_0\Big)(0,Z)=0. \tag{3-22}$$

Now by (1-43) and the formula (1-48), we have

$$(b_i^+\mathscr{P})(Z,Z')=0 \quad\text{and}\quad (b_i\mathscr{P})(Z,Z')=2\pi(\bar{z}_i-\bar{z}'_i)\mathscr{P}(Z,Z'). \tag{3-23}$$

Thus,

$$\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\Big[\bar{z}_i\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\Big]\mathscr{R}_{x_0}^k\mathscr{P}I_0\Big)(Z,Z') \tag{3-24}$$
$$=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\Big[\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\Big]\Big(\frac{b_i}{2\pi}+\bar{z}'_i\Big)\mathscr{P}I_0\Big)(Z,Z')$$
$$=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\Big[\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\Big]$$
$$\times\Big(\frac{1}{4\pi(2k+2+1)}\frac{b_i}{2\pi}+\frac{1}{4\pi(2k+2)}\bar{z}'_i\Big)\mathscr{P}I_0\Big)(Z,Z')$$
$$=\frac{1}{(4\pi)^j}\frac{1}{2^j j!}\Big(I_{2j}\Big[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\Big]\bar{z}'_i\mathscr{P}I_0\Big)(Z,Z')$$
$$+\frac{1}{(4\pi)^j}\frac{1}{2^k k!\prod_{k+1}^{j}(2\ell+1)}\Big(I_{2j}\Big[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\Big]\frac{b_i}{2\pi}\mathscr{P}I_0\Big)(Z,Z').$$

For the last two lines, we used that if $s\in N$, then $\mathscr{L}(b_i s)=4\pi b_i s$ (see Theorem 1.10). Thus, by (0-15) and (3-21)–(3-24),

$$A_k(0,Z)=\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\Big(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\Big[\bar{z}_i\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\Big]\mathscr{R}_{x_0}^k\mathscr{P}I_0\Big)(0,Z) \tag{3-25}$$
$$=(\mathrm{C}_j(j)-\mathrm{C}_j(k))I_{2j}\Big[\mathscr{R}_{x_0}^{j-k-1}\frac{\partial \mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\Big]\bar{z}_i\mathscr{P}(0,Z)I_0.$$

We know that $(\bar{z}_i\mathscr{P})^* = z_i\mathscr{P}$ and $\int_{\mathbb{C}^n} z_m\bar{z}_q e^{-\pi|z|^2}\,dZ = \frac{1}{\pi}\delta_{mq}$, so

$$(A_{k_1}A_{k_2}{}^*)(0,0) = \frac{1}{\pi}I_{2j}\Big[(\mathrm{C}_j(j)-\mathrm{C}_j(k_1))\mathscr{R}_{x_0}^{j-k_1-1}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k_1}\Big] \times\Big[(\mathrm{C}_j(j)-\mathrm{C}_j(k_2))\mathscr{R}_{x_0}^{j-k_2-1}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k_2}\Big]^* I_{2j}. \tag{3-26}$$

Finally,

$$\mathrm{I}_b(0,0) = \frac{1}{\pi}I_{2j}\Big[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\Big] \times\Big[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_{x_0}^{j-k-1}\frac{\partial\mathscr{R}_\cdot}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\Big]^* I_{2j}. \tag{3-27}$$

Computation of $\mathrm{I}_a(0,0)$ *and* $\mathrm{I}_c(0,0)$. First recall that $\mathrm{I}_c(0,0) = (\mathrm{I}_a(0,0))^*$, so we just need to compute $\mathrm{I}_a(0,0)$. By the definition of $\mathrm{I}_a(0,0)$, for it to be nonzero, it is necessary to have $j \geq 2$, which will be assumed in this paragraph. Let

$$A_{k,\ell} := I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-\ell-2}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{\ell}\mathscr{P}I_0, \tag{3-28}$$

the sum $\mathrm{I}_a(0,0)$ is then given by

$$\mathrm{I}_a(0,0) = \int_{\mathbb{R}^{2n}}\Big(\sum_{k,\ell}A_{k,\ell}(0,Z)\Big)\Big(\frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}_{x_0}^{j}\mathscr{P}I_0\Big)^*(Z,0)\,dv_{TX}(Z). \tag{3-29}$$

In the following, we will set

$$\tilde{b}_i := \frac{b_i}{2\pi}. \tag{3-30}$$

Using the same method as in (1-44) and (3-22)–(3-24), we find that there exist constants $C^1_{k,\ell}$, $C^2_{k,\ell}$ given by

$$\begin{aligned} C^1_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}}\frac{1}{2^{k+\ell+1}(k+\ell+1)!},\\ C^2_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}}\frac{1}{2^{\ell}\ell!\prod_{\ell+1}^{k+\ell+1}(2s+1)}, \end{aligned} \tag{3-31}$$

such that

$$
\begin{aligned}
(3\text{-}32)\quad & (\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell\mathscr{P}I_0 \\
&= \mathscr{L}_0{}^{-1}\bigg\{\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}_{x_0}^\ell C_{k,\ell}^1 z_i z_{i'} + \frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}_{x_0}^\ell C_{k,\ell}^1 z_{i'}(\tilde{b}_i+\bar{z}_i') \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}_{x_0}^\ell z_i(C_{k,\ell}^2\tilde{b}_{i'}+C_{k,\ell}^1\bar{z}_{i'}') \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}_{x_0}^\ell (\tilde{b}_i+\bar{z}_i')(C_{k,\ell}^2\tilde{b}_{i'}+C_{k,\ell}^1\bar{z}_{i'}')\bigg\}\mathscr{P}I_0 \\
&= \mathscr{L}_0{}^{-1}\bigg\{\frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}_{x_0}^\ell C_{k,\ell}^1 z_i z_{i'} \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\mathscr{R}_{x_0}^\ell C_{k,\ell}^1\Big(\tilde{b}_i z_{i'}+\frac{\delta_{ii'}}{\pi}+z_{i'}\bar{z}_i'\Big) \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}_{x_0}^\ell\Big(C_{k,\ell}^2\Big(\tilde{b}_{i'}z_i+\frac{\delta_{ii'}}{\pi}\Big)+C_{k,\ell}^1 z_i\bar{z}_{i'}'\Big) \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\mathscr{R}_{x_0}^\ell(C_{k,\ell}^2(\tilde{b}_i\tilde{b}_{i'}+\bar{z}_i'\tilde{b}_{i'})+C_{k,\ell}^1(\tilde{b}_i\bar{z}_{i'}'+\bar{z}_i'\bar{z}_{i'}'))\bigg\}\mathscr{P}I_0,
\end{aligned}
$$

Using Theorem 1.10, (1-44) and (3-23), we see that there exist constants $C^i_{j,k,\ell}$, $i = 3, \dots, 10$, such that

$$
\begin{aligned}
C^3_{j,k,\ell} &= C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s)}, \\
C^4_{j,k,\ell} &= C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s+1)}, \\
C^5_{j,k,\ell} &= C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s+1)}, \\
C^6_{j,k,\ell} &= C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s)},
\end{aligned}
\tag{3-33}
$$

and

$$
\begin{aligned}
(3\text{-}34)\quad & A_{k,\ell}(0,Z) \\
&= I_{2j}\bigg(\mathscr{R}_{x_0}^{j-k-\ell-2}\bigg\{\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial z_{i'}}(0)\Big(C^3_{j,k,\ell}\frac{\delta_{ii'}}{\pi}+C^4_{j,k,\ell}\tilde{b}_i z_{i'}\Big) \\
&\qquad + \frac{\partial\mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0)\Big(C^5_{j,k,\ell}\tilde{b}_{i'}z_i+\frac{\delta_{ii'}}{\pi}C^6_{j,k,\ell}\Big) + \frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\partial\mathscr{R}_\cdot}{\partial \bar{z}_{i'}}(0) \\
&\qquad \times(C^7_{j,k,\ell}\tilde{b}_i\tilde{b}_{i'}+C^8_{j,k,\ell}\bar{z}_i\tilde{b}_{i'}+C^9_{j,k,\ell}\tilde{b}_i\bar{z}_{i'}+C^{10}_{j,k,\ell}\bar{z}_i\bar{z}_{i'})\bigg\}\mathscr{R}_{x_0}^\ell\mathscr{P}I_0\bigg)(0,Z)
\end{aligned}
$$

$$= I_{2j}\mathscr{R}_{x_0}^{j-k-\ell-2}\bigg\{\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\frac{C^3_{j,k,\ell}-C^4_{j,k,\ell}}{\pi}$$
$$+\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\frac{C^6_{j,k,\ell}-C^5_{j,k,\ell}}{\pi}$$
$$+\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_{i'}}(0)(4C^7_{j,k,\ell}-2C^8_{j,k,\ell}-2C^9_{j,k,\ell}+C^{10}_{j,k,\ell})\bar{z}_i\bar{z}_{i'}\bigg\}\mathscr{R}_{x_0}^{\ell}\mathscr{P}(0,Z)I_0.$$

Now with $\int\bar{z}_i\bar{z}_{i'}\mathscr{P}(0,Z)\mathscr{P}(Z,0)\,dZ=0$, we can rewrite (3-29),

$$\text{(3-35)}\quad \mathrm{I}_a(0,0)=\frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{k,\ell}\mathscr{R}_{x_0}^{j-k-\ell-2}\bigg\{(C^3_{j,k,\ell}-C^4_{j,k,\ell})\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)$$
$$+(C^6_{j,k,\ell}-C^5_{j,k,\ell})\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\bigg\}\mathscr{R}_{x_0}^{\ell}(\mathscr{R}_{x_0}^{j})^*I_{2j}.$$

By (0-15), (3-31) and (3-33),

$$\text{(3-36)}\quad \begin{aligned}&C^3_{j,k,\ell}=\mathrm{C}_j(j),\quad C^4_{j,k,\ell}=\mathrm{C}_j(k+\ell+1),\\ &C^5_{j,k,\ell}=\mathrm{C}_j(\ell),\quad C^6_{j,k,\ell}=\mathrm{C}_j(\ell)\prod_{s=k+\ell+2}^{j}\Big(1+\frac{1}{2s}\Big).\end{aligned}$$

We can now write $\mathrm{I}_a(0,0)$ in (3-35) more precisely as

$$\text{(3-37)}\quad \frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\bigg\{(\mathrm{C}_j(j)-\mathrm{C}_j(q+1))\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{m}$$
$$+\mathrm{C}_j(m)\bigg[\prod_{q+2}^{j}\Big(1+\frac{1}{2s}\Big)-1\bigg]\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{m}\bigg\}(\mathscr{R}_{x_0}^{j})^*I_{2j}.$$

Computation of $\mathrm{II}(0,0)$. Recall that $\mathrm{II}(0,0)=\mathrm{II}_a(0,0)+(\mathrm{II}_a(0,0))^*$. The computation of $\mathrm{II}_a(0,0)$ is very similar to that of $\mathrm{I}_a(0,0)$, only simpler. We will follow the same method.

Let

$$B_k:=I_{2j}(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathscr{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^{k}\mathscr{P}I_0,$$

the sum $\mathrm{II}_a(0,0)$ is then given by

$$\text{(3-38)}\quad \mathrm{II}_a(0,0)=\int_{\mathbb{R}^{2n}}\Big(\sum_k B_k(0,Z)\Big)\Big(\frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}_{x_0}^{j}\mathscr{P}I_0\Big)^*(Z,0)\,dv_{TX}(Z).$$

Using (3-19), we can repeat what we have done for (3-32) and (3-34). We find that there is a constant C (which we do not need to compute) such that

$$B_k(0,Z) = I_{2j}\Big\{\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k\frac{\mathrm{C}_j(j)-\mathrm{C}_j(k)}{\pi} + \mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial\bar{z}_i\partial\bar{z}_{i'}}(0)\mathscr{R}_{x_0}^k C\frac{\bar{z}_i\bar{z}_{i'}}{2}\Big\}\mathscr{P}(0,Z)I_0. \tag{3-39}$$

Thus, we get

$$\mathrm{II}_a(0,0) = \frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{k=0}^{j-1}(\mathrm{C}_j(j)-\mathrm{C}_j(k))\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^k(\mathscr{R}_{x_0}^j)^*I_{2j}. \tag{3-40}$$

Conclusion. In order to conclude the proof of Theorem 0.7, we just need to put the pieces together. But before that, to write the formulas in a more intrinsic way, note that we trivialized $\Lambda^{0,\bullet}(T^*X)\otimes E$ with $\nabla^{\Lambda^{0,\bullet}\otimes E}$ and $\bar{w}_i = \sqrt{2}\frac{\partial}{\partial\bar{z}_i}$, so [Ma and Marinescu 2012, Equations (5.44), (5.45)] imply

$$\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0) = \frac{1}{\sqrt{2}}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x_0), \quad \frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0) = \frac{1}{\sqrt{2}}(\nabla_{w_i}^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x_0),$$
$$\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0) = -\frac{1}{4}(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}_{\cdot})(x_0). \tag{3-41}$$

With these remarks and equations (3-3), (3-17), (3-27), (3-37), (3-40) used in decomposition (3-2), we get Theorem 0.7.

4. The third coefficient in the asymptotic expansion when the first two vanish

In this section, we prove Theorem 0.8. Using (2-6), we know that

$$I_{2j}\boldsymbol{b}_{2j+2}I_{2j}(0,0) = I_{2j}\mathscr{F}_{4j+4}I_{2j}(0,0). \tag{4-1}$$

Here again, we will first decompose this into several terms in Section 4A, and then in Sections 4B, 4C and 4D we handle them separately.

Fix $j\in[\![1,n]\!]$ and suppose that

$$I_{2j}\boldsymbol{b}_{2j}I_{2j}(0,0) = I_{2j}\boldsymbol{b}_{2j+1}I_{2j}(0,0) = 0. \tag{4-2}$$

By Theorems 0.4 and 0.7, this is equivalent to

$$\begin{cases}\mathscr{R}_x^j = 0,\\ \mathcal{T}_0(j) = 0.\end{cases} \tag{4-3}$$

For every smoothing operator F acting on $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ that appears in this section, we will denote by $F(Z,Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

Moreover, recall that for every operator A we have

(4-4) $$\mathrm{Pos}[A] = AA^* \quad \text{and} \quad \mathrm{Sym}[A] = A + A^*.$$

4A. ***Decomposition of the computation.*** With the same reasoning as in Section 3A, we see that the nonzero terms $\int_\delta T_{\boldsymbol{r}}^\eta(\lambda)\,d\lambda$ satisfy $k = 2j,\ 2j+1$ or $2j+2$ in the decomposition (2-11) of $I_{2j}\mathscr{F}_{4j+4}I_{2j}$. Moreover, we can find the possible terms by adding one term to or modifying the subscript of the terms we mentioned in Section 3A. The list of possible terms is as follows:

(I) Terms satisfying $k = 2j+2$.

Here there are up to three indices $i \in \{j,\ j+1,\ j+2\}$ such that $\eta_i = N$. Moreover, the only $\mathcal{O}_\ell$ appearing are some $\mathcal{O}_2$. The possibilities are now

(I-a) $2j+2$ times $\mathcal{O}_2^{\pm 2}$,
(I-a) $2j$ times $\mathcal{O}_2^{\pm 2}$ and 2 times $\mathcal{O}_2^0$.

(II) Terms satisfying $k = 2j+1$.

Here, there are one or two indices $i \in \{j,\ j+1\}$ such that $\eta_i = N$, and there is exactly one $\mathcal{O}_\ell^0$ that appears in these terms. We regroup them in relation to the $\mathcal{O}_{r_i}$ that they contain:

(II-b) $2j$ times $\mathcal{O}_2^{\pm 2}$ and 1 time $\mathcal{O}_4^0$,
(II-b) $2j-1$ times $\mathcal{O}_2^{\pm 2}$, 1 time $\mathcal{O}_2^0$ and 1 time $\mathcal{O}_4^{\pm 2}$,
(II-b) $2j-1$ times $\mathcal{O}_2^{\pm 2}$, 1 time $\mathcal{O}_3^{\pm 2}$ and 1 time $\mathcal{O}_3^0$,
(II-b) $2j-2$ times $\mathcal{O}_2^{\pm 2}$, 1 time $\mathcal{O}_2^0$ and 2 times $\mathcal{O}_3^{\pm 2}$.

(III) Terms satisfying $k = 2j$.

Here, the i_0 such that $\eta_{i_0} = N$ is unique and equal to j, and no $\mathcal{O}_\ell^0$ appears in these terms. We regroup them in relation to the $\mathcal{O}_{r_i}$ that they contain:

(III-c) $2j-4$ times $\mathcal{O}_2^{\pm 2}$ and 4 times $\mathcal{O}_3^{\pm 2}$,
(III-c) $2j-3$ times $\mathcal{O}_2^{\pm 2}$, 2 times $\mathcal{O}_3^{\pm 2}$ and 1 time $\mathcal{O}_4^{\pm 2}$,
(III-c) $2j-2$ times $\mathcal{O}_2^{\pm 2}$ and 2 times $\mathcal{O}_4^{\pm 2}$,
(III-c) $2j-2$ times $\mathcal{O}_2^{\pm 2}$, 1 time $\mathcal{O}_3^{\pm 2}$ and 1 time $\mathcal{O}_5^{\pm 2}$,
(III-c) $2j-1$ times $\mathcal{O}_2^{\pm 2}$ and 1 time $\mathcal{O}_6^{\pm 2}$.

This list seems quite long, but fortunately most of the terms will ultimately vanish due to the fact that they are computed by means of some terms involved in $I_{2j}\boldsymbol{b}_{2j}I_{2j}$ and $I_{2j}\boldsymbol{b}_{2j+1}I_{2j}$.

In the sequel, the contribution to the third coefficient of the terms of type I-a), I-b), etc., will be denoted by $T_{\text{I-a)}}$, $T_{\text{I-b)}}$, etc.

4B. ***Terms of type*** I. We begin with an observation, whose proof is an easy extension of the computation (2-21), using the fact that $\mathscr{R}_x^j = 0$.

Lemma 4.1. *For any j-tuple $(a_1, \dots, a_j)$ of positive integers, we have*

$$X_{(a_1,\dots,a_j)} := I_{2j}\Bigg(\prod_{i=1}^{j} \mathscr{L}_0{}^{-a_i}\mathcal{O}_2^{+2}\Bigg)P^N = 0. \tag{4-5}$$

Terms of type I-a). In these terms, only some $\mathcal{O}_2^{\pm 2}$ appear. So there is either a unique i_0 such that $\eta_{i_0} = N$ which is then equal to j or $j+2$, or exactly two such i_0 which are then j and $j+2$.

Each term in the second case is a sum of terms of the form

$$-X_{(a_1,\dots,a_j)}\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-b}\mathcal{O}_2^{+2}X^*_{(a'_1,\dots,a'_j)} \tag{4-6}$$

with $a_i, a'_k, b \in \{1, 2\}$ (exactly one is equal to 2). By Lemma 4.1, these terms vanish.

Now, each term in the first case is equal or adjoint to a term of the form

$$I_{2j}\Bigg(\prod_{i=1}^{j+2} \mathscr{L}_0{}^{-1}\mathcal{O}_2^{\varepsilon_i}\Bigg)\mathscr{P}I_0(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^j\mathscr{P}I_0)^*, \tag{4-7}$$

where $\varepsilon_i \in \{-2, +2\}$ (exactly one of the ε_i is equal to -2). By Lemma 4.1, these terms vanish.

Finally, every term of type I-a) vanishes and $T_{\text{I-a)}} = 0$.

Terms of type I-b). Using Lemma 4.1 as above, we see that the only nonzero terms of this type satisfy the condition that before the first index i such that $\eta_i = N$ and after the last, there must be a $\mathcal{O}_2^0$ appearing. As a consequence, the cases where two or three η_i are equal to N lead to vanishing terms. We now deal with the terms where $\eta_{j+1} = N$ and for $i \neq j+1$, $\eta_i = N^\perp$. Such terms are of the form

$$\begin{aligned}(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}\mathscr{P})&\\ \times (I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k'}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k'}\mathscr{P})^*,&\end{aligned} \tag{4-8}$$

for $0 \le k, k' \le j$. By the computations in Section 3B, an in particular (3-16), we find

$$\begin{aligned}&I_{2j}((\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k})\mathscr{P}I_0\\ &= I_{2j}\mathscr{R}_x^{j-k}\bigg[\frac{1}{6}\bigg(\mathrm{C}_{j+1}(j+1) - \frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\bigg)r_x^X - \frac{\mathrm{C}_j(k)}{4\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x}\bigg]\mathscr{R}_x^k\mathscr{P}I_0.\end{aligned} \tag{4-9}$$

Observe that r^X commutes with $\mathscr{R}$ and $\mathscr{R}^j = 0$. So the contribution of the terms of type I-b) is finally $T_{\text{I-b)}} = \mathrm{Pos}[\mathcal{T}'_3(j)]$.

4C. *Terms of type* II.

Terms of type II-a). In these terms, there are either only $\mathcal{O}_2^{-2}$ appearing at the right of the first P^N or only $\mathcal{O}_2^{+2}$ appearing at the left of the last P^N. Either way, all these terms vanish by Lemma 4.1. Hence $T_{\text{II-a)}} = 0$.

Terms of type II-b). For these terms, there are two possibilities.

Firstly, there are two indices i such that $\eta_i = N$, and then they are equal to j and $j+1$. In this case, either before the first P^N or after the last, there appear j $\mathcal{O}_2^{+2}$ (or $\mathcal{O}_2^{-2}$), so all these terms vanish.

Secondly, there is a unique i_0 such that $\eta_{i_0} = N$ and it is equal to j or $j+1$. We denote by S_1 (resp. S_2) the sum of the terms for which $i_0 = j$ (resp. $i_0 = j+1$). Then $S_1 = S_2^*$ and

$$\begin{aligned}(4\text{-}10)\quad S_2 &= \sum_{k,\ell} \{I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k \mathscr{P} I_0\} \\ &\qquad\qquad \times \{I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0\}^* \\ &= \Big\{\sum_k I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathcal{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^k \mathscr{P} I_0\Big\} \\ &\qquad\qquad \times \Big\{\sum_\ell I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0\Big\}^*.\end{aligned}$$

By (3-16) and (3-40) we find that the contribution of the terms of type II-b), i.e., $S_1(0,0) + S_2(0,0)$, is $T_{\text{II-b)}} = \operatorname{Sym}[\mathcal{T}_2(j)\mathcal{T}_3'(j)^*]$.

Terms of type II-c). The computation is the same as for terms of type II-b), except that in the case of a unique i_0 such that $\eta_{i_0} = N$, we must replace $\mathcal{O}_4^{+2}$ by $\mathcal{O}_3^{+2}$ and $\mathcal{O}_2^0$ by $\mathcal{O}_3^0$ in (4-10). Recall that A_k has been defined in (3-20). By (3-25) and (4-3), we find that the contribution of the terms of type II-c) is the symmetric operator associated to

$$(4\text{-}11)\qquad \Big\{\sum_k A_k\Big\}\Big\{\sum_\ell I_{2j}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\ell}\mathscr{L}_0^{-1}\mathcal{O}_3^{0}(\mathscr{L}_0^{-1}\mathcal{O}_2^{+2})^\ell \mathscr{P} I_0\Big\}^*.$$

By (4-3) we get $T_{\text{II-c)}} = 0$.

Terms of type II-d). Here again, we have the same possibilities concerning the indices i such that $\eta_i = N$ as for terms of types II-b) or II-c). If there are two such indices, then they are equal to j and $j+1$ and between the two corresponding P^N we will have the term $\mathcal{O}_2^0$. By (3-5), these terms vanish.

We now suppose that there is a unique i_0 such that $\eta_{i_0} = N$. Then $i_0 = j$ or $j+1$. As $\mathscr{R}_x^j = 0$, any term in which the two $\mathcal{O}_3$ and the $\mathcal{O}_2^0$ appear on the same side of P^N will vanish. A term with one $\mathcal{O}_3$ at the left and one $\mathcal{O}_3$ at the right of P^N is equal or adjoint to

$$(4\text{-}12)\qquad I_{2j}\Big(\prod_{i=1}^{j+1}\mathscr{L}_0^{-1}\mathcal{O}_{a_i}^{\varepsilon_i}\Big)\mathscr{P} I_0 \times A_k^*,$$

where $a_i = 2$ or 3 and $\varepsilon_i = +2$ except for exactly one i_1 satisfying $a_{i_1} = 2$ (for which $\varepsilon_{i_1} = 0$). By (3-25) and (4-3), the sum of these terms vanishes.

Finally, the other possibility is that the two $\mathcal{O}_3$ appear on the same side of P^N, and $\mathcal{O}_2^0$ on the other side. Recall that $A_{k,\ell}$ has been defined in (3-28). The sum of the remaining terms is equal to

$$\operatorname{Sym}\Big[\Big\{\sum_{k,\ell} A_{k,\ell}\Big\}\Big\{\sum_m I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-m}\mathscr{L}_0{}^{-1}\mathcal{O}_2^0(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^m\mathscr{P}I_0\Big\}^*\Big]. \tag{4-13}$$

As a result, the contribution of terms of type II-d) is $T_{\text{II-d)}} = \operatorname{Sym}[\mathcal{T}_1(j)\mathcal{T}_3'(j)^*]$.

4D. *Terms of type* III. The computations rely on similar arguments as in Sections 4B and 4C. We will therefore give the contribution of each sub-type directly.

Terms of type III-a). The contribution is $T_{\text{III-a)}} = \operatorname{Pos}[\mathcal{T}_1(j)]$.

Terms of type III-b). The contribution is $T_{\text{III-b)}} = \operatorname{Sym}[\mathcal{T}_1(j)\mathcal{T}_2(j)^*]$

Terms of type III-c). The contribution is $T_{\text{III-c)}} = \operatorname{Pos}[\mathcal{T}_2(j)]$.

Terms of type III-d). The sum of all these terms vanishes, $T_{\text{III-d)}} = 0$.

Terms of type III-e). These terms vanish, so that $T_{\text{III-e)}} = 0$.

By all the computations in Sections 4B, 4C and 4D, we get Theorem 0.8.

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DETERMINANT RANK OF C^*-ALGEBRAS

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Dedicated to George A. Elliott on his seventieth birthday

Let A be a unital C^*-algebra and let $U_0(A)$ be the group of unitaries of A which are path-connected to the identity. Denote by $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. Let $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ be the homomorphism defined by sending u to $\mathrm{diag}(u, 1_{n-1})$. We study the problem of when the map $i_A^{(1,n)}$ is an isomorphism for all n. We show that it is always surjective and that it is injective when A has stable rank one. It is also injective when A is a unital C^*-algebra of real rank zero, or A has no tracial state. We prove that the map is an isomorphism when A is Villadsen's simple AH-algebra of stable rank $k > 1$. We also prove that the map is an isomorphism for all Blackadar's unital projectionless separable simple C^*-algebras. Let $A = \mathrm{M}_n(C(X))$, where X is any compact metric space. We note that the map $i_A^{(1,n)}$ is an isomorphism for all n. As a consequence, the map $i_A^{(1,n)}$ is always an isomorphism for any unital C^*-algebra A that is an inductive limit of the finite direct sum of C^*-algebras of the form $\mathrm{M}_n(C(X))$ as above. Nevertheless we show that there is a unital C^*-algebra A such that $i_A^{(1,2)}$ is not an isomorphism.

1. Introduction

Let A be a unital C^*-algebra and let $U(A)$ be the unitary group. Denote by $U_0(A)$ the normal subgroup which is the connected component of $U(A)$ containing the identity of A. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and by $CU(A)$ the closure of $DU(A)$. We will study the group $U_0(A)/CU(A)$. Recently this group has become an important invariant for the structure of C^*-algebras. It plays an important role in the classification of C^*-algebras (see [Elliott and Gong 1996; Nielsen and Thomsen 1996; Elliott 1997; Thomsen 1997; Gong 2002; Elliott et al. 2007; Lin 2007; 2011; Gong et al. 2015], for example). It was shown in [Lin 2007] that the map $U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is an isomorphism for all $n \geq 1$ if A is a unital simple C^*-algebra of tracial rank at most one (see also [Lin

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2010b, Corollary 3.5]). In general, when A has stable rank k, it was shown by Rieffel [1987] that the map $U(\mathrm{M}_k(A))/U_0(\mathrm{M}_k(A)) \to U(\mathrm{M}_{k+m}(A))/U_0(\mathrm{M}_{k+m}(A))$ is an isomorphism for all integers $m \geq 1$. In this case $U(\mathrm{M}_k(A))/U_0(\mathrm{M}_k(A)) = K_1(A)$. This fact plays an important role in the study of the structure of C^*-algebras, in particular those C^*-algebras of stable rank one, since it simplifies computations when K-theory involved. Therefore it seems natural to ask when the map $i_A^{(1,n)} : U_0(A)/CU(A) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is an isomorphism. It will also greatly simplify our understanding and usage of the group when $i_A^{(1,n)}$ is an isomorphism for all n. The main tool to study $U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$ is the de la Harpe–Skandalis determinant, studied early by K. Thomsen [1995] (henceforth abbreviated [Th]), which involves the tracial state space $T(A)$ of A. On the other hand, we observe that when $T(A) = \varnothing$, $U_0(A)/CU(A) = \{0\}$. So we focus our attention on the case $T(A) \neq \varnothing$. One of the authors was asked repeatedly if the map $i_A^{(1,n)}$ is an isomorphism when A has stable rank one.

It turns out that it is easy to see that the map $i_A^{(1,n)}$ is always surjective for all n. Therefore the issue is when $i_A^{(1,n)}$ is injective.

Definition 1.1. Let A be a unital C^*-algebra. Consider the homomorphism

$$i_A^{(m,n)} \colon U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

(induced by $u \mapsto \mathrm{diag}(u, 1_{n-m})$) for integers $n \geq m \geq 1$. The determinant rank of A is defined to be

$$\mathrm{Dur}\, A = \min\{m \in \mathbb{N} \mid i_A^{(m,n)} \text{ is isomorphism for all } n > m\}.$$

If no such integer exists, we set $\mathrm{Dur}\, A = \infty$.

We show that if $A = \lim_{n\to\infty} A_n$, then $\mathrm{Dur}\, A \leq \sup_{n\geq 1}\{\mathrm{Dur}\, A_n\}$. We prove that $\mathrm{Dur}\, A = 1$ for all C^*-algebras of stable rank one, which answers the question mentioned above. We also show that $\mathrm{Dur}\, A = 1$ for any unital C^*-algebra A with real rank zero. A closely related and repeatedly used fact is that the map $u \to u + (1-e)$ is an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$ when A is a unital simple C^*-algebra of tracial rank at most one and $e \in A$ is a projection (see [Lin 2007, Theorem 6.7; 2010b, Theorem 3.4]). We show in this note that this holds for any simple C^*-algebra of stable rank one.

Given Rieffel's early result mentioned above, one might be led to think that, when A has higher stable rank, or at least when $A = C(X)$ for higher-dimensional finite CW complexes, $\mathrm{Dur}\, A$ is perhaps large. On the other hand it was suggested (see [Th, Section 3]) that $\mathrm{Dur}\, A = 1$ may hold for most unital simple separable C^*-algebras. We found out, somewhat surprisingly, that the determinant rank of $\mathrm{M}_n(C(X))$ is always 1 for any compact metric space X and for any integer $n \geq 1$. This, together with previous mentioned result, shows that if $A = \lim_{n\to\infty} A_n$, where A_n is a finite direct sum of C^*-algebras of the form $\mathrm{M}_n(C(X))$, then Dur $A = 1$. Furthermore, we found out that Dur $A = 1$ for all of Villadsen's examples of unital simple AH-algebras A with higher stable rank. This research suggests that when A has an abundant amount of projections then Dur A is likely to be 1 (see Proposition 3.6(3)). In fact, we prove that if A is a unital simple AH-algebra with property (SP), then Dur $A = 1$. On the other hand, however, we show that if A is a unital projectionless simple C^*-algebra and $\rho_A(K_0(A)) = \mathbb{Z}$, then Dur $A = 1$. Furthermore, if A is one of Blackadar's examples of unital projectionless simple separable C^*-algebras with infinite many extremal tracial states, then Dur $A = 1$. Indeed, it seems that it is difficult to find any example of unital separable simple C^*-algebras for which Dur A is larger than 1. Nevertheless Proposition 3.12 below provides a necessary condition for Dur $A = 1$. In fact, we find that a certain unital separable C^*-algebra violates this condition, which, in turn, provides an example of a unital separable C^*-algebra A such that Dur $A > 1$.

2. Preliminaries

In this section, we list some notation and basic known facts for convenience, many of which are taken from [Th] and other sources.

Definition 2.1. Let A be a C^*-algebra. Denote by $\mathrm{M}_n(A)$ the $n \times n$ matrix algebra of over A. If A is not unital, we will use $\widetilde{A}$, the unitization of A, so suppose that A is unital. For u in $U_0(A)$, let $[u]$ be the class of u in $U_0(A)/CU(A)$.

We view A^n as the set of all $n \times 1$ matrices over A. Set

$$S_n(A) = \left\{ (a_1, \ldots, a_n)^T \in A^n \;\middle|\; \sum_{i=1}^{n} a_i^* a_i = 1 \right\},$$

$$\mathrm{Lg}_n(A) = \left\{ (a_1, \ldots, a_n)^T \in A^n \;\middle|\; \sum_{i=1}^{n} b_i a_i = 1 \text{ for some } b_1, \ldots, b_n \in A \right\}.$$

According to [Rieffel 1983; 1987], the topological stable rank and the connected stable rank of A are defined as

$$\operatorname{tsr} A = \min\{n \in \mathbb{N} \mid \mathrm{Lg}_m(A) \text{ is dense in } A^m \text{ for all } m \geq n\}$$

$$\operatorname{csr} A = \min\{n \in \mathbb{N} \mid U_0(\mathrm{M}_m(A)) \text{ acts transitively on } S_m(A) \text{ for all } m \geq n\}.$$

If no such integer exists, we set tsr $A = \infty$ and csr $A = \infty$. These notions are very useful tools in computing K-groups of C^*-algebras (see, e.g., [Rieffel 1987; Xue 2000; 2001; 2010]).

Definition 2.2. Let A be a C^*-algebra. Denote by A_{sa} (resp. A_+) the set of all self-adjoint (resp. positive) elements in A. Denote by $T(A)$ the tracial state space of A. Let $\tau \in T(A)$. We will also use the notation τ for the unnormalized trace

$\tau \otimes \mathrm{Tr}_n$ on $\mathrm{M}_n(A)$, where Tr_n is the standard trace for $\mathrm{M}_n(\mathbb{C})$. Every tracial state on $\mathrm{M}_n(A)$ has the form $(1/n)\tau$.

Definition 2.3. For $a, b \in A$, set $[a,b] = ab - ba$. Furthermore, set

$$[A, A] = \left\{ \sum_{j=1}^{n} [a_j, b_j] \,\middle|\, a_j, b_j \in A,\ j = 1, \dots, n,\ n \geq 1 \right\}.$$

Now, let A_0 denote the subset of A_{sa} consisting of elements of the form $x - y$ for $x, y \in A_{\mathrm{sa}}$ with $x = \sum_{j=1}^{\infty} c_j c_j^*$ and $y = \sum_{j=1}^{\infty} c_j^* c_j$ (convergent in norm) for some sequence $\{c_j\}$ in A. By [Cuntz and Pedersen 1979], A_0 is a closed subspace of A_{sa}.

Proposition 2.4 [Cuntz and Pedersen 1979; Thomsen 1995, Section 3]. *Let A be a C^*-algebra with unit* 1. *The following statements are equivalent*:

(1) $A_0 = A_{\mathrm{sa}}$.

(2) $1 \in A_0$.

(3) $T(A) = \varnothing$.

(4) $A = \overline{[A, A]}$.

(5) $A_{\mathrm{sa}} = \overline{\mathrm{span}\{[a^*, a] \mid a \in A\}}$.

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). If $T(A) \neq \varnothing$, then there is a tracial state τ on A. Since $1 \in A_0$, it follows that there is a sequence $\{a_j\}$ in A such that $b = \sum_{j=1}^{\infty} a_j^* a_j$ and $c = \sum_{j=1}^{\infty} a_j a_j^*$ are convergent in A and $1 = b - c$. Thus, $\tau(b) = \sum_{j=1}^{\infty} \tau(a_j^* a_j) = \tau(c)$ and $\tau(1) = \tau(b - c) = 0$, a contradiction since $\tau(1) = 1$.

(3) $\Longrightarrow$ (1). This follows from the proof of [Th, Lemma 3.1].

(4) $\Longleftrightarrow$ (5). Let $a, b \in A$ and write $a = a_1 + i a_2$ and $b = b_1 + i b_2$, where $a_1, a_2, b_1, b_2 \in A_{\mathrm{sa}}$. Then

$$[a, b] = [a_1, b_1] - [a_2, b_2] + i[a_2, b_1] + i[a_1, b_2]. \tag{2-1}$$

Put $c_1 = a_1 + i b_1$, $c_2 = a_2 + i b_2$, $c_3 = a_2 + i b_1$ and $c_4 = a_1 + i b_2$. Then, from (2-1), we get that

$$[a, b] = \frac{1}{2i}[c_1^*, c_1] - \frac{1}{2i}[c_2^*, c_2] + \frac{1}{2}[c_3^*, c_3] + \frac{1}{2}[c_4^*, c_4]. \tag{2-2}$$

So, by (2-2), (4) and (5) are equivalent.

(5) $\Longrightarrow$ (1). Let $x \in \mathrm{span}\{[a^*, a] \mid a \in A\}$. Then there are elements $a_1, \dots, a_k \in A$ and positive numbers $\lambda_1, \dots, \lambda_k$ such that $x = \sum_{i=1}^{j} \lambda_i [a_i^*, a_i] - \sum_{i=j+1}^{k} \lambda_i [a_i^*, a_i]$ for some $j \in \{1, \dots, k\}$. Put $c_i = \sqrt{\lambda_i}\, a_i$, $i = 1, \dots, j$ and $c_i^* = \sqrt{\lambda_i}\, a_i^*$ when

$i = j+1, \dots, k$. Then $x = \sum_{i=1}^{k} c_i^* c_i - \sum_{i=1}^{k} c_i c_i^* \in A_0$. Since A_0 is closed, we get that

$$A_{\mathrm{sa}} = \overline{\operatorname{span}\{[a^*, a] \mid a \in A\}} \subset \overline{A_0} = A_0 \subset A_{\mathrm{sa}}.$$

(1) $\Longrightarrow$ (5). According to the definition of A_0, every element $x \in A_0$ has the form $x = x_1 - x_2$, where $x_1 = \sum_{i=1}^{\infty} z_i^* z_i$ and $x_2 = \sum_{i=1}^{\infty} z_i z_i^*$. Thus, $x \in \overline{\operatorname{span}\{[a^*, a] \mid a \in A\}}$ and hence $A_{\mathrm{sa}} = \overline{\operatorname{span}\{[a^*, a] \mid a \in A\}}$. □

Combining Proposition 2.4 with Definition 2.2, we have:

Corollary 2.5. *Let A be a unital C^*-algebra with $A_0 = A_{\mathrm{sa}}$. Then* $(\mathrm{M}_n(A))_0 = (\mathrm{M}_n(A))_{\mathrm{sa}}$.

Let $a, b \in A_{\mathrm{sa}}$. Then, for any $n \ge 1$,

$$\exp(ia)\exp(ib)\Big(\exp\Big(-i\frac{a}{n}\Big)\exp\Big(-i\frac{b}{n}\Big)\Big)^n \in DU(A)$$

and $\exp(-i(a+b)) = \lim_{n\to\infty}(\exp(-ia/n)\exp(-ib/n))^n$ by the Trotter product formula [Masani 1981, Theorem 2.2]. So $\exp(ia)\exp(ib)\exp(-i(a+b)) \in CU(A)$. Consequently,

$$[\exp(ia)][\exp(ib)] = [\exp(i(a+b))] \quad \text{in } U_0(A)/CU(A). \tag{2-3}$$

The following is taken from the proof of [Th, Lemma 3.1].

Lemma 2.6. *Let $a \in A_{\mathrm{sa}}$.*

(1) *If $a \in A_0$, then* $[\exp(ia)] = 0$ *in* $U_0(A)/CU(A)$;

(2) *If $T(A) \neq \varnothing$ and $\tau(a) = \tau(b)$ for all $\tau \in T(A)$, then $a - b \in A_0$ and* $[\exp(ia)] = [\exp(ib)]$ *in* $U_0(A)/CU(A)$.

Combining Lemma 2.6(1) with Corollary 2.5, we have

Corollary 2.7. *If $T(A) = \varnothing$, then $U_0(\mathrm{M}_n(A)) = CU(\mathrm{M}_n(A))$ for $n \ge 1$.*

Definition 2.8. Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Let $PU_0^n(A)$ denote the set of all piecewise smooth maps $\xi : [0, 1] \to U_0(\mathrm{M}_n(A))$ with $\xi(0) = 1_n$, where 1_n is the unit of $\mathrm{M}_n(A)$. For $\tau \in T(A)$, the de la Harpe–Skandalis function Δ_τ^n on $PU_0^n(A)$ is given by

$$\Delta_\tau^n(\xi(t)) = \frac{1}{2\pi i}\int_0^1 \tau(\xi'(t)(\xi(t))^*)\,\mathrm{d}t \quad \text{for all } \xi \in PU_0^n(A).$$

Note that we use an unnormalized trace $\tau = \tau \otimes \mathrm{Tr}_n$ on $\mathrm{M}_n(A)$. This gives a homomorphism $\Delta^n : PU_0^n(A) \to \mathrm{Aff}(T(A))$, the space of all real affine continuous functions on $T(A)$.

We list some properties of $\Delta_\tau^n(\cdot)$:

Lemma 2.9 [de la Harpe and Skandalis 1984, Lemmas 1 and 3]. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Let $\xi_1, \xi_2, \xi \in PU_0^n(A)$. Then*:

(1) $\Delta_\tau^n(\xi_1(t)) = \Delta_\tau^n(\xi_2(t))$ *for all* $\tau \in T(A)$, *if* $\xi_1(1) = \xi_2(1)$ *and*

$$\xi_1 \xi_2^* \in U_0(\widetilde{(C_0(S^1, \mathrm{M}_n(A))}).$$

(2) *There are* $y_1, \dots, y_k \in \mathrm{M}_n(A)_{\mathrm{sa}}$ *such that* $\Delta_\tau^n(\xi(t)) = \sum_{j=1}^k \tau(y_j)$ *for all* $\tau \in T(A)$ *and* $\xi(1) = \exp(i2\pi y_1) \cdots \exp(i2\pi y_k)$.

