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SURFACES IN $\mathbb{R}^3_+$ WITH THE SAME GAUSSIAN CURVATURE INDUCED BY THE EUCLIDEAN AND HYPERBOLIC METRICS

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We show how to construct infinitely many immersions into the upper half-space such that the Gaussian curvatures induced from the ambient Euclidean and hyperbolic metrics coincide. We show how these immersions are related geometrically to classical minimal surfaces in Euclidean space and timelike minimal surfaces in Minkowski space.

1. Introduction

The typical scenario in problems about the geometry of submanifolds is usually given by a Riemannian manifold M and the search for a submanifold $S \subset M$ with some special geometric property with respect to the Riemannian ambient metric on M.

In the present work we will treat a problem that generalizes the above setting in the following sense. Instead of just one Riemannian metric on M, we will consider a pair of such metrics, say g_1 and g_2, and look for a submanifold $S \subset M$ with a special property that depends on both metrics g_1 and g_2. Of course, if the metrics g_1 and g_2 are arbitrary, it is hard to imagine that an interesting question will show up from this setup. However, assuming that g_1 and g_2 are in the same conformal class of metrics, we believe that there is a fertile and still unexplored field waiting for geometers. In this spirit, we will consider here an example that shows how it is possible to have a nice interplay between the geometric aspects induced by the two metrics g_1 and g_2. Specifically, we will treat the problem below.

Let S be an immersed surface in $\mathbb{R}^3_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$, and let ds_e^2 and ds_h^2 be the Euclidean and hyperbolic metrics on $\mathbb{R}^3_+$, respectively, given by

$$ds_e^2 = dx_1^2 + dx_2^2 + dx_3^2 \quad \text{and} \quad ds_h^2 = \frac{ds_e^2}{x_3^2}.$$

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We will denote by K_e and K_h the Gaussian curvatures of the metrics on S induced by ds_e^2 and ds_h^2, respectively.

Problem. Find surfaces immersed in $\mathbb{R}^3_+$ such that $K_h = K_e$.

To simplify our exposition, these surfaces will be called *isocurved* surfaces. To get a naive feeling of the problem, locally one can represent an isocurved surface as a graph $(u, v, \varphi(u, v))$, with $\varphi(u, v) > 0$ defined in a domain contained in $\partial\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$. The function φ must then be a solution of a Monge–Ampère equation; see (2-2).

This is a mixed type equation, and — according to the type of Monge–Ampère solution associated to it — we divide isocurved surfaces into three classes: elliptic, parabolic, and hyperbolic.

As often happens in the field of differential geometry of surfaces, it is very hard to find explicit solutions of PDEs describing geometric objects and clearly some kind of geometric method is necessary to find nontrivial examples of isocurved surfaces.

Our strategy to attack the problem is based on a surprising geometric relation between isocurved surfaces and minimal surfaces in $\mathbb{R}^3$ (for the elliptic case) and timelike minimal surfaces in Minkowski space $\mathbb{L}^3$ (for the hyperbolic case). This geometric relation, together with the well-known machinery to generate minimal surfaces, provides a relatively simple and efficient method to generate isocurved surfaces.

Our geometric construction stems from the curious fact that isocurved surfaces appear in a one parameter family of parallel surfaces in the sense of hyperbolic geometry. It is therefore natural to seek a method that enables one to consider this one parameter family of isocurved surfaces as a whole. This is achieved by looking at the congruence of geodesics of hyperbolic geometry that have isocurved surfaces as orthogonal surfaces.

We will present a technique to construct such congruences of geodesics by starting with an appropriate simply connected minimal surface. This technique yields an explicit constructive method to generate isocurved surfaces that are either elliptic or hyperbolic from minimal surfaces in $\mathbb{R}^3$ or $\mathbb{L}^3$.

We have organized this paper as follows. In Section 2 we present the simplest examples of isocurved surfaces and the PDE for isocurved graphs. We also show that isocurved surfaces appear in a one parameter family of parallel surfaces with respect to ds_h^2. In Section 3 we show how elliptic and hyperbolic isocurved surfaces can be constructed from minimal surfaces. We offer a detailed discussion about the elliptic case in Section 3A, and apply the theory to construct many examples in Section 3B. In Sections 3C and 3D we deal with the case of hyperbolic isocurved surfaces and give some examples. Section 4 is devoted to some final remarks about isocurved surfaces.

2. Basics about isocurved surfaces

2A. ***Simple examples.*** Throughout this work we will use (x_1, x_2, x_3) as coordinates in $\mathbb{R}^3$. We will also adopt the following terminology. A geodesic of hyperbolic geometry will be called an h-geodesic. We recall that in the upper half-space model ($\mathbb{R}^3_+$) these geodesics are represented either by circles orthogonal to $\partial\mathbb{R}^3_+$ or by vertical lines. Accordingly, the parallel surfaces in hyperbolic geometry will be called h-parallel surfaces.