Definition 2.10. Let A be a C^*-algebra with $T(A) \neq \varnothing$. Let $\mathrm{Aff}(T(A))$ be the set of all real continuous affine functions on $T(A)$. Define $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ by

$$\rho_A([p])(\tau) = \tau(p) \quad \text{for all } \tau \in T(A),$$

where $p \in \mathrm{M}_n(A)$ is a projection.

Define $P_n(A)$ to be the subgroup of $K_0(A)$ generated by projections in $\mathrm{M}_n(A)$. Denote by $\rho_A^n(K_0(A))$ the subgroup $\rho_A(P_n(A))$ of $\rho_A(K_0(A))$. In particular, $\rho_A^1(K_0(A))$ is the subgroup of $\rho_A(K_0(A))$ generated by the images of projections in A under the map ρ_A.

Definition 2.11. Let A be a unital C^*-algebra. Denote by $LU_0^n(A)$ the set of piecewise smooth loops in

$$U(\widetilde{C_0(S^1, \mathrm{M}_n(A))}).$$

Then, by Bott periodicity, $\Delta^n(LU_0^n(A)) \subset \rho_A(K_0(A))$. Denote by

$$\mathfrak{q}^n : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/\overline{\Delta^n(LU_0^n(A))}$$

the quotient map. Put $\overline{\Delta^n} = \mathfrak{q}^n \circ \Delta^n$. Since $\overline{\Delta^n}$ vanishes on $LU_0^n(A)$, we also use $\overline{\Delta^n}$ for the homomorphism from $U_0(\mathrm{M}_n(A))$ into $\mathrm{Aff}(T(A))/\overline{\Delta^n(LU_0^n(A))}$. An important fact that we will repeatedly use is that *the kernel of* $\overline{\Delta^n}$ *is exactly* $CU(\mathrm{M}_n(A))$, by [Th, Lemma 3.1]. In other words, if $u \in U_0(\mathrm{M}_n(A))$ and $\overline{\Delta^n}(u) = 0$, then $u \in CU(\mathrm{M}_n(A))$.

Corollary 2.12. *Let A be a unital C^*-algebra and let $u \in U_0(\mathrm{M}_n(A))$ for $n \geq 1$. Then there are an $a \in A_{\mathrm{sa}}$ and a $v \in CU(\mathrm{M}_n(A))$ such that*

$$u = \mathrm{diag}(\exp(i2\pi a), 1_{n-1})v$$

(*in the case $n = 1$, we define* $\mathrm{diag}(\exp(i2\pi a), 1_{n-1}) = \exp(i2\pi a)$).

Moreover, if there is a $u \in PU_0^n(A)$ with $u(1) = u$, we can choose a self-adjoint element a so that $\hat{a} = \Delta^n(u(t))$, where $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$.

Proof. Fix a piecewise smooth path $u(t) \in PU_0^n(A)$ with $u(0) = 1$ and $u(1) = u$. By Lemma 2.9(2), there are $a_1, a_2, \ldots, a_m \in \mathrm{M}_n(A)_{\mathrm{sa}}$ such that

$$u = \prod_{j=1}^{m} \exp(i2\pi a_j) \quad \text{and} \quad \Delta_\tau^n(u(t)) = \tau \sum_{j=1}^{m} a_j \quad \text{for all } \tau \in T(A).$$

Put $a_0 = \sum_{j=1}^{n} a_j$. Write $a_0 = (b_{i,j})_{n\times n}$. Define $a = \sum_{i=1}^{n} b_{i,i}$. Then $a \in A_{\mathrm{sa}}$. Moreover,

$$\overline{\Delta^n}(\mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u) = 0.$$

Thus, by [Th, Lemma 3.1], $\mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u \in CU(\mathrm{M}_n(A))$. Put $v = \mathrm{diag}(\exp(-i2\pi a), 1_{n-1})u$. Then $u = \mathrm{diag}(\exp(i2\pi a), 1_{n-1})v$. □

3. Determinant rank

Let A be a unital C^*-algebra. Consider the homomorphism

$$i_A^{(m,n)} : U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

for integers $n \geq m \geq 1$.

Proposition 3.1. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Then*

$$i_A^{(m,n)} : U_0(\mathrm{M}_m(A))/CU(\mathrm{M}_m(A)) \to U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A))$$

is surjective for $n \geq m \geq 1$.

Proof. It suffices to show that $i_A^{(1,n)}$ is surjective. Let $u \in U_0(\mathrm{M}_n(A))$. It follows from Corollary 2.12 that $u = \mathrm{diag}(\exp(i2\pi a), 1_{n-1})v$ for some $a \in A_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_n(A))$. Then $i_A^{(1,n)}([\exp(i2\pi a)]) = [u]$. □

Lemma 3.2. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Assume $u \in U_0(\mathrm{M}_m(A))$.*

(1) *If $\Delta^n(\mathrm{diag}(u(t), 1_{n-m}) \in \overline{\Delta^n(LU_0^n(A))}$ for some $n > m$, where $\{u(t) : t \in [0,1]\}$ is a piecewise smooth path with $u(0) = 1_m$ and $u(1) = u$, then, for any $\epsilon > 0$, there exist $a \in \mathrm{M}_m(A)_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $b \in \mathrm{M}_m(A)_{\mathrm{sa}}$, $v \in CU(\mathrm{M}_m(A))$ and $w \in LU_0^n(A)$ such that*

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \textit{and} \quad \tau(b) = \Delta_\tau^n(w(t)) \ \textit{for all } \tau \in T(A). \tag{3-1}$$

(2) *If $\Delta^m(u(t)) \in \overline{\rho_A(K_0(A))}$ for some $u \in PU_0^m(A)$ with $u(1) = u$, then, for any $\epsilon > 0$, there exist $a \in \mathrm{M}_m(A)_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $b \in \mathrm{M}_m(A)_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_m(A))$ such that*

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \textit{and} \quad \hat{b} \in \rho_A(K_0(A)), \tag{3-2}$$

where $\hat{b}(\tau) = \tau(b)$ for all $\tau \in T(A)$.

Proof. Let $\epsilon > 0$. For (1), there is a $w \in LU_0^n(A)$ such that

$$\sup\{|\Delta_\tau^n(u(t)) - \Delta_\tau^n(w(t))| : \tau \in T(A)\} < \epsilon/3\pi. \tag{3-3}$$

There is an $a_1 \in M_m(A)_{\text{sa}}$ by Corollary 2.12 such that

$$\tau(a_1) = \Delta_\tau^n(u(t)) - \Delta_\tau^n(w(t)) \quad \text{for all } \tau \in T(A). \tag{3-4}$$

Combining (3-3) with [Cuntz and Pedersen 1979] and the proof of [Th, Lemma 3.1], we can find $a \in \mathrm{M}_m(A)_{\text{sa}}$ such that $\tau(a) = \tau(a_1)$ for all $\tau \in T(A)$ and $\|a\| < \epsilon/2\pi$. There is also a $b \in A_{\text{sa}}$ such that $\tau(b) = -\Delta_\tau^n(w(t))$ for all $\tau \in T(A)$. Put

$$v(t) = \exp(-i2\pi bt)\exp(-i2\pi at)u(t) \quad \text{for } t \in [0, 1] \tag{3-5}$$

and $v = v(1)$. Then $\Delta^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(A)$. Then $u = \exp(i2\pi a)\exp(i2\pi b)v$.

For (2), there are an integer $n \geq m$ and projections $p, q \in \mathrm{M}_n(A)$ such that (for a piecewise smooth path $\{u(t) : t \in [0, 1]\}$ with $u(0) = 1_n$ and $u(1) = u$)

$$\|\Delta_\tau^m(u(t)) - \tau(p) + \tau(q)\| < \epsilon \quad \text{for all } \tau \in T(A). \tag{3-6}$$

Let $b \in \mathrm{M}_m(A)_{\text{sa}}$ such that $\tau(b) = \tau(p) - \tau(q)$ for all $\tau \in T(A)$ (see the proof above); there is an $a \in \mathrm{M}_m(A)_{\text{sa}}$ with $\|a\| < \epsilon$ such that

$$\tau(a) = \Delta_\tau^m(u(t)) - \tau(p) + \tau(q) \quad \text{for all } \tau \in T(A). \tag{3-7}$$

Let $v = u\exp(-i2\pi a)\exp(-i2\pi b)$ and $v(t) = u(t)\exp(-i2\pi at)\exp(-i2\pi bt)$. Then $\Delta_\tau^n(v(t)) = 0$. It follows from [Th, Lemma 3.1] that $v \in CU(\mathrm{M}_m(A))$. □

Let A be a unital C^*-algebra. Let Dur A be defined as in Definition 1.1. It follows from Corollary 2.7 that if $T(A) = \varnothing$ then Dur $A = 1$.

Proposition 3.3. *Let A be a unital C^*-algebra. Then, for any integer $n \geq 1$,*

$$\mathrm{Dur}(\mathrm{M}_n(A)) \leq \left\lfloor \frac{\mathrm{Dur}\, A - 1}{n} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is the integer part of x.

Proof. Note that $n(\lfloor(\mathrm{Dur}\, A - 1)/n\rfloor + 1) \geq \mathrm{Dur}\, A$. □

Theorem 3.4. *Let A be a unital C^*-algebra, and $I \subset A$ a closed ideal of A such that the quotient map $\pi : A \to A/I$ induces the surjective map from $K_0(A)$ onto $K_0(A/I)$. Then* $\mathrm{Dur}(A/I) \leq \mathrm{Dur}\, A$.

Proof. Let $m = \mathrm{Dur}\, A$ and $n > m$. Let $u \in U_0(\mathrm{M}_m(A/I))$ be a unitary such that $\mathrm{diag}(u, 1_{n-m}) \in CU(\mathrm{M}_n(A/I))$. We will show that $u \in CU(\mathrm{M}_m(A/I))$.

Let $\epsilon > 0$. By Lemma 3.2, without loss of generality we may assume that there are $a_1,\ b_1 \in (\mathrm{M}_m(A/I))_{\mathrm{sa}}$ such that

$$u = \exp(i2\pi a_1)\exp(i2\pi b_1)v, \\ v \in CU(\mathrm{M}_m(A/I)), \quad \|a_1\| < \epsilon \quad \text{and} \quad \tau(b_1) = \tau(q_1) - \tau(q_2), \tag{3-8}$$

where $q_1, q_2 \in M_K(A/I)$ are projections for some large $K \geq m$, for all $\tau \in T(A/I)$. By the assumption, without loss of generality we may assume further that there are projections $p_1, p_2 \in \mathrm{M}_K(A)$ such that $\pi_*([p_1 - [p_2]) = [q_1] - [q_2]$, where $\pi_* : K_0(A) \to K_0(A/I)$ is induced by π. Let $b_2 \in (\mathrm{M}_m(A))_{\mathrm{sa}}$ such that $\tau(b_2) = \tau(p_1) - \tau(p_2)$ for all $\tau \in T(A)$. There exists an $a \in (\mathrm{M}_m(A))_{\mathrm{sa}}$ such that $\pi_m(a) = a_1$, where $\pi_m : \mathrm{M}_m(A) \to \mathrm{M}_m(A/I)$ is the map induced by π. Then, by (3-8),

$$\pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2))u^* \in CU(\mathrm{M}_m(A/I)). \tag{3-9}$$

Put $u_1 = \pi_m(\exp(i2\pi a))\pi_m(\exp(i2\pi b_2))$. Let $w = \exp(i2\pi b_2)$. Then $\bar{\Delta}(w) = 0$. Since $m = \mathrm{Dur}\, A$, this implies that $w \in CU(M_m(A))$. It follows that $\pi_m(w) \in CU(\mathrm{M}_m(A/I))$, which implies by (3-9) that $\mathrm{dist}\big(u, CU(\mathrm{M}_m(A/I))\big) < \epsilon$. □

Theorem 3.5. *Let $A = \lim_{n\to\infty}(A_n, \phi_n)$ be a unital C^*-algebra, where each A_n is unital. Suppose that* $\mathrm{Dur}\, A_n \leq r$ *for all n. Then* $\mathrm{Dur}\, A \leq r$.

Proof. We write $\phi_{n_1,n_2} : A_{n_1} \to A_{n_2}$ for $\phi_{n_2} \circ \phi_{n_2-1} \circ \cdots \circ \phi_{n_1}$ and $\phi_{n_1,\infty} : A_{n_1} \to A$ for the map induced by the inductive limit system. Let $u \in U_0(\mathrm{M}_r(A))$ such that $u_1 = \mathrm{diag}(u, 1_{n-r}) \in CU(\mathrm{M}_n(A))$ for some $n > r$. Let $\epsilon > 0$. There is a $v \in DU(\mathrm{M}_n(A))$ such that

$$\|u_1 - v\| < \frac{\epsilon}{8n}. \tag{3-10}$$

Write $v = \prod_{j=1}^K v_j$, where $v_j = x_j y_j x_j^* y_j$ and $x_j, y_j \in U_0(\mathrm{M}_n(A))$ for $j = 1, 2, \ldots, K$. Choose a large $N \geq 1$ such that there are $v' \in U_0(\mathrm{M}_r(A_N))$ and $x_j', y_j' \in U_0(\mathrm{M}_n(A_N))$ such that

$$\|u - \phi_{N,\infty}(u')\| < \frac{\epsilon}{8nK} \quad \text{and} \quad \|\phi_{N,\infty}(x_j') - x_j\| < \frac{\epsilon}{8nK} \tag{3-11}$$

for $j = 1, 2, \ldots, K$. Then we have by (3-10) and (3-11)

$$\Big\|\phi_{N,\infty}(u_1') - \prod_{j=1}^K \phi_{N,\infty}(v_j')\Big\| < \frac{\epsilon}{4n}, \tag{3-12}$$

for $j = 1, 2, \ldots, K$, where $u_1' = \mathrm{diag}(u', 1_{n-r})$ and $v_j' = x_j' y_j' (x_j')^* (y_j')^*$. Then (3-12) implies that there is an $N_1 > N$ such that

$$\Big\|\phi_{N,N_1}(u_1') - \prod_{j=1}^K \phi_{N,N_1}(v_j')\Big\| < \frac{\epsilon}{2n}. \tag{3-13}$$

Put $U = \phi_{N,N_1}(u')$, $U_1 = \operatorname{diag}(U, 1_{n-r})$ and $w_j = \phi_{N,N_1}(v_j')$, $j = 1, 2, \ldots, K$. Note that $\phi_{N_1,\infty}(U) = \phi_{N,\infty}(u')$. There is an $a \in (\mathrm{M}_n(A_{N_1}))_{\mathrm{sa}}$ (by (3-13)) such that

$$U_1 = \exp(i2\pi a)\prod_{j=1}^{K} w_j \quad \text{and} \quad \|a\| < 2 \arcsin\frac{\epsilon}{8n}. \tag{3-14}$$

There is a $b \in (\mathrm{M}_r(A_{N_1}))_{\mathrm{sa}}$ such that

$$\tau(b) = \tau(a) \quad \text{for all } \tau \in T(A) \quad \text{and} \quad \|b\| < 2n \arcsin\frac{\epsilon}{8n}. \tag{3-15}$$

Put $W = \operatorname{diag}(U\exp(-i2\pi b), 1_{n-r})$; then $W \in CU(\mathrm{M}_n(A_{N_1}))$. Since $\operatorname{Dur} A_{N_1} \le r$, we conclude that $U\exp(-i2\pi b) \in CU(\mathrm{M}_r(A_{N_1}))$. It follows that

$$\phi_{N_1,\infty}(U\exp(-i2\pi b)) \in CU(\mathrm{M}_r(A)).$$

However, by (3-10), (3-11), (3-15),

$$\begin{aligned}\|u - \phi_{N_1,\infty}(U\exp(-i2\pi b))\| &\\ \le \|u - \phi_{N,\infty}(u')\| &+ \|\phi_{N_1,\infty}(U) - \phi_{N_1,\infty}(U\exp(-i2\pi b))\| \\ < \frac{\epsilon}{8nK} &+ \|1 - \exp(-i2\pi\phi_{N_1,\infty}(b))\| < \frac{\epsilon}{8nK} + \epsilon/4 < \epsilon.\end{aligned}$$

Therefore, $\operatorname{Dur} A \le r$. □

Proposition 3.6. *Let A be a unital C^*-algebra with $T(A) \ne \varnothing$. Let $a \in A_{\mathrm{sa}}$ and put $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$.*

1. *If* $\exp(2\pi i a) \in CU(A)$, *then* $\hat{a} \in \overline{\rho_A(K_0(A))}$.
2. *If $u \in U_0(A)$ and for some piecewise smooth path $\{u(t) : t \in [0,1]\}$ with $u(0) = 1$ and $u(1) = u$, $\Delta^1(u(t)) \in \rho_A^k(K_0(A))$ for some $k \ge 1$, then* $\operatorname{diag}(u, 1_{k-1}) \in CU(\mathrm{M}_k(A))$.
3. *If* $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$, *then* $\operatorname{Dur} A = 1$.

Proof. Part (1) follows from [Th].

(2) By applying Corollary 2.12, there exists a $v \in CU(A)$ such that

$$u = \exp(i2\pi a)v \qquad \text{and} \qquad \tau(a) = \Delta_\tau^1(u(t)) \text{ for all } \tau \in T(A).$$

So for any $\epsilon \in (0,1)$, there are projections $p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2} \in \mathrm{M}_k(A)$ such that

$$\sup\left\{\left|\sum_{j=1}^{m_1}\tau(p_j) - \sum_{j=1}^{m_2}\tau(q_j) - \tau(a)\right| : \tau \in T(A)\right\} < \frac{\arcsin(\epsilon/4)}{\pi}. \tag{3-16}$$

Set $b = \sum_{j=1}^{m_1} p_j - \sum_{j=1}^{m_2} q_j$ and $a_0 = \mathrm{diag}(a, \overbrace{0, 0, \ldots, 0}^{(k-1)})$. Then $a_0, b \in \mathrm{M}_k(A)_{\mathrm{sa}}$ and

$$|\tau(a_0) - \tau(b)| < \frac{\arcsin(\epsilon/4)}{k\pi} \quad \text{for all } \tau \in T(\mathrm{M}_k(A))$$

by (3-16). Thus, by the proof of [Th, Lemma 3.1], we have

$$\inf\{\|a_0 - b - x\| \mid x \in (\mathrm{M}_k(A))_0\} \\ = \sup\{|\tau(a_0 - b)| \mid \tau \in T(\mathrm{M}_k(A))\} \le \frac{\arcsin(\epsilon/4)}{k\pi}.$$

Choose $x_0 \in (\mathrm{M}_k(A))_0$ such that $\|a_0 - b - x_0\| < 2\arcsin(\epsilon/4)/k\pi$. Put $y_0 = a_0 - b - x_0$. Then $\|y_0\| \le 2\arcsin(\epsilon/4)/k\pi$. Put $u_1 = \mathrm{diag}(u, 1_{k-1})\exp(-i2\pi y_0)$. Define

$$w(t) = \mathrm{diag}(u(t), 1_{k-1})\exp(-i2\pi y_0 t)\prod_{j=1}^{m_1}\exp(-i2\pi p_j t)\prod_{j=1}^{m_2}\exp(i2\pi q_j t)$$

for $t \in [0, 1]$. Then $w(0) = 1$, $w(1) = u(1)\exp(-i2\pi y_0) = u_1$ and, moreover,

$$\begin{aligned}\Delta_\tau^k(w(t)) &= \tau(a) - \tau(y_0) - \Big(\sum_{j=1}^{m_1}\tau(p_j) - \sum_{j=1}^{m_2}\tau(q_j)\Big) \\ &= \tau(a) - \tau(a_0) + \tau(b) - \tau(x_0) - \tau(b) \\ &= \tau(a) - \tau(a_0) = 0 \qquad\qquad \text{for all } \tau \in T(A).\end{aligned}$$

It follows that $w(1) = u_1 \in CU(\mathrm{M}_k(A))$. Then

$$\|\mathrm{diag}(u, 1_{k-1}) - u_1\| = \|\exp(i2\pi y_0) - 1_k\| < \epsilon.$$

(3) Let $u \in U_0(A)$ such that $\mathrm{diag}(u, 1_{n-1}) \in CU(\mathrm{M}_n(A))$. Let $u(t)$ be a piecewise smooth path with $u(0) = 1$ and $u(1) = u$. Then

$$\Delta^1(u(t)) \in \overline{\rho_A(K_0(A))} = \overline{\rho_A^1(K_0(A))}.$$

By Part (2), $u \in CU(A)$. This implies that Dur $A = 1$. □

Proposition 3.7. *Let X be a compact metric space. Then* $\mathrm{Dur}\big(\mathrm{M}_n(C(X))\big) = 1$ *for all* $n \ge 1$.

Proof. By Proposition 3.3, it suffices to consider the case $A = C(X)$. One has

$$\rho_A^1(K_0(A)) = C(X, \mathbb{Z}) = \rho_A(K_0(A)).$$

It follows from Proposition 3.6(3) that Dur $A = 1$. □

Combining Theorem 3.5 with Proposition 3.7, we have:

Corollary 3.8. *Let* $A = \lim_{n\to\infty}(A_n, \phi_n)$, *where* $A_m = \bigoplus_{j=1}^{m(n)} \mathrm{M}_{k(n,j)}(X_{n,j})$ *and each* $X_{n,j}$ *is a compact metric space. Then* $\mathrm{Dur}\, A = 1$.

Theorem 3.9. *Let A be a unital C^*-algebra with real rank zero. Then* $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ *and* $\mathrm{Dur}\, A = 1$.

Proof. By Corollary 2.7, we may assume that $T(A) \neq \varnothing$. Since A is of real rank zero, by [Zhang 1990, Theorem 3.3], for any $n \geq 2$ and any nonzero projection $p \in \mathrm{M}_n(A)$, there are projections $p_1, \dots, p_n \in A$ such that $p \sim \mathrm{diag}(p_1, \dots, p_n)$ in $\mathrm{M}_n(A)$. Thus, $\tau(p) = \sum_{j=1}^n \tau(p_j)$ for all $\tau \in T(A)$ and, consequently, $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$. It follows from Proposition 3.6(3) that $\mathrm{Dur}\, A = 1$. □

Theorem 3.10. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. If* $\mathrm{csr}(C(S^1, A)) \leq n+1$ *for some* $n \geq 1$, *then* $\mathrm{Dur}\, A \leq n$.

Proof. Let $u \in U_0(\mathrm{M}_n(A))$ such that $\mathrm{diag}(u, 1_k) \in CU(\mathrm{M}_{n+k}(A))$ for some integer $k \geq 1$. Let $\{u(t) : t \in [0,1]\}$ be a piecewise smooth path with $u(0) = 1_n$ and $u(1) = u$. By [Th], $\Delta^{n+k}(\mathrm{diag}(u(t), 1_k)) \in \overline{\Delta^{n+k}(LU_0^{n+k}(A))}$. It follows from Lemma 3.2(1) that, for any $\epsilon > 0$, there are $a, b \in \mathrm{M}_n(A)_{\mathrm{sa}}$ and $v \in CU(\mathrm{M}_n(A))$ with $\|a\| < 2\arcsin(\epsilon/4)/\pi$ such that

$$u = \exp(i2\pi a)\exp(i2\pi b)v \quad \text{and} \quad \tau(b) = \Delta_\tau^{n+k}(w(t)) \text{ for all } \tau \in T(A), \tag{3-17}$$

where $w \in LU_0^{n+k}(A)$. Since $\mathrm{csr}(C(S^1, A)) \leq n+1$, by Proposition 2.6 of [Rieffel 1987] there is a $w_1 \in LU_0^n(A)$ such that $\mathrm{diag}(w_1, 1_{n+k})$ is homotopy to w. In particular, $\Delta_\tau^n(w_1(t)) = \Delta_\tau^{n+k}(w(t))$ for all $\tau \in T(A)$. Consider the piecewise smooth path

$$U(t) = \exp(-i2\pi at)\exp(i2\pi bt)w_1^*(t), \quad t \in [0,1].$$

Then $U(0) = 1_n$ and $U(1) = \exp(i2\pi b)$. We compute that $\Delta_\tau^n(U(t)) = 0$ for all $\tau \in T(A)$. It follows by [Th, Lemma 3.1] that $\exp(i2\pi b) \in CU(\mathrm{M}_n(A))$. By (3-17),

$$[u] = [\exp(i2\pi a)] \quad \text{in } U_0(\mathrm{M}_n(A))/CU(\mathrm{M}_n(A)),$$

Therefore $\mathrm{dist}\big(u, CU(\mathrm{M}_n(A))\big) \leq \|\exp(i2\pi a) - 1_n\| < \epsilon$. □

Corollary 3.11. *Let A be a unital C^*-algebra of stable rank one. Then* $\mathrm{Dur}\, A = 1$.

Proof. This follows from the inequality $\mathrm{csr}(C(S^1, A)) \leq \mathrm{tsr}\, A + 1$ (see [Rieffel 1983, Corollary 8.6]) and Theorem 3.10. □

We end this section with the following:

Proposition 3.12. *Let A be a unital C^*-algebra. Suppose that there is a projection* $p \in \mathrm{M}_2(A)$ *such that, for any* $x \in K_0(A)$ *with* $\rho_A(x) = \rho_A([p])$, *no unitary in* $U(\widetilde{C})$ *represents* x, *where* $C = C_0((0,1), A)$. *Then* $\mathrm{Dur}\, A > 1$.

Proof. There exists an $a \in A_+$ such that $\tau(a) = \rho_A([p])(\tau)$ for all $\tau \in T(A)$. Put $u = \exp(i2\pi a)$ and $v = \mathrm{diag}(u, 1)$. Then it follows from Proposition 3.6(2) that $v \in CU(\mathrm{M}_2(A))$. This implies that $i_A^{(1,2)}([u]) = 0$. Now we show that $u \notin CU(A)$. Let

$$w(t) = \exp(2i(1-t)\pi a) \quad \text{for all } t \in [0, 1].$$

Then $w(0) = u$ and $w(1) = 1_A$. If $u \in CU(A)$, then, by [Th, Lemma 3.1], there is a continuous and piecewise smooth path of unitaries $\xi \in \widetilde{C}$, where $C = C_0((0, 1), A)$, such that

$$\Delta_\tau(\xi(t)) = \tau(p) \quad \text{for all } \tau \in T(A). \tag{3-18}$$

The Bott map shows that the unitary ξ is homotopic to a projection loop which corresponds to some $x \in K_0(A)$ with $\rho_A(x) = \rho_A([p])$, which contradicts the assumption. □

4. Simple C^*-algebras

Let us begin with the following:

Theorem 4.1. *Let A be a unital infinite-dimensional simple C^*-algebra of real rank zero with $T(A) \neq \varnothing$. Then*

$$\overline{\rho_A^1(K_0(A))} = \mathrm{Aff}(T(A)) \quad \textit{and} \quad U_0(A) = CU(A).$$

Proof. Let $p \in A$ be a nonzero projection, let $\lambda = n/m$ with $n, m \in \mathbb{N}$ and let $\epsilon > 0$. Then by Zhang's half theorem (see [Lin 2010a, Lemma 9.4]), there is a projection $e \in A$ such that $\max_{\tau \in T(A)} |\tau(p) - n\tau(e)| < n\epsilon/m$. Thus,

$$\max_{\tau \in T(A)} |\lambda\tau(p) - m\tau(e)| < \epsilon,$$

and consequently $r\rho_A(p) \in \overline{\rho_A^1(K_0(A))}$ for all $r \in \mathbb{R}$.

Let $a \in A_{\mathrm{sa}}$. Since A has real rank zero, a is a limit of the form $\sum_{j=1}^k \lambda_j p_j$, where $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in A and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$. Therefore $\hat{a} \in \overline{\rho_A^1(K_0(A))}$ by the above argument, where $\hat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$. Since $\mathrm{Aff}(T(A)) = \{\hat{a} \mid a \in A_{\mathrm{sa}}\}$ by [Lin 2007, Theorem 9.3], it follows from Theorem 3.9 that

$$\mathrm{Aff}(T(A)) \subset \overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))} \subset \mathrm{Aff}(T(A)),$$

that is, $\mathrm{Aff}(T(A)) = \overline{\rho_A^1(K_0(A))}$.

Note that

$$\rho_A^1(K_0(A)) \subset \Delta^1(LU_0^1(A)) \subset \rho_A(K_0(A)) = \rho_A^1(K_0(A)).$$

So $\overline{\Delta^1(LU_0^1(A))} = \overline{\rho_A^1(K_0(A))} = \mathrm{Aff}(T(A))$. Thus, $\overline{\Delta^1} = 0$ (see Definition 2.11), and the assertion follows. □

For unital simple C^*-algebras, we have:

Theorem 4.2. *Let A be a unital infinite-dimensional simple C^*-algebra. Then* Dur $A = 1$ *if one of the following holds*:

(1) *A is not stably finite.*

(2) *A has stable rank one.*

(3) *A has real rank zero.*

(4) *A is projectionless and $\rho_A(K_0(A)) = \mathbb{Z}$ (with $\rho_A([1_A]) = 1$).*

(5) *A has property* (SP) *and has a unique tracial state.*

Proof. (1) In this case, there is a nonunitary isometry $u \in \mathrm{M}_k(A)$ for some $k \geq 2$. Since $\mathrm{M}_k(A)$ is also simple, every tracial state on $\mathrm{M}_k(A)$ is faithful if $T(A) \neq \varnothing$. This implies that $T(A) = \varnothing$. The assertion follows from Corollary 2.7.

(2) This follows from Corollary 3.11.

(3) This follows from Theorem 4.1 or Theorem 3.9.

(4) By the assumption, we have $\rho_A^1(K_0(A)) = \rho_A(K_0(A)) = \mathbb{Z}$. By Proposition 3.6, Dur $A = 1$.

(5) Let $\epsilon > 0$ and let $\tau \in T(A)$ be the unique tracial state. Let $k \geq 1$ be an integer and $p \in \mathrm{M}_k(A)$ a projection. Since A has (SP), there is a nonzero projection $q \in A$ such that $0 < \tau(q) < \frac{1}{2}\epsilon$ (see, for example, [Lin 2001, Lemma 3.5.7]). Then, there is an integer $m \geq 1$ such that $|m\tau(q) - \tau(p)| < \epsilon$. This implies that $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$. Therefore, by Proposition 3.6, Dur $A = 1$. □

For a unital simple C^*-algebra A, Theorem 4.2 indicates that the only case when Dur A might not be 1 is when A is stably finite and has stable rank greater than 1. The only example of this that we know so far is given by Villadsen [1999].

However, we have the following:

Theorem 4.3. *For each integer $n \geq 1$, there is a unital simple* AH-*algebra A with* tsr $A = n$ *such that* Dur $A = 1$.

Proof. Fix an integer $n > 1$. Let $A = \lim_{k\to\infty}(A_k, \phi_k)$ be the unital simple AH-algebra with tsr $A = n$ constructed by Villadsen [1999]. Then $A_1 = C(D^n)$. The connecting maps ϕ_k are "diagonal" maps. More precisely, $\phi_k(f) = \sum_{j=1}^{n(k)} f(\gamma_{k,j}) \otimes p_{k,j}$ for all $f \in A_k$, where $p_{k,1}$ is a trivial rank-1 projection, $A_{k+1} = \phi_k(\mathrm{id}_{A_k})\mathrm{M}_{(r(k)}(C(X_k))\phi_k(\mathrm{id}_{A_k})$ (for some large $r(n)$) for some spaces X_k, and $\gamma_{k,j} : X_{k+1} \to X_k$ is a continuous map (these are π_{i+1}^1 and some point evaluations as denoted in [Villadsen 1999, p. 1092]). Clearly A_1 contains a rank-1 projection. Suppose that A_k, as a unital hereditary C^*-subalgebra of

$M_{r(k)}(C(X_k))$, contains a rank-1 projection e_k (of $M_{r(k)}(C(X_k))$). Then, since $(\mathrm{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \le \phi_k(\mathrm{id}_{A_k})$, we have $(\mathrm{id}_{A_k} \circ \gamma_{k,1}) \otimes p_{k,1} \in A_{k+1}$. Then $e_k \circ \gamma_{k,1} \otimes p_{k,1} \in A_{k+1}$, which is a rank-1 projection.

The above shows every A_k contains a rank-1 projection.

Now let $p \in M_m(A)$ be a projection. We may assume that there is a projection $q \in M_m(A_{k_0+1})$ such that $\phi_{k_0+1,\infty}(q) = p$. Let $e_{k_0} \in A_{k_0+1}$ be a rank-1 projection. Then there is an integer $L \ge 1$ such that $L\tau(e_{k_0}) = \tau(q)$ for all $\tau \in T(A_{k_0+1})$. It follows that

$$L\tau(\phi_{k_0+1,\infty}(e_{k_0})) = \tau(p) \quad \text{for all } \tau \in T(A).$$

So $\rho_A^1(K_0(A)) = \rho_A(K_0(A))$ and hence $\mathrm{Dur}\, A = 1$ by Proposition 3.6. □

Theorem 4.4. *Let A be a unital simple* AH-*algebra with* (SP) *property. Then* $\mathrm{Dur}\, A = 1$.

Proof. By Proposition 3.1, it suffices to show that $i_A^{(1,n)}$ is injective, and by Proposition 3.6 it suffices to show that $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$.

Let p be a projection in $M_n(A)$. Since A is simple, $\inf\{\tau(p) \mid \tau \in T(A)\} = d > 0$. Given a positive number $\epsilon < \min\{\frac{1}{2}, \frac{1}{2}d\}$. Choose an integer $K \ge 1$ such that $1/K < \frac{1}{2}\epsilon$. Since A is a simple unital C^*-algebra with (SP), it follows from [Lin 2001, Lemma 3.5.7] that there are mutually orthogonal and mutually equivalent nonzero projections $p_1, p_2, \ldots, p_K \in A$ such that $\sum_{j=1}^{K} p_j \le p$. We compute that

(4-1) $$\tau(p_1) < \epsilon/2 \quad \text{and} \quad \tau(p_1) < d/K \qquad \text{for all } \tau \in T(A).$$

Since A is simple and unital, there are $x_1, x_2, \ldots, x_N \in A$ such that

$$\sum_{j=1}^{N} x_j^* p_1 x_j = 1_A.$$

Let $A = \varinjlim(A_m, \phi_m)$, where $A_m = \bigoplus_{i=1}^{r(m)} P_{m,j} M_{R(m,j)}(C(X_{m,j}))P_{n,j}$ for each m, $X_{n,j}$ is a connected finite CW-complex and $P_{m,j} \in M_{R(m,j)}(C(X_{m,j}))$ is a projection. Without loss of generality, we may assume that, there are projections $p_1' \in A_m$, $p' \in M_n(A_m)$ and elements $y_1, y_2, \ldots, y_N \in A_m$ such that $\phi_{m,\infty}(p_1') = p_1$, $\phi_{m,\infty}(y_j) = x_j$, $(\phi_{m,\infty} \otimes \mathrm{id}_{M_n})(p') = p$ and

(4-2) $$\left\| \sum_{j=1}^{N} y_j^* p_1' y_j - 1_A \right\| < 1.$$

Write p_1' and p' as

$$p_1' = p_{1,1}' \oplus p_{1,2}' \oplus \cdots \oplus p_{1,r(m)}' \quad \text{and} \quad p' = q_1 \oplus q_2 \oplus \cdots \oplus q_{r(m)},$$

where, for each $j = 1, \dots, r(m)$, $p'_{1,j} \in P_{m,j} \mathrm{M}_{R(m,j)}(C(X_{m,j}))P_{m,j}$ and $q_j \in \mathrm{M}_n(P_{m,j}\mathrm{M}_{R(m,j)}(C(X_{m,j}))P_{m,j})$ are projections. Note that (4-2) implies that $p'_{1,j} \neq 0$ for $j = 1, 2, \dots, r(m)$. Define

$$r_{1,j} = \operatorname{rank} p'_{1,j} \quad \text{and} \quad r_j = \operatorname{rank} q_j \qquad \text{for } j = 1, 2, \dots, r(m).$$

Then $r_j = l_j r_{1,j} + s_j$, where $l_j, s_j \geq 0$ are integers and $s_j < r_{1,j}$. It follows that

$$\left| t(p') - \sum_{j=1}^{r(m)} l_j t(p'_{1,j}) \right| < t(p'_1) \quad \text{for all } t \in T(A_m). \tag{4-3}$$

Define $q_{1,j} = \phi_{m,\infty}(p'_{1,j})$ for $j = 1, \dots, r(m)$. Then each $q_{1,j}$ is a projection in A. Note that for each $\tau \in T(A)$, $\tau \circ \phi_{m,\infty}$ is a tracial state on A_m. So, by (4-3),

$$\left| \tau(p) - \sum_{j=1}^{r(m)} l_j \tau(q_{1,j}) \right| < \tau(p_1) < \epsilon \quad \text{for all } \tau \in T(A).$$

This implies that $\overline{\rho_A^1(K_0(A))} = \overline{\rho_A(K_0(A))}$. □

Lemma 4.5. *Let A be a unital simple C^*-algebra with $T(A) \neq \varnothing$, and let $a \in A_+ \setminus \{0\}$. Then, for any $b \in A_{\mathrm{sa}}$, there is a $c \in \operatorname{Her} a$ such that $b - c \in A_0$.*

Proof. Since A is simple and unital, there are $x_1, x_2, \dots, x_m \in A$ such that $\sum_{j=1}^m x_j^* a x_j = 1_A$. Set $c = \sum_{j=1}^m a^{1/2} x_j b x_j^* a^{1/2}$. Then $c \in \operatorname{Her} a$ and

$$\tau(c) = \sum_{j=1}^m \tau(a^{1/2} x_j b x_j^* a^{1/2}) = \sum_{j=1}^m \tau(b x_j^* a x_j) = \tau(b) \quad \text{for all } \tau \in T(A).$$

It follows from Lemma 2.6(2) that $b - c \in A_0$. □

A special case of the following can be found in [Lin 2010b, Theorem 3.4]:

Theorem 4.6. *Let A be a unital simple C^*-algebra and let $e \in A$ be a nonzero projection. Consider the map $U_0(eAe)/CU(eAe) \to U_0(A)/CU(A)$ given by $i_e([u]) = [u + (1-e)]$. This map is always surjective, and is also injective if* $\operatorname{tsr} A = 1$.

Proof. To see that i_e is surjective, let $u \in U_0(A)$. Write $u = \prod_{k=1}^n \exp(i a_k)$ for $a_k \in A_{\mathrm{sa}}$, $k = 1, 2, \dots, n$. By Lemma 4.5, there are $b_1, \dots, b_n \in eAe$ such that $b_k - a_k \in A_0$. Put $w = e \prod_{k=1}^n \exp(i b_k)$. Then $w \in U_0(eAe)$. Set $v = w + (1-e)$. Then $v = \prod_{k=1}^n \exp(i b_k)$. Thus, by Lemma 2.6(1),

$$i_e([w]) = [v] = \sum_{k=1}^n [\exp(i b_k)] = \sum_{k=1}^n [\exp(i a_k)] = [u] \quad \text{in } U_0(A)/CU(A),$$

that is, i_e is surjective.

To see that i_e is injective when A has stable rank one, let $w \in U_0(eAe)$ such that $w + (1-e) \in CU(A)$. Since A is simple, there are $z_1, \dots, z_n \in A$ such that $1 - e = \sum_{j=1}^{n} z_j^* e z_j$. Set

$$X = \begin{bmatrix} ez_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ez_n & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{M}_n(A).$$

Then

$$\text{(4-4)} \qquad \mathrm{diag}(1-e, \overbrace{0, \dots, 0}^{n-1}) = X^* X, \quad XX^* \le \mathrm{diag}(\overbrace{e, e, \dots, e}^{n}).$$

Equation (4-4) indicates that $[1-e] \le n[e]$ in $K_0(A)$. Since tsr $A = 1$, we can find a projection $p \in \mathrm{M}_s(A)$ for some $s \ge n$ and a unitary $U \in \mathrm{M}_{s+1}(A)$ such that

$$\text{(4-5)} \qquad \mathrm{diag}(\overbrace{e, \dots, e}^{n}, \overbrace{0, \dots, 0}^{r}) = U \mathrm{diag}(1-e, p) U^*,$$

where $r = s - n + 1$. Write $v = w + (1-e)$ as $v = \left[\begin{smallmatrix} w & \\ & 1-e \end{smallmatrix}\right]$, and set

$$W = \begin{bmatrix} e & \\ & U \end{bmatrix} \quad \text{and} \quad Q = \mathrm{diag}(\overbrace{e, \dots, e}^{n}, \overbrace{0, \dots, 0}^{r}).$$

Then $W \mathrm{diag}(e, 1-e, p) \mathrm{M}_{s+2}(A) \,\mathrm{diag}(e, 1-e, p) W^* \subset \mathrm{M}_{n+1}(eAe) \oplus 0$ and

$$\text{(4-6)} \qquad W \begin{bmatrix} v & \\ & p \end{bmatrix} W^* = \begin{bmatrix} w & \\ & U \mathrm{diag}(1-e, p)\, U^* \end{bmatrix} = \mathrm{diag}(w, Q),$$

by (4-5). Note that $\mathrm{diag}(v, p) \in CU(\mathrm{diag}(e, 1-e, p) \mathrm{M}_{s+2}(A) \,\mathrm{diag}(e, 1-e, p))$. So, by (4-6),

$$\mathrm{diag}(w, \overbrace{e, \dots, e}^{n}) \in CU(\mathrm{M}_{n+1}(eAe)).$$

Since $\mathrm{tsr}(eAe) = 1$, it follows from Theorem 4.2(2) that $w \in CU(eAe)$. ☐

Lemma 4.7. *Let C be a nonunital C^*-algebra and $B = \widetilde{C}$. Assume $u_1, u_2, \dots, u_n \in U(\mathrm{M}_k(B))$ for some $k \ge 2$. Then, there are unitaries $u_1', u_2', \dots, u_n' \in \mathrm{M}_k(\widetilde{C})$ with $\pi_k(u_j') = 1_k$ and $w, z_j, \bar{u}_j \in U(\mathrm{M}_k(\mathbb{C}))$ for $j = 1, \dots, n$ such that*

$$\prod_{j=1}^{n} u_j = \Big(\prod_{j=1}^{n} u_j'\Big) w, \quad \textit{with } u_j' = z_j^* u_j \bar{u}_j^* z_j \ \textit{ for } j = 1, \dots, n,$$

$$w = \pi_k \prod_{j=1}^{n} u_j,$$

where $\pi(x+\lambda)=\lambda$ *for all* $x \in C$ *and* $\lambda \in \mathbb{C}$ *and* π_k *is the induced homomorphism of* π *on* $\mathrm{M}_k(B)$.

Moreover, if $u_j \in U_0(\mathrm{M}_k(B))$, *then we may assume that each* $u'_j \in U_0(\widetilde{\mathrm{M}_k(C)})$ *for* $j=1,\ldots,n$.

Proof. Put $\bar{u}_j=\pi_k(u_j)\in U(\mathrm{M}_k(\mathbb{C}))$. If $n=2$, then

$$\begin{aligned} u_1u_2 &= u_1\bar{u}_1^*(\bar{u}_1u_2\bar{u}_1^*)(\bar{u}_1\bar{u}_2^*\bar{u}_1^*)(\bar{u}_1\bar{u}_2\bar{u}_1^*\bar{u}_1)\\ &= u_1\bar{u}_1^*(\bar{u}_1u_2\bar{u}_1^*)(\bar{u}_1\bar{u}_2^*\bar{u}_1^*)(\bar{u}_1\bar{u}_2).\end{aligned}$$

Put $u'_1=u_1\bar{u}_1^*$, $u'_2=\bar{u}_1u_2\bar{u}_1^*\bar{u}_1\bar{u}_2^*\bar{u}_1^*$, $w_1=\bar{u}_1\bar{u}_2$, $z_1=1_k$, $z_2=\bar{u}_1$. Then

$$\pi_k(u'_1)=1_k,\quad \pi_k(u'_2)=\pi_k(\bar{u}_1(u_2\bar{u}_2^*)\bar{u}_1^*)=1_k,\quad w_1=\pi_k(u_1u_2).$$

Thus the lemma holds if $n=2$. Suppose that the lemma holds for s. Then

$$u_1u_2\cdots u_su_{s+1}=(u'_1u'_2\cdots u'_s)w_su_{s+1},$$

where $u'_j\in \mathrm{M}_k(\widetilde{C})$ are unitaries with $\pi_k(u'_j)=1_k$ and $u'_j=z_j^*u_j\bar{u}_j^*z_j$, where $z_j,\bar{u}_j\in U(\mathrm{M}_k(\mathbb{C}))$, $j=1,\ldots,s$ and $w_s=\pi_k\prod_{j=1}^s u_j$. It follows that

$$\prod_{j=1}^{s+1}u_j=\Bigl(\prod_{j=1}^{s}u'_j\Bigr)w_su_{s+1}w_s^*(w_s\bar{u}_{s+1}^*w_s^*)(w_s\bar{u}_{s+1}).$$

Put $u'_{s+1}=w_su_{s+1}w_s^*(w_s\bar{u}_{s+1}^*w_s^*)=w_s(u_{s+1}\bar{u}_{s+1}^*)w_s^*$, $z_{s+1}=w_s^*$ and $w_{s+1}=w_s\bar{u}_{s+1}$. Then

$$\begin{aligned}\pi_s(u'_{s+1})&=\pi_k(w_s)\pi(u_{s+1}\bar{u}_{s+1}^*)\pi_k(w_s^*)=1_k,\\ w_{s+1}=w_s\bar{u}_{s+1}&=\pi_k\biggl(\Bigl(\prod_{j=1}^{s}u_j\Bigr)u_{s+1}\biggr)=\pi_k\prod_{j=1}^{s+1}u_j.\end{aligned}$$

The first part of the lemma follows.

To see the second part, we first assume that $u_j=\exp(ia_j)$ for some $a_j\in(\mathrm{M}_k(B))_{\mathrm{sa}}$. Note that $\bar{u}_j=\exp(i\bar{a}_j)$, where $\bar{a}_j=\pi_k(a_j)\in(\mathrm{M}_k(\mathbb{C}))_{\mathrm{sa}}$, $j=1,\ldots,n$. Consider the path $u'_j(t)=\exp(ita_j)\exp(-it\bar{a}_j)$ for $t\in[0,1]$. Note that, for each $t\in[0,1]$ and $j=1,\ldots,n$,

$$\pi_k(\exp(ita_j)\exp(-it\bar{a}_j))=\exp(it\pi_k(a_j))\exp(-it\pi_k(a_j))=1_k.$$

It follows that $u'_j(t)\in\widetilde{\mathrm{M}_k(\mathbb{C})}$ for all $t\in[0,1]$ and $j=1,\ldots,n$. The case that $u_j=\exp\bigl(\prod_{k=1}^{m_j}(ia_k)\bigr)$ follows from this and what has been proved. □

Lemma 4.8. *Let C be a nonunital C^*-algebra and $B = \widetilde{C}$. Suppose that $z = aba^*b^*$, where $a, b \in U_0(\mathrm{M}_k(B))$. Then $z = yw$, where $y \in CU(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $w \in CU(\mathrm{M}_k(\mathbb{C}))$. Moreover, if $u = \prod_{j=1}^n z_j$, where each $z_j \in CU(\mathrm{M}_k(B))$, then $u = yv$, where $y \in CU(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $v \in CU(\mathrm{M}_k(\mathbb{C}))$.*

Proof. Let $\bar{a} = \pi_k(a)$ and $\bar{b} = \pi_k(b)$. Then $\bar{a}, \bar{b} \in U(\mathrm{M}_k(\mathbb{C}))$. It follows from Lemma 4.7 that for $j = 1, 2$ there are $a_j, b_j \in U_0(\widetilde{\mathrm{M}_k(\mathbb{C})})$ with $\pi_k(a_j) = \pi_k(b_j) = 1_k$ and $z_j \in U(\mathrm{M}_k(\mathbb{C}))$ such that

$$ab = a_1 b_1 w_1, \quad a_1 = a\bar{a}^*, \quad b_1 = z_1^* b\bar{b}^* z_1, \quad w_1 = \bar{a}\bar{b}, \tag{4-7}$$

$$ba = b_2 a_2 w_2, \quad b_2 = b\bar{b}^*, \quad a_2 = z_2^* a\bar{a}^* z_2, \quad w_2 = \bar{b}\bar{a}. \tag{4-8}$$

Set $x_1 = w_1 w_2^* z_2^*$ and $x_2 = w_1 w_2^* z_1$. Then $x_1, x_2 \in U_0(\mathrm{M}_k(\mathbb{C}))$ and

$$\begin{aligned} aba^*b^* &= a_1 b_1 (w_1 w_2^* z_2^* (a\bar{a}^*) z_2 w_2 w_1^*)(w_1 w_2^* (b\bar{b}^*) w_2 w_1^*)) w_1 w_2^* \\ &= a_1 b_1 (x_1 a_1^* x_1^*)(x_2^* b_1^* x_2) w_1 w_2^* \end{aligned}$$

by (4-7) and (4-8).

Write $a_1 = \prod_{j=1}^{m_1} \exp(i y_{1j})$ and $b_1 = \prod_{k=1}^{m_2} \exp(i y_{2k})$, where $y_{1j}, y_{2k} \in (\mathrm{M}_k(C))_{\mathrm{sa}}$, $j = 1, \dots, m_1$, $k = 1, \dots, m_2$. Let

$$y_{1j} = y_{1j}^+ - y_{1j}^- \quad \text{and} \quad y_{2k} = y_{2k}^+ - y_{2k}^-,$$

with $y_{1j}^+, y_{1j}^-, y_{2k}^+, y_{2k}^- \in (\mathrm{M}_k(C))_+$ for $j = 1, \dots, m_1$ and $k = 1, \dots, m_2$. Set

$$c_1 = \sum_{j=1}^{m_1} (y_{1j}^+ + x_1 y_{1j}^- x_1^*) + \sum_{k=1}^{m_2} (y_{2k}^+ + x_2 y_{2k}^- x_2^*),$$

$$c_2 = \sum_{j=1}^{m_1} (y_{1j}^- + x_1 y_{1j}^+ x_1^*) + \sum_{k=1}^{m_2} (y_{2k}^- + x_2 y_{2k}^+ x_2^*),$$

$$d_1 = \sum_{j=1}^{m_1} (y_{1j}^+ + y_{1j}^-) + \sum_{k=1}^{m_2} (y_{2k}^+ + y_{2k}^-),$$

$$d_2 = \sum_{j=1}^{m_1} (y_{1j}^- + y_{1j}^+) + \sum_{k=1}^{m_2} (y_{2k}^- + y_{2k}^+).$$

Then $c_1, c_2, d_1, d_2 \in (\mathrm{M}_2(C))_+$ and clearly $c_1 - d_1, c_2 - d_2 \in (\mathrm{M}_k(C))_0$. Therefore, $(c_1 - c_2) - (d_1 - d_2) \in (\mathrm{M}_k(C))_0$. Put $y = a_1 b_1 (x_1 a_1^* x_1^*)(x_2^* b_1^* x_2)$ and $w = w_1 w_2^*$. Then $y \in U_0(\widetilde{\mathrm{M}_k(C)})$ with $\pi_k(y) = 1_k$ and $w = \bar{a}\bar{b}\bar{a}^*\bar{b}^* \in DU_k(\mathbb{C})$. Moreover, in $U_0(\widetilde{\mathrm{M}_k(C)})/CU(\widetilde{\mathrm{M}_k(C)})$,

$$[y] = [\exp(i(c_1 - c_2))] = [\exp(i(d_1 - d_2))] = [a_1][b_1][a_1^*][b_1^*] = 0.$$

This proves the first part of the lemma. The second part follows. □

Theorem 4.9. *Let A be an infinite-dimensional unital simple C^*-algebra with $T(A) \neq \varnothing$ such that there is an $m \geq 1$, for every hereditary C^*-subalgebra C, with* $\mathrm{Dur}\,\widetilde{C} \leq m$. *Then* $\mathrm{Dur}\, A = 1$.

Proof. Let $n \geq 1$. By Proposition 3.1, it suffices to show that $i_A^{(1,n)}$ is injective. Let $u \in U_0(A)$ with $\mathrm{diag}(u, 1_{n-1}) \in CU(\mathrm{M}_n(A))$. Since A is simple and infinite-dimensional, we can find nonzero mutually orthogonal positive elements $c_1, \dots, c_m \in A$ and $x_1, \dots, x_m \in A$ such that

$$x_j^* x_j = c_1 \quad \text{and} \quad x_j x_j^* = c_j, \qquad j = 2, 3, \dots, m.$$

Put $\mathrm{Her}\, c_1 = C$ and $B = \widetilde{C}$. Then $\mathrm{Her}(c_1 + c_2 + \cdots + c_m) \cong \mathrm{M}_m(C)$. Note that $\mathrm{M}_m(B)$ is not isomorphic to a subalgebra of $\mathrm{M}_m(A)$.

By Lemma 4.5, we may assume, without loss of generality, that $u = \exp(2\pi i b)$ for some $b \in C_{\mathrm{sa}}$. Then, by Proposition 3.6(1), $\hat{b} \in \overline{\rho_A(K_0(A))}$.

Since A is simple and C is σ-unital, it follows from [Brown 1977, Theorem 2.8] that there is a unitary element W in $M(A \otimes \mathcal{K})$ (the multiplier algebra of $A \otimes \mathcal{K}$) such that $W^*(C \otimes \mathcal{K})W = A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra consisting of all compact operators on l^2. Note that since A is a unital simple C^*-algebra, every tracial state τ on C is the normalization of a tracial state restricted on C. Therefore

$$\hat{b} \in \overline{\rho_A(K_0(A))} = \overline{\rho_B(K_0(C))} \subset \overline{\rho_B(K_0(B))}. \tag{4-9}$$

Viewing b in $B_{s.a}$, consider $v = \exp(i2\pi b) \in U_0(B)$ and $v(t) = \exp(i2\pi t b)$, $t \in [0, 1]$. Then (4-9) implies that $\Delta^1(v(t)) \in \overline{\rho_B(K_0(B))}$. By Lemma 3.2(2), for any $\epsilon > 0$, there are $a \in B_{\mathrm{sa}}$ with $\|a\| < \epsilon$, $d \in B_{\mathrm{sa}}$ with $\hat{d} \in \rho_B(K_0(B))$ and $v_0 \in CU(B)$ such that

$$v = \exp(i2\pi a)\exp(i2\pi d)v_0. \tag{4-10}$$

Choose projections $p, q \in \mathrm{M}_n(B)$ for some $n > m$ such that for all $\tau \in T(B)$, $\tau(\mathrm{diag}(d, 0_{(n-1)\times(n-1)})) = \tau(p) - \tau(q)$. So $\mathrm{diag}(\exp(i2\pi d), 1_{n-1}) \in CU(\mathrm{M}_n(B))$ by Lemma 2.6(2). By assumption, $i_B^{(m,k)}$ is injective for all $k > m$. Therefore, we have $\mathrm{diag}(v, 1_{m-1}) \in CU(\mathrm{M}_m(B))$ by (4-10).

Let $\epsilon > 0$. Then there is a $v_1 \in DU(\mathrm{M}_m(B))$ such that $\|\mathrm{diag}(v, 1_{m-1}) - v_1\| < \frac{1}{2}\epsilon$. We may write $v_1 = \prod_{j=1}^r z_j$, where $z_j \in \mathrm{M}_m(B)$ is a commutator. It follows from Lemma 4.8 that there are $y \in CU(\widetilde{\mathrm{M}_m(C)})$ with $\pi_m(y) = 1_m$ and $w \in DU(\mathrm{M}_m(\mathbb{C}))$ such that $v_1 = yw$. Noting that $w = \pi_m(w) = \pi_m(v_1)$ and $\pi(v) = 1$, we have $\|1_m - w\| < \frac{1}{2}\epsilon$. Thus $\|\mathrm{diag}(v, 1_{m-1}) - y\| < \epsilon$. Set $v_0 = v - 1$ and $y_0 = y - 1_m$. Then

$$\begin{gathered} \mathrm{diag}(v_0, 0_{(m-1)\times(m-1)}),\, y_0 \in \mathrm{M}_m(C), \\ \|\mathrm{diag}(v_0, 0_{(m-1)\times(m-1)}) - y_0\| < \epsilon. \end{gathered} \tag{4-11}$$

By identifying $1_m + \mathrm{M}_m(C)$ with a unital C^*-subalgebra $1_A + \mathrm{Her}(c_1 + c_2 + \cdots + c_m)$ of A, we get that $\|\exp(i2\pi b) - y\| < \epsilon$ by (4-11). Since $y \in CU(\widetilde{\mathrm{M}_m(C)}) \subset CU(A)$ and hence $u \in CU(A)$, we have $\mathrm{Dur}\, A = 1$. □

Corollary 4.10. *Let A be a unital simple C^*-algebra. Suppose that there is an integer $K \geq 1$ such that* $\mathrm{csr}(C(S^1, C)) \leq K$ *for every hereditary C^*-subalgebra C. Then* $\mathrm{Dur}\, A = 1$.

Proof. It follows from Theorem 3.10 that $\mathrm{Dur}\, \widetilde{C} \leq \max\{K-1, 1\}$. Theorem 4.9 then applies. □

Definition 4.11. Let A be a C^*-algebra with $T(A) \neq \varnothing$. Define

$$\begin{aligned} D\big(\rho_A^1(K_0(A)), \rho_A(K_0(A))\big) &= \sup\big\{\mathrm{dist}\big(x, \rho_A^1(K_0(A))\big) \mid x \in \overline{\rho_A(K_0(A))}\big\} \\ &= \sup\big\{\mathrm{dist}\big(x, \rho_A^1(K_0(A))\big) \mid x \in \rho_A(K_0(A))\big\}. \end{aligned}$$

Theorem 4.12. *Let A be a unital simple C^*-algebra with $T(A) \neq \varnothing$ such that there is an $M > 0$ with $D\big(\rho_C^1(K_0(C)), \rho_C(K_0(C))\big) < M$ for all nonzero hereditary C^*-subalgebras C of A. Then* $\mathrm{Dur}\, A = 1$.

Proof. Let $u \in U_0(A)$ such that $\mathrm{diag}(u, 1_{n-1}) \in CU(\mathrm{M}_n(A))$. By Corollary 2.12, we may assume that $u = \exp(i2\pi a)$ for some $a \in A_{\mathrm{sa}}$. Then $\hat{a} \in \overline{\rho_A(K_0(A))}$ by Proposition 3.6(1).

Given $\epsilon > 0$, choose an integer $N \geq 1$ such that $M/N < \epsilon/2\pi$. There are mutually orthogonal nonzero positive elements $c_1, c_2, \ldots, c_N$ in A and elements $x_1, x_2, \ldots, x_N \in A$ such that

$$x_j^* x_j = c_1 \quad \text{and} \quad x_j x_j^* = c_j, \qquad j = 2, 3, \ldots, N. \tag{4-12}$$

Let $C = \mathrm{Her}\, c_1$ and $B = \widetilde{C}$. It follows from Lemma 4.5 that there is a $b \in C_{\mathrm{sa}}$ such that $a - b$ is in A_0, i.e., $\tau(a) = \tau(b)$ for all $\tau \in T(A)$. Therefore $[\exp(i2\pi a)] = [\exp(i2\pi b)]$ in $U_0(A)/CU(A)$ by Lemma 2.6(2).

Since A is a unital simple C^*-algebra and C is σ-unital, it follows from the proof of Theorem 4.9 that $\rho_C(b) \in \overline{\rho_C(K_0(C))}$. Therefore, by assumption, there are projections $p_1, p_2, \ldots, p_{k_1}, q_1, q_2, \ldots, q_{k_2} \in C$ such that

$$\sup_{\tau \in T(C)} \left| \tau(b) - \left(\sum_{i=1}^{k_1} \tau(p_i) - \sum_{j=1}^{k_2} \tau(q_j) \right) \right| < M.$$

Put $d = \sum_{i=1}^{k_1} p_i - \sum_{j=1}^{k_2} q_j$ and $f = b - d$. Then $\exp(i2\pi d) \in CU(A)$ by (2-3) and $[\exp(i2\pi f)] = [\exp(i2\pi b] \in U_0(A)/CU(A)$. Moreover, from

$$\inf\{\|f - x\| \mid x \in C_0\} = \sup\{|\tau(f)| \mid \tau \in T(C)\} < M$$

(see the proof of [Th, Lemma 3.1]), there are $f_0 \in C_0$ and $f_1 \in C_{\mathrm{sa}}$ with $\|f_1\| < M$ such that $f = f_1 + f_0$. By Lemma 2.6(1), $\exp(i2\pi f_0) \in CU(A)$. Since $f_1 \in C_{\mathrm{sa}}$, by (4-12), for $i = 1, 2, \ldots, N$ there are $g_i \in \mathrm{Her}\, c_i$ with

$$\|g_i\| \le \|f_1\|/N \quad \text{and} \quad \tau(g_i) = \tau(f_1/N) \ \text{ for all } \tau \in T(A). \tag{4-13}$$

Set $g = \sum_{i=1}^n g_i \in A$. Then, by (4-13),

$$\|\exp(i2\pi g) - 1_A\| < M/N < \epsilon \quad \text{and} \quad \overline{\Delta^1}(\exp(i2\pi f)\exp(-i2\pi g)) = 0. \tag{4-14}$$

So $\exp(i2\pi f)\exp(-i2\pi g) \in CU(A)$ and consequently $\mathrm{dist}(e^{i2\pi a}, CU(A)) < \epsilon$. □

Bruce Blackadar [1981] constructed three examples of unital simple separable nuclear C^*-algebras A, $A_\triangle$, A_H with no nontrivial projections. By [Blackadar 1981, Theorem 4.9], $K_0(A) = \mathbb{Z}$ with a unique tracial state. It follows from Theorem 4.2(4) that $\mathrm{Dur}\, A = 1$. We turn to his examples $A_\triangle$ and A_H, which may have rich tracial spaces. It should be also noted that, as Blackadar showed, when $\triangle$ is not trivial (for example), $\mathrm{M}_2(A_\triangle)$ has a projection p with $\tau(p) = 1$ for all $\tau \in T(A_\triangle)$. In particular, this implies that

$$\overline{\rho^1_{A_\triangle}(K_0(A_\triangle))} \ne \bar{\rho}_{A_\triangle}(K_0(A_\triangle)).$$

However, $\mathrm{Dur}\, A_\triangle = 1$ as shown below. It follows that there is a unitary $u \in \widetilde{C}$, where $C = C_0((0,1), A)$, which represents a projection q with $\tau(q) = 1$ for all $\tau \in T(A_\triangle)$.

Proposition 4.13. *Let B be a unital AF-algebra and σ an automorphism of B. Put $M_\sigma = \{f \in C([0,1], B) \mid f(1) = \sigma(f(0))\}$. Then* $\mathrm{Dur}\, M_\sigma = 1$.

Proof. Clearly, $T(M_\sigma) \ne \varnothing$. From the exact sequence of C^*-algebras

$$0 \longrightarrow C_0((0,1), B) \longrightarrow M_\sigma \longrightarrow B \longrightarrow 0,$$

we obtain the exact sequence of C^*-algebras

$$0 \longrightarrow C_0((0,1) \times S^1, B) \longrightarrow C(S^1, M_\sigma) \longrightarrow C(S^1, B) \longrightarrow 0. \tag{4-15}$$

Since B is an AF-algebra, it follows from [Nistor 1986, Corollary 2.11] that

$$\mathrm{csr}(C(S^1, B)) = \mathrm{csr}(C(S^1)) = 2,$$
$$\mathrm{csr}(C_0((0,1) \times S^1, B)) = \mathrm{csr}(C_0((0,1) \times S^1)) = 2,$$

and consequently, applying [Nagy 1987, Lemma 2] to (4-15), we get

$$\mathrm{csr}(C(S^1, M_\sigma)) \le \max\{\mathrm{csr}(C(S^1, B)), \mathrm{csr}(C_0((0,1) \times S^1, B))\} \le 2.$$

Therefore $\mathrm{Dur}\, A = 1$ by Theorem 3.10. □

Corollary 4.14. $\mathrm{Dur}\, A_\triangle = 1$ *and* $\mathrm{Dur}\, A_H = 1$.

Proof. Both C^*-algebras are of the form $\lim_{n\to\infty} A_n$, where each $A_n \cong M_\sigma$, where M_σ is as in Proposition 4.13, and thus $\operatorname{Dur} A_n = 1$. By Theorem 3.5, $\operatorname{Dur} A_\triangle = 1$ and $\operatorname{Dur} A_H = 1$. □

5. C^*-algebras with Dur $A > 1$

In this section, we will present a unital C^*-algebra C such that $\operatorname{Dur} C = 2$. In particular, we will show that there are C^*-algebras which satisfy the condition described in Proposition 3.12.

5.1. We first list some standard facts from elementary topology. We will give a brief proof of each fact for the reader's convenience.

Fact 1. *Let*

$$B_d(0) = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \le d\right\}.$$

Let $f : B_d(0) \times S^1 \to S^3 = \mathrm{SU}(2)$ be a continuous map which is not surjective. Then there is a homotopy

$$F : B_d(0) \times S^1 \times [0,1] \to S^3 = \mathrm{SU}(2)$$

such that $F(x, e^{i\theta}, 0) = f(x, e^{i\theta})$, $F(x, e^{i\theta}, s) = f(x, e^{i\theta})$ if $\|x\| = d$ (i.e., if $x \in \partial B_d(0)$) and $g(x, e^{i\theta}) = F(x, e^{i\theta}, 1)$ satisfies

$$g(0, e^{i\theta}) = F(0, e^{i\theta}, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{SU}(2) = S^3.$$

Proof. Assume that f misses a point $z \in S^3 = \mathrm{SU}(2)$ and that $z \neq \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SU}(2)$. Then $S^3 \setminus \{z\}$ is homeomorphic to $D^3 = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$, with the identity matrix mapping to $(0, 0, 0)$. Without loss of generality, we can assume that f is a map from $B_d(0) \times S^1$ to D^3. Let $F : B_d(0) \times S^1 \times [0,1] \to D^3$ be defined by

$$F(x, e^{i\theta}, s) = f(x, e^{i\theta}) \max\{1 - s, \|x\|/d\},$$

which satisfies the condition. □

Fact 2. *Let $f, g : S^4 \times S^1 \to \mathrm{SU}(n) \subset U(n) = U_n(\mathbb{C})$ (where $n \ge 2$) be continuous maps. If f is homotopic to g in $U(n)$, then they are also homotopic in* $\mathrm{SU}(n)$.

Proof. This follows from the fact that there is a continuous map $\pi : U(n) \to \mathrm{SU}(n)$ with $\pi \circ i = \mathrm{id}|_{\mathrm{SU}(n)}$, where $i : \mathrm{SU}(n) \to U(n)$ is inclusion. □

Fact 3. *Let $\xi \in S^4$ be the north pole. Suppose that $f, g : S^4 \times S^1 \to \mathrm{SU}(n)$ are two continuous maps such that*

$$f(\xi, e^{i\theta}) = 1_n = g(\xi, e^{i\theta})$$

for all $e^{i\theta} \in S^1$. If f and g are homotopic in $\mathrm{SU}(n)$, then there is a homotopy

$$F : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$$

such that $F(x, e^{i\theta}, 0) = f(x, e^{i\theta})$, $F(x, e^{i\theta}, 1) = g(x, e^{i\theta})$ for all $x \in S^4$, $e^{i\theta} \in S^1$ and $F(\xi, e^{i\theta}, t) = 1_n$ for all $e^{i\theta} \in S^1$.

Proof. Let $G : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$ be a homotopy between f and g. That is, $G(\cdot, \cdot, 0) = f$ and $G(\cdot, \cdot, 1) = g$. Let $F : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(n)$ be defined by

$$F(x, e^{i\theta}, t) = G(x, e^{i\theta}, t)(G(\xi, e^{i\theta}, t))^*.$$

Then F satisfies the condition. □

5.2. We will describe the projection $P \in \mathrm{M}_4(C(S^4))$ of rank two which represents the class of $(2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(S^4))$ as follows: One can regard S^4 as the quotient space $D^4/\partial D^4$, where

$$D^4 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \le 1\}.$$

It is standard to construct a unitary

$$\alpha : D^4 \to U_4(\mathbb{C}) = U(\mathrm{M}_4(\mathbb{C}))$$

such that $\alpha(0) = 1_4$ and such that, for any $(z, w) \in \partial D^4$ (i.e., $|z|^2 + |w|^2 = 1$),

$$\alpha(z, w) := \begin{bmatrix} z & w & 0 & 0 \\ -\bar{w} & \bar{z} & 0 & 0 \\ 0 & 0 & \bar{z} & -w \\ 0 & 0 & \bar{w} & z \end{bmatrix} \triangleq \begin{bmatrix} \beta(z, w) & 0 \\ 0 & \beta(z, w)^* \end{bmatrix},$$

where $\beta(z, w) = \left[\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right]$, for $(z, w) \in \partial D^4 = S^3$, represents the generator of $K_1(C(S^3))$. Define $P : S^4 \to U_4(\mathbb{C})$ by

$$P(z, w) \triangleq \alpha(z, w) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w).$$

Note that α is not defined as a function from $S^4 = D^4/\partial D^4$ to $U(4)$, but P is, since

$$P(z, w) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z, w) \in \partial D^4$$

and ∂D^4 is identified with the north pole $\xi \in S^4$. Hence $P(\xi) = \left[\begin{smallmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{smallmatrix}\right]$.

5.3. In the rest of the paper, for a compact metric space X with a given base point and a C^*-algebra A, by $C_0(X, A)$ we mean the C^*-algebra of the continuous functions from X to A which vanish at the base point (and $C_0(X, \mathbb{C})$ will be denoted by $C_0(X)$). (Most spaces we used here have an obvious base point, which we will not mention afterward.) Let $A = C_0(S^1, P\mathrm{M}_4C(S^4)P)$. Let $\widetilde{A}$ be the unitization of A. Let $B = C_0(S^1, C(S^4))$. Since A is a corner of $\mathrm{M}_4(B)$ and B is a corner of $\mathrm{M}_2(A)$ (note that a trivial projection of rank 1 is equivalent to a subprojection of $P \oplus P$), A is stably isomorphic to B. Let $\widetilde{B}$ be a unitization of B. Then $\widetilde{B} = C(S^4 \times S^1)$ and

$$K_1(\widetilde{A}) \cong K_1(A) \cong K_1(B) \cong K_1(\widetilde{B}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

5.4. For a unitary $u \in \mathrm{M}_4(C(S^4 \times S^1))$, in the identification of $[u] \in K_1(C(S^4 \times S^1))$ with $\mathbb{Z} \oplus \mathbb{Z}$, the first component corresponds to the winding number of

$$S^1 \hookrightarrow S^4 \times S^1 \xrightarrow{\det u} S^1 \subset \mathbb{C},$$

that is, the winding number of the map

$$e^{i\theta} \to \det u(\xi, e^{i\theta}),$$

where ξ is the north pole of S^4. Hence, if $u : S^4 \times S^1 \to \mathrm{SU}(n)$, then the first component of $[u] \in K_1(C(S^4 \times S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is automatically zero.

Lemma 5.5. *Let $u : S^4 \times S^1 \to \mathrm{SU}(2)$. Then $u \in \mathrm{M}_2(C(S^4 \times S^1))$ represents the zero element in $K_1(C(S^4 \times S^1))$. In other words, if $u \in \mathrm{SU}_n(S^4 \times S^1)$ represents a nonzero element in K-theory, then $n \geq 3$.*

Proof. Let $f : S^4 \times S^1 \to S^5$ be the standard quotient map sending $\{\xi\} \times S^1 \cup S^4 \times \{1\}$ to a single point. Consider $u : S^4 \times S^1 \to \mathrm{SU}(2)$. Without loss of generality, assume $u(\xi, 1) = 1_2 \in \mathrm{SU}(2)$. Then $u|_{S^4 \times \{1\}} : S^4 \to \mathrm{SU}(2) = S^3$ represents an element in $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $u^2|_{S^4 \times \{1\}} : S^4 \to \mathrm{SU}(2) = S^3$ is homotopically trivial, with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Evidently, $u^2|_{\{\xi\} \times S^1} : S^1 \to S^3 = \mathrm{SU}(2)$ is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed point. Consequently

$$u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1} : S^4 \times \{1\} \cup \{\xi\} \times S^1 \to S^3$$

is homotopically trivial with $(\xi, 1) \in S^4 \times S^1$ as a fixed base point. There is a homotopy

$$F : (S^4 \times \{1\} \cup \{\xi\} \times S^1) \times [0, 1] \to S^3$$

with $F(\cdot\,, 0) = u^2|_{S^4 \times \{1\} \cup \{\xi\} \times S^1}$ and

$$F(x, 1) = 1_2 \quad \text{for all } x \in S^4 \times \{1\} \cup \{\xi\} \times S^1.$$

The following is a well-known easy fact: For any relative CW complex (X, Y) $(Y \subset X)$, any continuous map $Y \times I \cup X \times \{0\} \to Z$ (where Z is any other CW complex) can be extended to a continuous map $X \times I \to Z$.

Hence, there is a homotopy $G : (S^4 \times S^1) \times [0, 1] \to S^3$ with $G(\cdot, 0) = u^2$, and $G|_{S^4\times\{1\}\cup\{\xi\}\times S^1\times[0,1]} = F$. Let $v : S^4 \times S^1 \to \mathrm{SU}(2)$ be defined by $v(x) = G(x, 1)$; then $[v] = [u^2] \in K_1(C(S^4 \times S^1))$ and v maps $S^4 \times \{1\} \cup \{\xi\} \times S^1$ to $1_2 \in \mathrm{SU}(2)$. Consequently, v passes to a map

$$v_1 : S^5 \triangleq S^4 \times S^1 / S^4 \times \{1\} \cup \{\xi\} \times S^1 \to S^3 = \mathrm{SU}(2)$$

and represents an element in $\pi_5(S^3) = \mathbb{Z}/2\mathbb{Z}$. Hence $v_1^2 : S^5 \to S^3$ is homotopically trivial, and therefore v^2 is as well. So we have

$$4[u] = 2[u^2] = 2[v] = [v^2] = 0 \in K_1(C(S^4 \times S^1)),$$

which implies $[u] = 0 \in K_1(C(S^4 \times S^1))$. □

Remark 5.6. In the proof of Lemma 5.5, we in fact proved the following fact: For any $u : S^4 \times S^1 \to \mathrm{SU}(2)$, the map $u^4 : S^4 \times S^1 \to \mathrm{SU}(2)$ is homotopically trivial.

5.7. Note that $P \in \mathrm{M}_4(C(S^4))$ can be regarded as a projection in $\mathrm{M}_4(C(S^4 \times S^1))$, still denoted by P, i.e., for fixed $x \in S^4$, $P(x, \cdot)$ is a constant projection along the S^1 direction. Then

$$K_1(A) \cong K_1(\widetilde{A}) \cong K_1(C(S^4 \times S^1)) \cong K_1(P\mathrm{M}_4(C(S^4 \times S^1))P), \tag{5-1}$$

where $A = C_0(S^1, P\mathrm{M}_4(C(S^4))P)$ is defined in Section 5.2. Let

$$E = \{(\zeta, u) : \zeta \in S^4 \times S^1, u \in \mathrm{M}_4(\mathbb{C}) \text{ with } P(x)uP(x) = u, u^*u = uu^* = P(x)\},$$
$$SE = \{(\zeta, u) \in E : \det(P(x)uP(x) + (1_4 - P(x)) = 1\}.$$

Then $E \to S^4 \times S^1$ and $SE \to S^4 \times S^1$ are fiber bundles with fibers $U(2)$ and $\mathrm{SU}(2)$, respectively. Also the unitaries in $P\mathrm{M}_4(C(S^4 \times S^1))P$ correspond bijectively to the cross-sections of a bundle $E \to S^4 \times S^1$. For this reason, we will call a unitary (of $P\mathrm{M}_4(C(S^4 \times S^1))P$) with determinant 1 everywhere a cross-section of a bundle $SE \to S^4 \times S^1$.

Theorem 5.8. *If $u \in P\mathrm{M}_4(C(S^4 \times S^1))P$ has determinant 1 everywhere, i.e., if u is a cross-section of $SE \to S^4 \times S^1$, then $[u] = 0$ in $K_1(P\mathrm{M}_4(C(S^4 \times S^1))P)$.*

Proof. Note that $SE \to S^4 \times S^1$ is a smooth fiber bundle over the smooth manifold $S^4 \times S^1$. By a standard result in differential topology, u is homotopic to a C^∞-section. Without loss of generality, we may assume that u itself is smooth. Identify the north pole $\xi \in S^4$ with $0 \in \mathbb{R}^4$ and a neighborhood of ξ with $B_\epsilon(0) \subset \mathbb{R}^4$ for $\epsilon > 0$. Since $B_\epsilon(0)$ is contractible, $SE|_{B_\epsilon(0)\times S^1}$ is a trivial bundle. Note that the projection $P \in \mathrm{M}_4(C(S^4 \times S^1))$ is constant along S^1, hence $SE \cong SE|_{S^4\times\{1\}} \times S^1$

and $SE|_{B_\epsilon(0)\times S^1} \cong SE|_{B_\epsilon(0)\times\{1\}} \times S^1$; in other words, the fiber is constant along S^1 and $SE|_{B_\epsilon(0)\times\{1\}}$ is trivial and isomorphic to $(B_\epsilon(0)\times\{1\})\times \mathrm{SU}(2)$. There is a smooth bundle isomorphism

$$\gamma : SE|_{B_\epsilon(0)\times S^1} \to (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2). \tag{5-2}$$

Then

$$\gamma\circ u|_{B_\epsilon(0)\times S^1} : B_\epsilon(0)\times S^1 \to (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2)$$

is a smooth map with

$$\pi_1\circ(\gamma\circ u)|_{B_\epsilon(0)\times S^1} = \mathrm{id}_{B_\epsilon(0)\times S^1},$$

where $\pi_1 : (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2) \to B_\epsilon(0)\times S^1$ is the projection onto the first coordinate. Define $\phi = \pi_2\circ(\gamma\circ u|_{B_\epsilon(0)\times S^1})$, where $\pi_2 : (B_\epsilon(0)\times S^1)\times \mathrm{SU}(2) \to \mathrm{SU}(2)$ is the projection onto the second coordinate. Since ϕ is smooth, $\phi|_{\{\xi\}\times S^1}$ is not onto $\mathrm{SU}(2)$ (note $\dim(\mathrm{SU}(2)) = 3$ and $\dim(S^1) = 1$). Therefore, if ϵ is small enough, $\phi|_{B_\epsilon(0)\times S^1}$ is not onto. By Fact 1 of Section 5.1, ϕ is homotopic to a constant map $\phi_1 : B_\epsilon(0)\times S^1 \to \mathrm{SU}(2)$ with

$$\phi_1(\{\xi\}\times S^1) = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} \quad\text{and}\quad \phi|_{\partial B_\epsilon(0)\times S^1} = \phi_1|_{\partial B_\epsilon(0)\times S^1}, \tag{5-3}$$

via a homotopy $F : (B_\epsilon(0)\times S^1)\times[0,1] \to \mathrm{SU}(2)$ with $F(x, e^{i\theta}, t)$ constant with respect to t if $x\in\partial B_\epsilon(0)$.

Let $u_1 : B_\epsilon(0)\times S^1 \to SE$ be the cross-section defined by

$$u_1(x, e^{i\theta}) = \gamma^{-1}((x, e^{i\theta}), \phi_1(x, e^{i\theta})) \in SE.$$

Then $u_1(x, e^{i\theta}) = u(x, e^{i\theta})$ if $x\in\partial B_\epsilon(0)$. We can extend u_1 to $S^4\times S^1$ by defining

$$u_1(x, e^{i\theta}) = u(x, e^{i\theta}) \quad \text{if } (x, e^{i\theta}) \notin B_\epsilon(0)\times S^1.$$

Hence u_1 is a section of SE with

$$u_1(\xi, e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2\\ 0_2 & 0_2\end{bmatrix} = P(\xi) \quad \text{for all } e^{i\theta}\in S^1.$$

Moreover, u_1 is homotopic to u by a homotopy that is constant on $(S^4\setminus B_\epsilon(0))\times S^1$ (on which $u_1 = u$) and that agrees with F on $B_\epsilon(0)\times S^1$. Hence $[u] = [u_1] \in K_1(P\mathrm{M}_4(C(S^4\times S^1))P)$. Recall that S^4 is obtained from

$$D^4 = \{(z, w)\in\mathbb{C}^2 \mid |z|^2 + |w|^2 \le 1\}$$

by identifying

$$\partial D^4 = \{(z, w)\in\mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

with the north pole $\xi \in S^4$. Recall that $P \in \mathrm{M}_4(C(S^4))$ (viewed as a projection in $\mathrm{M}_4(C(S^4 \times S^1))$ constant along the S^1 direction) is defined as

$$P(z, w) = \alpha(z, w) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w),$$

where $\alpha(z, w)$ is defined as in Section 5.2.

Define

$$v(z, w, e^{i\theta}) = \alpha^*(z, w) u_1(z, w, e^{i\theta}) \alpha(z, w).$$

Then we have that

(i) $$v(z, w, e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z, w) \in \partial D^4,$$

and therefore v can be regarded as a map from $S^4 \times S^1$ to $\mathrm{M}_4(\mathbb{C})$. Moreover,

(ii) $$v(z, w, e^{i\theta}) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} v(z, w, e^{i\theta}) \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \quad \text{for all } (z, w, e^{i\theta}) \in S^4 \times S^1.$$

By considering the upper-left corner of v (still denoted by v), we obtain a unitary $v : S^4 \times S^1 \to \mathrm{SU}(2)$. By Lemma 5.5 and Remark 5.6, v^4 is homotopically trivial. Furthermore, by Fact 3 of Section 5.1, there is a homotopy $F : S^4 \times S^1 \times [0, 1] \to \mathrm{SU}(2)$ such that

(iii) $$F(z, w, e^{i\theta}, 0) = v^4(z, w, e^{i\theta}) \qquad \text{for all } (z, w) \in S^4, e^{i\theta} \in S^1,$$

(iv) $$F(\xi, e^{i\theta}, t) = 1_2 \qquad \text{for all } e^{i\theta} \in S^1,$$

(v) $$F(z, w, e^{i\theta}, 1) = 1_2 \qquad \text{for all } (z, w) \in S^4, e^{i\theta} \in S^1.$$

Define $G : D^4 \times S^1 \times [0, 1] \to \mathrm{M}_4(\mathbb{C})$ by

$$G(z, w, e^{i\theta}, t) = \alpha(z, w) \begin{bmatrix} F(z, w, e^{i\theta}, t) & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w).$$

Then, by (iv), for $(z, w) \in \partial D^4$ we have

$$G(z, w, e^{i\theta}, t) = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}.$$

Hence G defines a map (still denoted by G) from $S^4 \times S^1 \times [0, 1] \to \mathrm{M}_4(\mathbb{C})$. Furthermore $G(z, w, e^{i\theta}, t) \in P(z, w) M_4(\mathbb{C}) P(z, w)$, and

$$G(z, w, e^{i\theta}, 0) = \alpha(z, w) \begin{bmatrix} v^4 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \alpha^*(z, w) = u_1^4.$$

That is, G defines a homotopy between u_1^4 and the unit $P \in P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$. Consequently $[u_1^4] = 0$ and $[u_1] = 0 \in K_1\big(P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P\big)$. Moreover, $[u] = 0 \in K_1(C(S^4 \times S^1))$, as desired. □

5.9. We identify $P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$ as a corner of $\mathrm{M}_4(C(S^4 \times S^1))$; then $K_1\big(P(\mathrm{M}_4(C(S^4 \times S^1)))P\big)$ is isomorphic to $K_1(C(S^4 \times S^1)) = \mathbb{Z} \oplus \mathbb{Z}$ naturally. Let $a \in P\big(\mathrm{M}_4(C(S^4 \times S^1))\big)P$ be defined by

$$a(x, e^{i\theta}) = e^{i\theta} P(x).$$

On the other hand, a could also be regarded as a unitary in $\mathrm{M}_4(C(S^4 \times S^1))$ as $a(x, e^{i\theta}) = e^{i\theta} P(x) + (1_4 - P(x))$. Then $[a] = (2, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_1(C(S^4 \times S^1))$, since $[a]$ is the image of $[P] \in K_0(C(S^4))$ under the exponential map

$$K_1(C(S^4)) \to K_1\big(C_0(S^1, C(S^4))\big),$$

and $[P] = (2, 1) \in K_0(C(S^4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 5.10. *No element* $(1, k) \in K_1(C(S^4 \times S^1))$ *can be realized by a unitary* $b \in P\mathrm{M}_4(C(S^4 \times S^1))P$.