We now start our study of isocurved surfaces with a description of the simplest examples. The trivial example is a horizontal plane $x_3 = \text{const.} > 0$ (horosphere in hyperbolic geometry), since $K_e = K_h = 0$ for any such plane. In fact, any surface that is flat with respect to the Euclidean and hyperbolic metrics is isocurved. For instance, if we consider the part of vertical right circular cone with vertex at $\partial\mathbb{R}^3_+$ that lies in the region $x_3 > 0$, it is well-known that $K_e = K_h = 0$ for this surface. A not so obvious example of this type is provided by the surface with horizontal rulings orthogonal to a tractrix contained in a vertical plane and asymptotic to $\partial\mathbb{R}^3_+$; see Figure 1.

An example that is not flat is simply a (round) sphere properly placed in $\mathbb{R}^3_+$. To see that this is true, consider any sphere in $\mathbb{R}^3_+$. Of course, K_e is invariant with respect to vertical Euclidean translations, but such motion clearly changes K_h. Note that for a given sphere with fixed Euclidean radius, if we vertical translate upwards, then K_h increases without limit as we go up and if we translate downwards then K_h tends to zero as we approach $\partial\mathbb{R}^3_+$. So, by continuity, there is a specific placement of the sphere such that $K_e = K_h$.

A particular feature of all these examples is the following: if S is one of the examples cited above, then the h-parallel surfaces are also examples.

This is clear for the horizontal planes and the circular cones, and a simple computation shows that it is also true for spheres and the ruled example. In fact, as we shall see in Theorem 2.2, this property is valid for any isocurved surface.

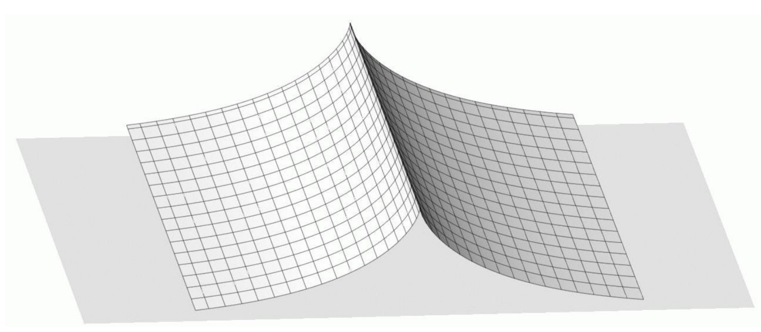

Figure 1. Flat ruled surface generated by a tractrix.

2B. ***The PDE for isocurved graphs.*** We will now present a PDE that is related to isocurved surfaces. The straightforward approach we adopt here is to consider an isocurved surface that is the graph of some function defined in a domain in $\partial\mathbb{R}^3_+$ and study the corresponding PDE that has φ as a solution.

To derive this PDE, we start by recalling that K_h and K_e are related by

$$K_h = x_3^2 K_e + 2H_e x_3 n_3 + n_3^2 - 1, \tag{2-1}$$

where x_3 is the third coordinate in $\mathbb{R}^3_+$, n_3 is the third coordinate of a unit normal vector in the Euclidean sense, and H_e is the mean curvature with respect to the Euclidean metric. Equation (2-1) follows easily from the well-known expression relating the principal curvatures of an immersed surface in $\mathbb{R}^3_+$ with respect to ds_e^2 and ds_h^2 [López 2001] and the Gauss equation

$$K_{\text{ext}} = K_h + 1,$$

relating K_h to the extrinsic curvature K_{ext} of a surface induced by the ambient metric ds_h^2.

Using the well-known expressions for K_e, H_e, n_3 for a graph $(u, v, \varphi(u, v))$ over a domain in the boundary of $\mathbb{R}^3_+$, we conclude that φ is a solution of the PDE:

$$(1-\varphi^2)\frac{\det \nabla^2\varphi}{1+|\nabla\varphi|^2} - \varphi\frac{(1+\varphi_v^2)\varphi_{uu} - 2\varphi_u\varphi_v\varphi_{uv} + (1+\varphi_u^2)\varphi_{vv}}{1+|\nabla\varphi|^2} + |\nabla\varphi|^2 = 0, \tag{2-2}$$

where $\det \nabla^2\varphi = \varphi_{uu}\varphi_{vv} - \varphi_{uv}^2$ and $|\nabla\varphi|^2 = \varphi_u^2 + \varphi_v^2$.

Equation (2-2) is a Monge–Ampère mixed type PDE. It turns out that the analytical classification of such equations can be interpreted geometrically, and the method to construct isocurved surfaces depends on the type of the equation. It is then appropriate to recall the classification of the solutions of this class of PDEs. A nice discussion about Monge–Ampère equations from a geometric point of view can be found in [Ivey and Landsberg 2003].