Proof. We argue by contradiction. Assume $b \in P\mathrm{M}_4(C(S^4 \times S^1))P$ satisfies $[b] = (1, k) \in K_1(P\mathrm{M}_4(C(S^4 \times S^1)P))$. Without loss of generality, we assume that $b(\xi, 1) = P$. Then

$$[b^2 a^*] = (0, 2k - 1) \in K_1(PM_4(C(S^4 \times S^1))P).$$

In particular, the map

$$e^{i\theta} \to \det \begin{bmatrix} P(\xi)(b^2 a^*)(\xi, e^{i\theta}) P(\xi) & 0 \\ 0 & 1_4 - P(\xi) \end{bmatrix}_{8\times 8}$$

has winding number 0. That is, it is homotopically trivial. Hence

$$(x, e^{i\theta}) \xrightarrow{h} \det \begin{bmatrix} P(\xi)(b^2 a^*)(x, e^{i\theta}) P(\xi) & 0 \\ 0 & 1_4 - P(\xi) \end{bmatrix}_{8\times 8}$$

defines a map $h : S^4 \times S^1 \to S^1$ such that $h_* : \pi_1(S^4 \times S^1) \to \pi_1(S^1)$ is the zero map. Hence there is a lifting $\tilde{h} : S^4 \times S^1 \to \mathbb{R}$ with $h(x, e^{i\theta}) = \exp(i\tilde{h}(x, e^{i\theta}))$. Define a unitary $b_1 \in P\mathrm{M}_4(C(S^4 \times S^1))P$ by $b_1(x, e^{i\theta}) = \exp(i\,\tfrac{1}{2}\tilde{h}(x, e^{i\theta}))P(x)$. Then $[b_1] = 0 \in K_1(C(S^4 \times S^1))$, and $b^2 a^* b_1^* \in U(P\mathrm{M}_4 C(S^4 \times S^1)P)$ has determinant 1 everywhere. By Theorem 5.8, $[b^2 a^* b_1^*] = 0 \in K_1(C(S^4 \times S^1))$. On the other hand,

$$[b^2 a^* b_1^*] = [b^2 a^*] = (0, 2k - 1) \neq 0 \in K_1(C(S^4 \times S^1)),$$

which is a contradiction. □

Remark 5.11. Similarly, we can show that for any unitary $u \in P\mathrm{M}_4(C(S^4 \times S^1))P$, $[u] = l[a] = (2l, l) \in K_1(C(S^4 \times S^1))$ for some $l \in \mathbb{Z}$.

Corollary 5.12. *Let $A = C_0(S^1, PC(S^4)P)$, and let $\widetilde{A}$ be the unitization of A. Then there is no unitary $u \in \widetilde{A}$ such that $[u] = (1, k) \in K_1(A)$. In particular, no unitary u can correspond to a rank-1 projection in $M_4(C(S^4))$.*

Proof. Note that we may view P as a projection in $M_4(C(S^4 \times S^1))$ which is constant along the direction of S^1 (Section 5.7). So we may view $\widetilde{A}$ as a unital C^*-subalgebra of $PM_4(C(S^4 \times S^1))P$. Thus, by the identification (5-1), Theorem 5.10 applies. □

Theorem 5.13. *Let $A = PM_4(C(S^4))P$. Then* $\mathrm{Dur}\, A = 2$.

Proof. There is a projection $e \in M_2(A)$ which is unitarily equivalent to a rank-1 projection in $M_8(C(S^4))$ corresponding to $(1, 0) \in K_0(C(S^4))$. Let $C = C_0((0, 1), A)$. By Corollary 5.12, there is no unitary in $\widetilde{C}$ which represents a rank-1 projection. It follows from Proposition 3.12 that $\mathrm{Dur}\, A > 1$.

However, since $\rho_C\big(K_0(M_2(C))\big) = \frac{1}{2}\mathbb{Z}$ and $M_2(C)$ contains a rank-1 projection (with trace $\frac{1}{2}$), by Proposition 3.6(3), $\mathrm{Dur}(M_2(C)) = 1$. It follows that $\mathrm{Dur}\, C = 2$. □

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MOTION BY MIXED VOLUME PRESERVING CURVATURE FUNCTIONS NEAR SPHERES

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In this paper we investigate the flow of hypersurfaces by a class of symmetric functions of the principal curvatures with a mixed volume constraint. We consider compact hypersurfaces without boundary that can be written as a graph over a sphere. The linearisation of the resulting fully nonlinear PDE is used to prove a short-time existence theorem for hypersurfaces that are sufficiently close to a sphere and, using centre manifold analysis, the stability of the sphere as a stationary solution to the flow is determined. We will find that for initial hypersurfaces sufficiently close to a sphere, the flow will exist for all time and the hypersurfaces will converge exponentially fast to a sphere. This result was shown for the case where the symmetric function is the mean curvature and the constraint is on the $(n+1)$-dimensional enclosed volume by Escher and Simonett (1998).

1. Introduction

Given a sufficiently smooth hypersurface $\Omega_0 = X_0(M^n) \subset \mathbb{R}^{n+1}$ that is compact without boundary, where M^n is an n-dimensional manifold, we are interested in finding a family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ such that

(1)
$$\frac{\partial X}{\partial t} = (h_k - F(\kappa))\nu_{\Omega_t}, \quad X(\cdot, 0) = X_0, \quad h_k = \frac{1}{\int_{M^n} E_{k+1}\, d\mu_t} \int_{M^n} F(\kappa) E_{k+1}\, d\mu_t,$$

where $\kappa = (\kappa_1, \ldots, \kappa_n)$, κ_i are the principal curvatures of the hypersurface $\Omega_t = X(M^n, t) = X_t(M^n)$, ν_{Ω_t} and $d\mu_t$ are the outward pointing unit normal and induced measure of Ω_t, respectively, and k is a fixed integer between -1 and $n-1$. Here E_l denotes the l-th elementary symmetric function of the principal curvatures:

$$E_l = \begin{cases} 1 & \text{if } l = 0, \\ \sum_{1 \le i_1 < \cdots < i_l \le n} \kappa_{i_1}\kappa_{i_2}\ldots\kappa_{i_l} & \text{if } l = 1, \ldots, n, \end{cases}$$

and $F(\kappa)$ is a given smooth, symmetric function that satisfies $(\partial F/\partial\kappa_i)(\kappa_0) > 0$, where $\kappa_0 = (1/R, \ldots, 1/R)$ for some fixed $R \in \mathbb{R}^+$. The flow can be seen to

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preserve the $(n-k)$-th mixed volume of the hypersurface (see Corollary 2.2). Note that while such a quantity is usually only defined for convex hypersurfaces, there is an obvious extension to all hypersurfaces (see Section 2).

This flow has been studied previously in [McCoy 2005]. There it was proved that under some additional conditions on F, for example homogeneity of degree one and convexity or concavity, initially convex hypersurfaces admit a solution for all time and the hypersurfaces converge to a sphere as $t \to \infty$. This result had previously been proved for the specific case where $F(\kappa) = H$, the mean curvature, in [McCoy 2004] and, if in addition, $k = -1$ (in which case the flow is the well-known volume preserving mean curvature flow), in [Huisken 1987]. Other results for the volume preserving mean curvature flow include average mean convex hypersurfaces with initially small traceless second fundamental form converging to spheres (see [Li 2009]) and hypersurfaces that are graphs over spheres with a height function close to zero, in a certain function space, converging to spheres (see [Escher and Simonett 1998b]). Techniques similar to those in this paper were used to study volume preserving mean curvature flow for hypersurfaces close to a cylinder in [Hartley 2013] and spherical caps in [Abels et al. 2015].

The situation where F has homogeneity greater than one has been considered in [Cabezas-Rivas and Sinestrari 2010]. There it was proved that if $k = -1$ and $F(\kappa) = H_m^{\beta}$, with $m\beta > 1$ and $H_m = \binom{n}{m}^{-1} E_m$ the m-th mean curvature, the flow takes initially convex hypersurfaces that satisfy a pinching condition to spheres; the pinching condition is of the form $E_n > CH^n > 0$, where C is a constant depending on the parameters of the flow.

The main result of this paper is:

Theorem 1.1. *Let F be a smooth, symmetric function of the principal curvatures satisfying $(\partial F/\partial \kappa_a)(\kappa_0) > 0$ for $a = 1, \dots, n$ and some $R \in \mathbb{R}^+$. If Ω_0 is a graph over the sphere $\mathcal{S}_R^n$ with height function sufficiently small in $h^{2+\alpha}(\mathcal{S}_R^n)$, $0 < \alpha < 1$* (see Section 2), *then its flow by (1) exists for all time and converges exponentially fast to a sphere as $t \to \infty$, with respect to the $h^{2+\alpha}(\mathcal{S}_R^n)$-topology.*

Part (c) of the main result in [Escher and Simonett 1998b] proves a similar result for the specific case of the volume preserving mean curvature flow. Some differences include that Escher and Simonett are able to use the quasilinear nature of the flow to prove the hypersurfaces are smooth after the initial time. This also allows them to obtain convergence with respect to the $C^l(\mathcal{S}_R^n)$-topology, for any fixed l, and only requires the initial height function to be small in $h^{1+\alpha}(\mathcal{S}_R^n)$. In contrast, the current paper deals with flows that are, in general, fully nonlinear, so the methods we use require the initial height function to be small in $h^{2+\alpha}(\mathcal{S}_R^n)$, which gives a condition on its curvature, and only give convergence in the $h^{2+\alpha}(\mathcal{S}_R^n)$-topology. The key theorems for nonlinear flow appear in [Lunardi 1995] and have been included in the Appendix for the reader's ease.

In Section 2 of this paper we convert the flow (1) to a PDE for the graph function and also introduce the spaces and notation that will be used throughout the paper. The section ends with a corollary proving the flow preserves a certain mixed volume. In Section 3 we consider the problem as an ODE on Banach spaces and determine the linearisation of the speed. This leads to a new short-time existence theorem for the flow that includes some initially $h^{2+\alpha}(\mathcal{S}^n_R)$ hypersurfaces. In the final section the eigenvalues of the linearised operator are determined and a centre manifold is constructed. The proof of the main result is finished by showing that the centre manifold consists entirely of spheres and is exponentially attractive.

We note here that the $(n-k)$-th mixed volumes, for $k \geq 1$, are only well defined for convex hypersurfaces (see [Andrews 2001]). However, we will refer to the flow (1) as mixed volume preserving for any Ω_0, with the understanding that it preserves a quantity that coincides with the $(n-k)$-th mixed volume when Ω_0 is convex (see Corollary 2.2).

2. Notation and preliminaries

In this paper we consider $M^n = \mathcal{S}^n_R$, a given sphere of radius R, and hypersurfaces that are normal graphs over $\mathcal{S}^n_R$, $\boldsymbol{X}_\rho(\boldsymbol{p}) = \boldsymbol{p} + \rho(\boldsymbol{p})\nu_{\mathcal{S}^n_R}(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{S}^n_R$. The volume form on such a hypersurface will be denoted by $d\mu_\rho$ and we let $\mu(\rho)$ be the function such that $d\mu_\rho = \mu(\rho)\, d\mu_0$. We now proceed as in [Escher and Simonett 1998b] and convert the flow to an evolution equation for the height function $\rho : \mathcal{S}^n_R \times [0, T) \to \mathbb{R}$. Up to a tangential diffeomorphism the flow (1) is equivalent to solving the PDE

$$\frac{\partial \rho}{\partial t} = \sqrt{1 + \frac{R^2}{(R+\rho)^2}|\nabla \rho|^2}\big(h_k(\rho) - F(\kappa_\rho)\big), \quad \rho(\,\cdot\,, 0) = \rho_0, \tag{2}$$

where $h_k(\rho) = \int_{\mathcal{S}^n_R} E_{k+1}(\rho) F(\kappa_\rho)\, d\mu_\rho \big/ \int_{\mathcal{S}^n_R} E_{k+1}(\rho)\, d\mu_\rho$, κ_ρ is the principal curvature vector of the hypersurface defined by $\rho(\,\cdot\,, t)$, and ∇ denotes the gradient on $\mathcal{S}^n_R$ (see [McCoy 2005]).

The graph functions ρ are chosen in the little Hölder spaces, $h^{l+\alpha}(\mathcal{S}^n_R)$, for $\alpha \in (0, 1)$, $l \in \mathbb{N}$. These spaces are defined for an open set $U \subset \mathbb{R}^n$ and a multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with $|\beta| = \sum_{i=1}^n \beta_i$ as follows:

$$h^\alpha(\bar{U}) = \left\{\rho \in C^\alpha(\bar{U}) : \lim_{r \to 0} \sup_{\substack{x, y \in \bar{U} \\ 0 < |x-y| < r}} \frac{|\rho(x) - \rho(y)|}{|x-y|^\alpha} = 0\right\},$$

$$h^{l+\alpha}(\bar{U}) = \big\{\rho \in C^{l+\alpha}(\bar{U}) : D^\beta \rho \in h^\alpha(\bar{U}) \text{ for all } \beta, |\beta| = l\big\},$$

where D is the derivative operator on $\mathbb{R}^n$ and C^α, $C^{l+\alpha}$ are the Hölder spaces (see [Lunardi 1995]). The norm on the little Hölder space $h^{l+\alpha}$ is inherited from $C^{l+\alpha}$.

The little Hölder spaces can be extended to $\mathcal{S}^n_R$ by means of an atlas. In addition, it is known that the little Hölder spaces are the continuous interpolation spaces between themselves (see [Guenther et al. 2002, Equation 19]), that is, for real numbers $0 < \alpha < \beta$ we have

$$(3)\qquad \left(h^\alpha(\mathcal{S}^n_R), h^\beta(\mathcal{S}^n_R)\right)_\theta = h^{(\beta-\alpha)\theta+\alpha}(\mathcal{S}^n_R),$$

provided $(\beta-\alpha)\theta+\alpha \notin \mathbb{Z}$, where $(\cdot\,,\cdot)_\theta$ is an interpolation functor for each $\theta \in (0, 1)$, defined for $Y \subset X$ as

$$(X, Y)_\theta = \left\{x \in X : \lim_{t\to 0^+} t^{-\theta} K(t, x, X, Y) = 0\right\},$$
$$\text{where } K(t, x, X, Y) = \inf_{a\in Y}(\|x-a\|_X + t\|a\|_Y).$$

We will often abuse notation and use ρ to represent both a function on $\mathcal{S}^n_R \times [0, T)$ and the mapping from $[0, T)$ to a space of functions such that $\rho(t) = \rho(\cdot\,, t)$ for all $t \in [0, T)$. In this regard we define the spaces $C(I, X)$ and $C^k(I, X)$ consisting of continuous and continuously k-differentiable functions from an interval $I \subset \mathbb{R}$ to a Banach space X. They have the norms $\|\rho\|_{C^k(I,X)} = \sum_{j=0}^k \sup_{t\in I} \|\rho^{(j)}(t)\|_X$.

For an operator between function spaces $G : Y \to \tilde{Y}$ we denote the Fréchet derivative by ∂G. A linear operator, $A : Y \subset X \to X$, is called *sectorial* if there exist $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $\omega \in \mathbb{R}$ and $M > 0$ such that

(i) $\rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega,\ |\arg(\lambda-\omega)| < \theta\}$,

(ii) $\|R(\lambda, A)\|_{\mathcal{L}(X,X)} \leq \dfrac{M}{|\lambda-\omega|}$ for all $\lambda \in S_{\theta,\omega}$.

Here $\rho(A)$ is the resolvent set, $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator, and $\|\cdot\|_{\mathcal{L}(X,X)}$ is the standard linear operator norm (see [Lunardi 1995]).

For all closed, compact hypersurfaces $\Omega \subset \mathbb{R}^{n+1}$, we define the quantity

$$V_l(\Omega) = \begin{cases} \left((n+1)\binom{n}{l}\right)^{-1} \int_M E_{n-l}\, d\mu & \text{if } l = 0, \dots, n, \\ \operatorname{Vol}(\Phi) & \text{if } l = n+1, \end{cases}$$

where Φ is the $(n+1)$-dimensional region contained inside Ω; for convex hypersurfaces this agrees with the mixed volumes.

Lemma 2.1. *For a family of hypersurfaces Ω_t satisfying (1), the Weingarten map, volume form and mixed volumes satisfy the evolution equations*

$$\frac{\partial h^i_j}{\partial t} = g^{im}\nabla_m\nabla_j F - (h_k - F)h^i_m h^m_j, \qquad \frac{\partial(d\mu)}{\partial t} = (h_k - F)H\, d\mu,$$
$$\frac{dV_l}{dt} = \begin{cases} 0 & \text{if } l = 0, \\ \binom{n+1}{l}^{-1} \int_M E_{n+1-l}(h_k - F)\, d\mu & \text{if } l = 1, \dots, n+1. \end{cases}$$

Proof. The first two equations are well known; see [Andrews 1994], for example. The last equation, except for the $l = n+1$ case, which can be found in [Cabezas-Rivas and Sinestrari 2010], is given in Lemma 4.3 of [McCoy 2005] for the case where the Ω_t are convex hypersurfaces. McCoy uses the definition of mixed volumes of convex hypersurfaces (see [Andrews 2001]), which is not valid unless the hypersurface is convex. To obtain the result for all solutions to the flow we use the following identity, found in Equation (5.86) of [Gerhardt 2008]:

$$\frac{\partial E_{a+1}}{\partial h^i_j} = E_a \delta^j_i - h^j_q \frac{\partial E_a}{\partial h^i_q}, \tag{4}$$

where $a = 0, \dots, n$ (in the $a = n$ case we use the convention $E_{n+1} = 0$). Now if we take the divergence of this identity, we obtain

$$\begin{aligned}
g^{im}\nabla_m\left(\frac{\partial E_{a+1}}{\partial h^i_j}\right) &= g^{jm}\nabla_m E_a - g^{im}\nabla_m h^j_q \frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right)\\
&= g^{jm}\nabla_m h^p_q \frac{\partial E_a}{\partial h^p_q} - g^{im} g^{jp}\nabla_m h_{pq}\frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right)\\
&= g^{jm} g^{pi}\nabla_m h_{iq}\frac{\partial E_a}{\partial h^p_q} - g^{im} g^{jp}\nabla_p h_{mq}\frac{\partial E_a}{\partial h^i_q} - g^{im} h^j_q \nabla_m\left(\frac{\partial E_a}{\partial h^i_q}\right)\\
&= -h^j_q g^{im}\nabla_i\left(\frac{\partial E_a}{\partial h^m_q}\right),
\end{aligned}$$

using the Codazzi equation to get to the second last line. Since $g^{im}\nabla_i\left(\frac{\partial E_0}{\partial h^m_j}\right)$ vanishes, we see this equation implies

$$g^{im}\nabla_i\left(\frac{\partial E_a}{\partial h^m_j}\right) = 0 \quad \text{for all } a = 0, \dots, n.$$

We can now derive the evolution equation:

$$\begin{aligned}
&(n+1)\binom{n}{l}\frac{dV_l}{dt}\\
&\quad = \int_M \frac{\partial E_{n-l}}{\partial t} + (h_k - F) H E_{n-l}\, d\mu\\
&\quad = \int_M \frac{\partial E_{n-l}}{\partial h^i_j}\frac{\partial h^i_j}{\partial t} + (h_k - F) H E_{n-l}\, d\mu\\
&\quad = \int_M \frac{\partial E_{n-l}}{\partial h^i_j} g^{im}\nabla_m\nabla_j F - (h_k - F)\frac{\partial E_{n-l}}{\partial h^i_j} h^i_m h^m_j + (h_k - F) H E_{n-l}\, d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \nabla_m \left(\frac{\partial E_{n-l}}{\partial h^i_j} g^{im} \nabla_j F \right) \\
&\qquad + (h_k - F) h^i_m \left(\frac{\partial E_{n+1-l}}{\partial h^i_m} - E_{n-l} \delta^m_i \right) + (h_k - F) H E_{n-l} \, d\mu \\
&= (n+1-l) \int_M (h_k - F) E_{n+1-l} \, d\mu,
\end{aligned}
$$

where the second last line is due to (4) and the last line is due to the homogeneity of E_{n+1-l}. □

Corollary 2.2. *For a compact hypersurface without boundary, Ω_0, the flow (1) preserves the value of V_{n-k}, i.e., $V_{n-k}(\Omega_t) = V_{n-k}(\Omega_0)$ as long as the flow exists.*

3. Graphs over spheres

The flow in (2) can be considered as an ordinary differential equation between Banach spaces. Set $0 < \alpha < 1$ and define

$$
G : h^{2+\alpha}(\mathcal{S}^n_R) \to h^\alpha(\mathcal{S}^n_R),
$$
$$
G(\rho) := L(\rho)(h_k(\rho) - F(\kappa_\rho)), \quad L(\rho) := \sqrt{1 + \frac{R^2}{(R+\rho)^2} |\nabla \rho|^2}.
$$

The flow (2) is then rewritten as

$$
\rho'(t) = G(\rho(t)), \quad \rho(0) = \rho_0 \in h^{2+\alpha}(\mathcal{S}^n_R). \tag{5}
$$

Lemma 3.1. *The linearisation of G about zero is given by*

$$
\partial G(0) u = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \left(\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R} \right) u - \frac{n}{R^2} ⨍_{\mathcal{S}^n_R} u \, d\mu_0 \right),
$$

for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$.

Note that only the derivative of $F(\kappa)$ with respect to κ_1 appears in this formula for convenience, since $(\partial F / \partial \kappa_1)(\kappa_0) = (\partial F / \partial \kappa_i)(\kappa_0)$ for all $i = 1, \dots, n$. We also use the notation $⨍_{M^n} f \, d\mu := \int_{M^n} f \, d\mu \big/ \int_{M^n} d\mu$.

Proof. Firstly note $L(0) = 1$ and $\partial L(0) = 0$. By linearising the curvature function, we find

$$
\partial F(\kappa_\rho) \big|_{\rho=0} = \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i}(\kappa_\rho) \partial \kappa_i(\rho) \bigg|_{\rho=0} = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \sum_{i=1}^n \partial \kappa_i(0) = \frac{\partial F}{\partial \kappa_1}(\kappa_0) \partial H(0).
$$

It follows that for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$,

$$\begin{aligned}
&\partial h_k(0)u \\
&\quad = \partial\biggl(\frac{1}{\int_{\mathcal{S}^n_R} E_{k+1}(\rho)\mu(\rho)\,d\mu_0}\int_{\mathcal{S}^n_R} E_{k+1}(\rho)F(\kappa_\rho)\mu(\rho)\,d\mu_0\biggr)\bigg|_{\rho=0} u \\
&\quad = \frac{1}{\bigl(\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0\bigr)^2}\biggl(\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0\,\partial\biggl(\int_{\mathcal{S}^n_R} E_{k+1}(\rho)F(\kappa_\rho)\mu(\rho)\,d\mu_0\biggr)\bigg|_{\rho=0} u \\
&\qquad\qquad - \int_{\mathcal{S}^n_R} E_{k+1}(0)F(\kappa_0)\,d\mu_0\,\partial\biggl(\int_{\mathcal{S}^n_R} E_{k+1}(\rho)\mu(\rho)\,d\mu_0\biggr)\bigg|_{\rho=0} u\biggr) \\
&\quad = \frac{1}{\int_{\mathcal{S}^n_R} E_{k+1}(0)\,d\mu_0} \\
&\qquad \times\biggl(\int_{\mathcal{S}^n_R}\Bigl(E_{k+1}(0)\,\partial F(\kappa_\rho)\big|_{\rho=0}u + F(\kappa_0)\,\partial\bigl(E_{k+1}(\rho)\mu(\rho)\bigr)\big|_{\rho=0}u\Bigr)d\mu_0 \\
&\qquad\qquad - F(\kappa_0)\int_{\mathcal{S}^n_R}\partial\bigl(E_{k+1}(\rho)\mu(\rho)\bigr)\big|_{\rho=0}u\,d\mu_0\biggr) \\
&\quad = \frac{\partial F}{\partial\kappa_1}(\kappa_0)\fint_{\mathcal{S}^n_R}\partial H(0)u\,d\mu_0.
\end{aligned}$$

It was shown in [Escher and Simonett 1998a] that

$$\partial H(0) = -\Bigl(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\Bigr),$$

so combining these results gives, for $u \in h^{2+\alpha}(\mathcal{S}^n_R)$,

$$\partial G(0)u = \frac{\partial F}{\partial\kappa_1}(\kappa_0)\biggl(\Bigl(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\Bigr)u - \fint_{\mathcal{S}^n_R}\Bigl(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\Bigr)u\,d\mu_0\biggr). \tag{6}$$

The divergence theorem gives the result. □

Lemma 3.2. *For any α_0 such that $0 < \alpha_0 < \alpha$, there exists a neighbourhood, O_1, of $0 \in h^{2+\alpha}(\mathcal{S}^n_R)$ such that the operator $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of a sectorial operator $A_\rho : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$ for all $\rho \in O_1$.*

Proof. We set $\bar{G} : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$ with $\bar{G}(\rho) := L(\rho)(h_k(\rho) - F(\kappa_\rho))$ so that with $A_\rho = \partial\bar{G}(\rho)$ it is clear that $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of A_ρ. It remains to show that there exists O_1 such that A_ρ is sectorial for $\rho \in O_1$.

As $\partial H(0) = -(n/R^2 + \Delta_{\mathcal{S}^n_R})$ is a uniformly elliptic operator on a compact manifold without boundary, its negative is sectorial as a map from $h^{2+\alpha_0}(\mathcal{S}^n_R)$; see [Guenther et al. 2002, Lemma 3.4], for example. Now the operator $A_0 : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$, defined by

$$A_0 u = \Bigl(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\Bigr)u - \frac{n}{R^2}\fint_{\mathcal{S}^n_R} u\,d\mu_0,$$

is sectorial by the perturbation result in Proposition 2.4.1(ii) of [Lunardi 1995], since the map $u \mapsto -(n/R^2) ⨍_{\mathcal{S}^n_R} u\, d\mu_0$ is in $\mathcal{L}(h^{2+\alpha_0}(\mathcal{S}^n_R), h^{2+\alpha_0}(\mathcal{S}^n_R))$. This then implies, by Proposition 2.4.2 of [Lunardi 1995], that $A_\rho = A_0 + (\partial\bar{G}(\rho) - \partial\bar{G}(0))$ is sectorial for all ρ in a neighbourhood of zero, $O_2 \subset h^{2+\alpha_0}(\mathcal{S}^n_R)$. The result follows by setting $O_1 = O_2 \cap h^{2+\alpha}(\mathcal{S}^n_R)$. □

Theorem 3.3. *There are constants* $\delta, r > 0$ *such that if* $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)} \le r$, *then* (5) *has a unique maximal solution*:

$$\rho \in C\big([0,\delta), h^{2+\alpha}(\mathcal{S}^n_R)\big) \cap C^1\big([0,\delta), h^{\alpha}(\mathcal{S}^n_R)\big).$$

Proof. This existence theorem is a result of Theorem A.1, which is Theorem 8.4.1 in [Lunardi 1995], by setting $\bar{u} = 0$. In order to satisfy the assumption of the theorem it must be shown that there exists a neighbourhood of zero, $O \subset h^{2+\alpha}(\mathcal{S}^n_R)$, such that G and ∂G are continuous on O and for every $\rho \in O$ the operator $\partial G(\rho)$ is the part in $h^\alpha(\mathcal{S}^n_R)$ of a sectorial operator $A : h^{2+\alpha_0}(\mathcal{S}^n_R) \to h^{\alpha_0}(\mathcal{S}^n_R)$.

As in [Andrews and McCoy 2012, Remark 1], since F is a smooth symmetric function of the principal curvatures, it is also a smooth function of the elementary symmetric functions, which depend smoothly on the components of the Weingarten map. We now consider a neighbourhood of zero, O_3, such that if $\rho \in O_3$, then $\int_{\mathcal{S}^n_R} E_{k+1}(\rho)\, d\mu_\rho > 0$ and $\rho(\boldsymbol{p}) > -R$ for all $\boldsymbol{p} \in \mathcal{S}^n_R$ (note if $k = -1$ the former is always satisfied). It is easily seen that the Weingarten map depends smoothly on $\rho \in O_3 \subset h^{2+\alpha}(\mathcal{S}^n_R)$, so that G depends smoothly on $\rho \in O_3$. The sectorial condition was established in Lemma 3.2 for a neighbourhood O_1, so the proof is complete by setting $O = O_3 \cap O_1$. □

4. Stability around spheres

As we are considering the flow locally about $\rho = 0$, it is convenient to rewrite (5) highlighting the dominant linear part:

$$\rho'(t) = \partial G(0)\rho(t) + \tilde{G}(\rho(t)), \qquad \tilde{G}(u) := G(u) - \partial G(0)u. \tag{7}$$

Lemma 4.1. *The spectrum*, $\sigma(\partial G(0))$, *of* $\partial G(0)$ *consists of a sequence of isolated nonpositive eigenvalues where the multiplicity of the* 0 *eigenvalue is* $n+2$.

Proof. This follows from [Escher and Simonett 1998b], as $\partial G(0)$ is a positive constant multiple of the linear operator in that paper. To be exact, we calculate all the elements of the spectrum. Since $h^{2+\alpha}(\mathcal{S}^n_R)$ is compactly embedded in $h^\alpha(\mathcal{S}^n_R)$, the spectrum consists entirely of eigenvalues. To characterise the spectrum we first look at the spectrum of the L^2-self adjoint operator:

$$\tilde{A}u = \frac{\partial F}{\partial \kappa_1}(\kappa_0)\left(\frac{n}{R^2} + \Delta_{\mathcal{S}^n_R}\right)u.$$

The eigenvalues of the spherical Laplacian are well known to be $-l(l+n-1)/R^2$ for $l \in \mathbb{N} \cup \{0\}$ with eigenfunctions the spherical harmonics of order l, denoted by $Y_{l,p}$, $1 \le p \le M_l$, where

$$M_l = \begin{cases} \binom{l+n}{n} - \binom{l+n-2}{n} & \text{if } l \ge 2, \\ \binom{l+n}{n} & \text{if } l \in \{0, 1\}. \end{cases}$$

Therefore the eigenfunctions of $\tilde{A}$ are also the spherical harmonics with eigenvalues

$$\xi_l = \frac{\partial F}{\partial \kappa_1}(\kappa_0)\left(\frac{n}{R^2} - \frac{l(l+n-1)}{R^2}\right) = -\frac{\partial F}{\partial \kappa_1}(\kappa_0)\frac{(l-1)(l+n)}{R^2}.$$

Returning to the spectrum of $\partial G(0)$, $Y_{0,1} = 1$ is still an eigenfunction but with eigenvalue $\lambda_0 = 0$. The operator $\partial G(0)$ is also self-adjoint with respect to the L^2 inner product on $h^{2+\alpha}(\mathcal{S}^n_R)$. Therefore we need only consider eigenfunctions orthogonal to $Y_{0,1}$ in order to characterise the remainder of the spectrum. This means that for an eigenfunction u we assume

$$\int_{\mathcal{S}^n_R} u \, d\mu_0 = 0,$$

and hence by Lemma 3.1, $\partial G(0)u = \tilde{A}u$. The remaining eigenfunctions of $\partial G(0)$ are then the remaining eigenfunctions of $\tilde{A}$, with the same eigenvalues. So the spectrum of $\partial G(0)$ consists of the eigenvalues

$$\lambda_l = \begin{cases} 0 & \text{if } l = 0, \\ -\frac{\partial F}{\partial \kappa_1}(\kappa_0)\frac{l(l+n+1)}{R^2} & \text{if } l \in \mathbb{N}, \end{cases}$$

with eigenfunctions

$$u_{l,p} = \begin{cases} Y_{0,1} & \text{if } l = p = 0, \\ Y_{l+1,p} & \text{if } l \in \mathbb{N} \cup \{0\},\ 1 \le p \le M_{l+1}. \end{cases}$$

The multiplicity of the 0 eigenvalue is then $M_1 + 1 = n + 2$. □

In what follows, we set P to be the projection from $h^\alpha(\mathcal{S}^n_R)$ onto the $\lambda = 0$ eigenspace given by

$$Pu := \sum_{p=0}^{n+1} \langle u, u_{0,p} \rangle u_{0,p},$$

where we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product on $h^\alpha(\mathcal{S}^n_R)$. Because $\partial G(0)$ is self-adjoint with respect to this inner product, clearly $P \partial G(0)u = \partial G(0) Pu = 0$ for every $u \in h^{2+\alpha}(\mathcal{S}^n_R)$. Due to this, $h^{2+\alpha}(\mathcal{S}^n_R)$ can be split into the subspaces

$X^c = P(h^\alpha(\mathcal{S}^n_R))$ and $X^s = (I-P)(h^{2+\alpha}(\mathcal{S}^n_R))$, called the *centre subspace* and *stable subspace*, respectively. We are now in a position to apply Theorem A.3, which is Theorem 9.2.2 in [Lunardi 1995].

Theorem 4.2. *For any $l \in \mathbb{N}$, there is a function $\gamma \in C^{l-1}(X^c, X^s)$ such that $\gamma^{(l-1)}$ is Lipschitz continuous, $\gamma(0) = \partial\gamma(0) = 0$, and $\mathcal{M}^c = \text{graph}(\gamma)$ is a locally invariant manifold for (7) of dimension $n+2$.*

Note that by *locally invariant* it is meant that there exists a ball around zero, $B_r(0) \subset X^c$ with $r > 0$, such that if $\rho_0 \in \text{graph}(\gamma|_{B_r(0)})$ then the solution to (7) is in $\text{graph}(\gamma|_{B_r(0)})$ for all time or until $P\rho(t) \notin B_r(0)$. We now set

$$\mathcal{S} := \big\{\rho \in h^{2+\alpha}(\mathcal{S}^n_R) : \text{graph}(\rho) \text{ is a sphere}\big\}.$$

Lemma 4.3. *$\mathcal{M}^c$ coincides with the set $\mathcal{S}$ in a neighbourhood of zero, $\Lambda \subset h^{2+\alpha}(\mathcal{S}^n_R)$.*

Proof. By Theorem 2.4 in [Simonett 1995], the equation $y'(t) = \partial G(0)|_{X^s}\, y(t) + f(t)$ has a unique continuous, bounded solution for any continuous, bounded $f : \mathbb{R} \to (I-P)(h^\alpha(\mathcal{S}^n_R))$. Furthermore the solution is given by $y(t) = (Kf)(t)$, with $K \in \mathcal{L}\big(BC_\eta\big(\mathbb{R}, (I-P)(h^\alpha(\mathcal{S}^n_R))\big), BC_\eta(\mathbb{R}, X^s)\big)$ for any $\eta \in [0, -\lambda_1)$, where

$$BC_\eta(\mathbb{R}, X) := \Big\{g \in C(\mathbb{R}, X) : \|g\|_\eta := \sup_{t\in\mathbb{R}} \exp(-\eta|t|)\|g(t)\|_X < \infty\Big\}.$$

This is the key condition that allows us to apply Theorem 2.3 in [Vanderbauwhede and Iooss 1992] and conclude that $\mathcal{M}^c$ contains all equilibria of (7) with $P\rho_0 \in B_r(0)$. It was shown in [Escher and Simonett 1998b] that (along with $\mathcal{M}^c$) $\mathcal{S}$ is locally a graph over X^c, so since $\mathcal{S} \cap (B_r(0) \times X^s) \subset \mathcal{M}^c$, we conclude that $\mathcal{S}$ and $\mathcal{M}^c$ coincide locally. Note that while [Vanderbauwhede and Iooss 1992] proves the existence of a centre manifold differently than [Lunardi 1995], the two manifolds can be seen to be equal over $B_r(0)$, possibly making r smaller. □

We now prove the main result.

Proof of Theorem 1.1. By Proposition A.4, which is Proposition 9.2.4 in [Lunardi 1995], when $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ is small enough we obtain the decay in (11), with $x(t) = P\rho(t)$ and $y(t) = (I-P)\rho(t)$, for any $\omega \in (0, -\lambda_1)$ and as long as $P\rho(t) \in B_r(0)$. However, by using (11) evaluated at $t = 0$, we obtain

$$\begin{aligned}\|\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)} &\le \|P\rho_0\|_{h^\alpha(\mathcal{S}^n_R)} + \|P\rho_0 - \bar{x}\|_{h^\alpha(\mathcal{S}^n_R)}\\ &\le \|P\rho_0\|_{h^\alpha(\mathcal{S}^n_R)} + C(\omega)\|(I-P)\rho_0 - \gamma(P\rho_0)\|_{h^{2+\alpha}(\mathcal{S}^n_R)},\end{aligned}$$

and since γ is Lipschitz and P is bounded, this leads to a bound of the form $\|\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)} \le C(\omega)\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$. Therefore we can ensure that $\bar{x} \in P(\Lambda) \cap B_r(0)$ by taking $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ small enough, and Lemma 4.3 then implies that the function

$\bar{x}+\gamma(\bar{x})$ defines a sphere. Hence $\bar{x}+\gamma(\bar{x})$ is a stationary solution to (2), which in turn means that $z(t)=\bar{x}$ is the solution to (12). So we can restate (11) as

$$\text{(8)}\quad \|P\rho(t)-\bar{x}\|_{h^\alpha(\mathcal{S}^n_R)}+\|(I-P)\rho(t)-\gamma(\bar{x})\|_{h^{2+\alpha}(\mathcal{S}^n_R)} \le C(\omega)e^{-\omega t}\|(I-P)\rho_0-\gamma(P\rho_0)\|_{h^{2+\alpha}(\mathcal{S}^n_R)},$$

for as long as $P\rho(t)\in B_r(0)$. However, using this bound and our bound for $\bar{x}$, it follows that $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<C(\omega)\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ as long as $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<r$. By choosing $\|\rho_0\|_{h^{2+\alpha}(\mathcal{S}^n_R)}$ small enough, we can therefore ensure $\|P\rho(t)\|_{h^\alpha(\mathcal{S}^n_R)}<r/2$ for all $t\ge 0$. Thus (8) is true for all $t\ge 0$, and this proves that $\rho(t)$ converges to $\bar{x}+\gamma(\bar{x})$ as $t\to\infty$, which is the height function of a sphere. □

Corollary 4.4. *Let Ω_0 be a graph over a sphere with height ρ_0 such that the solution, $\rho(t)$, to the flow (2) with initial condition ρ_0 exists for all time and converges to zero. Suppose further that $(\partial F/\partial\kappa_i)|_{\kappa_{\rho(t)}}>0$ for all $t\in[0,\infty)$ and $i=1,\dots,n$. Then there exists a neighbourhood, O, of ρ_0 in $h^{2+\alpha}(\mathcal{S}^n_R)$, $0<\alpha<1$, such that for every $u_0\in O$ the solution to (2) with initial condition u_0 exists for all time and converges to a function near zero whose graph is a sphere.*

Proof. This follows by the same arguments given in [Guenther et al. 2002] for the Ricci flow. First we set $U\subset h^{2+\alpha}(\mathcal{S}^n_R)$ to be the neighbourhood of zero given in Theorem 1.1. Since $\rho(t)$ converges to zero in the $h^{2+\alpha}$-topology, there exists a time T such that $\rho(T)\in U$ and, as U is open, there exists an open ball $B_\epsilon(\rho(T))\subset U$ of radius ϵ centred at $\rho(T)$. The condition that $(\partial F/\partial\kappa_i)|_{\kappa_{\rho(t)}}>0$ for all $t\in[0,\infty)$ and $i=1,\dots,n$ ensures that the operator $L(\rho)F(\kappa_\rho)$ is elliptic around the point $\rho(t)$ for every $t\in[0,\infty)$ (see [Andrews 1994]). As the global term is in $\mathcal{L}(h^{2+\beta}(\mathcal{S}^n_R),h^\alpha(\mathcal{S}^n_R))$ for any $\beta<\alpha$, we can use Proposition 2.4.1(i) in [Lunardi 1995] to conclude that the linear operator $\partial G(\rho(t))$ is sectorial for all $t\in[0,T]$, and hence in a neighbourhood of each point. By Theorem A.2, which is Theorem 8.4.4 in [Lunardi 1995], the flow depends continuously on the initial condition in a neighbourhood of ρ_0. Therefore there exists a ball $B_\delta(\rho_0)$ such that if $u_0\in B_\delta(\rho_0)$, then the solution, $u(t)$, to (2) with initial condition u_0 exists for $t\in[0,T]$ and $u(T)\in B_\epsilon(\rho(T))$. Since $u(T)$ is in U, by Theorem 1.1, the solution to (2) with initial condition $u(T)$ converges to a function near zero that defines a sphere. By uniqueness of the flow we get the result. □

Appendix: Key theorems

In this appendix we restate the key theorems from [Lunardi 1995] using the notation of this paper. In the following, E_1, E_0 and E will represent Banach spaces with $E_1\subset E_0\subset E$.

Theorem A.1 [Lunardi 1995, Theorem 8.4.1]. *Let $O_1 \subset E_1$ be a neighbourhood of 0 and let $G : O_1 \to E_0$ and $\partial G : O_1 \to \mathscr{L}(E_1, E_0)$ be continuous. Assume that for every $v \in O_1$, the operator $\partial G(v) : E_1 \to E_0$ is the part in E_0 of a sectorial operator $A : D \subset E \to E$ such that $E_0 \simeq (E, D)_\theta$ and $E_1 \simeq \{x \in D : Ax \in (E, D)_\theta\}$, for some $\theta \in (0, 1)$. Then for every $\bar{u} \in O_1$ there are $\delta > 0$, $r > 0$ such that if $\|u_0 - \bar{u}\|_{E_1} \le r$, then the problem*

$$u'(t) = G(u(t)), \quad 0 \le t \le \delta, \quad u(0) = u_0 \tag{9}$$

has a unique solution $u \in C([0, \delta], E_1) \cap C^1([0, \delta], E_0)$.