Consider a PDE of the form

$$A(\varphi_{uu}\varphi_{vv} - \varphi_{uv}) + B\varphi_{uu} + 2C\varphi_{uv} + D\varphi_{vv} + E = 0, \tag{2-3}$$

where A, B, C, D, and E are functions of u, v, φ, φ_u, and φ_v, and define the quantity $\Delta = AE - BD + C^2$. Equation (2-3) (or its solutions for some authors) is called *elliptic* if $\Delta < 0$, *hyperbolic* if $\Delta > 0$, and *parabolic* if $\Delta = 0$.

The proposition below shows that the criteria for deciding whether (2-1) is elliptic, parabolic, or hyperbolic admits a geometric interpretation.

Proposition 2.1. *Let S be an isocurved surface that is the graph of a function $\varphi(u, v)$ defined in a domain of the plane $x_3 = 0$. For points $(u, v, \varphi(u, v))$ such that the normal vector to S is not vertical, i.e., $|\nabla\varphi| \neq 0$, let $\rho(u, v)$ be the Euclidean radius of the circle orthogonal to $x_3 = 0$ that represents the geodesic in hyperbolic* *geometry passing through* $(u, v, \varphi(u, v))$ *and orthogonal to* S *at this point. Then the solution* φ *of* (2-2) *is*

$$\begin{aligned} \textit{elliptic} &\iff \textit{either } \rho > 1 \textit{ or } |\nabla\varphi| = 0,\\ \textit{parabolic} &\iff \rho = 0,\\ \textit{hyperbolic} &\iff \rho < 1. \end{aligned}$$

Proof. Writing (2-2) in the form (2-3) yields $A = (1 - \varphi^2)$, $B = -\varphi(1 + \varphi_v)$, $C = \varphi\varphi_v\varphi_u$, $D = -\varphi(1+\varphi_u)$, and $E = (1+\varphi_u^2+\varphi_v^2)(\varphi_u^2+\varphi_v^2)$. Direct computation shows that

$$\Delta = -\big(1 + |\nabla\varphi|^2\big)\big(\varphi^2(1 + |\nabla\varphi|^2) - |\nabla\varphi|^2\big).$$

Thus φ is elliptic at points where $|\nabla\varphi| = 0$. For points such that $|\nabla\varphi| \neq 0$, an elementary computation shows that ρ is given by

$$\rho = \frac{\varphi\sqrt{1 + |\nabla\varphi|^2}}{|\nabla\varphi|}.$$

To finish the proof just use the above expressions for ρ and Δ. □

2C. ***An invariance property of isocurved surfaces.*** Even though the Euclidean and hyperbolic metrics enter on equal footing in the definition of isocurved surfaces, the following property suggests that these surfaces might have some alternative geometric description only in terms of hyperbolic geometry.

Theorem 2.2. *Let* S *be an isocurved surface and* S^t *be the h-parallel surface at distance* t. *If* S^t *is smooth, then it is also an isocurved surface.*

Proof. The equation that defines an isocurved surface can be rewritten as

$$K_{\text{ext}}(1 - x_3^2) - 2H_h n_3 + n_3^2 + x_3^2 = 0, \tag{2-4}$$

where K_{ext} and H_h denote the extrinsic and mean curvatures with respect to the hyperbolic metric, respectively. So, we have to prove that if (2-4) holds for S, then

$$K^t_{\text{ext}}(1 - (x^t_3)^2) - 2H^t_h n^t_3 + (n^t_3)^2 + (x^t_3)^2 = 0, \tag{2-5}$$

holds for S^t, where the quantities with upper index t are relative to S^t. To do this, it is convenient to start by writing S and S^t in the hyperboloid model to obtain manageable expressions in the upper half-space model.

Let $P = (P_1, P_2, P_3, P_4)$ and $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ be the vector position and unit normal field of S in the hyperboloid model of $\mathbb{H}^3$. Then the parallel surface S^t in this model has vector position and unit normal field given by

$$P^t = \cosh t P + \sinh t\eta, \quad \eta^t = \sinh t P + \cosh t\eta.$$

Now, consider the map

$$\Phi(u_1,u_2,u_3,u_4)=\Big(\frac{u_2}{u_1-u_4},\frac{u_3}{u_1-u_4},\frac{1}{u_1-u_4}\Big),$$

that maps the upper sheet of the hyperboloid $-u_1^2+u_2^2+u_3^2+u_4^4=-1$ onto the upper half-space. After some computations we obtain the following expressions for the third coordinates of the position and Euclidean unit normal vector of S and S^t:

$$x_3=\frac{1}{P_1-P_4}, \qquad n_3=-\frac{\eta_1-\eta_4}{P_1-P_4}, \tag{2-6}$$

$$x_3^t=\frac{1}{c(P_1-P_4)+s(\eta_1-\eta_4)}, \qquad n_3^t=-\frac{s(P_1-P_4)+c(\eta_1-\eta_4)}{c(P_1-P_4)+s(\eta_1-\eta_4)}, \tag{2-7}$$

where $c=\cosh t$ and $s=\sinh t$.

We also need expressions for K_{ext}^t and H_h^t as functions of K_{ext} and H_h:

$$K_{\text{ext}}^t=\frac{c^2K_{\text{ext}}-2H_hsc+s^2}{s^2K_{\text{ext}}-2H_hsc+c^2}, \tag{2-8}$$

$$H_h^t=\frac{(c^2+s^2)H_h-sc(K_{\text{ext}}+1)}{s^2K_{\text{ext}}-2H_hsc+c^2}. \tag{2-9}$$

See (2-8) and (2-9) for more details about [Tenenblat 1998, Proposition 3.2, page 24].

Now, using (2-6), Equation (2-4) is equivalent to

$$K_{\text{ext}}((P_1-P_4)^2-1)+2H_h(P_1-P_4)(\eta_1-\eta_4)+(\eta_1-\eta_4)^2+1=0. \tag{2-10}$$

Using (2-7), (2-8), and (2-9), and performing some computations, we conclude that the left hand side of (2-5) vanishes if and only if

$$\begin{aligned}K_{\text{ext}}\big[(P_1-P_4)^2(c^2-s^2)^2-c^2+s^2\big]&\\ +2H_h(P_1-P_4)(\eta_1-\eta_4)(c^2-s^2)^2&\\ +(c^2-s^2)\big[(c^2-s^2)(\eta_1-\eta_4)^2+1\big]&=0.\end{aligned}$$

Using the fact that $c^2-s^2=1$, we see that the expression above coincides with (2-10) and so S^t is isocurved. □

3. Isocurved surfaces from minimal surfaces

We are now in position to present a geometric method that allows us to construct infinitely many nontrivial examples of isocurved surfaces. Actually, our method of construction works only for elliptic and hyperbolic isocurved surfaces and we don't have a general method to construct examples of parabolic isocurved surfaces.

The basic ingredient of our method is a simply connected minimal surface in $\mathbb{R}^3$ (for elliptic isocurved surfaces) or simply connected timelike minimal surface in $\mathbb{L}^3$ (for hyperbolic isocurved surfaces). For both cases, starting with a minimal surface, we will show how to construct a congruence of geodesics of hyperbolic space (h-geodesics) that has isocurved surfaces as orthogonal surfaces. Since there are slight variations between the two cases, we will discuss them separately.

3A. ***Elliptic isocurved surfaces.*** We first introduce a process to induce from a given simply connected immersed oriented surface Σ in $\mathbb{R}^3$ a congruence of h-geodesics C_Σ whose elements are represented either by circles orthogonal to the plane $x_3 = 0$ or by vertical lines in $\mathbb{R}^3_+$.

A circle orthogonal to the plane $x_3 = 0$ is determined by its (Euclidean) center σ that lies in the plane, its radius R, and a horizontal unit vector e_1 in the Euclidean sense that together with $e_3 = (0, 0, 1)$ defines the vertical plane where the circle lies. The case of vertical lines can be treated as a limiting case.

For simplicity, we will consider only the h-geodesics that are circles. In other words, from now on we will assume that for every $p \in \Sigma$, the tangent plane $T_p\Sigma$ is not horizontal. From the surface Σ with unit normal vector field N, we will define the congruence of h-geodesics as follows.

Let P_{hor} be the orthogonal projection onto the horizontal plane $x_3 = 0$, let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the $\frac{\pi}{2}$ counterclockwise rotation in this plane, and let $N = (n_1, n_2, n_3)$ be a unit vector field normal to Σ.

For $p \in \Sigma$ we define the center of the circle $\sigma(p)$ as

$$\sigma(p) = P_{\text{hor}}(p), \tag{3-1}$$

the direction $e_1(p)$ as

$$e_1(p) = \frac{J\big(P_{\text{hor}}(N(p))\big)}{\big|P_{\text{hor}}(N(p))\big|}, \tag{3-2}$$

and the radius $R(p)$ as

$$R(p) = \frac{1}{\big|P_{\text{hor}}(N(p))\big|}. \tag{3-3}$$

We now pose the question: what is the condition on the surface Σ such that the congruence of h-geodesics C_Σ admits orthogonal surfaces?

If we knew in advance that C_Σ defines a distribution of planes (the planes of the distribution being the ones orthogonal to the h-geodesics), then we could use the geometric version of the Frobenius theorem to verify the condition for the existence of integral surfaces to this distribution.

However, even though the statements that follow become a bit clumsy, it is interesting to consider the general situation where C_Σ does not necessarily define a distribution of planes. As we shall see, this procedure allows the consideration of self-intersections and singularities for isocurved surfaces in a natural way.

So we ask a weaker question, namely, about the existence of a differentiable map $Y : \Sigma \to \mathbb{R}^3_+$ defined by

$$Y = \sigma + R(\cos\theta\, e_1 + \sin\theta\, e_3), \tag{3-4}$$

where σ, e_1, and R are defined by (3-1), (3-2), and (3-3), respectively, and $\theta : \Sigma \to \mathbb{R}$ is an unknown differentiable function such that, for any point $p \in \Sigma$ where Y is an immersion, the tangent plane of Y at $Y(p)$ is orthogonal to the geodesic of C_Σ associated to p. If Y satisfies the condition above we say that Y has the *orthogonal property.*

The geometric condition above can be rephrased in terms of a system of Frobenius type for the unknown function θ. The next proposition shows that, up to a degenerate case, the function θ exists if and only if Σ is a minimal surface.