Theorem A.2 [Lunardi 1995, Theorem 8.4.4]. *Let G be as in Theorem A.1. For every $\bar{u} \in O_1$ and for every $\bar{\tau} \in (0, \tau(\bar{u}))$, where $\tau(v)$ is the maximal time of a solution to (9) with $u_0 = v$, there is $r > 0$ such that if $\|u_0 - \bar{u}\|_D \le r$, then $\tau(u_0) \ge \bar{\tau}$ and the mapping*

$$\Phi : B_r(\bar{u}) \to C([0, \bar{\tau}], E_1) \cap C^1([0, \bar{\tau}], E_0), \quad \Phi(v) = u(\,\cdot\,; v),$$

where $u(\,\cdot\,; v)$ solves (9) with $u_0 = v$, is continuously differentiable with respect to v. If in addition G is k times continuously differentiable or analytic, then so is Φ.

We now set $E_0 = (E, D)_\theta$, $E_1 = \{x \in D : Ax \in (E, D)_\theta\}$ for some $\theta \in (0, 1)$, and let O_1 be a neighbourhood of $0 \in E_1$. For a finite-dimensional space X we also define $\eta : X \to \mathbb{R}$ to be a cutoff function such that $0 \le \eta(x) \le 1$ for all $x \in X$, $\eta(x) = 1$ if $\|x\|_X \le 1$, and $\eta(x) = 0$ if $\|x\|_X \ge 2$.

Theorem A.3 [Lunardi 1995, Theorem 9.2.2]. *Let $A : D \subset E \to E$ be a sectorial operator such that $\sigma(A)\backslash\mathbb{R}_-$ consists of a finite number of isolated eigenvalues, each with finite algebraic multiplicity. Let $\tilde{G} \in C^1(O_1, E_0)$ be a nonlinear function such that $\tilde{G}(0) = 0$ and $\partial\tilde{G}(0) = 0$. Then there exists $r_1 > 0$ such that for $r \le r_1$ there is a Lipschitz continuous function $\gamma : P(E_0) \to (I - P)(E_1)$ such that the graph of γ is invariant for the system*

$$\begin{aligned} x'(t) &= A|_{P(E_0)} x(t) + P\tilde{G}\left(\eta\left(\frac{x(t)}{r}\right)x(t) + y(t)\right), \quad x(0) = x_0 \in P(E_0), \\ y'(t) &= A|_{(I-P)(E_1)} y(t) + (I - P)\tilde{G}\left(\eta\left(\frac{x(t)}{r}\right)x(t) + y(t)\right), \\ &\qquad\qquad\qquad y(0) = y_0 \in (I - P)(E_1), \end{aligned} \tag{10}$$

where P is the spectral projection associated with the set of nonnegative eigenvalues. If in addition $\tilde{G}$ is k times continuously differentiable, with $k \ge 2$, then there exists

$r_k > 0$ *such that if* $r < r_k$, *then* $\gamma \in C^{k-1,1}$ *and for* $x \in P(E_0)$,

$$\partial\gamma(x)\left(A|_{P(E_0)}x + P\tilde{G}\left(\eta\left(\frac{x}{r}\right)x + \gamma(x)\right)\right) = A|_{(I-P)(E_1)}\gamma(x) + (I-P)\tilde{G}\left(\eta\left(\frac{x}{r}\right)x + \gamma(x)\right).$$

Proposition A.4 [Lunardi 1995, Proposition 9.2.4]. *Take* A *and* $\tilde{G}$ *as in Theorem A.3. For every* $\omega \in (0, \omega_-)$, *where* $\omega_- = -\sup\{\Re(\lambda) : \lambda \in \sigma(A) \cap \mathbb{R}_-\}$, *there is* $C(\omega) > 0$ *such that if* $\|x_0\|_{E_0}$ *and* $\|y_0\|_{E_1}$ *are small enough, then there exists* $\bar{x} \in P(E_0)$ *such that for all* $t \geq 0$,

$$\|x(t) - z(t)\|_{E_0} + \|y(t) - \gamma(z(t))\|_{E_1} \leq C(\omega)\exp(-\omega t)\|y_0 - \gamma(x_0)\|_{E_1}, \tag{11}$$

where $(x(t), y(t))$ *is the solution to* (10) *and* $z(t)$ *is the solution to*

$$z'(t) = A|_{P(E_0)}z(t) + P\tilde{G}\left(\eta\left(\frac{z(t)}{r}\right)z(t) + \gamma(z(t))\right), \quad z(0) = \bar{x}. \tag{12}$$

Note that throughout the paper we considered, for $0 < \alpha_0 < \alpha < 1$, the spaces $E_1 = h^{2+\alpha}(\mathcal{S}^n_R)$, $E_0 = h^{\alpha}(\mathcal{S}^n_R)$ and $E = h^{\alpha_0}(\mathcal{S}^n_R)$, with D, the domain of a linear operator A, given by $h^{2+\alpha_0}(\mathcal{S}^n_R)$. The characterisation of $h^{\alpha}(\mathcal{S}^n_R)$ as an interpolation space between $h^{\alpha_0}(\mathcal{S}^n_R)$ and $h^{2+\alpha_0}(\mathcal{S}^n_R)$ is given in (3).

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HOMOMORPHISMS ON INFINITE DIRECT PRODUCTS OF GROUPS, RINGS AND MONOIDS

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We study properties of a group, abelian group, ring, or monoid B which (a) guarantee that every homomorphism from an infinite direct product $\prod_I A_i$ of objects of the same sort onto B factors through the direct product of finitely many ultraproducts of the A_i (possibly after composition with the natural map $B \to B/Z(B)$ or some variant), and/or (b) guarantee that when a map does so factor (and the index set has reasonable cardinality), the ultrafilters involved must be principal.

A number of open questions and topics for further investigation are noted.

1. Introduction

A direct product $\prod_{i \in I} A_i$ of infinitely many nontrivial algebraic structures is in general a "big" object: it has at least continuum cardinality, and if the operations of the A_i include a vector-space structure, it has at least continuum dimension. But there are many situations where the set of homomorphisms from such a product to a fixed object B is unexpectedly restricted.

The poster child for this phenomenon is the case where the objects are abelian groups, and B is the infinite cyclic group. In that situation, if the index set I is countable (or, indeed, of less than an enormous cardinality — some details are recalled in Section 4), then every homomorphism $\prod_{i \in I} A_i \to B$ factors through the projection of $\prod_{i \in I} A_i$ onto the product of finitely many of the A_i. An abelian group B which, like the infinite cyclic group, has this property, is called "slender". Slender groups have been completely characterized [Nunke 1961], and slender modules over general rings have been studied.

Recent work [Bergman and Nahlus 2011 and 2012; Bergman 2014] on factorization properties of homomorphisms on infinite direct products of not-necessarily-associative algebras (motivated by the case of Lie algebras) has turned up interesting variants on the above sort of behavior.

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First, it turns out that in that context, a useful way to prove every surjective homomorphism $\prod_{i\in I} A_i \to B$ factors through finitely many of the A_i is by proving (a) that every such homomorphism factors through the product of finitely many *ultraproducts* of the A_i, and also (b) that whenever one has a map that factors in that way, the ultrafilters involved must be principal. In this note, we shall consider each of conditions (a) and (b) on an object B as of separate interest.

Secondly, we found that in many cases, though one cannot say that every surjective homomorphism from a direct product to B will itself factor in one of these ways, one can say that for every such homomorphism $\prod_{i\in I} A_i \to B$, the induced homomorphism $\prod_{i\in I} A_i \to B/Z(B)$ so factors, where $Z(B)$ denotes the zero-multiplication ideal, $\{b \in B \mid bB = Bb = \{0\}\}$ (which for B a Lie algebra is the center of B). In the next section, we shall get similar results for groups, with $Z(B)$ the center of the group B. (Note that these statements do not say that every surjective homomorphism $\prod_{i\in I} A_i \to B/Z(B)$ factors as stated; such a factorization is asserted only when the homomorphism $\prod_{i\in I} A_i \to B/Z(B)$ can be lifted to a homomorphism $\prod_{i\in I} A_i \to B$.) Maalouf [2014] abstracts this property, and strengthens some of the results of the papers cited.

In the classical case of abelian groups (and its generalization to modules), the condition on an object B that every homomorphism from an infinite product *onto* B yield a factorization through finitely many of the A_i, and the corresponding condition for homomorphisms *into* B, are equivalent. Indeed, from any homomorphism $\prod_{i\in I} A_i \to B$, one can get, in an obvious way, a surjective homomorphism $B \times \prod_{i\in I} A_i \to B$, and the original homomorphism factors through finitely many of the A_i if and only if that surjective map factors through B and finitely many A_i. This observation uses implicitly the fact that one can add homomorphisms of abelian groups — in this case, the map $B \times \prod_{i\in I} A_i \to B$ induced by the given map on the one hand, and the projection to B on the other. But one cannot do this for homomorphisms of noncommutative groups, of algebras, etc.; so for these, the condition involving arbitrary maps and the condition involving surjective maps are not equivalent. In these cases, the condition on B defined in terms of surjective homomorphisms is the more informative. Once one has characterized those B for which all surjective homomorphisms $\prod_{i\in I} A_i \to B$ yield such a factorization, one can, if one wishes, characterize the B with the corresponding property for general homomorphisms as the objects all of whose subobjects have the the property for surjections.

In stating results of the sort we shall obtain, one has a choice between (i) saying that if a structure B does *not* have one or another of a list of “messy” properties, then every homomorphism from an infinite direct product onto B leads to a certain kind of factorization, or (ii) the contrapositive statement, that if there exists a homomorphism onto B that does not so factor, then B has one of those messy properties. Each approach has its plusses and minuses; here I have followed (ii), because it seems more straightforward to understand how a non-factorable map forces B to have a messy property than to show that the absence of certain messy properties implies that all maps factor; and also because some of the conditions on B come in several versions, and I find it easier to parse a statement having a single hypothesis and several conclusions than one with several alternative hypotheses giving a single conclusion. (But the above choice also has its awkward aspects; I can't say which is really best.)

In Sections 2–4, we shall study the case where our structures are not-necessarily-abelian groups, in Sections 5–7, abelian groups, then, briefly, in Section 8 and Section 9, rings and monoids. In Section 10 we note why lattices are likely to be another case worth examining.

For a short review, for the nonspecialist, of the concepts of filter, ultrafilter and ultraproduct, see [Bergman and Nahlus 2011, Appendix A]; and for measurable cardinals κ, and κ-complete ultrafilters, which come up in Sections 4–5 below, [ibid., Appendix B]. For detailed developments of these concepts see, e.g., [Chang and Keisler 1990] or [Comfort and Negrepontis 1974].

We remark that there is in the literature a concept of "noncommutative slender group" that is quite different from the subject of Sections 2–4 below. The concept so named can be arrived at by regarding the infinite direct product in the definition of a slender abelian group as a *completed direct sum*, and using in the noncommutative case, instead of the direct product, an analogously completed noncommutative coproduct. For work on that topic see [Shelah and Strüngmann 2001] and references given there.

2. Factoring group homomorphisms through finitely many ultraproducts.

Let $(G_i)_{i\in I}$ be a family of groups. By the *support* of an element $g = (g_i)_{i\in I} \in \prod_{i\in I} G_i$, we will understand the set

$$\operatorname{supp}(g) = \{i \in I \mid g_i \neq e\} \subseteq I. \tag{1}$$

Given any subset $S \subseteq I$, we shall identify $\prod_{i\in S} G_i$ in the obvious way with the subgroup of $\prod_{i\in I} G_i$ consisting of elements whose support is contained in S. In particular, for $g \in \prod_{i\in I} G_i$, the statement $g \in \prod_{i\in S} G_i$ will mean $\operatorname{supp}(g) \subseteq S$, and the statement $g \in G_i$ will mean $\operatorname{supp}(g) \subseteq \{i\}$.

Whereas the theory of slender abelian groups is based on delicate structural properties of those groups, most of our results on nonabelian groups will be based on a much simpler observation: Elements of $\prod_{i\in I} G_i$ with disjoint supports centralize one another. As a quick example, it is not hard to see that if B is a simple nonabelian group, and we have any surjective homomorphism $f : \prod_{i\in I} G_i \to B$, then for each $S \subseteq I$, the map f must annihilate one of the mutually centralizing subgroups $\prod_{i \in S} G_i$ and $\prod_{i \in S-I} G_i$. From this one can deduce that the subsets $S \subseteq I$ such that f factors through the projection $\prod_{i \in I} G_i \to \prod_{i \in S} G_i$ form an ultrafilter (principal or nonprincipal) on I.

In the opposite direction, however, if we take for B a cyclic group of prime order p (thus losing the leverage provided by noncommutativity), and let all the G_i be copies of that group, then by linear algebra over the field of p elements, there exist homomorphisms $\prod_{i \in I} G_i \to B$ that send every G_i onto B, and hence don't factor through any proper subproduct $\prod_{i \in S} G_i$.

As indicated in the introduction, we shall get around the problem created (as above) by commutativity by composing homomorphisms $\prod_{i \in I} G_i \to B$ with the quotient map $B \to B/Z(B)$, where $Z(B)$ is the center of B. Given a homomorphism $f : \prod_{i \in I} G_i \to B$, the key to our considerations will be the family of subsets

(2) $$\begin{aligned} \mathcal{F} = \{S \subseteq I \mid{} & \text{the composite map } \textstyle\prod_{i \in I} G_i \to B \to B/Z(B) \\ & \text{factors through the projection } \textstyle\prod_{i \in I} G_i \to \prod_{i \in S} G_i\} \\ = \{S \subseteq I \mid{} & f(\textstyle\prod_{i \in I-S} G_i) \subseteq Z(B)\}. \end{aligned}$$

It is easy to see that $\mathcal{F}$, so defined, is a *filter* on I, and that if we write

(3) $$\pi : B \to B/Z(B)$$

for the quotient map, then $\mathcal{F}$ is the largest filter such that $\pi f : \prod_{i \in I} G_i \to B/Z(B)$ factors through the reduced product $\prod_{i \in I} G_i/\mathcal{F}$. (The above observation, and the next few, do not yet use the fact that we are working with a map of the form πf, but only that we are considering a homomorphism on a product group. The fact that our map has the form πf will become significant starting with Lemma 1 below.)

If the filter $\mathcal{F}$ of (2) is a finite intersection of distinct ultrafilters, $\mathcal{U}_0 \cap \cdots \cap \mathcal{U}_{n-1}$, then $\prod_{i \in I} G_i/\mathcal{F} \cong \prod_{i \in I} G_i/\mathcal{U}_0 \times \cdots \times \prod_{i \in I} G_i/\mathcal{U}_{n-1}$, so πf factors through the projection to that product; and conversely, if πf factors through the projection to such a product, then $\mathcal{F}$ is the intersection of some subset of the $\mathcal{U}_k$ (the minimal set of $\mathcal{U}_k$ allowing such a factorization). In this connection, we recall

(4) [Bergman 2014, Lemma 1.3, (3)$\Longleftrightarrow$(5)] *A filter $\mathcal{F}$ on a set I can be written as the intersection of finitely many ultrafilters on I if and only if for every partition of I into countably many sets J_m $(m \in \omega)$, there is at least one $m \in \omega$ such that $I - J_m \in \mathcal{F}$.*

Here and below, we make the conventions that a partition may include one or more instances of the empty set, and that the intersection of the empty family of filters on a set is the set of all subsets of that set, i.e., the improper filter. (These conventions are needed to make various statements correct in degenerate cases.)

Let us note what (4) tells us about homomorphisms on direct product groups.

Lemma 1. *Let $f : \prod_{i\in I} G_i \to B$ be a homomorphism from a direct product of groups G_i to a group B, which is surjective; or more generally, such that the composite $\pi f : \prod_{i\in I} G_i \to B \to B/Z(B)$ is surjective. Then the following conditions are equivalent.*

(5) *$\pi f : \prod_{i\in I} G_i \to B/Z(B)$ does* not *factor through the natural map $\prod_{i\in I} G_i \to \prod_{i\in I} G_i/\mathcal{U}_0 \times \cdots \times \prod_{i\in I} G_i/\mathcal{U}_{n-1}$ for any finite family $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$ of ultrafilters on I.*

(6) *There exists a partition of I into countably many subsets $J_0, J_1, \ldots,$ such that each subgroup $\prod_{i\in J_n} G_i \subseteq \prod_{i\in I} G_i$ contains a pair of elements x_n, y_n whose images in B under f do not commute.*

Proof. The easy direction is (6) $\Longrightarrow$ (5). The fact that $f(x_n)$ and $f(y_n)$ do not commute tells us, in particular, that $f(x_n) \notin Z(B)$. Hence for $\mathcal{F}$ defined by (2) (noting in particular the last line thereof), $I - J_n \notin \mathcal{F}$. Since this is true for each n, (4) tells us that the filter $\mathcal{F}$ is not a finite intersection of ultrafilters, giving (5).

To get the converse, note that if (5) holds, equivalently, if $\mathcal{F}$ is not a finite intersection of ultrafilters, then by (4) we can partition I into subsets $J_0, J_1, \ldots,$ none of whose complements lies in $\mathcal{F}$; i.e., by the last line of (2), such that each $\prod_{i\in J_n} G_i$ contains an element x_n which is mapped by f to a noncentral element of B. Fixing n, this says that there exists an element $b \in B$ which does not commute with $f(x_n)$. I claim we can take such a b to be the image of an element $y \in \prod_{i\in I} G_i$ under f. Indeed, if f is surjective, this is immediate. If instead we have the weaker hypothesis that $\pi f : \prod_{i\in I} G_i \to B \to B/Z(B)$ is surjective, then we can choose $y \in \prod_{i\in I} G_i$ whose image under f is congruent to b modulo $Z(B)$. Since multiplication by an element of $Z(B)$ does not affect what members of B an element commutes with, $f(y)$ does not commute with $f(x_n)$.

Let us now write $y = y_n y'$, where $y_n \in \prod_{i\in J_n} G_i$ while $y' \in \prod_{i\in I-J_n} G_i$. Then y' commutes with x_n, since they have disjoint supports in our product group. Hence $f(y')$ commutes with $f(x_n)$; hence if $f(y_n)$ also commuted with $f(x_n)$, then $f(y) = f(y_n)f(y')$ would commute with $f(x_n)$, contradicting our choice of y. Hence, rather, $x_n, y_n \in \prod_{i\in J_n} G_i$ have images in B which do not commute, giving (6). □

We can now get the first of our results showing that any group B admitting a map f satisfying (5) must be "big".

Theorem 2. *Let B be a group such that there exist a family of groups $(G_i)_{i\in I}$, and a group homomorphism $f : \prod_{i\in I} G_i \to B$, for which the induced homomorphism $\pi f : \prod_{i\in I} G_i \to B/Z(B)$ does not factor through the projection of $\prod_{i\in I} G_i$ to the product of finitely many ultraproducts of the G_i. Then B contains families of elements $(a_S)_{S\subseteq\omega}$, $(b_S)_{S\subseteq\omega}$, indexed by the subsets S of ω, such that:*

(7) *All the elements* a_S $(S \subseteq \omega)$ *commute with one another, and all the elements* b_S $(S \subseteq \omega)$ *likewise commute with one another.*

(8) *For* S *and* T disjoint *subsets of* ω*, one has* $a_S a_T = a_{S\cup T}$*,* $b_S b_T = b_{S\cup T}$*, and* $a_S b_T = b_T a_S$.

(9) *For subsets* S *and* T *of* ω *with* $\mathrm{card}(S\cap T) = 1$*,* $a_S b_T \neq b_T a_S$.

Proof. Given G_i and f as in the hypothesis, i.e., satisfying (5), Lemma 1 gives us sets $J_n \subseteq I$ and elements x_n, y_n $(n \in \omega)$ as in (6). Let $H_n = \prod_{i\in J_n} G_i \subseteq \prod_{i\in I} G_i$ $(n \in \omega)$, so that we can regard $\prod_{i\in I} G_i$ as $\prod_{n\in\omega} H_n$, the x_n and y_n as elements of that group with singleton supports, and f as a homomorphism $\prod_{n\in\omega} H_n \to B$.

For each subset $S \subseteq \omega$, let x_S be the element of $\prod_{n\in\omega} H_n$ whose component at n is x_n if $n \in S$, and e otherwise, and let elements y_S be obtained similarly from the y_n. It is easy to see that any two elements x_S and x_T commute with one another in $\prod_{n\in\omega} H_n$, and similarly for the y's; and that for S and T disjoint, $x_S x_T = x_{S\cup T}$, $y_S y_T = y_{S\cup T}$, and $x_S y_T = y_T x_S$. Hence, letting $a_S = f(x_S)$, $b_S = f(y_S)$, we get (7) and (8).

For general S and T, the commutator $[x_S, y_T]$ will have n-th component $[x_n, y_n]$ if $n \in S\cap T$, and e otherwise. So if $S\cap T$ is exactly $\{n\}$ for some $n \in \omega$, then $f([x_S, y_T]) = f([x_n, y_n])$, which by choice of x_n and y_n is not e, giving (9). □

By restricting the elements b_T that we consider, we can get a clearer view of the behavior of the elements a_S:

Corollary 3. *In the situation of Theorem 2, an element* a_S $(S \subseteq \omega)$ *commutes with an element* $b_{\{n\}}$ $(n \in \omega)$ *if and only if* $n \notin S$*. Thus, the elements* a_S *exhibit all possible combinations of which members of the countable set* $\{b_{\{n\}} \mid n \in \omega\}$ *they commute with. Hence they are distinct modulo* $Z(B)$*; so their images in* $B/Z(B)$ *generate a commutative subgroup of continuum cardinality.*

Proof. The first sentence is immediate from (8) and (9), and clearly implies the second. Since multiplication by a member of $Z(B)$ does not affect what elements a member of B commutes with, elements which can be distinguished by the latter properties are necessarily distinct modulo $Z(B)$. The group generated by the a_S is commutative in view of (7), hence so is the image of that group in $B/Z(B)$. □

Above we have obtained "element-theoretic" consequences of the existence of a map $\prod_{i\in I} A_i \to B$ that does not factor through finitely many ultrafilters. There are also "subgroup-theoretic" consequences. We shall find it convenient to state some of these, not in terms of image subgroups $\pi f\big(\prod_{i\in S} G_i\big) \subseteq B/Z(B)$, but in terms of the inverse images $f\big(\prod_{i\in S} G_i\big)Z(B)$ of those subgroups in B. Let us start by noting some general properties of this construction, independent of whether πf factors through finitely many ultraproducts.

Lemma 4. *Let B be a group, $(G_i)_{i\in I}$ a family of groups, and $f : \prod_{i\in I} G_i \to B$ a homomorphism which is surjective (or more generally, satisfies $B = f\big(\prod_{i\in I} G_i\big)Z(B)$). For every subset $S \subseteq I$, let*

(10) $B_S = f\big(\prod_{i\in S} G_i\big)Z(B)$, *a normal subgroup of B.*

Then:

(11) $B_\varnothing = Z(B)$, $B_I = B$, *and for $S, T \subseteq I$, one has $B_S B_T = B_{S\cup T}$ and $B_S \cap B_T = B_{S\cap T}$.*

(12) *For $S, T \subseteq I$, the centralizer of B_T in B_S is B_{S-T}.*

Hence (again writing $\pi : B \to B/Z(B)$ for the quotient map),

(13) *For* disjoint *subsets $S, T \subseteq I$, $\pi(B_{S\cup T})$ is the direct product of its subgroups $\pi(B_S)$ and $\pi(B_T)$.*

Moreover,

(14) *If $(S_k)_{k\in K}$ is a family of pairwise disjoint subsets of I, and we let $S = \bigcup_{k\in K} S_k$, then the map $\pi(B_S) \to \prod_{k\in K} \pi(B_{S_k})$ determined by the projections $\pi(B_S) \to \pi(B_{S_k})$ (which by (13) is an isomorphism if K is finite) is always surjective.*

Proof. That each B_S is normal in B, as asserted in (10), follows from the normality of $\prod_{i\in S} G_i$ in $\prod_{i\in I} G_i$, and the centrality of $Z(B)$ in B.

The first three equalities of (11) are immediate, as is the direction $B_S \cap B_T \supseteq B_{S\cap T}$ of the final equality. Before proving the reverse inclusion, let us note a case of (12) which is also immediate:

(15) If S and T are disjoint subsets of I, then B_S and B_T centralize one another.

To get the remaining part of (11), $B_S \cap B_T \subseteq B_{S\cap T}$, consider an element of the left-hand side, which we may write

(16) $f(u)z_1 = f(v)z_2$, where $u \in \prod_{i\in S} G_i$, $v \in \prod_{i\in T} G_i$, and $z_1, z_2 \in Z(B)$.

Let us write $u = u'u''$, where $u' \in \prod_{i\in S\cap T} G_i$ and $u'' \in \prod_{i\in S-T} G_i$. Thus our element (16) becomes $f(u')f(u'')z_1$. Since $u' \in \prod_{i\in S\cap T} G_i$, if we can show that $f(u'') \in Z(B)$, then (16) will lie in $f\big(\prod_{i\in S\cap T} G_i\big)Z(B) = B_{S\cap T}$, as required.

Thus, we need to show that $f(u'')$ centralizes $B = B_I = B_{S-T} B_{I-(S-T)}$. Since $f(u'') \in B_{S-T}$, it certainly centralizes $B_{I-(S-T)}$. On the other hand if we write the equation in (16) as

$$f(u')f(u'')z_1 = f(v)z_2, \quad \text{equivalently,} \quad f(u'') = f(u')^{-1}f(v)z_2z_1^{-1},$$

we see that all the factors on the right lie in B_T, hence centralize B_{S-T}. Hence so does $f(u'')$, completing the proof of the last assertion of (11).

We can now easily prove (12). By (11), B_{S-T} is contained in B_S, and by (15), it centralizes B_T, so we need only show that conversely, any element of B_S that centralizes B_T lies in B_{S-T}. As in the preceding argument, we can write our element of B_S as $f(u')f(u'')z$, where $u' \in \prod_{i\in S\cap T} G_i$ and $u'' \in \prod_{i\in S-T} G_i$. This time, we need to prove that $f(u') \in Z(B)$. Now since $f(u')f(u'')z$ centralizes B_T, and $f(u'')$ and z automatically do, we see that $f(u')$ centralizes B_T. Also, since $f(u') \in B_{S\cap T}$, and $S \cap T$ is disjoint from $I - T$, $f(u')$ centralizes B_{I-T}. Hence it centralizes $B_T B_{I-T} = B$, so it lies in $Z(B)$, as claimed.

The conclusion (13) follows easily from (12) and (11).

To establish (14), we take an element of $\prod_{k\in K} \pi(B_{S_k})$, lift its component in each $\pi(B_{S_k})$ to an element of $\prod_{i\in S_k} G_i$, and regard these together as giving an element of $\prod_{i\in S} G_i$; note that the image of this element in $\pi(B_S)$ has the desired property. □

Note that in the situation of the above lemma, the subgroups B_S need not be distinct for distinct $S \subseteq I$. For instance, if we take a family $(G_i)_{i\in I}$ of noncommutative groups and an ultrafilter $\mathcal{U}$ on I, let $B = \prod_{i\in I} G_i/\mathcal{U}$, and let $f : \prod_{i\in I} G_i \to B$ be the quotient map, then the above construction gives only two distinct subgroups of B: $B_S = B$ if $S \in \mathcal{U}$, and $B_S = Z(B)$ otherwise.

We shall now get a factorization-through-ultraproducts result from the above lemma. Let us (following [Bergman 2014, §4.3]) call subgroups B', B'' of a group B *almost direct factors* if $B = B'B''$, and each of B', B'' is the centralizer in B of the other. A subgroup $B' \subseteq B$ belonging to such a pair (equivalently, such that B' is its own double centralizer in B, and B is the product of B' and its centralizer) will thus be called an *almost direct factor of B*. We shall say B has *chain condition on almost direct factors* if the partially ordered set of almost direct factors of B has ascending chain condition, equivalently (since that partially ordered set is self-dual under the operation of taking centralizers), if it has descending chain condition. (As noted in [ibid.], these are the analogs for groups of definitions first made for algebras in [Bergman and Nahlus 2011, §6].)

Observe that in the situation treated in Lemma 4, statements (12) and (11) show that for every $S \subseteq I$, the subgroups B_S, B_{I-S} are a pair of almost direct factors of B. We deduce:

Theorem 5 (cf. [Bergman 2014, Proposition 4.1]). *Let B be a group, and suppose that there exist a family of groups $(G_i)_{i\in I}$ and a homomorphism $f : \prod_{i\in I} G_i \to B$ such that the induced homomorphism $\pi f : \prod_{i\in I} G_i \to B/Z(B)$ is surjective and does not factor through the natural projection of $\prod_{i\in I} G_i$ to any finite product of ultraproducts of the G_i.*

Then B does not have chain condition on almost direct factors. In fact, it has a family of almost direct factors order-isomorphic to the lattice 2^ω, and forming a sublattice of the lattice of subgroups of B.

Proof. Given $(G_i)_{i\in I}$ with the indicated non-factorization property, let $J_0, J_1, \ldots$ be as in Lemma 1. To every subset S of ω, let us associate the subgroup $B_{\bigcup_{n\in S} J_n}$.

From Lemma 4 we see that each of these subgroups is an almost direct factor of B, and that the lattice relations among the subsets of ω are also satisfied by the corresponding subgroups; so it will suffice to show that *non-inclusions* of subsets of ω yield non-inclusions of subgroups. If $S \not\subseteq T$, take $m \in S - T$. By our assumption on the J_n, the subgroup B_{J_m} is not self-centralizing, hence though it centralizes $B_{\bigcup_{n\in T} J_n}$, it does not centralize $B_{\bigcup_{n\in S} J_n}$; so the latter is not contained in the former. □

Neither of the conclusions of Theorem 2 and Theorem 5 implies the other. To get examples of these non-implications, let G be a simple group.

If we embed G^ω in any simple overgroup B, then B inherits from G^ω families of elements a_S, b_S as in Theorem 2; but being simple, B has no nontrivial almost direct decompositions, hence it satisfies chain condition on almost direct factors, i.e., fails to satisfy the conclusion of Theorem 5.

On the other hand, if we take for B the group $\bigoplus_\omega G$ of elements of G^ω having finite support, and let $B_S = \bigoplus_S G$ for each $S \subseteq \omega$, we find that these subgroups satisfy (11)–(13), hence constitute a system of almost direct factors lattice-isomorphic to 2^ω, as in Theorem 5. But if G is countable, B will also be so, so it cannot satisfy the conclusion of Theorem 2.

So neither of these groups B admits a surjective homomorphism f from a direct product group such that πf (which in both cases would be f, since $Z(B)$ is trivial) fails to factor through finitely many ultraproducts. However, in the first case, only Theorem 5 rules this out, while in the second, only Theorem 2 does.

Though the above example with $B = \bigoplus_\omega G$ satisfies (11)–(13), it does not satisfy (14), as can be seen by taking for the S_k the singleton subsets of ω. One may ask whether for any group B, every system of subgroups B_S ($S \subseteq I$) of B that satisfies all of (11)–(14) arises as in Lemma 4.

The answer is still negative. For instance, suppose B is a group which has trivial center, and which cannot be written as a homomorphic image of a nonprincipal ultraproduct of a family of groups indexed by ω. (We shall see in Section 4 that the free group on two generators, among many others, cannot be so written.) Suppose we take a nonprincipal ultrafilter $\mathcal{U}$ on ω, and define $B_S \subseteq B$ to be all of B whenever $S \in \mathcal{U}$, and $\{e\}$ otherwise. It is not hard to verify that this family satisfies (11)–(14), but that if it arose as in Lemma 4 (with ω for I), then B would be a homomorphic image of $\prod_{n\in\omega} G_n/\mathcal{U}$, contradicting our choice of B.

We record a special case of Theorem 5 for easy application to some later examples.

Corollary 6 (to Theorem 5 and its proof). *Suppose B is a group with trivial center, and having no nontrivial direct product decomposition. Then every homomorphism from a direct product group $\prod_{i\in I} G_i$ onto B factors through a single ultraproduct $\prod_{i\in I} G_i/\mathcal{U}$ of the G_i.* □

3. Further examples

Theorem 5 shows that a group B which admits a surjective homomorphism from an infinite direct product group that does not factor through finitely many ultraproducts looks, itself, in some ways, like an infinite direct product — at least after we divide out $Z(B)$. The next example shows that this behavior of $B/Z(B)$ can coexist with very un-product-like behavior in $Z(B)$.

Example 7. *Groups B and G and a homomorphism $f : G^{\omega} \to B$ such that the induced subgroups B_S ($S \subseteq \omega$) are all distinct, but such that the center of each of the given copies of G in G^{ω} is mapped isomorphically to $Z(B) \neq \{e\}$; and which also show that in (9), the hypothesis* $\mathrm{card}(S \cap T) = 1$ *cannot be weakened to merely say that $S \cap T$ is nonempty and finite.*

Construction and proof. Let k be a field, and G the Heisenberg group over k; that is, the multiplicative group of upper triangular 3×3 matrices with 1's on the main diagonal; equivalently, the group of 3-tuples of elements of k under the multiplication $(a, a', a'')(b, b', b'') = (a + b, a' + b', a'' + b'' + a'b)$. Clearly, the countable power group G^{ω} can be described as the group of 3-tuples of elements of the power ring k^{ω} under the operation given by the same formula.

Let us now take the k-vector-space homomorphism $s : \bigoplus_{\omega} k \to k$ which for each n acts on the n-th direct summand by $1 \mapsto s_n$, for some specified elements $s_n \in k - \{0\}$, and by linear algebra, let us extend s to a vector-space homomorphism $\sigma : k^{\omega} \to k$. Let B be the homomorphic image of G^{ω} gotten by dividing $Z(G^{\omega}) = k^{\omega}$ by $\ker(\sigma)$. This can be described as

(17) $\quad k^{\omega} \times k^{\omega} \times k, \quad$ under the operation

$$(a, a', a'')(b, b', b'') = (a + b, a' + b', a'' + b'' + \sigma(a'b)).$$

I claim that $Z(B) = \{0\} \times \{0\} \times k$. To see this, let us first show that every $(a, b, c) \in B$ with $a \neq 0$ is noncentral. Choose n such that a has n-th component $a_n \neq 0$, and take $b' \in k^{\omega}$ to have 1 in the n-th position and 0 in all others. Then we find that the commutator of (a, b, c) and $(0, b', 0)$ is $(0, 0, s_n a_n) \neq e$. The analogous argument shows (a, b, c) noncentral if $b \neq 0$. Elements $(0, 0, c)$ are clearly central, so we get the asserted description of $Z(B)$, and we see that this is the image in B of the center of each of our copies of G in G^{ω}, and indeed, of the center of $G^S \subseteq G^{\omega}$ whenever $\varnothing \neq S \subseteq \omega$.

So though the images in $B/Z(B)$ of these subgroups G^S are the corresponding factors $(k \times k)^S \subseteq (k \times k)^{\omega}$, when we look at the images in $Z(B)$ of their centers, the distinctions among them disappear.

To get the final assertion of this example, let us partition ω into the singletons $J_n = \{n\}$, so that in the notation of the proof of Theorem 2, each H_n is G. For each n, let $x_n = (1, 0, 0)$, $y_n = (0, 1, 0)$ in H_n, and let us use these to construct elements $a_S, b_T \in B$ as in that proof. Then if S and T are subsets of ω which intersect in a finite set $\{n_0, \ldots, n_{d-1}\}$, we see that in B the commutator $[a_S, b_T]$ is $(0, 0, s_{n_0} + \cdots + s_{n_{d-1}})$. If $d = 1$ this is necessarily a nonidentity element, as stated in (9); but if S and T intersect in more than one element, this may or may not be true, depending on the choice of the s_n. (In particular, if the field k is finite, then whatever the s_n, there must be some nonempty family of $\leq \mathrm{card}(k)$ s_n's that sum to zero.) So the restriction $\mathrm{card}(S \cap T) = 1$ in (9) cannot be dropped. □

In the above example, the focus was on the part of the map going into $Z(B)$; the map $G^\omega \to B/Z(B)$ was a straightforward homomorphism of direct products. But this is not always the case; that is, the maps which (14) shows to be surjective need not, in general, be isomorphisms. For instance, in the example mentioned immediately after the proof of Lemma 4, where the G_i were arbitrary noncommutative groups, and f was the map $\prod_{i \in I} G_i \to \bigl(\prod_{i \in I} G_i\bigr)/\mathcal{U}$, for $\mathcal{U}$ an ultrafilter on I, if $\mathcal{U}$ is nonprincipal and we take for the S_k all the singletons $\{i\}$ ($i \in I$), so that $S = I$, then each $\pi(B_{S_k})$ is trivial, but $\pi(B_S)$ is not.

One can, of course, modify this example to get one which also has the property that every G_i has nontrivial image in $B/Z(B)$:

Example 8. *A group homomorphism $\prod_{n \in \omega} G_n \to B$ where all the $B_{\{n\}}/Z(B)$ are nonzero* (*so that all the B_S are distinct*), *but not all the surjections of* (14) *are isomorphisms.*

Construction. Let G_n ($n \in \omega$) be groups with trivial centers, each having a proper nontrivial normal subgroup $N_n \lhd G_n$ such that G_n/N_n also has trivial center. Let $\mathcal{U}$ be any nonprincipal ultrafilter on ω, let $H = \bigl(\prod_{n \in \omega} G_n\bigr)/\mathcal{U}$, and let $f : \prod_{n \in \omega} G_n \to H \times \prod_{n \in \omega} G_n/N_n$ be the map obtained from the obvious homomorphisms $\prod_{n \in \omega} G_n \to H$ and $\prod_{n \in \omega} G_n \to \prod_{n \in \omega} G_n/N_n$. Let B be the image of f.

For $S \subseteq \omega$, what does B_S look like? This depends on whether or not $S \in \mathcal{U}$. If not, we see that $B_S = \prod_{n \in S} G_n/N_n$; in particular, for every $n \in \omega$ we have $B_{\{n\}} = G_n/N_n$. Thus for $S \notin \mathcal{U}$, the group B_S can be identified with $\prod_{n \in S} B_{\{n\}}$. However, when $S \in \mathcal{U}$, the H-component of B_S will be the full group H, which carries structure from the normal subgroups N_n which is ignored by each group $B_{\{n\}}$; so in these cases, the natural map $B_S \to \prod_{n \in S} B_{\{n\}}$ is not one-to-one. □

One can generalize the above construction by replacing $\mathcal{U}$ with an arbitrary filter $\mathcal{F}$, though the description of the groups B_S is more complicated to state when S is neither a member of $\mathcal{F}$ nor the complement of one. And, of course, one can set up examples based on more than one system of normal subgroups and more than one filter.

In the above example, though the system of subgroups B_S described does not have the property that the maps of (14) are isomorphisms, the group B has other systems of subgroups that can be shown to have that property. Here is an example having no such family.

Example 9. *A group B having elements a_S and b_S $(S \subseteq \omega)$ satisfying (7)–(9), and distinct subgroups B_S $(S \subseteq \omega)$ satisfying (10)–(13), but having no such system of distinct subgroups also satisfying (14) for any infinite family of disjoint nonempty sets $(s_k)_{k\in K}$; so that B cannot admit a surjective homomorphism from a direct product group which does not factor through the product of finitely many ultraproducts.*

Construction and sketch of proof. Let G be an infinite simple group, and B the subgroup of G^ω consisting of those ω-tuples assuming only finitely many distinct values in G. If we choose a pair of noncommuting elements $x, y \in G$, and for each $n \in \omega$ let x_n be the element x of the n-th copy of G, and y_n the element y thereof, then we see that the elements x_S and y_S $(S \subseteq \omega)$, constructed as in Theorem 2, will lie in B, and, renamed a_S and b_S, will satisfy (7)–(9). Similarly, if we let B_S be the subgroup of B consisting of elements with support in S, then (11)–(13) are immediate.

I will now sketch why B admits no system of nontrivial almost direct factors B_{S_k} and B_S satisfying (14) for any infinite K. Note that B has trivial center, so that almost direct factors are simply direct factors. Now it is easy to verify using the simplicity of G that if B has a direct product decomposition $B = B' \times B''$, then for each $n \in \omega$, one of B', B'' has as n-th coordinates all members of G, while the other has only e in that coordinate. From this one can deduce that every such decomposition has the form $B' = B_S$, $B'' = B_{\omega - S}$ for some $S \subseteq \omega$. We can now combine the "finitely many distinct values" condition in the definition of B with the fact that G is infinite to see the impossibility of an infinite family of nontrivial almost direct factors B_{S_k} satisfying the surjectivity condition (14).

Lemma 4 and the method of proof of Theorem 5 now show that every homomorphism from a direct product onto B must factor through finitely many ultraproducts. □

(For some other results on the subgroup of a power group G^I consisting of the elements with only finitely many distinct coordinates — though for abelian groups — see [Bergman 1972].)

4. Conditions forcing the ultrafilters to be principal

We have obtained conditions that force group homomorphisms $\prod_{i\in I} G_i \to B$ to factor through the direct product of finitely many ultraproducts of the G_i. When can we say that any map that so factors must in fact factor through the product of finitely many G_i; i.e., that the ultrafilters involved must be principal?

Here set-theoretic considerations come in. If κ is a *measurable cardinal*, then sets I of cardinality $\geq \kappa$ admit nonprincipal *κ-complete* ultrafilters; that is, ultrafilters closed under all $<\kappa$-fold intersections. (Two quick terminological notes:

(i) The condition of being closed under countable intersections, which by the above definition is $\aleph_1$-completeness, is also called *countable completeness*. (ii) We shall follow the definition of measurable cardinal used in [Chang and Keisler 1990], which counts $\aleph_0$ as measurable; so we will write "uncountable measurable cardinal" for what many authors simply call a measurable cardinal.)

If κ is an uncountable measurable cardinal and I a set of cardinality $\geq \kappa$, and we take a family $(G_i)_{i\in I}$ of groups (or more generally, of any sort of algebraic structures defined by finitely many finitary operations) whose cardinalities have a common bound $<\kappa$, then their ultraproducts with respect to κ-complete ultrafilters behave very much as do ordinary ultraproducts of finite groups with a common finite bound on their orders; to wit, every such ultraproduct is isomorphic to one of the G_i. Hence, if there exists such a cardinal κ, then *every* group B of cardinality $<\kappa$ can be represented as an ultrapower of itself with respect to a nonprincipal κ-complete ultrafilter $\mathcal{U}$. So for every such B we get a surjective homomorphism $B^I \to B$ which factors through the ultrapower $B^I/\mathcal{U}$ but not through finitely many projection maps — which seems to be bad news for the type of result we are hoping for.