Theorem 3.1. *Let Σ be an oriented simply connected surface in the upper half-plane such that $T_p\Sigma$ is not horizontal for every $p \in \Sigma$. The function θ that appears in the expression of the map Y given by* (3-4) *and such that Y has the orthogonal property exists if and only if Σ is either a minimal surface or vertical cylinder over a planar curve in the plane $\partial\mathbb{R}^3_+$.*

Proof. First suppose that Σ is the graph of a function ψ defined in a domain $\Omega \subset \partial\mathbb{R}^3_+$. A local chart for Σ is given by

$$X(u, v) = (u, v, \psi(u, v)).$$

The function θ can be written in these local coordinates as a function $\theta : \Omega \to \mathbb{R}$. Since we are assuming that Y has the orthogonal property, we have

$$\langle dY, -\sin\theta e_1 + \cos\theta e_3\rangle = 0,$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the usual inner product of Euclidean space.

Writing $\eta = -\sin\theta\, e_1 + \cos\theta\, e_3$, we find

$$\begin{aligned} 0 &= \langle Y_u, \eta\rangle \\ &= \langle \sigma_u + R_u(\cos\theta\, e_1 + \sin\theta\, e_3) + R(\cos\theta\, e_1 + \sin\theta\, e_3)_u, \eta\rangle \\ &= \langle \sigma_u + (R_u\cos\theta - R\theta_u\sin\theta)e_1 + R\cos\theta(e_1)_u + R\theta_u\cos\theta e_3, \eta\rangle \\ &= R\theta_u - \sin\theta\langle\sigma_u, e_1\rangle, \end{aligned}$$

and in the same fashion,

$$0 = \langle Y_v, \eta\rangle = R\theta_v - \sin\theta\langle\sigma_v, e_1\rangle.$$

We conclude that

$$\theta_u = \frac{\sin\theta}{R}\langle\sigma_u, e_1\rangle, \quad \theta_v = \frac{\sin\theta}{R}\langle\sigma_v, e_1\rangle. \tag{3-5}$$

The change of variables $\sin\theta = 1/\cosh\beta$ and $\cos\theta = \tanh\beta$, transforms (3-5) into

$$\beta_u = -\frac{\langle \sigma_u, e_1\rangle}{R}, \quad \beta_v = -\frac{\langle \sigma_v, e_1\rangle}{R}, \tag{3-6}$$

and the Frobenius condition for the integrability of the system is

$$\left(\frac{\langle \sigma_u, e_1\rangle}{R}\right)_v = \left(\frac{\langle \sigma_v, e_1\rangle}{R}\right)_u. \tag{3-7}$$

Since

$$\sigma(u, v) = (u, v, 0), \tag{3-8}$$

$$R = \frac{\sqrt{1 + |\nabla\psi|^2}}{|\nabla\psi|}, \tag{3-9}$$

$$e_1 = \frac{(-\psi_v, \psi_u)}{|\nabla\psi|}, \tag{3-10}$$

Equation (3-7) implies that

$$(1+\psi_v^2)\psi_{uu} - 2\psi_u\psi_v\psi_{uv} + (1+\psi_v^2)\psi_{uu} = 0,$$

that is, $\theta(u, v)$ exists if and only if Σ is a minimal surface.

Given a general parametrization for Σ, a long but straightforward computation shows that the integrability condition for θ is always satisfied for points where the normal vector N is horizontal, and — as we have seen — if N is not horizontal, then the mean curvature must vanish. So there two possibilities for Σ: either it is a minimal surface or it is a vertical cylinder over a plane curve in $\partial\mathbb{R}^3_+$. □

Remark 3.2. If Σ in our construction is a vertical cylinder over a planar curve, then it is easy to check that map Y is not an immersion anywhere. On the other hand, as we shall see later on, if Σ is a minimal surface, we cannot guarantee in general that the Y is an immersion everywhere.

From Theorem 3.1, we may start from a minimal surface Σ and construct C_Σ that admits orthogonal surfaces. But it is not clear if these orthogonal surfaces have any geometric property. We will now show that, if the map Y associated to C_Σ is an immersion, then $Y(\Sigma)$ is an isocurved surface.

To prove this, we will start with an arbitrary immersed oriented surface S in $\mathbb{R}^3_+$ and find the conditions on S such that S is orthogonal to the h-geodesics of C_Σ induced by some surface Σ via our geometric method. This leads us to our next result.

Theorem 3.3. *Let S be an immersed surface in $\mathbb{R}^3_+$. If S is orthogonal to the h-geodesics of the congruence C_Σ induced by an oriented immersed surface Σ, then S is an isocurved surface.*

Proof. Let $p \in S$. Without loss of generality, we can assume that T_pS is not vertical. In fact, if we had T_pS vertical, then we could replace S with S_t, i.e., the parallel surface to S at distance t with respect to the hyperbolic metric. For t small enough, S_t is also an immersed surface and the tangent plane $T_{p_t}S_t$ is not vertical. For the following argument, we will also assume that T_pS is not horizontal, and we will treat this situation as a limiting case at the end of the proof.

In a neighborhood of p, the surface S is the graph of a function φ defined in a domain Ω in the plane $x_3 = 0$. We will use u and v as coordinates and write this graph as

$$Y(u, v) = (u, v, \varphi(u, v)).$$

Now we reverse the steps in our geometric construction that induces a congruence of h-geodesics from a given oriented surface. In other words, we will compute the center σ, the radius R, and the direction e_1 for the family of h-geodesics that are orthogonal to S.

The direction e_1 is the projection of the normal vector of S in the plane $x_3 = 0$:

$$e_1 = \frac{(-\varphi_u, -\varphi_v)}{|\nabla\varphi|}.$$

To find the center σ we proceed as follows. The normal field to S is given by

$$\eta = (-\varphi_u, -\varphi_v, 1) = |\nabla\varphi| e_1 + e_3.$$

Let $\tilde{J} : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation of $\frac{\pi}{2}$ radians in the plane spanned by e_1 and e_3. Consider the line starting from $(u, v, \varphi(u, v))$ in the direction of $\tilde{J}\eta$, parametrized by $\ell(t) = Y + t\tilde{J}\eta$. This line intersects $\partial\mathbb{R}^3_+$ at σ, so we obtain

$$\sigma(u, v) = \left(u - \frac{\varphi\varphi_u}{|\nabla\varphi|^2}, v - \frac{\varphi\varphi_v}{|\nabla\varphi|^2}, 0\right). \tag{3-11}$$

Finally, the radius R is the Euclidean distance from σ to $Y(u, v)$, that is,

$$R = \frac{\varphi\sqrt{1 + |\nabla\varphi|^2}}{|\nabla\varphi|}. \tag{3-12}$$

If S is orthogonal to the h-geodesics of C_Σ, then there is a differentiable function ψ defined in Ω such that Σ is locally parametrized by the map X given by

$$X = \sigma + \psi e_3. \tag{3-13}$$

Let N be the normal field of Σ and θ be the angle between N and the horizontal plane. From our geometric construction, we have that the projection of N into the horizontal plane is $-Je_1$ and $R = (\cos\theta)^{-1}$. Thus,

$$N = \cos\theta(-Je_1) + \sin\theta e_3 = -\frac{1}{R}Je_1 + \frac{\sqrt{R^2-1}}{R}e_3.$$

The orthogonality condition $\langle dX, N\rangle = 0$ yields the following system for ψ:

$$\psi_u = \frac{\langle Je_1, \sigma_u\rangle}{\sqrt{R^2-1}}, \quad \psi_v = \frac{\langle Je_1, \sigma_v\rangle}{\sqrt{R^2-1}}. \tag{3-14}$$

The Frobenius integrability condition for this system is given by

$$\left(\frac{\langle Je_1, \sigma_u\rangle}{\sqrt{R^2-1}}\right)_v - \left(\frac{\langle Je_1, \sigma_v\rangle}{\sqrt{R^2-1}}\right)_u = 0. \tag{3-15}$$

Using equations (3-2), (3-11), (3-12), and performing some computations, we obtain

$$\frac{\langle Je_1, \sigma_u\rangle}{\sqrt{R^2-1}} = -\frac{\varphi_v\varphi_u^2 + \varphi\varphi_u\varphi_{uv} - \varphi\varphi_v\varphi_{uu} + \varphi_v^3}{\sqrt{\varphi^2(1+|\nabla\varphi|^2) - |\nabla\varphi|^2|\nabla\varphi|^2}}, \tag{3-16}$$

$$\frac{\langle Je_1, \sigma_v\rangle}{\sqrt{R^2-1}} = \frac{\varphi_u\varphi_v^2 + \varphi\varphi_v\varphi_{uv} - \varphi\varphi_u\varphi_{vv} + \varphi_u^3}{\sqrt{\varphi^2(1+|\nabla\varphi|^2) - |\nabla\varphi|^2|\nabla\varphi|^2}}. \tag{3-17}$$

After the substitution of the expressions given by (3-16) and (3-17) into (3-15) and some straightforward computations, we see that (3-15) is in fact equivalent to (2-2). So, the function ψ exists if and only if S is an isocurved elliptic surface.