However, it is known that if uncountable measurable cardinals exist, they must be quite enormous [Chang and Keisler 1990, Theorem 4.2.14], and that if the standard set theory, ZFC, is consistent, it is consistent with the nonexistence of such cardinals. Hence it would be reasonable to work under the assumption that no uncountable measurable cardinals exist, or, if they exist, to restrict our index sets to cardinalities less than all such cardinals.

The next observation shows that when doing the spade-work of our investigation, we can in fact restrict attention to the case where our index set is countable.

Lemma 10. *If B is a group, then the following conditions are equivalent.*

(18) *B is a homomorphic image of an ultraproduct of a family of groups indexed by an arbitrary set I, with respect to some ultrafilter $\mathcal{U}$ on I that is not countably complete, equivalently, that is not κ-complete for any uncountable measurable cardinal κ.*

(19) *B is a homomorphic image of an ultraproduct of a family of groups indexed by ω, with respect to a nonprincipal ultrafilter on ω.*

The same is true with "groups" replaced by objects of any other variety of finitary algebras, in the sense of universal algebra.

Proof. The equivalence referred to in (18) follows from the fact that any countably complete ultrafilter must be κ-complete for some uncountable measurable cardinal κ [Chang and Keisler 1990, Proposition 4.2.7].

Since a nonprincipal ultrafilter on ω is not countably complete, we have (19) $\Longrightarrow$ (18). On the other hand, it is easy to show that if $\mathcal{U}$ is a non-countably-complete ultrafilter on a set I, then I can be partitioned as $\bigcup_{n\in\omega} J_n$ where no J_n belongs to $\mathcal{U}$. In this situation we find that $\{S \subseteq \omega \mid \bigcup_{n\in S} J_n \in \mathcal{U}\}$ is a nonprincipal ultrafilter $\mathcal{U}'$ on ω, and that given groups G_i $(i \in I)$, the natural map $\prod_{i\in I} G_i \to \prod_{i\in I} G_i/\mathcal{U}$ factors through $\prod_{n\in\omega}\bigl(\prod_{i\in J_n} G_i\bigr)/\mathcal{U}'$. Hence, writing $\prod_{i\in J_n} G_i = H_n$, we see that if, as in (18), B is a homomorphic image of $\prod_{i\in I} G_i/\mathcal{U}$, then it is also a homomorphic image of $\prod_{n\in\omega} H_n/\mathcal{U}'$, giving (19).

The final assertion is clear. (The assumption that our algebras are finitary is needed to insure that algebra structures are induced on ultraproducts of such algebras.) □

So below, it will suffice to examine which groups are homomorphic images of nonprincipal ultraproducts of countable families of groups. For brevity, we shall call an ultraproduct of a countable family a "countable ultraproduct".

My first guess was that if B was such a homomorphic image, then the cardinality of $B/Z(B)$ would have to be either finite or at least the cardinality of the continuum. But Tom Scanlon suggested the following counterexample.

Lemma 11 (T. Scanlon, personal communication). *Let B be the semidirect product of the additive group $\mathbb{Q}$ of rational numbers, and the 2-element group $\{\pm 1\}$, determined by the multiplicative action of the latter on the former.* (*I.e.*, *B has underlying set $\{\pm 1\} \times \mathbb{Q}$, and multiplication $(\alpha, a)(\beta, b) = (\alpha\beta, \beta a + b)$.*)

Then every ultrapower of B admits a homomorphism onto B. Hence though $B = B/Z(B)$ is countable, *it is a homomorphic image of a nonprincipal countable ultraproduct of groups.*

Proof. Clearly, the only elements of B that commute with $(-1, 0)$ are those with second component 0, while the only elements that commute with $(1, 1)$ are those with first component 1; so $Z(B) = \{e\}$, justifying the formula $B = B/Z(B)$.

It is easy to see that for any ultrafilter $\mathcal{U}$ on any index set I, the ultrapower $B^I/\mathcal{U}$ will be the semidirect product of $\{\pm 1\}$ and $\mathbb{Q}^I/\mathcal{U}$ determined by the natural action of the former group on the latter. Now $\mathbb{Q}^I/\mathcal{U}$, like $\mathbb{Q}$, is a nontrivial torsion-free divisible group, i.e., a nontrivial $\mathbb{Q}$-vector-space, and, as such, admits a surjective homomorphism $\varphi : \mathbb{Q}^I/\mathcal{U} \to \mathbb{Q}$. The map $B^I/\mathcal{U} \to B$ given by $(\alpha, \beta) \mapsto (\alpha, \varphi(\beta))$ is easily seen to be a surjective homomorphism, as claimed. □

By Corollary 6, every homomorphism from a direct product group $\prod_{i\in I} G_i$ onto the above group B factors through a single ultraproduct of the G_i; but the above result shows that (even when the index set is countable) the ultrafilter involved need not be principal.

In fact, the only condition I know that guarantees factorization through finitely many of the G_i is based on requiring appropriate abelian subgroups of B to satisfy similar factorization properties as abelian groups. The key observation is:

Lemma 12. *Suppose B is a homomorphic image of a nonprincipal countable ultraproduct of groups, $\left(\prod_{n\in\omega} G_n\right)/\mathcal{U}$. Then every element $b \in B$ lies in a homomorphic image within B of $\mathbb{Z}^\omega/\mathcal{U}$, a nonprincipal countable ultrapower of $\mathbb{Z}$.*

Proof. Given $b \in B$, let b be the image of $(g_n)_{n\in\omega} \in \prod_{n\in\omega} G_n$. Then the homomorphism $\gamma : \mathbb{Z}^\omega \to \prod_{n\in\omega} G_n$ taking $(m_n)_{n\in\omega}$ to $(g_n^{m_n})_{n\in\omega}$ induces a homomorphism $\gamma' : \mathbb{Z}^\omega/\mathcal{U} \to \left(\prod_{n\in\omega} G_n\right)/\mathcal{U}$, with which it forms a commuting square. Hence the composite map $\mathbb{Z}^\omega \to \prod_{n\in\omega} G_n \to \prod_{n\in\omega} G_n/\mathcal{U} \to B$, which carries $(1, 1, \dots) \in \mathbb{Z}^\omega$ to b, factors through $\mathbb{Z}^\omega/\mathcal{U}$; so b lies in a homomorphic image of that group. □

To see that this puts strong restrictions on groups B admitting such homomorphisms, note that every slender abelian group, in particular, the infinite cyclic group, has the property of *not* being a homomorphic image of a nonprincipal countable ultrapower of $\mathbb{Z}$. We will see wider classes of abelian groups with this property in the next section.

Though this note emphasizes the separate conditions that maps from infinite products yield factorizations through finitely many ultraproducts, and that the ultraproducts in all such factorizations are principal, let us record how the above lemma allows one to combine results of the former sort obtained in Section 2 above, and results of the latter sort for abelian groups, which will be obtained in Sections 5–6, to give sufficient conditions for all maps from a direct product of groups to factor through finitely many projection maps.

Theorem 13. *Let B be a group with the property that for every homomorphism from a direct product group,*

(20) $f : \prod_{i\in I} G_i \to B$ *such that the composite homomorphism* $\pi f : \prod_{i\in I} G_i \to B/Z(B)$ *is surjective,*

the map πf factors through the projection to finitely many ultraproducts of the G_i (cf. Section 2 above).

Suppose, moreover, that for every almost direct factor $B' \neq Z(B)$ of B, the group $B'/Z(B)$ contains at least one element b which does not lie in any homomorphic image therein of a nonprincipal countable ultraproduct of copies of $\mathbb{Z}$ (cf. Sections 5–7 below).

Then for every homomorphism (20) such that card(I) *is less than every uncountable measurable cardinal (if any such cardinals exist), the composite $\pi f : \prod_{i\in I} G_i \to B/Z(B)$ factors through the product of finitely many of the G_i.*

Proof. Given a homomorphism (20) satisfying the indicated bound on card(I), let us factor πf through a direct product $\prod_{i\in I} G_i/\mathcal{U}_0 \times \dots \times \prod_{i\in I} G_i/\mathcal{U}_{m-1}$, where $\mathcal{U}_0, \dots, \mathcal{U}_{m-1}$ are distinct ultrafilters on I. Without loss of generality, we may assume that each $\prod_{i\in I} G_i/\mathcal{U}_k$ has nontrivial image in $B/Z(B)$. Choosing a partition $I = J_0 \cup \dots \cup J_{m-1}$ with $J_k \in \mathcal{U}_k$, we get, by Lemma 4, an almost direct decomposition of B into subgroups B_{J_k}. Now suppose one of our ultrafilters $\mathcal{U}_k$ were not principal. By our assumption on the cardinality of I, $\mathcal{U}_k$ is not κ-complete for any uncountable measurable cardinal κ, hence by Lemma 10, (18) $\implies$ (19), $B_{J_k}/Z(B)$ satisfies the hypothesis of Lemma 12. But since B_{J_k} is an almost direct factor of B, by assumption $B_{J_k}/Z(B)$ has an element b whose properties contradict the conclusion of that lemma. So, rather, every $\mathcal{U}_k$ must be principal, say generated by a singleton $\{n_k\} \subseteq J_k$. Hence our factorization through

$$\Big(\prod_{i\in I} G_i\Big)/\mathcal{U}_0 \times \cdots \times \Big(\prod_{i\in I} G_i\Big)/\mathcal{U}_{m-1}$$

is in fact a factorization through $G_{n_0} \times \cdots \times G_{n_{m-1}}$. □

Quick examples of groups B to which the above result applies are free groups on more than one generator, and the infinite dihedral group. Indeed, since both groups have trivial center, the "$/Z(B)$" in the statement can be ignored, and since neither has a nontrivial direct product decomposition, it suffices to verify that each has an element b not contained in any homomorphic image of a nonprincipal countable ultraproduct of copies of $\mathbb{Z}$. In a free group, every nontrivial abelian subgroup is infinite cyclic, hence slender, so any nonidentity element can serve as such a b. In the dihedral group $D = \langle x, y \mid x^2 = e = y^2 \rangle$, the element $b = xy$ generates an infinite cyclic subgroup which is its own centralizer, again establishing the hypothesis of the theorem. Another class of examples is noted in:

Corollary 14 (to Lemma 12). *Let X be an infinite set, and B a group of permutations of X having a cyclic subgroup $\langle b \rangle$ whose action on X has exactly one infinite orbit (no restriction being assumed on the number of finite orbits of $\langle b \rangle$). Then the centralizer of b in B admits a homomorphism to $\mathbb{Z}$ taking b to* 1. *Hence B is not a homomorphic image of a nonprincipal ultraproduct of a countable family of groups.*

In particular, this is true if B is the full symmetric group on X, or more generally, if for some filter $\mathcal{F}$ on X not consisting entirely of cofinite subsets, B is the group of permutations of X whose fixed sets belong to $\mathcal{F}$.

Proof. If two permutations a and b of a set X commute, it is easy to see that a will carry orbits of $\langle b \rangle$ to orbits of $\langle b \rangle$, and, of course, the image orbits will have the same cardinalities as the original orbits. Hence, if $\langle b \rangle$ has a unique infinite orbit Y, then a must carry Y to itself; and it is easy to verify that it must act on Y by some power b^{n_a} of b. The function $a \mapsto n_a$ now gives the desired homomorphism of the centralizer of b onto $\mathbb{Z}$. Hence, every commutative subgroup of B containing b admits a homomorphism onto $\mathbb{Z}$, so as in the other examples discussed above, B is not a homomorphic image of a nonprincipal ultraproduct of a countable family of groups.

Now if $\mathcal{F}$ is a filter on X containing a set W which is not cofinite, we can take a countably infinite subset $Y \subseteq X - W$, and let b be a permutation which has Y as an orbit, and fixes all other points of X. This gives the final assertion of the corollary. The full symmetric group on X is the particular case where $\mathcal{F}$ is the improper filter on X. □

(With a little more work, one can get a result similar to the first paragraph of the above corollary under the weaker assumption that $\langle b\rangle$ has at least one but only finitely many distinct infinite orbits, say $\langle b\rangle x_0, \ldots, \langle b\rangle x_{d-1} \subseteq X$. In this case, for each a centralizing b, we find that $ax_i = b^{n_{a,i}} x_{\pi_a(i)}$ $(0 \leq i < d)$ for some permutation π_a of $\{0, \ldots, d-1\}$ and integers $n_{a,0}, \ldots, n_{a,d-1}$. It is then easy to verify that the map $a \mapsto \sum_i n_{a,i}$ is a homomorphism from the centralizer of b to $\mathbb{Z}$, which carries b to d.)

The next result, in contrast, gives a large class of groups that *do* admit surjective homomorphisms from nonprincipal countable ultraproducts. The construction appears to be well known, but I have not been able to find a reference.

Proposition 15. *If a group B admits a compact Hausdorff group topology, then for any set I and any ultrafilter $\mathcal{U}$ on I, there exists a homomorphism $B^I/\mathcal{U} \to B$ left-inverse to the natural embedding $B \to B^I \to B^I/\mathcal{U}$ (where the first arrow is the diagonal map).*

Hence, every group B admitting a compact Hausdorff group topology is a homomorphic image of a nonprincipal countable ultraproduct of groups; hence so is every homomorphic image of such a group.

These statements hold, more generally, with groups replaced by the objects of any variety of finitary algebras, in the sense of universal algebra.

Sketch of proof. Fix a compact Hausdorff group topology on B. Given $x \in B^I$, let us associate to each $S \in \mathcal{U}$ the set $X_S = \{x_i \mid i \in S\} \subseteq B$. These sets clearly have the finite intersection property, hence so do their closures. On the other hand, with the help of the definition of ultrafilter and the Hausdorffness of our topology, it is easy to verify that those closures can have no more than one common point. Hence by compactness, the system of sets X_S must converge to a single point of B. It is immediate that the map associating to x the limit point of this system depends only on the image of x in $B^I/\mathcal{U}$, and so induces a map $B^I/\mathcal{U} \to B$, and it is easy to verify that this is a homomorphism with the asserted properties.

The statements in the second paragraph of the lemma clearly follow. The final generalization holds by the same reasoning. □

By the above result, such a B is a homomorphic image of $B^I/\mathcal{U}$ for *every* ultrafilter $\mathcal{U}$ on every set I. This suggests the following question, where for simplicity we limit ourselves to $I = \omega$.

Question 16. *If $\mathcal{U}, \mathcal{U}'$ are nonprincipal ultrafilters on ω, can every group B which can be written as a homomorphic image of an ultraproduct of groups with respect* *to $\mathcal{U}$ also be written as a homomorphic image of an ultraproduct of groups with respect to $\mathcal{U}'$?*

Question 17. *If the answer to Question 16 is negative, is it at least true that for any two ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ on ω, there exists an ultrafilter $\mathcal{U}''$ on ω such that every group which can be written as a homomorphic image of an ultraproduct of groups with respect to $\mathcal{U}$* or *with respect to $\mathcal{U}'$ can be written as a homomorphic image of an ultraproduct with respect to $\mathcal{U}''$?*

If Question 17 has a positive answer, one can deduce that the class of groups which can be written as homomorphic images of nonprincipal countable ultraproducts of groups is closed under finite direct products.

Proposition 15 also leads one to wonder whether every group B which can be written as a homomorphic image of a nonprincipal countable ultraproduct of groups can in fact be written as a homomorphic image of a nonprincipal countable ultrapower $B^\omega/\mathcal{U}$ of itself, via a left inverse to the natural embedding $B \to B^\omega/\mathcal{U}$. The answer is negative; we shall see in the second paragraph after Lemma 28 that there exist abelian groups for which this is not true.

Let us note a couple of groups B for which the results of this section do not, as far as I can see, give us any information.

Question 18. *Can either of the following groups be written as a homomorphic image of a nonprincipal ultraproduct of a countable family of groups?*

(i) *An infinite finitely generated Burnside group?*

(ii) *The group of those permutations of an infinite set that move only finitely many elements? (Contrast Corollary 14.)*

Let us also record, since we know no counterexample,

Question 19. *Is the converse to Lemma 12 true? That is, if $\mathcal{U}$ is an ultrafilter on ω, and B is a group such that every $b \in B$ lies in a homomorphic image within B of $\mathbb{Z}^\omega/\mathcal{U}$, must B be a homomorphic image of an ultraproduct group $\prod_{i\in\omega} G_i/\mathcal{U}$?*

A positive answer seems extremely unlikely. It would imply, in particular, that every torsion group was such a homomorphic image for every $\mathcal{U}$. (So it would imply positive answers to both parts of Question 18.)

We remark that the results we have obtained so far show that the two sorts of properties of an object B that we are considering in this note — (a) that surjective homomorphisms from direct products onto B yield factorizations through finitely many ultraproducts, and (b) that when one has such a factorization, and the index set of the product is countable, the ultraproducts involved must be principal — are independent, for groups. Theorem 13 gave us examples satisfying both (a) and (b), such as the free group on more than one generator, and the infinite dihedral group.

Any infinite direct product of free groups on more than one generator will still satisfy (b) (since a homomorphism from a nonprincipal countable ultraproduct group *into* such a product will have trivial composite with the projection onto each factor, hence must be trivial), but will fail to satisfy (a), by virtue of *being* an infinite direct product. Examples satisfying (a) but not (b) are given by groups satisfying the hypotheses of both Corollary 6 and Proposition 15; for instance, finite simple groups. Finally, infinite direct products of such examples satisfy neither (a) nor (b).

(Incidentally, if a map $\prod_{i\in I} G_i \to B$ factors as

$$\prod_{i\in I} G_i \to \Big(\prod_{i\in I} G_i\Big)/\mathcal{U}_0 \times \cdots \times \Big(\prod_{i\in I} G_i\Big)/\mathcal{U}_{m-1} \to B,$$

one or more of the factors $\big(\prod_{i\in I} G_i\big)/\mathcal{U}_k$ may be irrelevant to the factorization, i.e., may map trivially to B. In condition (b) in the above discussion, we understand the phrase "the ultraproducts involved" to exclude such "irrelevant" factors; if we did not, (b) could never hold.)

5. Abelian groups

We have seen that in the study of homomorphisms on products of nonabelian groups, the analogous questions for abelian groups are important. We now turn to that case.

Although, as just noted, the two sorts of condition we are interested in are independent for nonabelian groups, we shall find that this is not true of the corresponding conditions on abelian groups.

First, some notation, language, and basic observations.

Definition 20. *In Sections 5–7, we shall use additive notation in abelian groups.*

In groups $\mathbb{Z}^\omega$, $(\mathbb{Z}/p\mathbb{Z})^\omega$, etc., we shall write δ_n $(n \in \omega)$ for the element having 1 *in the n-th position and* 0 *in all other positions.*

An abelian group B is called slender *if it is torsion-free, and every homomorphism $f : \mathbb{Z}^\omega \to B$ annihilates all but finitely many of the δ_n.*

The above definition of a slender abelian group is standard, but the condition that B be torsion-free is redundant: no B with torsion satisfies the condition on homomorphisms. For in such a B, we can choose an element b of prime order p, define the homomorphism $\bigoplus_{n\in\omega} \mathbb{Z}/p\mathbb{Z} \to \langle b\rangle$ taking each δ_n to b, extend this, by linear algebra over the field $\mathbb{Z}/p\mathbb{Z}$, to a homomorphism $(\mathbb{Z}/p\mathbb{Z})^\omega \to \langle b\rangle$, and precompose with the natural map $\mathbb{Z}^\omega \to (\mathbb{Z}/p\mathbb{Z})^\omega$, to get a map $\mathbb{Z}^\omega \to B$ that does not annihilate any δ_n.

The condition of slenderness is stronger than it looks. Indeed, our statement of that condition in Section 1 implicitly incorporated the following striking complementary fact.

(21) [Fuchs 1973, fact (f) on p. 159] *If B is a slender abelian group, then the only homomorphism $f : \mathbb{Z}^\omega \to B$ which annihilates all the elements δ_n $(n \in \omega)$ is* 0.

We can now prove:

Proposition 21. *The following conditions on an abelian group B are equivalent.*

(22) *There exists a surjective homomorphism $f : \prod_{i\in I} A_i \to B$ from the direct product of a family of abelian groups to B, which does not factor through the natural map from $\prod_{i\in I} A_i$ to the direct product of finitely many countably complete ultraproducts of the A_i.*

(23) *There exists a surjective homomorphism $f : \prod_{n\in\omega} A_n \to B$ from the direct product of a countable family of abelian groups to B, which does not factor through the projection of $\prod_{n\in\omega} A_n$ to the direct product of finitely many ultraproducts* (*principal or nonprincipal*) *of the A_n.*

(24) *B is not slender.*

These are also equivalent to the variants of conditions (22) and (23) without the assumption that f be surjective.

Proof. We start with the final sentence. Conditions (23) and (22) certainly imply the corresponding statements without the condition of surjectivity. Conversely (as noted in Section 1), if we have an example of either of those conditions minus the surjectivity restriction, we can get one satisfying that condition by passing from the given map $\prod_{i\in I} A_i \to B$ to the obvious surjective map $B \times \prod_{i\in I} A_i \to B$.

Let us now show that (24) $\Longrightarrow$ (23)$\Longrightarrow$ (22) $\Longrightarrow$ (24).

Given (24), take a map $f : \mathbb{Z}^\omega \to B$ witnessing the failure of slenderness, i.e., carrying infinitely many of the δ_n to nonzero values. If f factored through a product of finitely many ultrapowers, $\mathbb{Z}^\omega/\mathcal{U}_0 \times \cdots \times \mathbb{Z}^\omega/\mathcal{U}_{m-1}$, then the only elements δ_n which could have nonzero image under f would be those such that one of the $\mathcal{U}_k$ was the principal ultrafilter generated by $\{n\}$, of which there can be at most finitely many. So there is no such factorization, so f witnesses (23) (in its version without the hypothesis of surjectivity).

Clearly, (23) $\Longrightarrow$ (22).

Given f as in (22), we shall prove (24) by considering two cases. First suppose that f can be factored through a product of finitely many ultraproducts $\prod_{i\in I} A_i/\mathcal{U}_0 \times \cdots \times \prod_{i\in I} A_i/\mathcal{U}_{m-1}$, but that not all the $\mathcal{U}_k$ can be taken countably complete. Note that f is the *sum* of homomorphisms f_k $(k = 0, \dots, m-1)$ that factor through the respective ultraproducts $\prod_{i\in I} A_i/\mathcal{U}_k$, and we can drop from this sum, and hence from our factorization, any factors $\prod_{i\in I} A_i/\mathcal{U}_k$ such that f_k is zero. Hence for some k with $\mathcal{U}_k$ *not* countably complete, we must have a *nonzero* homomorphism $f_k : \prod_{i\in I} A_i/\mathcal{U}_k \to B$. The statement that $\mathcal{U}_k$ is not countably complete is equivalent to saying that there exists a partition $I = J_0 \cup \cdots \cup J_n \cup \cdots$ such that none of the J_n lie in $\mathcal{U}_k$; in other words, such that for each n, $f_k | \prod_{i\in J_n} A_i = 0$. If we regard f_k as a map $\prod_{n\in\omega}(\prod_{i\in J_n} A_i) \to B$, the fact that it is nonzero means that we can choose an element $(x_n)_{n\in\omega} \in \prod_{n\in\omega}(\prod_{i\in J_n} A_i)$ which f_k sends to a nonzero element of B, though we know that it takes each x_n to 0. Using this element $(x_n)_{n\in\omega}$, let us construct a map $\mathbb{Z}^\omega \to B$ by taking each $(d_n)_{n\in\omega} \in \mathbb{Z}^\omega$ to $(d_n x_n)_{n\in\omega}$, and applying f_k to this ω-tuple. This gives a homomorphism $\mathbb{Z}^\omega \to B$ which is zero on each δ_n, but not on $(1, \ldots, 1, \ldots)$. Thus, by (21), B is not slender. (Alternatively, we can get a direct contradiction to the definition of slenderness by choosing f_k and $(x_n)_{n\in\omega}$ as above, and mapping $(d_n)_{n\in\omega} \in \mathbb{Z}^\omega$ to $f_k((\sum_{m<n} d_m)x_n)_{n\in\omega} \in B$.)

There remains the case where f cannot be factored through any product of finitely many ultraproducts of the A_i. Then the filter $\mathcal{F}$ of subsets $S \subseteq I$ such that f can be factored through $\prod_{i\in S} A_i$ is not a finite intersection of ultrafilters, so by (4) there exists a partition $I = J_0 \cup \cdots \cup J_n \cup \cdots$ such that $I - J_n \notin \mathcal{F}$ for all n; in other words, such that each $\prod_{i\in J_n} A_i \subseteq \prod_{i\in I} A_i$ has nonzero image under f. Choosing an x_n in each $\prod_{i\in J_n} A_i$ with nonzero image, we construct as in the preceding case a homomorphism $\mathbb{Z}^\omega \to B$. This time, that homomorphism will be nonzero on every δ_n, showing that B does not satisfy the definition of slenderness. □

Slender abelian groups have been precisely characterized [Nunke 1961; Fuchs 1973, Proposition 95.2]: they are the abelian groups which have no torsion elements, and contain no embedded copies of either $\mathbb{Q}$, or the group of p-adic integers for any prime p, or $\mathbb{Z}^\omega$.

In the statement of the above proposition, note that condition (22) is formally weaker than (23) in two ways: it allows an arbitrary index set I, and it excludes factorization only through *countably complete* ultraproducts (which in the context $I = \omega$ of (23) would mean principal ultraproducts, i.e., the given groups A_n). Since (22) and (23) are equivalent, they are also equivalent to two intermediate conditions: the one obtained from (23) by replacing “ultraproducts (principal or nonprincipal)” by “principal ultraproducts”, and the one obtained from (22) by deleting the words “countably complete”.

We can deduce from these observations that the two sorts of conditions on an abelian group B that we are interested in — namely, (a) that maps to B from infinite direct products factor through finitely many ultraproducts, and (b) that in the case of a countable product, if we have a such a factorization, the ultrafilters involved are all principal — are not independent; precisely, that (a) implies (b). Indeed, (a) is equivalent to the negation of the version of (22) without the “countably complete” condition, which by the above observations is equivalent to the negation of the version of (23) in which the ultrafilters are assumed principal, i.e., the statement that every homomorphism from a countable product into B factors through finitely many A_n, which clearly entails (b). Bringing in (24), we see that both (a) and (a)$\wedge$(b) are equivalent to slenderness.

On the other hand, the three cases not excluded by the implication (a) $\implies$ (b) all do occur. Slender groups, such as $\mathbb{Z}$, satisfy both (a) and (b). An infinite direct product of nontrivial slender groups, e.g., $\mathbb{Z}^\omega$, satisfies (b) but not (a). Finally, any nonprincipal countable ultraproduct of nontrivial abelian groups will not satisfy (b), hence, since (a) $\implies$ (b), it will satisfy neither.

Having characterized the abelian groups B that satisfy (a), it remains to characterize the larger class satisfying (b). As preparation, we shall first study the abelian groups that are homomorphic images of a *single* nonprincipal countable ultraproduct. We will need a few more definitions from the theory of infinite abelian groups.

Definition 22 [Fuchs 1970; Rotman 2009]. *A subgroup B of an abelian group A is called* pure *if for every positive integer n, $B \cap nA = nB$.*

An abelian group B is said to be algebraically compact *if for every overgroup $A \supseteq B$ in which B is pure, B is a direct summand in A; equivalently* [Fuchs 1970, Theorem 38.1], *if for every set X of group equations in constants from B and B-valued variables, such that every finite subset of X has a solution in B, the whole set X has a solution in B.*

An abelian group B is said to be cotorsion *if for every overgroup $A \supseteq B$ such that A/B is torsion-free (a stronger condition than B being pure in A), B is a direct summand in A.*

Of the two definitions of algebraic compactness quoted above, the first is the one commonly used. I include the second because it motivates the name of the condition. The theorem cited for their equivalence establishes several other diverse conditions as also equivalent to algebraic compactness; below, I shall pull these out of a hat as needed.

The cotorsion abelian groups clearly include the algebraically compact abelian groups. In fact, they are precisely the *homomorphic images* of such groups [Fuchs 1970, Proposition 54.1], a fact called on in condition (30) in the next result.

Proposition 23. *For B an abelian group, the following conditions are equivalent.*

(25) *There exists a family of abelian groups $(A_i)_{i\in I}$ and a non-countably-complete ultrafilter $\mathcal{U}$ on I such that B is a homomorphic image of the ultraproduct $\prod_{i\in I} A_i/\mathcal{U}$.*

(26) *There exists a countable family of abelian groups $(A_n)_{n\in\omega}$ and a nonprincipal ultrafilter $\mathcal{U}$ on ω such that B is a homomorphic image of the ultraproduct $\prod_{n\in\omega} A_n/\mathcal{U}$.*

(27) *There exists a countable family of abelian groups $(A_n)_{n\in\omega}$ and a filter $\mathcal{F}$ on ω which is not contained in any principal ultrafilter (i.e., which satisfies $\bigcap_{S\in\mathcal{F}} S = \varnothing$), such that B is a homomorphic image of $\prod_{n\in\omega} A_n/\mathcal{F}$.*

(28) *There exists a countable family of abelian groups* $(A_n)_{n\in\omega}$ *such that* B *is a homomorphic image of the reduced product* $\big(\prod_{n\in\omega} A_n\big)/\bigoplus_{n\in\omega} A_n$.

(29) B *is a homomorphic image of an abelian group* C *admitting a compact Hausdorff group topology.*

(30) B *is a cotorsion abelian group; i.e., a homomorphic image of an algebraically compact abelian group.*

Proof. We shall show (25) $\Longrightarrow$ (26) $\Longrightarrow$ (27) $\Longrightarrow$ (28) $\Longrightarrow$ (30) $\Longrightarrow$ (29) $\Longrightarrow$ (25).

In the situation of (25), the fact that $\mathcal{U}$ is not countably complete implies that we can find a partition $I = \bigcup_{n\in\omega} J_n$ such that no J_n belongs to $\mathcal{U}$. Let us again write $\prod_{i\in I} A_i = \prod_{n\in\omega}\big(\prod_{i\in J_n} A_i\big)$. As in the proof of Lemma 10, if we let $\mathcal{U}' = \{S \subseteq \omega \mid \bigcup_{n\in S} J_n \in \mathcal{U}\}$, we find that $\mathcal{U}'$ is a nonprincipal ultrafilter on ω, yielding a factorization of the map from our product group to our original ultraproduct as

$$\prod_{i\in I} A_i \to \prod_{n\in\omega}\Big(\prod_{i\in J_n} A_i\Big)/\mathcal{U}' \to \prod_{i\in I} A_i/\mathcal{U}.$$

Since B is a homomorphic image of $\prod_{i\in I} A_i/\mathcal{U}$, it is a homomorphic image of the factoring object, proving (26).

We get (26) $\Longrightarrow$ (27) by taking $\mathcal{F} = \mathcal{U}$.

Given (27), note that since the filter $\mathcal{F}$ on ω is not contained in a principal ultrafilter, it contains the complement of every singleton, hence it contains the *Fréchet* filter $\mathcal{C}$ of complements of finite sets. So the quotient map

$$\prod_{n\in\omega} A_n \to \prod_{n\in\omega} A_n/\mathcal{F} \quad \text{factors through} \quad \prod_{n\in\omega} A_n/\mathcal{C} = \Big(\prod_{n\in\omega} A_n\Big)\Big/\bigoplus_{n\in\omega} A_n,$$

giving (28).

Given (28), we call on [Fuchs 1970, Corollary 42.2] which says that every group of the form $\big(\prod_{n\in\omega} A_n\big)/\bigoplus_{n\in\omega} A_n$ is algebraically compact, yielding (30).

For the step (30) $\Longrightarrow$ (29), we call on [Fuchs 1970, Theorem 38.1] (or on [Rotman 2009, Theorem 7.42]) which, among the equivalent conditions for an abelian group to be algebraically compact, includes that of being a *direct summand* in an abelian group that admits a compact Hausdorff group topology. So an algebraically compact abelian group is, in particular, a *homomorphic image* of an abelian group admitting such a topology, hence so is any homomorphic image of an algebraically compact group.

Finally, by Proposition 15 above, any abelian group A admitting a compact Hausdorff group topology can be written as a homomorphic image of its ultrapower $A^I/\mathcal{U}$ for any ultrafilter $\mathcal{U}$ on any set I. So choosing a $\mathcal{U}$ which is not countably complete (e.g., any nonprincipal ultrafilter on $I = \omega$), we get (29) $\Longrightarrow$ (25). □

We note that for a nonzero abelian group B, the equivalent conditions of Proposition 23 imply those of Proposition 21. Indeed, thinking in terms of the conditions (a) and (b) that we have been discussing, if we write (b_1) for the case of (b) where there is only a single ultraproduct involved (i.e., the condition that if there exists a nonzero homomorphism from an ultraproduct group $\prod_{n\in\omega} A_n/\mathcal{U}$ onto B, then the ultrafilter $\mathcal{U}$ is principal), then we have (a) $\Longrightarrow$ (b) $\Longrightarrow$ (b_1), so $\neg(\mathrm{b}_1) \Longrightarrow \neg$(a); moreover, we see that for $B \neq \{0\}$, (26) is equivalent to $\neg(\mathrm{b}_1)$, while we have previously noted that the conditions of Proposition 21 are equivalent to $\neg$(a). (Alternatively, it not hard to see directly that for $B \neq \{0\}$, an example witnessing (26) also witnesses (22).) Choosing the equivalent conditions of the two propositions that have standard names, these observations say that for $B \neq \{0\}$, (30) $\Longrightarrow$ (24); in other words, no nonzero cotorsion abelian group is slender. Since the class of cotorsion abelian groups is closed under homomorphic images, this in fact gives:

Corollary 24. *No cotorsion abelian group has a nonzero slender homomorphic image.* □

Corollary 24 allows us to apply Theorem 13 to many variants of the examples immediately following it. For instance, one of those was the infinite dihedral group, i.e., the semidirect product arising from the natural action of $\{\pm 1\}$ on the slender group $\mathbb{Z}$. I claim that we can replace $\mathbb{Z}$ in that example by any abelian group A without 2-torsion that has $\mathbb{Z}$ as a homomorphic image; for instance, $\mathbb{Z}^\omega$, or $\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$ for any odd n. Indeed, taking a homomorphism from such an abelian group A onto $\mathbb{Z}$, and any $b \in A$ that maps to a generator of $\mathbb{Z}$ under that homomorphism, we see from the above corollary that no subgroup B of A containing b is cotorsion, equivalently, by Proposition 23, that no such subgroup B satisfies (26); hence b satisfies the condition of the second paragraph of Theorem 13. (The assumption that A has no 2-torsion keeps the center of the semidirect product trivial, to avoid complicating our considerations.) In Section 6 we will obtain more information on which abelian groups are cotorsion.

We can now answer the question of which abelian groups B have the property we called (b) in our earlier discussion, namely, that any map f from a countable direct product of abelian groups A_n onto B which factors through finitely many ultrafilters in fact factors through the projection to the product of finitely many of the A_n. We shall see that this is true if and only if B contains no nontrivial cotorsion subgroup. Although the class of cotorsion abelian groups is difficult to describe exactly, a simple criterion is known for an abelian group to be *cotorsion-free*, i.e., to contain no nontrivial cotorsion subgroup: It is that the group be torsion-free, and contain no copy of the additive group of $\mathbb{Q}$, nor of the p-adic integers for any prime p [Dugas and Göbel 1982, Theorem 2.4 (1) $\Longrightarrow$ (4)]. (So it is like the condition characterizing slenderness, but without the exclusion of subgroups isomorphic to $\mathbb{Z}^\omega$.) This condition is also equivalent to that of containing no nonzero algebraically compact subgroup: it implies the latter because every algebraically compact group is cotorsion, while the reverse implication holds because $\mathbb{Q}$, and the groups of p-adic integers, and all finite abelian groups, are algebraically compact. As is usual in this note, the statement below will be the contrapositive of the version suggested by this discussion.

Theorem 25. *The following conditions on an abelian group B are equivalent.*

(31) *There exist a set I, a family of abelian groups $(A_i)_{i\in I}$, and a surjective homomorphism $f : \prod_{i\in I} A_i \to B$ such that f factors through the product of finitely many ultraproducts $\prod_{i\in I} A_i/\mathcal{U}_k$, but does not factor through the product of finitely many* countably complete *ultraproducts.*

(32) *There exist a countable family $(A_n)_{n\in\omega}$ of abelian groups and a surjective homomorphism $f : \prod_{n\in\omega} A_n \to B$ such that f factors through the product of finitely many ultraproducts $\prod_{n\in\omega} A_n/\mathcal{U}_k$, but does not factor through the product of finitely many of the A_n.*

(33) *B has a nontrivial cotorsion subgroup; equivalently* (*by the result from* [Dugas and Göbel 1982] *quoted above*)*, B either has nonzero elements of finite order, or contains a copy of the additive group of $\mathbb{Q}$, or contains a copy of the additive group of the p-adic integers for some prime p; equivalently, B has a nontrivial algebraically compact subgroup.*

These conditions are also equivalent to the variants of (32) *and* (31) *without the assumption that f be surjective.*

Proof. The equivalence of (31) and (32) to the corresponding conditions without the assumption of surjectivity is seen as in the first paragraph of the proof of Proposition 21. We shall use those variants to prove (33) $\Longrightarrow$ (32) $\Longrightarrow$ (31) $\Longrightarrow$ (33).

Assuming (33), let $C \subseteq B$ be a nonzero cotorsion subgroup. Then our earlier result (30) $\Longrightarrow$ (26) gives a surjective homomorphism $\prod_{n\in\omega} A_n/\mathcal{U} \to C$ for a family of abelian groups A_n and a nonprincipal ultrafilter $\mathcal{U}$, which we regard as a nonzero homomorphism into B. Since $\mathcal{U}$ is not principal, f annihilates each of the A_n, so it cannot be factored through the product of finitely many of these, giving (32).

Clearly, (32) $\Longrightarrow$ (31), since the countably complete ultrafilters on ω are the principal ultrafilters.

Assuming (31), let $f : \prod_{i\in I} A_i \to B$ be a homomorphism that factors through a product of ultraproducts $\prod_{i\in I} A_i/\mathcal{U}_0 \times \cdots \times \prod_{i\in I} A_i/\mathcal{U}_{m-1}$, but not through such a product in which all the $\mathcal{U}_k$ are countably complete. As noted in the proof of Proposition 21, the given factorization is equivalent to an expression of f as the sum of maps that factor $\prod_{i\in I} A_i \to \prod_{i\in I} A_i/\mathcal{U}_k \to B$, and if any of these maps are zero, we can drop them, leaving a factorization with all these maps nonzero, and which, by choice of f, must still have at least one with $\mathcal{U}_k$ not countably complete. So there exists a nonzero map $g : \prod_{i\in I} A_i/\mathcal{U} \to B$ for some non-countably-complete ultrafilter $\mathcal{U}$ on I. Our earlier result (25) $\Longrightarrow$ (30) now tells us that the nonzero image of g is a cotorsion submodule of B, proving (33). □

We have not yet said much about algebraically compact groups, except that the cotorsion groups are their homomorphic images. We record:

Lemma 26. *The following conditions on an abelian group B are equivalent.*

(34) *For every proper filter $\mathcal{F}$ on a nonempty set I, the natural embedding $B \to B^I/\mathcal{F}$ has a left inverse.*

(35) *There exists a nonprincipal ultrafilter $\mathcal{U}$ on ω such that the natural embedding $B \to B^\omega/\mathcal{U}$ has a left inverse.*

(36) *B is algebraically compact.*

Proof. For any filter $\mathcal{F}$ on a set I, the natural embedding $B \to B^I/\mathcal{F}$ is easily seen to be pure, so the definition of algebraic compactness gives (36) $\Longrightarrow$ (34). Clearly, (34) $\Longrightarrow$ (35).

To show that (35) $\Longrightarrow$ (36), we use the result [Eklof 1973, third sentence of §2], that a nonprincipal countable ultrapower of any abelian group B is algebraically compact. Hence (35) implies that B is a direct summand in an algebraically compact abelian group, from which one easily sees that it itself is algebraically compact. □

Since the cotorsion abelian groups are the homomorphic images of the algebraically compact ones, the above result shows that the analog of Question 16 has a positive answer for abelian groups. (This can also be seen from the proof of Proposition 23, where the closing step (29) $\Longrightarrow$ (25) allows us to choose $\mathcal{U}$ essentially arbitrarily.)

Another interesting necessary and sufficient condition for B to be algebraically compact, obtained (in the more general context of modules) as [Jensen and Lenzing 1989, Theorem 7.1(vi)], is that for every set I, the summation map $\bigoplus_{i\in I} B \to B$ extend to a map $B^I \to B$.

6. More on algebraically compact and cotorsion abelian groups

The distinction between the class of cotorsion abelian groups and its subclass, the algebraically compact abelian groups, is a subtle one. It follows from the definitions that every cotorsion abelian group B that is torsion-free is algebraically compact [Fuchs 1970, Corollary 54.5]. The only example I have found in the literature of a cotorsion abelian group that is not algebraically compact, that of [Rotman 2009, Proposition 7.48(ii)], is described as an Ext of other groups, rather than explicitly. (It is known that for any abelian groups A and A', $\mathrm{Ext}(A, A')$ is cotorsion [Fuchs

1970, Theorem 54.6], [Rotman 2009, Corollary 7.47].) Let us begin this section by constructing a more explicit example.

We will use the characterization of an algebraically compact abelian group as an abelian group B such that whenever a system of equations has the property that all its finite subsystems have solutions in B, then the whole system has such a solution. An easy example of an infinite system of equations is the following, where p is a prime, x_0 is a given element of B, and $x_1, \dots, x_n, \dots$ are to be found.

$$x_0 = px_1, \quad x_1 = px_2, \quad \dots, \quad x_{n-1} = px_n, \quad \dots. \tag{37}$$

The necessary condition for algebraic compactness that this system yields is:

Lemma 27. *If B is an algebraically compact group and p a prime, then the subgroup $B' = \bigcap_{n\in\omega} p^n B \subseteq B$ is p-divisible, i.e., satisfies $pB' = B'$.*

Proof. Suppose $x_0 \in B'$. Let us fix $n \geq 0$, and choose $x_n \in B$ such that $x_0 = p^n x_n$. If we now let $x_m = p^{n-m} x_n$ for $0 < m < n$, we see that $x_0, \dots, x_n$ satisfy the first n equations of (37). Since we can do this for any n, every finite subfamily of (37) has a solution, so algebraic compactness implies that we can choose $x_1, \dots, x_n, \dots$ satisfying the full set of equations. For such $x_1, \dots, x_n, \dots$ we see that x_1 also belongs to B'; so $x_0 \in pB'$, as required. □

(It is also not hard to prove the above lemma from the definition of algebraic compactness in terms of pure extensions: given algebraically compact B, and $x_0 \in B'$, let B^+ be the extension of B gotten by adjoining new generators $x_1, \dots, x_n, \dots$ and the relations (37). It is straightforward to show that B embeds in B^+, and from the fact that $x_0 \in B'$, one can deduce that B is pure in B^+. Hence the definition of algebraic compactness says that there exists a retraction of B^+ onto B, i.e., a solution to (37) in B; hence, as above, $x_0 = px_1 \in pB'$.)

So let us try to construct a cotorsion abelian group B with an element that we force to lie in B', without creating any apparent reason why it should lie in pB'. To do this, let $\mathbb{Z}_p$ denote the additive group of p-adic integers, which is algebraically compact by Proposition 15 and Lemma 26; within its countable power $\mathbb{Z}_p^\omega$, let δ_n be, as usual, the element with 1 in the n-th coordinate and 0 in all others; and for a first try, let B be the factor group of $\mathbb{Z}_p^\omega$ by the subgroup generated by the elements

$$\delta_0 - p^n \delta_n \quad (n \in \omega). \tag{38}$$

Letting x be the image of δ_0 in B, we clearly have $x \in B'$.

But this group is messy, making it hard to see whether some $y \in B'$ might satisfy $x = py$. It becomes nicer if we impose (38) as $\mathbb{Z}_p$-module relations rather than just as additive group relations. If we then change coordinates in $\mathbb{Z}_p^\omega$, so that the elements $\delta_n - p\delta_{n+1}$ become the new δ_n (namely, we map $(a_n)_{n\in\omega}$ to $(\sum_{m\leq n} p^{n-m} a_m)_{n\in\omega}$), the resulting construction takes the form shown in the next lemma.

Lemma 28. *Let p be a prime number, and B the group $\mathbb{Z}_p^{\omega}/\bigoplus_{n\in\omega} p^n\mathbb{Z}_p$. Then B is cotorsion, but fails to satisfy the conclusion of Lemma 27; hence B is not algebraically compact.*

Proof. As a homomorphic image of an algebraically compact group, B is cotorsion.

To see the failure of the conclusion of Lemma 27, let $x \in B$ be the image of $(p^n)_{n\in\omega} \in \mathbb{Z}_p^{\omega}$. (Note that the above coordinates p^n are "ghosts", in the sense that any *finite* set of them may, by the definition of B, be changed to 0 without changing the element x.) For each $n > 0$, if we let $x_n \in B$ be the image of the element of $\mathbb{Z}_p^{\omega}$ whose coordinate in position m is 0 for $m < n$, and p^{m-n} for $m \geq n$, then we see that $x = p^n x_n$. Hence $x \in B'$.

Now let y be any element satisfying $x = py$. Writing y as the image of $(a_n)_{n\in\omega} \in \mathbb{Z}_p^{\omega}$, we see from the definition of B that for all but finitely many n we must have $a_n = p^{n-1}$. (And note that coordinates with this property are not "ghosts"!) But for any n such that this equation holds, we can see by looking at the n-th coordinate that $y \notin p^n B$. So $y \notin B'$; and since we have shown this for all y with $x = py$, we have $x \notin pB'$. Since $x \in B'$, this shows that $B' \neq pB'$. □

(L. Fuchs (personal communication) points out another way to see that the above group B is not algebraically compact: by noting that its torsion subgroup $\bigoplus_{n\in\omega} \mathbb{Z}_p/p^n\mathbb{Z}_p$ is not torsion-complete, and calling on [Fuchs 1973, Theorem 68.4, (ii) $\Longrightarrow$ (i)].)

Note that any group B which, like the one constructed above, is cotorsion but not algebraically compact is, by the former fact, a homomorphic image of a nonprincipal countable ultraproduct of groups, but by Lemma 26 (35) $\Longrightarrow$ (36), does not admit a left inverse to a diagonal embedding $B \to B^{\omega}/\mathcal{U}$, confirming the assertion made in the second paragraph after Question 17.

Let us obtain, next, some restrictions on the class of cotorsion abelian groups. These will allow us to deduce that many sorts of groups are not cotorsion, and so give more examples to which we can apply Theorem 13. In the next lemma we combine the fact that the cotorsion groups are the homomorphic images of the algebraically compact groups with another of the criteria for algebraic compactness given in [Fuchs 1970, Theorem 38.1], namely, that an abelian group C is algebraically compact if and only if it is *pure-injective*, meaning that for any pure subgroup A_0 of an abelian group A_1, every homomorphism $A_0 \to C$ extends to a homomorphism $A_1 \to C$. In an earlier version of this note, I asked whether the direction "(39) $\Longrightarrow$ cotorsion" in the lemma held; I am indebted to K. M. Rangaswamy and Manfred Dugas for (independently) showing me why it does.

Lemma 29. *An abelian group B is cotorsion if and only if it satisfies*

(39) *For every abelian group A having a pure subgroup F which is free abelian, every homomorphism $F \to B$ extends to a homomorphism $A \to B$.*

Proof. Assuming B cotorsion, let us write it as a homomorphic image of an algebraically compact abelian group C. Since F is free, we can lift the given map $F \to B$ to a map $F \to C$, and then, since C is algebraically compact, equivalently, pure-injective, we can extend that lifted map to a map $A \to C$. Composing with our map $C \to B$, we get the desired extension to A of the given map $F \to B$.

Conversely, assuming (39), write B as a homomorphic image of a free abelian group F. Now by [Fuchs 1970, §38, Exercise 8, p. 162], every abelian group embeds as a pure subgroup in a group admitting a compact Hausdorff group topology; let A be such an overgroup of F. (For an explicit embedding in this case, let $\widehat{\mathbb{Z}}$ denote the completion of $\mathbb{Z}$ with respect to its subgroup topology. Then $\mathbb{Z}$ is a pure subgroup of the compact group $\widehat{\mathbb{Z}}$, so writing $F = \bigoplus_I \mathbb{Z}$, we see that F is pure in the compact group $\widehat{\mathbb{Z}}^I$.) By (39), our homomorphism of F onto B extends to a homomorphism of A onto B, so by Proposition 23, (29) $\Longrightarrow$ (30), B is cotorsion. □

Our first application of this result will show that in a cotorsion abelian group B, highly divisible elements abound; for instance, that if p_1 and p_2 are distinct primes, then every element of B is the sum of an element divisible by all powers of p_1 and an element divisible by all powers of p_2. To state the result in greater generality, let us, for any set P of primes, write $\mathbb{Z}[P^{-1}]$ for the subring of $\mathbb{Q}$ consisting of elements whose denominators lie in the multiplicative monoid generated by P, and call an element x of an abelian group A *P-divisible* if it lies in the image of a homomorphism from the additive group of $\mathbb{Z}[P^{-1}]$ to A. We shall call an abelian group P-divisible if all its elements are.

Proposition 30. *If B is a cotorsion abelian group, and $P_0, \ldots, P_{m-1}$ are sets of prime numbers such that $P_0 \cap \cdots \cap P_{m-1} = \varnothing$, then every element $b \in B$ can be written $b_0 + \cdots + b_{m-1}$, where for each j, b_j is P_j-divisible. Equivalently, B is a sum of subgroups $B_0 + \cdots + B_{m-1}$ such that each group B_j is P_j-divisible.*

Proof. Let A be the additive group of $\mathbb{Z}[P_0^{-1}] \times \cdots \times \mathbb{Z}[P_{m-1}^{-1}]$, and F the infinite cyclic subgroup thereof generated by $(1, \ldots, 1)$. That the inclusion $F \subseteq A$ is pure follows from the fact that $P_0 \cap \cdots \cap P_{m-1} = \varnothing$. Indeed, if an element $d(1, \ldots, 1) \in F$ is not divisible in F by some positive integer n, then d is not divisible by n, so n has a prime power factor p^i not dividing d. Choosing k such that $p \notin P_k$, we see that the k-th coordinate of $d(1, \ldots, 1)$ is not divisible by p^i in $\mathbb{Z}[P_k^{-1}]$, so in A, $d(1, \ldots, 1)$ is not divisible by p^i, hence not divisible by n.

Hence by Lemma 29, for any $b \in B$, the map $F \to B$ taking $(1, \ldots, 1)$ to b extends to A, giving a representation of b as the sum of the images of the elements $(0, \ldots, 1, \ldots, 0)$, each of which is P_j-divisible for some j. The equivalence of this result to the final statement of the lemma follows from the fact that for any set P of primes, the P-divisible elements of an abelian group form a subgroup. □

As a quick illustration, consider the group $\mathbb{Z}_p$ of p-adic integers, which we have seen is algebraically compact, and hence cotorsion. That group is P-divisible for P the set of all primes other than p. Given $P_0, \dots, P_{m-1}$ as in Proposition 30, at least one P_j will fail to contain p, so $\mathbb{Z}_p$ is P_j-divisible for that j, confirming the conclusion of the proposition.

Of course, the much smaller group of rational numbers with denominators relatively prime to p (of which the group $\mathbb{Z}_p$ is a completion) is P-divisible for the same set P, and so also satisfies the conclusion of Proposition 30. However, that group is not cotorsion. Indeed, from the characterization of slender abelian groups recalled immediately after the proof of Proposition 21, every abelian group which is torsion-free and which contains no copy of $\mathbb{Q}$ and has less than continuum cardinality is slender, hence, if nonzero, is non-cotorsion.

The next result generalizes the above restriction on cotorsion groups.

Proposition 31. *If B is a cotorsion abelian group such that $dB \neq \{0\}$ for every positive integer d, but $\bigcap_{d\in\mathbb{Z},\, d>0} dB = \{0\}$, then B has at least continuum cardinality.*

Proof. We shall construct a homomorphism $\bigoplus_\omega \mathbb{Z} \to B$, extend it to a map $\mathbb{Z}^\omega \to B$ by Lemma 29, and show that under the extended map, continuum many elements of $\mathbb{Z}^\omega$ have distinct images. We begin by carefully selecting the elements to which to send the free generators of $\bigoplus_\omega \mathbb{Z}$.

I claim that we can choose positive integers $d_0, d_1, \dots$, each a multiple of the one before, and elements $b_0, b_1, \dots \in B$, such that for each $n \in \omega$, we have $d_n b_n \notin d_{n+1} B$. We start with $d_0 = 1$, and b_0 any nonzero element of B. Assuming that for some $n \geq 0$, d_n and b_n have been chosen with $d_n b_n \neq 0$, the hypothesis $\bigcap_{d\in\mathbb{Z},\, d>0} dB = \{0\}$ allows us to choose $d_{n+1} > 0$ such that $d_n b_n \notin d_{n+1} B$. Replacing d_{n+1} by a proper multiple if necessary, we may assume $d_n | d_{n+1}$. Using the fact that $d_{n+1} B \neq \{0\}$, we can then choose b_{n+1} such that $d_{n+1} b_{n+1} \neq 0$. Continuing recursively, we get $d_0, d_1, \dots$ and $b_0, b_1, \dots$ with the asserted properties.

We now map $\bigoplus_\omega \mathbb{Z}$ to B by sending each δ_n to b_n. Since $\bigoplus_\omega \mathbb{Z}$ is a pure subgroup of $\mathbb{Z}^\omega$, Lemma 29 allows us to extend this map to a homomorphism $f : \mathbb{Z}^\omega \to B$, which still carries each δ_n to b_n.

For each $\varepsilon = (\varepsilon_n)_{n\in\omega} \in \{0, 1\}^\omega$, let εd denote $(\varepsilon_0 d_0, \dots, \varepsilon_n d_n, \dots) \in \mathbb{Z}^\omega$. I claim that distinct strings ε yield distinct elements $f(\varepsilon d) \in B$. Indeed, for $\varepsilon \neq \varepsilon'$, let $n \in \omega$ be the least index such that $\varepsilon_n \neq \varepsilon'_n$, and let us write

$$f(\varepsilon d) = f(\varepsilon_0 d_0, \dots, \varepsilon_n d_n, 0, 0, \dots) + f(0, \dots, 0, \varepsilon_{n+1} d_{n+1}, \varepsilon_{n+2} d_{n+2}, \dots).$$

If we compare this with the corresponding expression for $f(\varepsilon' d)$, we see that the left-hand summands in these expressions differ by exactly $f(d_n \delta_n)$, i.e., $d_n b_n$, which by assumption does not lie in $d_{n+1} B$; while the right-hand summands *do* lie in

$d_{n+1}B$, since for all $m \geq n$ we have $d_{n+1}|d_m$. Hence $f(\varepsilon d) - f(\varepsilon' d) \neq 0$; so we indeed have continuum many distinct elements of B. □

As an application, it is easy to deduce that no subgroup B of $\prod_{\text{primes } p} \mathbb{Z}/p\mathbb{Z}$ which is infinite, but of less than continuum cardinality, can be cotorsion. Hence, if we take such a subgroup with no 2-torsion, containing an element b of infinite order, its semidirect product with ± 1 will again be a group to which Theorem 13 applies.

On the other hand, we saw in Lemma 11 that for the semidirect product of $\{\pm 1\}$ with the group $\mathbb{Q}$, the conclusion of Theorem 13 fails; and Proposition 15 shows the same for the semidirect product of $\{\pm 1\}$ with any *finite* abelian group. In fact, $\mathbb{Q}$ and all finite abelian groups are cotorsion; the next result includes these statements as special cases. It is curious that its formulation is analogous to that of Proposition 15, but the reasoning is quite different.

Proposition 32 (cf. [Fuchs 1970, p. 178, last paragraph of Notes]). *Let B be an abelian group which is divisible, or is of finite exponent, or more generally, is the sum of a divisible group and one of finite exponent; or, still more generally, is the underlying additive group of an injective module over some ring R. Then for any set I and any ultrafilter $\mathcal{U}$ on I, there is a group homomorphism $B^I/\mathcal{U} \to B$ left-inverse to the natural embedding $B \to B^I \to B^I/\mathcal{U}$.*

Hence by Lemma 26, (35) $\implies$ (36), every such B is algebraically compact, and so in particular is cotorsion.

Proof. First suppose B has the property introduced above by the words "still more generally". Then the maps $B \to B^I \to B^I/\mathcal{U}$ are R-module homomorphisms whose composite is an embedding. The injectivity of B as an R-module thus yields the desired left inverse map. Taking $I = \omega$ and $\mathcal{U}$ nonprincipal, we conclude that B is algebraically compact (by Lemma 26, (35) $\implies$ (36)).

It remains to show that the various sorts of abelian groups named are indeed injective modules over appropriate rings. Any divisible abelian group is an injective $\mathbb{Z}$-module by [Lam 1999, Proposition 3.19]. An abelian group B of finite exponent n can be written as a direct product of free $\mathbb{Z}/d\mathbb{Z}$-modules as d ranges over the divisors of n; and each of the rings $\mathbb{Z}/d\mathbb{Z}$ is self-injective, so that its free modules are injective by [ibid., Corollary 3.13(1) and Theorem 3.46(4) $\implies$ (2)]. Finally, if B is the sum of a divisible subgroup D and a subgroup E of finite exponent, then the injectivity of D over $\mathbb{Z}$ allows us to split it off as a direct summand, and the complementary summand will be a homomorphic image E' of E, hence again of finite exponent. We can now make $B = D \oplus E'$ a module over the direct product R of $\mathbb{Z}$ and finitely many rings $\mathbb{Z}/d\mathbb{Z}$, in such a way that the component over each of these factor rings is injective over that ring. The group B will then be injective over R. □

I do not know the answer to:[1]

[1] See note added in proof, page 493.

Question 33. *For an abelian group B to be cotorsion, is it sufficient that every homomorphism $\bigoplus_\omega \mathbb{Z} \to B$ extend to a homomorphism $\mathbb{Z}^\omega \to B$? (In other words, in Lemma 29, is condition (39) equivalent to the special case where the inclusion $F \subseteq A$ is $\bigoplus_\omega \mathbb{Z} \subseteq \mathbb{Z}^\omega$?)*

The following example shows that the converse of Corollary 24 is not true: a group B with no nonzero slender homomorphic image need not be cotorsion.

Lemma 34. *Within the group $A = \prod_{\text{primes } p} \mathbb{Z}/p\mathbb{Z}$, let u be the element having 1 in every coordinate, and let B consist of all elements $b \in A$ such that $db = nu$ for some integer n and nonzero integer d (mnemonic for "numerator" and "denominator").*

Then B is a countable subgroup of A, such that every cotorsion subgroup of B is torsion (so that B is not *itself cotorsion), but the factor-group of B by its torsion subgroup is isomorphic to $\mathbb{Q}$, and so is cotorsion.*

Hence B has no nonzero slender homomorphic images.

Proof. That B is a subgroup of A is immediate. It is countable because each $b \in B$ is determined by any choice of n and d satisfying $db = nu$, together with the coordinates of b at the finitely many primes dividing d.

By the observation following the proof of Proposition 31, cotorsion subgroups of B are finite, hence are torsion.

On the other hand, the factor group of B by its torsion subgroup is isomorphic to $\mathbb{Q}$ via the map sending the image of each $b \in B$ to the common value of $n/d \in \mathbb{Q}$ for all relations $db = nu$ satisfied by b; and $\mathbb{Q}$, being divisible, is cotorsion by Proposition 32.

Since slender groups are torsion-free, a homomorphism f from B to a slender group must annihilate the torsion subgroup of B, hence $f(B)$ must be a homomorphic image of $\mathbb{Q}$, hence by Corollary 24 must be zero. □

Here is a question of a different flavor.

Question 35. *If an abelian group B can be written as a homomorphic image of a nonprincipal countable ultraproduct of* not necessarily abelian *groups G_n, must it be a homomorphic image of a nonprincipal countable ultraproduct of* abelian *groups, i.e., must it be cotorsion?*

The reason this question is nontrivial is that abelianization does not commute with ultraproducts. For instance, let G be a group which is perfect (satisfies $G = [G, G]$) but which for each n has an element x_n that cannot be written as the product of fewer than n commutators. (The latter property is called "infinite commutator width"; for examples of such G see [Muranov 2007].) Then no nonprincipal ultrapower $G^\omega/\mathcal{U}$ will be perfect, because for such a family of elements x_n, the image of $(x_n)_{n\in\omega} \in G^\omega$ in $G^\omega/\mathcal{U}$ will not be a product of finitely many commutators. Hence the abelianization B of $\prod_{n\in\omega} G_n/\mathcal{U}$ is a nontrivial abelian group satisfying the hypothesis of Question 35, but there is no obvious candidate for a representation of B as in the conclusion of that question.

We can, however, prove a weak result in the direction of a positive answer.

Lemma 36. *If an abelian group B can be written as a homomorphic image of a nonprincipal countable ultraproduct $\prod_{n\in\omega} G_n/\mathcal{U}$ of not necessarily abelian groups, then B is a directed union of cotorsion abelian subgroups.*

Proof. By Lemma 12, every cyclic subgroup of B is contained in a cotorsion subgroup. Now the class of cotorsion abelian groups, as characterized by any of (27), (28) or (29), is easily seen to be closed under finite direct sums, hence since it is closed under homomorphic images, it is closed under finite sums in abelian overgroups, so the cotorsion subgroups of such an overgroup form a directed system. □

But not every directed union of cotorsion groups is cotorsion. For instance, every torsion abelian group is the directed union of its finite subgroups, which are cotorsion by Proposition 32; but by Proposition 31, the group $\bigoplus_{\text{primes } p} \mathbb{Z}/p\mathbb{Z}$ is not cotorsion.

We remark that Questions 19 and 35 cannot both have positive answers, since as noted earlier, a positive answer to Question 19 would make every torsion group, including the abovementioned group $\bigoplus_{\text{primes } p} \mathbb{Z}/p\mathbb{Z}$, a homomorphic image of a countable ultraproduct of (not necessarily abelian) groups. But as we also said earlier, a positive answer to Question 19 seems highly unlikely.

A noticeable difference between our results on general groups in Sections 2–4 and our results on abelian groups in the above three sections is that in the former we composed maps $\prod_{i\in I} G_i \to B$ with the natural map $B \to B/Z(B)$ before looking at factorization properties, but we have done nothing of the sort for abelian groups. It might be of interest to see whether one can improve the results of these sections by composing homomorphisms $\prod_{i\in I} A_i \to B$ with the map $B \to B/X(B)$ for some natural choice of $X(B)$, such as the torsion subgroup of B, the subgroup of divisible elements, their sum, or the sum of all cotorsion subgroups of B. (Lemma 34 shows that for the last of these choices, $B/X(B)$ may not itself be cotorsion-free; but this need not be a problem; cf. the fact that for a nonabelian group B, the group $B/Z(B)$ need not have trivial center.) In the opposite direction, it might be possible to strengthen the results of Sections 2–4 by dividing B, not by $Z(B)$, but by a smaller subgroup $X(Z(B))$ for one of the above constructions X. I leave these ideas for others to explore.

7. Some related questions that have been studied

The direct sum $\bigoplus_{i\in I} A_i$ of a family of abelian groups — or more generally, of a family of modules over any ring R — is their *coproduct* in the category of abelian groups or R-modules; hence for such objects, their coproduct can be regarded as the subgroup or submodule of elements of finite support in their direct product $\prod_{i\in I} A_i$. Now in any category, a homomorphism from a coproduct of objects A_i to an arbitrary object B is determined simply by choosing a homomorphism from each A_i to B. So the phenomena we have been investigating in the last two sections can be looked at as consequences of the fact that not every such map on a coproduct of abelian groups can be extended consistently to the elements of $\prod_{i\in I} A_i$ with infinite supports. The slender modules are those modules B for which this restriction on maps to B is so strong that it can only be satisfied by maps that factor through the product of finitely many of the A_i.

Dually, one gets homomorphisms from an abelian group or R-module B to a direct product $\prod_{j\in J} C_j$ simply by choosing a homomorphism into each C_j; but if we wish to map B into the coproduct $\bigoplus_{j\in J} C_j \subseteq \prod_{j\in J} C_j$, we face the problem of choosing those homomorphisms so that the resulting map takes each element of B to an element of finite support. The question of which modules B have the property that the only way to achieve this is by mapping into a finite subsum of $\bigoplus_{j\in J} C_j$ is answered by El Bashir, Kepka and Němec in Proposition 4.1 of [El Bashir et al. 2003]; that paper also studies the corresponding questions for colimit constructions other than coproducts.

Several workers, beginning with Chase [1962a; 1962b], have looked at the two-headed situation of module homomorphisms $f : \prod_{i\in I} A_i \to \bigoplus_{j\in J} C_j$. Here one may ask when every such map is a sum of one homomorphism which factors through the projection of $\prod_{i\in I} A_i$ onto a finite subproduct, and another which factors through the inclusion of a finite subsum in $\bigoplus_{j\in J} C_j$. Just as, in studying nonabelian groups in Sections 2–4, we found it desirable to divide out by $Z(B)$ to avoid certain easy ways that maps could involve infinitely many factors, so in the results of this sort, two adjustments turn out to be useful: dividing out by submodules of "highly divisible" elements of the C_j, and multiplying the given homomorphism by some nonzero ring element d; which essentially means restricting it to $\prod_{i\in I} dA_i$. Thus, [Chase 1962b, Theorem 1.2], more or less the starting point for the development of the subject, says, if restricted to the case where the base ring is $\mathbb{Z}$ and where a certain filter of principal right ideals in the statement of that theorem consists of *all* the nonzero ideals of $\mathbb{Z}$, that given any homomorphism of abelian groups $f : \prod_{n\in\omega} A_n \to \bigoplus_{i\in I} C_i$, there exists an integer $d > 0$ such that when f is applied to $\prod_{n\in\omega} dA_n$, and followed by the factor map $\bigoplus_{i\in I} C_i \to \bigoplus_{i\in I}(C_i / \bigcap_{e>0} eC_i)$, it carries the product of some cofinite subfamily of the dA_n into the sum of a finite subfamily of the $C_i / \bigcap_{e>0} eC_i$.

That result (in its general module-theoretic form) is strengthened in [Dugas and Zimmermann-Huisgen 1981, Theorem 2] to allow products $\prod_{i\in I} A_i$ over any index set I of cardinality less than all uncountable measurable cardinals, and to remove the requirement that the right ideals considered be principal, while in [Bergman

2006, Theorem 9], the direct product is replaced by a general inverse limit. For further related work, see references in the first paragraph of p. 46 of that paper.

(It is curious that the proof of the abovementioned theorem from [Chase 1962b], and that of Proposition 31 of this note, use virtually the same construction, but for very different purposes: in Chase's paper, to obtain a contradiction by constructing an element whose image in the direct sum would have infinitely many nonzero components; in Proposition 31, to get continuum many distinct elements in the image of our map.)

Turning back to the results of the three preceding sections, it would, of course, be desirable to investigate the corresponding questions with abelian groups replaced by modules over a general ring R. In [Jensen and Lenzing 1989, Chapters 7–8], algebraically compact modules over general R are studied, but cotorsion modules are not mentioned. (There *are* numerous MathSciNet results for "cotorsion module", but I have not had time to examine them.) One might also take a hint from [Chase 1962a; 1962b], and see whether one gets nonobvious variants of our results if one considers those B such that all homomorphisms $\prod_{i\in I} A_i \to B$ acquire the factorization properties we are looking for *after* multiplying $\prod_{i\in I} A_i$ by some integer (or ideal), and/or dividing B by an appropriate subgroup (or submodule) of highly divisible elements. (This is related to the suggestion in the last paragraph of the preceding section.)

8. Rings

As mentioned in Section 1, the results of this note were inspired by investigations of factorization properties of homomorphisms on direct products of not-necessarily-associative algebras over an infinite field [Bergman and Nahlus 2011; 2012; Bergman 2014; Maalouf 2014]. In those papers, the assumption of infinite base field was used to show that, under appropriate bounds on the vector-space dimension of B, the ultrafilters occurring had to be principal.

If we look at not-necessarily-associative *rings*, without assuming a structure of algebra over a field, then as we shall see below, we can still get results analogous to the main results of Section 2 (on when maps must factor through finitely many ultraproducts) and those of Section 4 (saying that such ultraproducts must be principal under appropriate assumptions on the additive structure of B). Between these we shall insert Proposition 39, which will say that if our rings have unit, then the absence of factorization through finitely many ultraproducts implies the existence of an associative commutative subring of B with the cardinality of the continuum, of an explicitly describable form, over which B becomes an algebra. I will not repeat here the results on algebras over an infinite field from the papers cited above; and having spent many words on those papers, I will be brief in this section.

In a direct product ring $\prod_{i\in I} R_i$, we define the support of an element $x=(x_i)_{i\in I}$ to be $\{i\in I \mid x_i\neq 0\}$. Whereas in Sections 2–4, our basic tool was the *commutativity* of elements with disjoint supports in a product group, and the phenomenon that this tool could not handle was avoided by dividing out by the center, $Z(B)$, the corresponding tool in the four works cited above was the fact that ring elements with disjoint supports have product zero; and the ideal one had to divide out by (which was also denoted $Z(B)$) was the zero-multiplication ideal. In this section, for B a ring, we will, as in those papers, write

$$Z(B) \;=\; \{b\in B \mid bB = Bb = 0\}. \tag{40}$$

As in Sections 2–4, we let $\pi : B\to B/Z(B)$ be the quotient homomorphism.

There was one commutativity result in Theorem 2 above that arose for a reason other than that elements in different factors of a direct product commute, namely, (7), which followed from the fact that every element commutes with itself. Thus, the analog of that one statement, (41) below, again concerns commutativity, rather than zero products.

The obvious analog of Lemma 1 holds for rings, and yields the following analog of Theorem 2.

Theorem 37. *Let B be a ring (understood here to mean an abelian group given with an arbitrary bilinear multiplication $B\times B\to B$), and suppose there exist a family $(R_i)_{i\in I}$ of rings, and a surjective ring homomorphism $f:\prod_{i\in I} R_i\to B$, such that the induced homomorphism $\pi f:\prod_{i\in I} R_i\to B/Z(B)$ does not factor through the natural map from $\prod_{i\in I} R_i$ to any finite product of ultraproducts of the R_i. Then B contains families of elements (a_S), (b_S), indexed by the subsets $S\subseteq\omega$, such that:*

(41) *All the elements a_S $(S\subseteq\omega)$ commute with one another, and all the elements b_S likewise commute with one another.*

(42) *For* disjoint *subsets S and T of ω, we have $a_S+a_T=a_{S\cup T}$, $b_S+b_T=b_{S\cup T}$, and $0=a_Sa_T=a_Sb_T=b_Sa_T=b_Sb_T$.*

(43) *For subsets S and T of ω with* $\operatorname{card}(S\cap T)=1$*, we have $a_Sb_T\neq 0$.* □

One gets from this the obvious analog of Corollary 3, which I will not write down, only noting one minor way in which the statement is weaker than that corollary: in a nonassociative ring, a family of pairwise commuting elements need not generate a commutative subring, so the assertion of commutativity in the last sentence of Corollary 3 disappears here.

One likewise has the analog of Theorem 5. Namely, following [Bergman and Nahlus 2011, Definitions 13 and 15], we define an almost direct decomposition of a ring B as an expression $B=B'+B''$, where B' and B'' are ideals of B, each of which is the 2-sided annihilator of the other; and we shall say that B has chain condition on almost direct factors if every chain of such ideals is finite. Then we get:

Theorem 38 (cf. [Bergman and Nahlus 2011, Proposition 16]). *Let B again be a ring such that there exist a family of rings $(R_i)_{i\in I}$, and a surjective ring homomorphism $f:\prod_{i\in I} R_i \to B$ such that the induced homomorphism $\pi f:\prod_{i\in I} R_i \to B/Z(B)$ does not factor through the natural map of $\prod_{i\in I} R_i$ to any finite direct product of ultraproducts of the R_i.*

Then B does not have chain condition on almost direct factors. In fact, it has a family of almost direct factors order-isomorphic to the lattice 2^ω, and forming a sublattice of the lattice of ideals of B.

So far we have not assumed our rings unital, since that hypothesis is unnatural for many important classes of nonassociative rings. The next result shows how in the unital case, the above theorems can be simplified and strengthened. For unital rings B we have $Z(B)=0$, so $B/Z(B)$ everywhere becomes B. Moreover, we can take each of the systems of elements $x_n, y_n \in \prod_{i\in J_n} R_i$ from which we obtain the elements a_S and b_S in Theorem 37 to consist of the multiplicative identity elements of the rings $\prod_{i\in J_n} R_i$. With a little further work, we shall get:

Proposition 39. *Under the common hypotheses of Theorem 37 and 38, if the rings B and R_i are unital, with homomorphisms preserving multiplicative identity elements, then B is a faithful unital algebra over a commutative associative unital subring of the form $\prod_{n\in\omega} \mathbb{Z}/d_n\mathbb{Z}$, where each d_n is a nonnegative integer $\neq 1$. Moreover, one can take all but one of the d_n to be equal, and that one to be a multiple (not necessarily proper) of the common value of the others.*

Sketch of proof. As in the proof of Theorem 2, the non-factorization of f tells us that we can partition I into subsets J_n $(n\in\omega)$ such that for each n, $f\left(\prod_{i\in J_n} R_i\right)\neq\{0\}$. (Here we regard the direct product of each subfamily of the R_i as an ideal of $\prod_{i\in I} R_i$. Each of these ideals has a multiplicative identity element, generally different from that of $\prod_{i\in I} R_i$.) For each $S\subseteq\omega$, let x_S denote the multiplicative identity element of $\prod_{i\in\bigcup_{n\in S} J_n} R_i \subseteq \prod_{i\in I} R_i$, and let $a_S = f(x_S)$. We see that the operations of multiplication by x_S and $x_{\omega-S}$ are idempotent endomorphisms of the additive group of $\prod_{i\in I} R_i$, which give the projection homomorphisms to the two factors of its decomposition as

$$\Big(\prod_{i\in\bigcup_{n\in S} J_n} R_i\Big)\times\Big(\prod_{i\in\bigcup_{n\in\omega-S} J_n} R_i\Big).$$

Hence their images a_S and $a_{\omega-S}$ likewise determine a direct product decomposition of the ring B.

Now for every $S\subseteq\omega$, let c_S denote the nonnegative integer such that the additive subgroup of B generated by a_S is isomorphic to $\mathbb{Z}/c_S\mathbb{Z}$ (the characteristic of the ring $f\left(\prod_{i\in\bigcup_{n\in S} J_n} R_i\right)$). Note that for $\varnothing\neq T\subseteq S$, we have $1\neq c_T\,|\,c_S$.

The behavior of c_S as a function of S can be complicated; but with the help of the Noetherian property of the integers, we can find a family of subsets of ω on which that function has an easy description. Namely, let us choose, among all *infinite* $S \subseteq \omega$, one which gives a maximal value for the ideal $c_S\mathbb{Z}$. Then for every infinite subset $T \subseteq S$, we necessarily have $c_T = c_S$. Hence, let us partition S into countably many infinite subsets, $T_0, \dots, T_m, \dots$, and use these to partition ω into subsets S_m, where for $m > 0$, we let $S_m = T_m$, while we let $S_0 = (\omega - S) \cup T_0$. Thus, for $m > 0$ we have $c_{S_m} = c_S$ by choice of S, while $c_{S_0} = \mathrm{lcm}(c_{\omega-S}, c_{T_0})$, a multiple of $c_{T_0} = c_S$.

Let us now map the ring $\mathbb{Z}^\omega$ into $\prod_{i\in I} R_i$ by sending each element $(e_m)_{m\in\omega}$ to the element whose value on each factor $\prod_{n\in S_m}\big(\prod_{i\in J_n} R_i\big)$ is e_m times the multiplicative identity element, and then apply the map $f : \prod_{i\in I} R_i \to B$. I claim that the image of $\mathbb{Z}^\omega$ in B will be isomorphic to $\prod_{m\in\omega} \mathbb{Z}/d_m\mathbb{Z}$, where $d_m = c_{S_m}$. Indeed, it is easy to verify that an element of $\mathbb{Z}^\omega$ that goes to zero under the componentwise map into $\prod_{m\in\omega} \mathbb{Z}/d_m\mathbb{Z}$ goes to zero in B, as a result of our choice of S and the S_m. Conversely, if an element $(e_m)_{m\in\omega} \in \mathbb{Z}^\omega$ does not have zero image in $\prod_{m\in\omega} \mathbb{Z}/d_m\mathbb{Z}$, we can choose an m_0 such that e_{m_0} is not divisible by d_{m_0}; and taking the image, in B, of the ring relation $(e_m)_{m\in\omega}\delta_{m_0} = e_{m_0}\delta_{m_0}$ in $\mathbb{Z}^\omega$, we see that the image of $(e_m)_{m\in\omega}$ in B is also nonzero.

Finally, the fact that every ring R is a $\mathbb{Z}$-algebra, and that if R has a multiplicative identity element 1_R, its $\mathbb{Z}$-algebra structure is induced by the operations of multiplication in R by members of $\mathbb{Z}\cdot 1_R$, easily leads to the result that $\prod_{n\in S_m}\big(\prod_{i\in J_n} R_i\big)$ is a $\mathbb{Z}^\omega$-algebra, and that this structure leads to a structure of $\prod_{m\in\omega} \mathbb{Z}/d_m\mathbb{Z}$-algebra on its homomorphic image B. □

Going back to not-necessarily-unital rings, and turning to the question of when finitely many ultraproducts through which a map factors must all be principal, we can combine Theorem 25 with the idea of Lemma 12 to get the following result.

Theorem 40. *Suppose B is a ring which admits a surjective homomorphism from a direct product ring, $f : \prod_{i\in I} R_i \to B$, such that the composite*

$$\pi f : \prod_{i\in I} R_i \to B \to B/Z(B)$$

factors through the product of finitely many ultraproducts of the R_i, but not through the product of finitely many countably complete ultraproducts. (*So if* card(I) *is less than all uncountable measure cardinals, if any exist, the latter condition simply says that πf does not factor through any finite product of the A_i.*)

Then the additive group of $B/Z(B)$ has a nonzero cotorsion subgroup; equivalently, it either contains nonzero elements of finite order, or a copy of the additive group of $\mathbb{Q}$, or a copy of the additive group of the p-adic integers for some prime p.

□

9. Monoids

In studying homomorphisms from direct product monoids onto a monoid B, it is useful to assume some cancellation condition on B. One that will suffice for our present purposes is

$$xy = x \implies y = e \quad \text{for } x, y \in B. \tag{44}$$

Note that (44) implies that one-sided inverses are two-sided, since if $xy = e$, we get $xyx = x$, which by (44) gives $yx = e$.

We shall consider two sorts of obstruction to mapping infinite products onto B in ways that indiscriminately merge the factors. On the one hand, there is the same effect of noncommutativity that we took advantage of in the case of groups. On the other hand, noninvertible elements create restrictions. For instance, though linear algebra shows that the additive group $\mathbb{Q}$ admits homomorphisms from the additive group $\mathbb{Q}^\omega$ that behave arbitrarily on $\bigoplus_\omega \mathbb{Q}$, it is not hard to show that, writing $\mathbb{Q}^{\geq 0}$ for the additive monoid of nonnegative rational numbers, it is impossible to have a homomorphism $(\mathbb{Q}^{\geq 0})^\omega \to \mathbb{Q}^{\geq 0}$ that acts in a nonzero way on infinitely many of the summands of $\bigoplus_\omega \mathbb{Q}^{\geq 0}$.

In our factorization results for groups, we divided out by the center of B; in the case of monoids, we will divide out by the group of *central invertible* elements. There are two versions of this concept: $Z(U(B))$, the center of the group of units (invertible elements) of B, and $U(Z(B))$, the group of units of the center; the former may be larger than the latter. It is $U(Z(B))$, the smaller of the two, that we will divide out by (though the other will make a brief appearance in a proof). Note that since $U(Z(B))$ consists of B-centralizing invertible elements, one can speak (without distinguishing right from left) of the orbits of B under multiplication by that group, and the set of such orbits forms a factor monoid $B/U(Z(B))$. Clearly, the noninvertible elements of this factor monoid are precisely the cosets of the noninvertible elements of B. We shall write π for the projection map $B \to B/U(Z(B))$.

Given a monoid homomorphism $f : \prod_{i\in I} M_i \to B$, we define the analog of the filter $\mathcal{F}$ of (2), namely

$$\begin{aligned}\mathcal{F} = \bigl\{S \subseteq I \mid{} & \text{the composite map } \textstyle\prod_{i\in I} M_i \to B \to B/U(Z(B)) \\ & \text{factors through the projection } \textstyle\prod_{i\in I} M_i \to \prod_{i\in S} M_i\bigr\} \\ = \bigl\{S \subseteq I \mid{} & f\bigl(\textstyle\prod_{i\in I-S} M_i\bigr) \subseteq U(Z(B))\bigr\}.\end{aligned} \tag{45}$$

The version of Lemma 1 that we will use for monoids is not, as for rings, a carbon copy of that lemma, so we shall give the statement and proof. (But we will cut corners where the method of proof is the same; so the reader might want to review the proof of Lemma 1 before beginning this one.) We do not yet assume the cancellativity condition (44).

Lemma 41. *Let $f : \prod_{i\in I} M_i \to B$ be a homomorphism from a direct product of monoids M_i to a monoid B, which is surjective (or more generally, such that the homomorphism $\pi f : \prod_{i\in I} M_i \to B \to B/U(Z(B))$ is surjective). Then the following two conditions are equivalent.*

(46) *The homomorphism $\pi f : \prod_{i\in I} M_i \to B \to B/U(Z(B))$ does not factor through the natural map $\big(\prod_{i\in I} M_i\big)/\mathcal{U}_0 \times \cdots \times \big(\prod_{i\in I} M_i\big)/\mathcal{U}_{n-1}$ for any finite family of ultrafilters $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$ on I.*

(47) *There exists a partition of I into countably many subsets $J_0, J_1, \ldots,$ such that either*

(47a) *each submonoid $\prod_{i\in J_n} M_i \subseteq \prod_{i\in I} M_i$ contains a pair of invertible elements x_n, y_n whose images under f do not commute in B,*

or

(47b) *each submonoid $\prod_{i\in J_n} M_i \subseteq \prod_{i\in I} M_i$ contains an element z_n whose image in B is noninvertible.*

Proof. To get (47) $\Longrightarrow$ (46), note that in the situation of (47a), since for each n, $f(x_n)$ and $f(y_n)$ are invertible and do not commute in B, they do not lie in $Z(U(B))$. Hence in particular $f(x_n) \notin U(Z(B))$, so by the final line of (45), $I - J_n \notin \mathcal{F}$. That this implies (46) is seen as in Lemma 1.

If, rather, we are in the situation of (47b), then the fact that the $f(z_n)$ are nonunits implies that they do not lie in $U(Z(B))$, giving the same result for the same reason.

The proof of the converse begins, as for Lemma 1, with the observation that (46) implies that there exists a partition of I into countably many subsets $J_0, J_1, \ldots,$ such that each $\prod_{i\in J_n} M_i$ contains elements mapped by f to elements of B not in $U(Z(B))$. Let $L_n = \prod_{i\in J_n} M_i$, so that $\prod_{i\in I} M_i = \prod_{n\in\omega} L_n$. Clearly, it will either be true that for infinitely many n, the submonoid $f(L_n) \subseteq B$ contains a noninvertible element, or that for infinitely many n, that submonoid consists entirely of invertible elements.

In the former case, those J_n such that $f(L_n)$ contains a noninvertible element of B will constitute a partition of some subset $J \subseteq I$ into countably many subsets. If we enlarge one of these sets by throwing in the complementary set $I - J$, we get a partition of I of the sort described in (47b).