To finish the proof, recall that we have assumed that $p \in S$ was such that T_pS was not horizontal. Suppose now that T_pS is horizontal, then either there is a neighborhood U of p in S such that the tangent plane is horizontal for any point of U, or there is a sequence of points $\{p_n\}$ of S converging to p such that $T_{p_n}S$ is not horizontal. For the first situation U is part of a horizontal plane, so it is an isocurved surface. For the second case, since the isocurved condition is satisfied for all the points in $\{p_n\}$, by continuity, it must also be satisfied at p, so S is an isocurved surface. □

Remark 3.4. The proof of Theorem 3.3 shows that if we start with an elliptic isocurved surface, then the map (3-13) is well defined but it is not necessarily an immersion. In the case where X is an immersion, it is also minimal due to Theorem 3.1.

A careful analysis shows that if we define

$$g_{11} = \langle X_u, X_u\rangle, \quad g_{12} = \langle X_u, X_v\rangle, \quad g_{22} = \langle X_v, X_v\rangle,$$
$$h_{11} = \langle X_{uu}, N\rangle, \quad h_{12} = \langle X_{uv}, N\rangle, \quad h_{22} = \langle X_{vv}, N\rangle,$$

then

$$g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11} = 0.$$

In other words, if we start with an elliptic isocurved surface S, we can always locally associate to S a *generalized* minimal surface in the following sense: we have the two maps X and N satisfying the equation above, but not necessarily $g_{11}g_{22} - g_{12}g_{21} > 0$. We will exhibit an example of this situation in the next subsection.

3B. ***Examples of elliptic isocurved surfaces.*** We now wish to apply the relation between minimal and isocurved surfaces to generate explicit examples of isocurved surfaces. Some apparent difficulties arise if we stick to the local analysis using graphs that we have adopted until now. Using graphs, we would have to start with an explicit minimal graph and then have the trouble to integrate the system (3-6). Since there aren't many explicit minimal graphs, our method would not be very useful to generate new examples.

Fortunately, the function β that appears in (3-6) has a nice geometric interpretation and this frees us from the graph representation. In fact, if we consider a simply connected minimal surface Σ, we can associate to it the so-called conjugate surface Σ^*. It turns out that β is the height function (i.e., x_3 coordinate) of Σ^*. This can be seen quite easily by using the expressions (3-8), (3-9), and (3-10) to rewrite (3-6) as follows:

$$\beta_u = \frac{\varphi_v}{\sqrt{1+|\nabla\varphi|^2}}, \quad \beta_v = -\frac{\varphi_u}{\sqrt{1+|\nabla\varphi|^2}}. \tag{3-18}$$

It is well-known that the system above coincides with the one for the conjugate height function; see, for instance, [Mazet et al. 2007].

Now that we don't need graphs anymore, we will generate examples with minimal surfaces constructed using the classic Weierstrass representation. For the examples below, let (f, g) be the Weierstrass data for the starting simply connected minimal surface Σ. The conjugate surface Σ^* is defined by the data (if, g).

Example (Rotational isocurved surfaces). Let $z \in \mathbb{C}$, $z = x + iy$ and consider $f(z) = i\,\mathrm{e}^z$ and $g(z) = c\,\mathrm{e}^{-z}$, where $c \in \mathbb{R}$, $c \neq 0$. The corresponding minimal surface is a helicoid, and the associated isocurved surface is a surface of revolution (with respect to the x_3-axis). For instance, choosing $c = 2$ yields the following parametrized surface.

$$X(x, y) = (\alpha(x)\sin y, \alpha(x)\cos y, \gamma(x)),$$

where

$$\alpha(x) = \frac{1}{4}\frac{(-\mathrm{e}^{5x} + 8\mathrm{e}^{3x} + 45\mathrm{e}^{x} - 8\mathrm{e}^{-x} + 16\mathrm{e}^{-3x})\sin y}{(4\mathrm{e}^{-2x}+1)(\mathrm{e}^{4x}+1)},$$

and

$$\gamma(x) = \frac{1}{2}\frac{\mathrm{e}^{3x}\sqrt{1+8\mathrm{e}^{-2x}+16\mathrm{e}^{-4x}}}{\mathrm{e}^{4x}+1}.$$

This surface is seen in Figure 2. Note that even though we have started with a smooth minimal surface, the corresponding isocurved surfaces has singularities.

Example (The isocurved surface associated to a point). If we choose $f(z) = 0$ and $g(z) = z$ and use the Weierstrass representation we don't get a minimal surface; the result is just the point $(0, 0, 0)$ together with the Gauss map defined by $g(z)$.

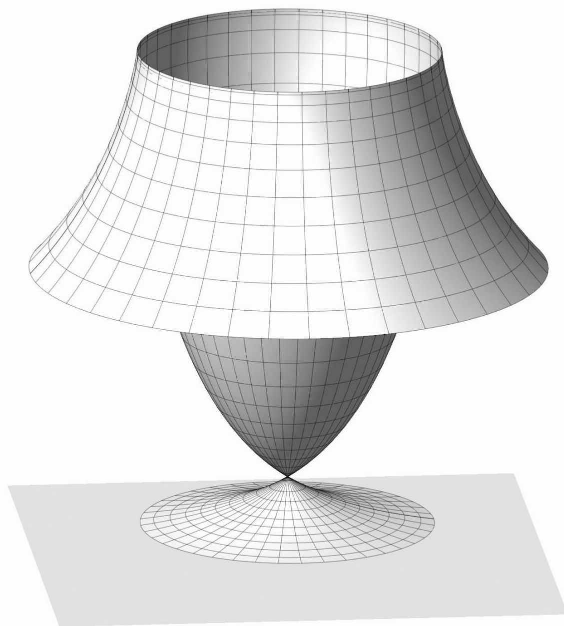

Figure 2. An isocurved surface of revolution.