If, on the other hand, there are infinitely many n such that $f(L_n)$ consists entirely of invertible elements of B, then for each such n, let us choose an $x_n \in L_n$ such that $f(x_n) \notin U(Z(B))$, and then a $y \in \prod_{m\in\omega} L_m$ such that $f(y)$ does not commute with $f(x_n)$. As in the proof of Lemma 1, we can obtain from y an element $y_n \in L_n$ such that $f(y_n)$ still does not commute with $f(x_n)$. By assumption, $f(L_n)$ consists of invertible elements, so $f(x_n)$ and $f(y_n)$ belong to $U(B)$. Thus, we have a partition of some $J \subseteq I$ into countably many subsets as in (47a). Again tacking $I - J$ onto one of these, we can take this to be a partition of the whole set I. □

This leads to an analog of Theorem 2 which, similarly, has two alternative conclusions. We shall describe one of these by referring to that earlier theorem, and spell out the other.

Theorem 42. *Let B be a monoid satisfying the cancellation condition (44), and suppose there exists a family $(M_i)_{i\in I}$ of monoids, and a monoid homomorphism $f : \prod_{i\in I} M_i \to B$ such that the induced homomorphism $\pi f : \prod_{i\in I} M_i \to B/U(Z(B))$ does not factor through any finite product of ultraproducts of the M_i.*

Then either

(a) *the group $U(B)$ satisfies the hypothesis, and hence the conclusions, of Theorem 2,*

or

(b) *B contains a family of elements (a_S) indexed by the subsets $S \subseteq \omega$ and satisfying the following conditions:*

(48) *$a_\varnothing = e$, and all the elements a_S ($S \subseteq \omega$) commute with one another.*

(49) *For disjoint sets $S,\ T \subseteq \omega$, one has $a_S a_T = a_{S\cup T}$.*

(50) *For sets $S \subsetneq T \subseteq \omega$, a_T is a right multiple of a_S, but a_S is not a right multiple of a_T.*

Proof. The two cases of (47) will yield the two alternative conclusions shown. It is easy to verify that (47a) yields conclusion (a).

In case (47b), let $L_n = \prod_{i\in J_n} M_i$, and take elements $z_n \in L_n$ with noninvertible images in B. For each $S \subseteq \omega$, let a_S be the image under f of the element of $\prod_{n\in S} L_n \subseteq \prod_{n\in\omega} L_n$ whose n-th coordinate is z_n for each $n \in S$. Then (48) and (49) are immediate, and the first assertion of (50) follows from (49) applied to S and $T - S$.

To get the final assertion of (50), choose any $n \in T - S$, and note that by (49), we have

$$a_T \ = \ a_S \ a_{T-(S\cup\{n\})} \ a_{\{n\}}. \tag{51}$$

If we also had $a_S = a_T b$ for some $b \in B$, then substituting this into the right-hand-side of (51) and canceling a_T by (44), we could conclude that $a_{\{n\}}$ was left invertible, hence by the observation following (44), invertible, contradicting our choice of z_n. □

In case (b) of Theorem 42, we cannot say, as we can in case (a), that distinct sets S yield distinct elements $a_S \in B$. For instance, let B be the factor monoid of the additive monoid $(\mathbb{Z}^{\geq 0})^\omega$ by the relation that equates elements x and y if there is some $n \geq 0$ such that x and y agree at all but the first n coordinates, and such that the sum of the entries at those first n coordinates is the same for x and y. Then B is a cancellative abelian monoid with trivial group of units, and the quotient map $f : (\mathbb{Z}^{\geq 0})^\omega \to B$ does not annihilate any of the δ_n (defined in $(\mathbb{Z}^{\geq 0})^\omega$ as in $\mathbb{Z}^\omega$). Hence (47b) holds for this f, with the J_n taken to be the singletons $\{n\}$, and $z_n = \delta_n$. But defining the a_S in terms of these as in the proof of Theorem 42, we find that for finite subsets $S, T \subseteq \omega$ of the same cardinality, we have $a_S = a_T$ in B; so the a_S are not all distinct.

Nevertheless, in the situation of Theorem 42(b) we always get continuum many distinct a_S. For by (50), distinct *comparable* sets give distinct elements; and the partially ordered set of subsets of any countably infinite set has chains of the order type of the real numbers. (Indeed, the countable set of rational numbers has the chain of Dedekind cuts, and any countably infinite set can be put in bijective correspondence with the rationals.) Thus, we get:

Corollary 43. *In the situation of conclusion* (b) *of Theorem 42*, *B has a set of mutually commuting noninvertible elements which form*, *under the relation of divisibility*, *a chain with the order type of the real numbers. In particular*, B (*and in fact*, $B/U(Z(B))$ *has at least the cardinality of the continuum.* □

The results proved above are far from optimal. For instance, the conclusions of Theorem 42 and Corollary 43 are consistent with B being the additive monoid $\mathbb{R}^{\geq 0}$ of nonnegative real numbers; but that case is easy to exclude. Indeed, suppose $B = \mathbb{R}^{\geq 0}$ admitted a map as in the hypothesis of Theorem 42. By the proof of Lemma 41, since B has trivial group of units, we must have a homomorphism $f : \prod_{n\in\omega} L_n \to B$ such that each L_n has an element x_n making $f(x_n)$ a positive real number. By the Archimedean property of the reals, we can modify our choices of x_n so that for each n we have $f(x_n) \geq 1$. Thus, when we construct elements $a_S \in B$ as in the proof of the theorem, we get $a_{\{n\}} \geq 1$ for each n, from which it follows that for any $S \subseteq \omega$ of $\geq m$ elements ($m \in \omega$), $a_S \geq m$. For S infinite, this gives a contradiction; so B admits no such map. It is not clear to me what the best assertion that can be gotten by this technique is.

Let us also note that in place of the two-way subdivision of the sets J_n used in the proof of Lemma 41, we could (at least if we assumed full cancellativity rather than just (44)) have used a three-way subdivision, noting that for each n, $f(L_n)$ either contains noncommuting *invertible* elements of B, or contains noncommuting *non*invertible elements, or contains a central noninvertible element. (Cancellativity is needed to show that if a nonunit x and a unit u fail to commute, then so do the two nonunits x and xu.) So there must be infinitely many n for which one of these statements holds, and we can deduce a three-alternative conclusion: either, as before, we have invertible elements $a_S, b_S \in B$ indexed by the subsets $S \subseteq \omega$

which can be distinguished by their commutativity relations, or we have elements $a_S, b_S \in B$ which, except for $a_\varnothing, b_\varnothing$, are *noninvertible*, and satisfy the same relations and can be distinguished in the same way, or we have *central* elements a_S which satisfy (48)–(50).

Though one could define "almost direct factors" for monoids, as for groups, using submonoids that are each other's centralizers, there doesn't seem to be an analogous way to "split" a monoid based on noninvertible central elements; so I have not attempted to formulate an analog of Theorem 5. I leave further exploration of these questions to those better versed than I in the study of monoids.

One can also consider for *semigroups* the same factorization properties studied here for monoids. Since the above constructions involved elements of direct products defined to have the value e on complements of given subsets S of our index set, the absence of identity elements should lead to changes in what can be proved.

10. Lattices: a case worth looking at

One other class of mathematical structures suggests itself, to which similar methods might be applicable — lattices. Just as a direct product decomposition of a group or monoid leads to certain pairs of elements that must commute, and a direct product decomposition of a ring leads to certain pairs of elements that must have zero product, so a direct product decomposition of a lattice leads to certain 3-tuples of elements that must satisfy distributivity relations. Perhaps this observation can be used to get lattice-theoretic analogs of some of the results of this note.

(In [Bergman 2014, §5] I speculate on very general properties of a variety of algebras that would allow one to get such results; but I am not confident that that approach will go anywhere.)

Note added in proof

Jan Šaroch (personal communication) has obtained a positive answer to Question 33, which applies more generally to a module B over any countable ring R such that R^ω is a flat Mittag-Leffler module.

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THE VIRTUAL FIRST BETTI NUMBER OF SOLUBLE GROUPS

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We show that if a group G is finitely presented and nilpotent-by-abelian-by-finite, then there is an upper bound on $\dim_{\mathbb{Q}} H_1(M, \mathbb{Q})$, where M runs through all subgroups of finite index in G.

1. Introduction

The *virtual first betti number* of a finitely generated group G is defined as

$$\mathrm{vb}_1(G) = \sup\{\dim H_1(S, \mathbb{Q}) \mid S \leq G \text{ of finite index}\}.$$

A group is said to be *large* if it has a subgroup of finite index that maps onto a nonabelian free group. If G is large then $\mathrm{vb}_1(G) = \infty$. It is easy to find finitely generated groups G that are not large but have $\mathrm{vb}_1(G) = \infty$. For example, in the metabelian group $\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid [a, t^{-n}at^n] = 1 \text{ for all } n\rangle$, the subgroup $S_m < \mathbb{Z} \wr \mathbb{Z}$ generated by t^m and the conjugates of a has index m and $H_1(S_m, \mathbb{Z}) = \mathbb{Z}^{m+1}$. In contrast, no example is known of a *finitely presented* group that is not large but has $\mathrm{vb}_1(G) = \infty$ (see [Button 2010; Lackenby 2010]). Since amenable groups do not contain nonabelian free subgroups, one might hope to resolve this issue by finding a finitely presented amenable group with $\mathrm{vb}_1(G) = \infty$, but this seems to be a nontrivial matter.

We shall prove in this paper that for large classes of finitely presented soluble groups $\mathrm{vb}_1(G)$ is always finite. One would like to prove that the same is true for all finitely presented soluble groups, but here one faces the profound difficulty of deciding which soluble groups admit finite presentations; this is unknown even for abelian-by-polycyclic and nilpotent-by-abelian groups.

In the case of metabelian groups, finite presentability is completely understood in terms of the Bieri–Strebel invariant [Bieri and Strebel 1980]. Some sufficient conditions for finite presentability of nilpotent-by-abelian groups were considered by McIsaac [1984] and later Groves [1991]. In the case of S-arithmetic nilpotent-by-abelian groups G one knows more thanks to the work of Abels [1987]: if G is an extension of a nilpotent group N by an abelian group Q then G is finitely

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presented if and only if it is of type FP_2, which it is if and only if $H_2(N, \mathbb{Z})$ is finitely generated as a $\mathbb{Z}Q$-module (where the Q action is induced by conjugation) and[1] G/N' is finitely presented as a group. The first of these conditions is an easy consequence of the fact that $\mathbb{Z}Q$ is a Noetherian ring, and the second is a corollary of a result in [Bieri and Strebel 1980] that every metabelian quotient of a group of type FP_2 that does not contain noncyclic free subgroups is finitely presented. The case where G is an extension of an abelian normal subgroup A by a polycyclic group Q was approached by Brookes and Groves who studied modules over crossed products of a division ring by a free abelian group; see [Brookes and Groves 1995; 2000; 2002].

Given this background, the natural place to begin our investigation into the virtual first betti number of finitely presented soluble groups is in the setting of metabelian groups. Using methods from commutative algebra, we prove (Theorem 4.3) that if G is finitely presented and metabelian, then $\mathrm{vb}_1(G)$ is finite. (The hypothesis that one actually needs to impose on G is somewhat weaker than finite presentability; see Remark 6.5.) The metabelian case is used in the proof of our main theorem, which is the following.

Theorem A. *Let G be a finitely presented group. If G is nilpotent-by-abelian-by-finite, then $\mathrm{vb}_1(G)$ is finite.*

Our proof of this theorem relies on the fact that all metabelian quotients of soluble groups of type FP_2 are finitely presented [Bieri and Strebel 1980, Theorem 5.5], as well as a technical result concerning the homology of subgroups of finite index (Proposition 6.2). Groves, Kochloukova and Rodrigues [Groves et al. 2008, Theorem A] proved that if an abelian-by-polycyclic group G is of type FP_3 then it is nilpotent-by-abelian-by-finite, in which case $\mathrm{vb}_1(G)$ is finite by Theorem A. The same is true of all soluble groups of type FP_∞, because they are constructible [Kropholler 1986], hence nilpotent-by-abelian-by-finite, but in this case stronger finiteness results were already known: constructible soluble groups are obtained from the trivial group by finite sequences of ascending HNN extensions and finite extensions, from which it follows that they have finite Prüfer rank (i.e., there is an upper bound on the number of generators for the finitely generated subgroups).

It is natural to wonder if Theorem A might remain true when the field of rationals $\mathbb{Q}$ in the definition of virtual betti number is replaced with other coefficient fields, such as the field with p elements F_p. We shall see in Section 5 that it does not.

Conjecture. If G is finitely presented and soluble, then $\mathrm{vb}_1(G)$ is finite.

It is difficult to construct finitely presented soluble groups that are not nilpotent-by-abelian-by-finite. The examples provided by the constructions of Robinson and Strebel [1982] all satisfy the conjecture.

[1] Throughout this article, H' denotes the commutator subgroup of a group H.

While editing the final version of this work, we learnt that Andrei Jaikin-Zapirain has, in unpublished work, also proved Theorem A in the metabelian case. Higher dimensional analogues of Theorem A are considered in the forthcoming PhD thesis of Fatemeh Mokari.

2. Preliminary results

2A. ***Preliminaries on finitely presented metabelian groups.*** We fix a short exact sequence of groups $A \rightarrowtail G \twoheadrightarrow Q$, where A and Q are abelian and G is finitely generated. The action of G on A by conjugation induces an action of Q, which enables us to regard A as a right $\mathbb{Z}Q$-module. Because G is finitely generated and Q is finitely presented, A is finitely generated as a $\mathbb{Z}Q$-module.

Associated to a nonzero real character $\chi : Q \to \mathbb{R}$ one has the monoid

$$Q_\chi = \{g \in Q \mid \chi(g) \geq 0\}.$$

The character sphere $S(Q)$ is the set of equivalence classes in $\mathrm{Hom}(Q, \mathbb{R}) \smallsetminus \{0\}$ under the relation that identifies $\chi_1 \sim \chi_2$ if $\chi_1 = \lambda\chi_2$ for some $\lambda > 0$. We write $[\chi]$ for the class of χ. Following [Bieri and Strebel 1980], let

$$\Sigma_A(Q) = \{[\chi] \mid A \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module}\}.$$

By definition, the $\mathbb{Z}Q$-module A is 2-*tame* if $\Sigma_A(Q)^c = S(Q) \smallsetminus \Sigma_A(Q)$ contains no pair of antipodal points. According to [op. cit., Theorem 5.4], G is finitely presented if and only if A is a 2-tame $\mathbb{Z}Q$-module, and this happens precisely when G is of homological type FP_2. We refer the reader to [Bieri 1981] for general results concerning groups of type FP_m. If $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ is an exact sequence of finitely generated $\mathbb{Z}Q$-modules, then $\Sigma_{A_2}(Q)^c = \Sigma_{A_1}(Q)^c \cup \Sigma_{A_3}(Q)^c$ (see [Bieri and Strebel 1980, Proposition 2.2]), hence every quotient of a 2-tame $\mathbb{Z}Q$-module is 2-tame.

2B. ***Tensor products and finite presentability.*** Let R be a noetherian commutative ring with unit 1 and let W be a finitely generated RQ-module. As above, we have a Sigma invariant $\Sigma_W(Q) = \{[\chi] \mid W \text{ is finitely generated as an } RQ_\chi\text{-module}\}$, and W is defined to be 2-tame as an RQ-module if $\Sigma_W^c(Q) = S(Q) \setminus \Sigma_W(Q)$ has no pair of antipodal points.

The question of when the tensor square $W \otimes_R W$ is finitely generated as an RQ-module (with Q acting diagonally) is addressed in [Bieri and Groves 1985], where it is shown that $[\chi]$ lies in $\Sigma_W^c(Q)$ if and only if the ring $S = RQ/\operatorname{ann}_{RQ}(W)$ admits a real valuation $v : S \to \mathbb{R} \cup \{\infty\}$ (in the sense of Bourbaki) that extends χ and is such that the restriction v_0 of v to the image $\bar{R}$ of R in S is nonnegative and discrete. By [loc. cit.], $W \otimes_R W$ is finitely generated as an RQ-module if and only if there is no pair of antipodal elements $[\chi], [-\chi] \in \Sigma_W^c(Q)$ that can be lifted to valuations of S that have the same restriction v_0 to $\bar{R}$, with v_0 discrete and nonnegative. (These last conditions on v_0 are automatic if $\bar{R}$ is $\mathbb{Z}$.)

Returning to the context of Section 2A, we apply these general considerations with $W = A \otimes \mathbb{Q}$ and $R = \mathbb{Q}$, in which case $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We deduce that if there exists a group extension $A \rightarrowtail G \twoheadrightarrow Q$, with G finitely presented, then $W = A \otimes \mathbb{Q}$ is 2-tame as a $\mathbb{Q}Q$-module, and $W \otimes_R W \cong (A \otimes_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action.

We shall also need a refinement of this observation that involves the annihilator $\mathrm{ann}_{\mathbb{Z}Q}(A)$ of A in $\mathbb{Z}Q$, which we denote I. Bieri and Strebel [1981, (1.3)] prove that

$$\Sigma_A(Q) = \Sigma_{\mathbb{Z}Q/I}(Q).$$

Thus if A is 2-tame as a $\mathbb{Z}Q$-module, then so is $\mathbb{Z}Q/I$.

Lemma 2.1. *If there exists a group extension $A \rightarrowtail G \twoheadrightarrow Q$ with A and Q abelian and G finitely presented, and $I = \mathrm{ann}_{\mathbb{Z}Q}(A)$, then $(\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} (\mathbb{Z}Q/I) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action.*

2C. ***Preliminaries on commutative algebra.*** We will need the following basic facts from commutative algebra; for details see, for example, [Bourbaki 1961–1965; Atiyah and Macdonald 1969; Eisenbud 1995]. Let Q be a finitely generated abelian group and recall that the *Krull dimension* of a commutative ring is the supremum of the lengths of all chains of prime ideals in the ring.

(1) The radical $\sqrt{J}$ of each ideal $J \lhd \mathbb{Q}Q$ is the intersection of the finitely many prime ideals that contain J and are minimal subject to this condition.

(2) Finite dimensional $\mathbb{Q}$-algebras are Artinian and thus have Krull dimension 0.

Throughout, if R is a commutative ring and m a positive integer, then R^m will denote the subring generated by m-th powers, *except* that $\mathbb{Z}^n$ and $\mathbb{Q}^n$ will denote Cartesian powers. Where no ring is specified, tensor products are assumed to be taken over $\mathbb{Z}$.

3. A finiteness result in commutative algebra

Lemma 2.1 assures us that the following theorem applies to the modules that arise from short exact sequences $N \rightarrowtail G \twoheadrightarrow \mathbb{Z}^n$ associated to finitely presented metabelian groups.

Theorem 3.1. *Let $Q \cong \mathbb{Z}^n$ be a group and let $S = \mathbb{Z}Q/I$ be a commutative ring such that $(S \otimes_{\mathbb{Z}} S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal Q-action. Then,*

$$\sup_m \dim_{\mathbb{Q}}(S \otimes_{\mathbb{Z}Q^m} \mathbb{Q}) < \infty.$$

Proof. Let $B = S \otimes \mathbb{Q} = \mathbb{Q}Q/J$ and for each positive integer m define $J_m \lhd \mathbb{Q}$ to be $(J, Q^m - 1)$ and

$$B_m := B \otimes_{\mathbb{Q}Q^m} \mathbb{Q} = \mathbb{Q}Q/J_m \cong S \otimes_{\mathbb{Z}Q^m} \mathbb{Q}.$$

As $\mathbb{Q}Q/(Q^m - 1)$ is finite dimensional over $\mathbb{Q}$, so is $B_m = \mathbb{Q}Q/J_m$. Hence B_m has Krull dimension 0; i.e., every prime ideal in B_m is a maximal one. Therefore, the finite collection of primes ideals $P_{m,t}$ whose intersection is $\sqrt{B_m}$ are the only prime ideals in $\mathbb{Q}Q$ above J_m, and each of the quotients $\mathbb{Q}Q/P_{m,t}$ is a field.

We shall establish the theorem by proving the following:

Claim 1. *There exist only finitely many fields F such that for some $m \geq 1$ (depending on F) the field F is a quotient of B_m.*

Claim 1 provides an integer m_0 such that if a field F is a quotient of B_m then the natural map $\mathbb{Q}Q \to F$ factors through $\mathbb{Q}Q/(Q^{m_0} - 1)$.

Claim 2. *If m_0 divides m then $J_m = J_{mr}$ for every $r \in \mathbb{N}$.*

To see that the theorem follows from these claims, note that for an arbitrary positive integer m we have $J_m \supseteq J_{mm_0} = J_{m_0}$, whence

$$\begin{aligned}\dim_{\mathbb{Q}}(\mathbb{Q}Q/J_m) &\leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/J_{m_0}) \leq \dim_{\mathbb{Q}}(\mathbb{Q}Q/(Q^{m_0} - 1)) \\ &= \dim_{\mathbb{Q}} \mathbb{Q}[Q/Q^{m_0}] = m_0^n.\end{aligned}$$

Proof of Claim 1. Our hypothesis on S implies that $B \otimes_{\mathbb{Q}} B$ is finitely generated as $\mathbb{Q}Q$-module via the diagonal Q-action, by d elements say. Let F be a field quotient of B_m and let $\theta : \mathbb{Q}Q \to F$ be the canonical projection; so $Q^m - 1 \subseteq \ker(\theta)$. Then, $\theta(Q)$ is a finitely generated multiplicative subgroup of F^* that has finite exponent and F, being finite dimensional over $\mathbb{Q}$, embeds in $\mathbb{C}$. Hence $\theta(Q)$ is a finite cyclic group, generated by a root of unity, ϵ of order s, say. Thus we obtain a subgroup $H < Q$ such that Q/H is cyclic of order s and $H - 1 \subseteq \ker(\theta)$. Now, $F \cong \mathbb{Q}[x]/(f)$, where f is the minimal polynomial of ϵ over $\mathbb{Q}$. And f is an irreducible factor of $x^s - 1$ in $\mathbb{Q}[x]$, whose zeroes are distinct roots of unity with order precisely s. Thus $\dim_{\mathbb{Q}} F = \deg(f) = \varphi(s)$, where φ is Euler's totient function. On the other hand, $F \otimes_{\mathbb{Q}} F$ is an epimorphic image of the $\mathbb{Q}Q$-module $B \otimes_{\mathbb{Q}} B$ and the action of Q on $F \otimes_{\mathbb{Q}} F$ factors through the action of Q/H, so $F \otimes_{\mathbb{Q}} F$ is generated as a $\mathbb{Q}[Q/H]$-module by d elements. Hence

$$\varphi(s)^2 = (\dim_{\mathbb{Q}} F)^2 = \dim_{\mathbb{Q}}(F \otimes_{\mathbb{Q}} F) \leq d \dim_{\mathbb{Q}} \mathbb{Q}[Q/H] = ds.$$

An elementary calculation shows that $\varphi(n)/\sqrt{n} \to \infty$ as $n \to \infty$, so for fixed d there are only finitely many possible values of s and ϵ. Let b be a natural number such that the order of ϵ is at most b. Then, the order of ϵ is a divisor of $m_0 = b!$ and

$$F \text{ is a quotient of } \mathbb{Q}Q/(Q^{m_0} - 1).$$

Since $\mathbb{Q}Q/(Q^{m_0}-1)$ is finite dimensional over $\mathbb{Q}$ it has Krull dimension 0, so has only finitely many prime ideals and finitely many field quotients. This completes the proof of Claim 1. □

Proof of Claim 2. Since m_0 divides m we have $J_m \subseteq J_{m_0}$, so the prime ideals containing J_{m_0} also contain J_m. On the other hand, we saw earlier that for each of the prime ideals $P_{m,i}$ containing J_m, the quotient $F_i := \mathbb{Q}Q/P_{m,i}$ is a field. By definition, m_0 is such that $\mathbb{Q}Q \to F_i$ factors through $\mathbb{Q}Q/(Q^{m_0}-1)$, and therefore $P_{m,i}$ (which already contains $J \subset J_m$) contains $J_{m_0} = (J, Q^{m_0}-1)$. The radical of J_m is the intersection of the prime ideals containing it, so

$$\sqrt{J_m} = \sqrt{J_{m_0}}.$$

Arguing by induction on r, Claim 2 will follow if we can prove that for every prime number p we have $J_m = J_{mp}$, which is equivalent to the assertion that $q^m - 1 \in J_{mp}$ for all $q \in Q$.

We now fix $q \in Q$. From the preceding argument, $\sqrt{J_m} = \sqrt{J_{mp}}$. In particular, $Q^m - 1 \subseteq J_m \subseteq \sqrt{J_m} = \sqrt{J_{mp}}$, so there is a natural number s (over which we have no control) such that

$$(q^m - 1)^s \in J_{mp}. \tag{3-1}$$

As $Q^{mp} - 1 \subseteq J_{mp}$, we also have

$$q^{mp} - 1 \in J_{mp}. \tag{3-2}$$

Let $g(x)$ be the greatest common divisor of $x^{pm}-1$ and $(x^m-1)^s$ in $\mathbb{Q}[x]$. In characteristic zero, the polynomial $x^{pm}-1$ has no repeated roots, so neither does $g(x)$. Since $g(x)$ divides $(x^m-1)^s$, it must actually divide $x^m - 1$, so in fact $g(x) = x^m - 1$. From (3-1), (3-2) and Bézout's lemma, we have $g(q) \in J_{pm}$. Since $q \in Q$ is arbitrary, this implies that $J_{mp} = J_m$. □

4. The main theorem for metabelian groups

In this section we prove that all finitely presented metabelian groups have finite virtual first betti number. The proof relies on the finiteness theorem proved in the previous section and two technical lemmas, the first of which is a simple observation about commensurable groups.

Lemma 4.1. *Let G be a group. If $G_0 < G$ is a subgroup of finite index, then* $\mathrm{vb}_1(G) = \mathrm{vb}_1(G_0)$.

Proof. By definition, $\mathrm{vb}_1(G) = \sup_M \dim H_1(M, \mathbb{Q})$, where the supremum is taken over finite-index subgroups of G. If M has finite index in G_0, then it also has finite index in G, so $\mathrm{vb}_1(G) \geq \mathrm{vb}_1(G_0)$. Conversely, if S has finite index in G, then

$S_0 = G_0 \cap S$ has finite index in G_0, and since it also has finite index in S, we have $\dim H_1(S_0, \mathbb{Q}) \geq \dim H_1(S, \mathbb{Q})$, so $\mathrm{vb}_1(G_0) \geq \mathrm{vb}_1(G)$. □

Lemma 4.2. *Let $A \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with A and Q abelian and let n be the torsion-free rank of Q. Then*:

(a) *Writing $[G, A] = \langle \{[g, a] = g^{-1}a^{-1}ga \mid g \in G, a \in A\} \rangle$, we have*

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

In the split case, $G = A \rtimes Q$, we have $H_1(G, \mathbb{Q}) \cong (G/[G, A]) \otimes_{\mathbb{Z}} \mathbb{Q}$, and

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

(b) *If G_m is a subgroup of finite index in G and Q_m is the image of G_m in Q, then*

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$

In the split case, $G_m = (A \cap G_m) \rtimes Q_m$, equality is attained:

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n.$$

(c) *If $G = A \rtimes Q$ and $\mathcal{B}$ denotes the set of subgroups of finite index in Q, then*

$$\mathrm{vb}_1(G) = \sup_{S \in \mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n.$$

Proof. (a) As $[G, A] \subseteq [G, G]$, we see that $H_1(G, \mathbb{Z}) = G/[G, G]$ is a quotient of $G/[G, A]$. So from the central extension $A/[G, A] \rightarrowtail G/[G, A] \twoheadrightarrow Q$, we get

$$\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + \dim_{\mathbb{Q}}(Q \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(A/[G, A] \otimes \mathbb{Q}) + n.$$

If $G = A \rtimes Q$ then, using that A, Q are abelian and A is normal in G, we get $[G, G] = [AQ, AQ] = [Q, A] \subseteq [G, A] \subseteq [G, G]$, hence $[G, G] = [G, A]$ and $A/[G, A] \rightarrowtail G/[G, G] \twoheadrightarrow Q$ is an exact sequence of abelian groups.

(b) We consider the short exact sequence $A_m \rightarrowtail G_m \twoheadrightarrow Q_m$, where $A_m = A \cap G_m$. From part (a) we have

$$\dim_{\mathbb{Q}} H_1(G_m, \mathbb{Q}) \leq \dim_{\mathbb{Q}}(A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) + n, \tag{4-1}$$

with equality if the sequence splits. Furthermore, since A/A_m is finite we have

$$0 = \mathrm{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \quad \text{and} \quad (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0.$$

Thus there is an exact sequence (part of the long exact sequence in Tor associated to $A \cap G_m \rightarrowtail A \twoheadrightarrow A/(A \cap G_m)$)

$$0 = \mathrm{Tor}_1^{\mathbb{Z}Q_m}(A/A_m, \mathbb{Q}) \to A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to A \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to (A/A_m) \otimes_{\mathbb{Z}Q_m} \mathbb{Q} = 0,$$

whence $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \cong A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$. Thus, we may replace $A_m \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$ in (4-1) by $A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$, and (b) is proved.

(c) From the first part of (b) we have

$$\mathrm{vb}_1(G) \leq \sup_{S\in\mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n,$$

and to obtain the reverse inequality, we use the second part of (b)

$$\sup_{S\in\mathcal{B}} \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}S} \mathbb{Q}) + n = \sup_{S\in\mathcal{B}} \dim_{\mathbb{Q}} H_1(A \rtimes S, \mathbb{Q}),$$

noting that $A \rtimes S$ has finite index in G. □

Theorem 4.3. *Let $A \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with A and Q abelian. If G is finitely presented then its virtual first betti number* $\mathrm{vb}_1(G)$ *is finite.*

Proof. By passing to a subgroup of finite index in Q and replacing G by the inverse image of this subgroup, we may assume that Q is free abelian. Lemma 4.1 assures us that it is enough to consider this case, and Lemma 4.2(b) tells us that we will be done if we can establish an upper bound on $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q})$ as Q_m ranges over the subgroups of finite index in Q.

Recall that A is finitely generated as a $\mathbb{Z}Q$-module, say by d elements. Thus, denoting the annihilator $\mathrm{ann}_{\mathbb{Z}Q}(A) = \{\lambda \in \mathbb{Z}Q \mid A\lambda = 0\}$ by I, we have an epimorphism of $\mathbb{Z}Q$-modules

$$(\mathbb{Z}Q/I)^{[d]} = \mathbb{Z}Q/I \oplus \cdots \oplus \mathbb{Z}Q/I \to A$$

that induces an epimorphism of $\mathbb{Q}$-vector spaces

$$\big((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q}\big)^{[d]} = (\mathbb{Z}Q/I)^{[d]} \otimes_{\mathbb{Z}Q_m} \mathbb{Q} \to A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}.$$

Thus,

$$\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) \leq d \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q})$$

and it suffices to show that

$$\sup_m \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) < \infty.$$

For every m there is a natural number α_m such that $Q^{\alpha_m} \subseteq Q_m$, and $\mathbb{Z}Q/I \otimes_{\mathbb{Z}Q_m} \mathbb{Q}$ is a quotient of $\mathbb{Z}Q/I \otimes_{\mathbb{Z}Q^{\alpha_m}} \mathbb{Q}$. Thus,

$$\dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q_m} \mathbb{Q}) \leq \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q^{\alpha_m}} \mathbb{Q}),$$

and we have reduced to showing that

$$\sup_s \dim_{\mathbb{Q}}((\mathbb{Z}Q/I) \otimes_{\mathbb{Z}Q^s} \mathbb{Q}) < \infty.$$

The theorem now follows from Lemma 2.1 and Theorem 3.1. □

5. Characteristic p case

In this section we shall construct examples which show that the restriction to fields of characteristic 0 in Theorem A is essential, even in the metabelian case.[2] To this end, we consider the mod p *virtual first betti number* of a finitely generated group G,

$$\mathrm{vb}_1{}^{(p)}(G) = \sup\{\dim H_1(S, F_p) \mid S < G \text{ of finite index}\}.$$

Proposition 5.1. *For every prime p there exist finitely presented metabelian groups Γ such that $\mathrm{vb}_1{}^{(p)}(\Gamma)$ is infinite.*

Proof. Let Q be a free abelian group with generators x and y and let $A = F_p Q/I$, where I is the ideal of $F_p Q$ generated by $y - x^2 + x - 1$. Then,

$$A \cong F_p\Big[x, x^{-1}, \frac{1}{x^2-x+1}\Big].$$

Consider

$$A_m = A \otimes_{\mathbb{Z}Q^{p^m}} F_p \cong F_p Q/(I, Q^{p^m} - 1).$$

Since $(x^2 - x + 1)^{p^m} - 1 = x^{2p^m} - x^{p^m} + 1 - 1 = x^{p^m}(x^{p^m} - 1)$, we have

$$\begin{aligned} A_m &= F_p\Big[x, x^{-1}, \frac{1}{x^2-x+1}\Big] \Big/ \big(x^{p^m} - 1, (x^2 - x + 1)^{p^m} - 1\big) \\ &= F_p\Big[x, x^{-1}, \frac{1}{x^2-x+1}\Big] \Big/ \big(x^{p^m} - 1\big) \end{aligned}$$

is the localisation

$$B_m S^{-1}$$

where $B_m = F_p[x, x^{-1}]/(x^{p^m} - 1)$ and S is the image of $\{(x^2 - x + 1)^j\}_{j \geq 1}$ in B_m. Note that $x^{p^m} - 1$ and $x^2 - x + 1$ do not have a common root in any finite field extension of F_p, for if z were a common root we would have $1 = z^{2p^m} = (z-1)^{p^m} = z^{p^m} - 1 = 0$, which is a contradiction. Thus the polynomials $x^{p^m} - 1$ and $(x^2 - x + 1)^j$ are coprime in $F_p[x, x^{-1}]$; i.e., they generate the whole ring as an ideal, and so the elements of S are invertible in B_m. Therefore $B_m S^{-1} = B_m$ and

$$\dim_{F_p} A_m = \dim_{F_p} B_m S^{-1} = \dim_{F_p} B_m = p^m.$$

Now define

$$\Gamma = A \rtimes Q \quad \text{and} \quad \Gamma_m = A \rtimes Q^{p^m}.$$

Then, as in the split case of Lemma 4.2(b) (with coefficients in F_p in place of $\mathbb{Q}$),

$$\dim_{F_p} H_1(\Gamma_m, F_p) = \dim_{F_p} A_m + 2 = p^m + 2,$$

[2]John Wilson [1998] proved that the dimension of $H_1(S, F_p)$ can grow at most like the square root of the index $[G : S]$. Jack Button [2010] exhibited a finitely presented soluble group that exhibits this growth for all p.

which tends to infinity with m.

By the calculation [Bieri and Strebel 1981, Theorem 5.2] of $\Sigma_A(Q)$ for $A = F_p Q/I$, where the ideal I is 1-generated, or by the link between $\Sigma_A^c(Q)$ and valuation theory (as described in Section 2B), we have

$$\Sigma_A^c(Q) = \{[\chi_1], [\chi_2], [\chi_3]\},$$

with

$$\chi_1(x) = 0, \qquad \chi_2(x) = 1, \qquad \chi_3(x) = -1,$$
$$\chi_1(y) = 1, \qquad \chi_2(y) = 0, \qquad \chi_3(y) = -2.$$

Thus, A is 2-tame as a $\mathbb{Z}Q$-module, and by the classification of finitely presented metabelian groups in [Bieri and Strebel 1980], Γ is finitely presented. □

Corollary 5.2. *There exists a finitely presented metabelian group G such that for the class $\mathscr{A}$ of all subgroups of finite index in G,*

$$\sup_{M \in \mathscr{A}} d(M) = \infty,$$

where $d(M)$ is the minimal number of generators of M.

Proof. Immediate, since $d(M) \geq \dim_{F_p} H_1(M, F_p)$. □

It is natural to wonder if the lack of finiteness exhibited in the preceding proposition might be avoided by restricting to subgroups whose index is coprime to p. The following refinement shows that this is not the case.

Proposition 5.3. *Let p be a prime. There exist finitely presented metabelian groups G such that*

$$\sup\{\dim_{F_p} H_1(S, F_p) \mid S \in \mathscr{A}_p\} = \infty,$$

where

$$\mathscr{A}_p = \{S \leq G \mid [G : S] \text{ is finite and coprime to } p\}.$$

Proof. Let $A = F_p[x, x^{-1}, (x+1)^{-1}]$ and let Q be a free abelian group of rank 2 whose generators x_1, x_2 act on A as multiplication by x and $x+1$, respectively. We consider the group $G = A \rtimes Q$. As an $F_p[Q]$-module, $A \cong F_p[Q]/I$ where I is the ideal generated by $x_2 - x_1 - 1$, and the argument given in the preceding proposition shows that $\Sigma_A(Q)^c$ consists of precisely 3 points, no pair of which is antipodal. Therefore, G is finitely presented.

Let F be a finite field with p^r elements, $r \geq 2$. Let w be a generator of the multiplicative group $F^* = F \setminus \{0\}$. Let Q_r be the kernel of the homomorphism $Q \to F^*$ defined by $x_1 \mapsto w$ and $x_2 \mapsto w+1$. Let $G_r = A \rtimes Q_r$ and note that $|G/G_r| = |Q/Q_r| = p^r - 1$ is coprime to p.

The ring epimorphism $A \to F$ sending x to w provides an epimorphism of the underlying additive groups which extends to a group epimorphism $A \rtimes Q_r \to F \times \mathbb{Z}^2$.

Since $\dim_{F_p} F = r$, it follows that $\dim_{F_p} H_1(G_r, F_p) \geq r + 2$. And, $r \geq 2$ was arbitrary. □

6. Beyond the metabelian case

In this section we shall prove Theorem A, but first we present a consequence of Theorem 4.3 that describes what one can deduce about towers of finite-index subgroups above the commutator subgroup in amenable and related groups.

Proposition 6.1. *Let G be a group of type* FP_2 *that does not contain a nonabelian free group and let $\mathscr{C}$ be the set of finite-index subgroups in G that contain the commutator subgroup G'. Then*, $\sup_{M\in\mathscr{C}} \dim_{\mathbb{Q}} H_1(M, \mathbb{Q}) < \infty$.

Proof. By [Bieri and Strebel 1980, Theorem 5.5], G/G'' is finitely presented. Since $M \supseteq G'$, we have $M' \supseteq G''$ and can replace G by G/G'' and M by MG''/G'' without changing $H_1(M, \mathbb{Q})$. Then we can apply Theorem 4.3. □

Our proof of Theorem A relies on the following proposition, which is of interest in its own right.

Proposition 6.2. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups*, *where N is nilpotent*, *Q is abelian and G is finitely generated. Let G_n be a subgroup of finite index in G and let $\overline{G}_n$ be the image of G_n in the metabelian group G/N'. Then*,

$$\dim_{\mathbb{Q}} H_1(G_n, \mathbb{Q}) = \dim_{\mathbb{Q}} H_1(\overline{G}_n, \mathbb{Q}).$$

Proof. We argue using the Malcev completion $j_N : N \to N^*$ [Malcev 1949]. According to [Quillen 1969, Appendix A, Corollary 3.8], for any nilpotent group N, the homomorphism $j_N : N \to N^*$ is characterized up to isomorphism by the following properties:

(a) N^* is nilpotent and uniquely divisible.

(b) $\ker j_N$ is the torsion subgroup of N.

(c) For every $x \in N^*$, there is a positive integer n such that $x^n \in N$.

In any nilpotent group, the set $\sqrt{S}$ of elements that have powers in a fixed subgroup S is a subgroup. It follows that, for every subgroup $M < N$, the map $M \to \sqrt{j_N(M)}$ satisfies properties (a) to (c). Thus we may identify M^* with $\sqrt{j_N(M)} < N^*$. If $M < N$ has finite index, then $M^* = \sqrt{j_N(M)} = N^*$. And $(N^*)' = (N')^*$.

With these facts in hand, for all subgroups of finite index $G_n < G$ we have $(G_n')^* \supseteq ((G_n \cap N)')^* = ((G_n \cap N)^*)' = (N^*)' = (N')^*$. Thus $(G_n' N')^* = (G_n')^*$, and from (c) we deduce that $G_n'(N' \cap G_n)/G_n'$ is torsion. As $G_n'(N' \cap G_n)/G_n'$ is the kernel of the canonical epimorphism $G_n/G_n' \to G_n N'/G_n' N'$, we have

$$H_1(G_n, \mathbb{Q}) \cong (G_n/G_n') \otimes \mathbb{Q} \cong (G_n N'/G_n' N') \otimes \mathbb{Q} \cong H_1(\overline{G}_n, \mathbb{Q}). \qquad \square$$

Theorem 6.3. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. If N is nilpotent, Q is abelian and G is of type* FP_2, *then the virtual first betti number* $\mathrm{vb}_1(G)$ *is finite.*

Proof. In the light of Proposition 6.2, this follows directly from Theorem 4.3 and the fact [Bieri and Strebel 1980, Theorem 5.5] that G/N' is a finitely presented metabelian group. □

Corollary 6.4 (Theorem A). *If a group G is nilpotent-by-abelian-by-finite and of type* FP_2, *then* $\mathrm{vb}_1(G)$ *is finite.*

Proof. Let G_0 be a subgroup of finite index in G such that G_0 is nilpotent-by-abelian. Then, G_0 has type FP_2, so $\mathrm{vb}_1(G_0)$ is finite, by Theorem 6.3, and hence, so is G, by Lemma 4.1. □

Remark 6.5. We did not use the full force of finite presentability in establishing Theorem A: in fact, it is enough to assume that G has a subgroup of finite index G_0 in which there is a nilpotent subgroup $N \lhd G_0$ such that $Q = G_0/N$ is free abelian and, writing $A = N/N'$, the $\mathbb{Q}Q$-module $A \otimes A \otimes \mathbb{Q}$, with diagonal action, should be finitely generated. These requirements follow from the finite presentability of G_0/N' but are strictly weaker.

Corollary 6.6. *Every soluble group of type* FP_∞ *has finite virtual first betti number.*

Proof. Soluble groups S of type FP_∞ are constructible and hence nilpotent-by-abelian-by-finite [Kropholler 1986]. □

Corollary 6.7. *Every abelian-by-polycyclic group of type* FP_3 *has finite virtual first betti number.*

Proof. By the main result of [Groves et al. 2008], abelian-by-polycyclic groups of type FP_3 are nilpotent-by-abelian-by-finite. □

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