However, this data, a surface in the space of contact elements of $\mathbb{R}^3$, is enough to use our machinery. For this simple example the associated isocurved surface is the elliptic region of the vertical circular right cone parametrized as

$$X(r,t) = \left(\frac{\tanh k(1+r^2)}{2r}\sin t,\ -\frac{\tanh k(1+r^2)}{2r}\cos t,\ \frac{1+r^2}{2r\cosh k}\right),$$

where $z = r\,\mathrm{e}^{it}$ and $k \in \mathbb{R}$ is a parameter used to describe the family of parallel surfaces in hyperbolic geometry.

Example (A 1-periodic isocurved surface). With $f(z) = z$ and $g(z) = 1/z$, the associated isocurved surface is parametrized by

$$X(r,t) = X_1(r,t) + X_2(r,t),$$

where

$$X_1(r,t) = \frac{(r^2+1)(\mathrm{e}^{2r\sin t}-1)}{2r(\mathrm{e}^{2r\sin t}+1)}(\sin t, \cos t, 0),$$

and

$$X_2(r,t) = \left(-\frac{2\ln r + r^2(1-2(\cos t)^2)}{4},\ -\frac{t + r^2\sin t\cos t}{2},\ \frac{(r^2+1)\mathrm{e}^{r\sin t}}{r(\mathrm{e}^{2r\sin t}+1)}\right).$$

This example is invariant under Euclidean translations in the x_2 direction by a multiple of π. A view of part of this surface is shown in Figure 3.

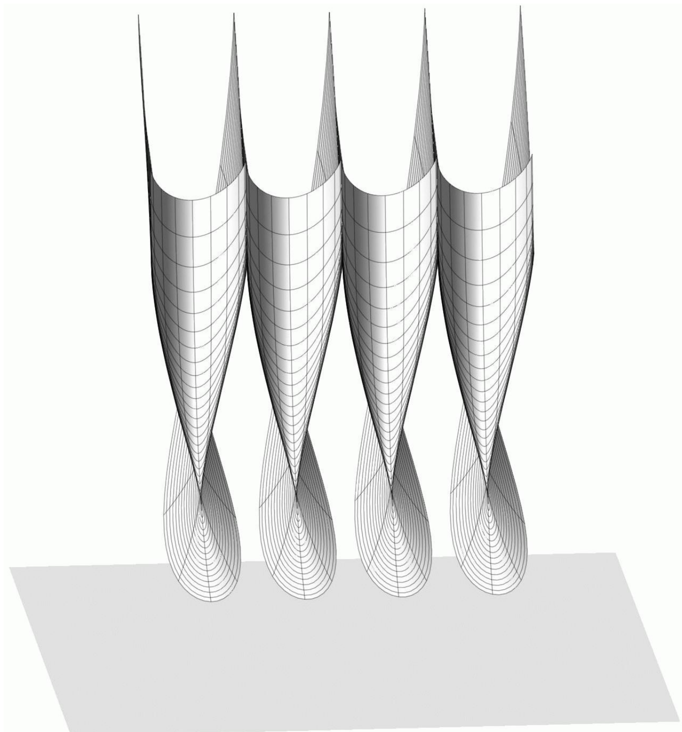

Figure 3. A piece of a 1-periodic isocurved surface.

Example (A Scherk-type isocurved surface). We end our list of examples showing the isocurved surface obtained from Scherk's minimal surface that can be written as the graph of the function

$$\varphi(x, y) = \ln \frac{\cos y}{\cos x}.$$

It is known — see [Nitsche 1989] — that the conjugate function in this case is

$$\varphi^*(x, y) = \arcsin(\sin x \ \sin y).$$

Using our geometric method we obtain the map

$$X(x, y) = (x - \Lambda_1 \sin y \cos x,\ y - \Lambda_1 \sin x \cos y,\ \Lambda_2),$$

where

$$\Lambda_1 = \frac{\sqrt{\cos^2 x + \cos^2 y - \cos^2 x \cos^2 y}\ \tanh(\arcsin(\sin x \ \sin y))}{\sin^2 x \cos^2 y + \sin^2 y \cos^2 x},$$

$$\Lambda_2 = \frac{\sqrt{\cos^2 x + \cos^2 y - \cos^2 x \cos^2 y}}{\cosh(\arcsin(\sin x \ \sin y))\sqrt{\cos^2 x + \cos^2 y - 2\cos^2 x \cos^2 y}}.$$

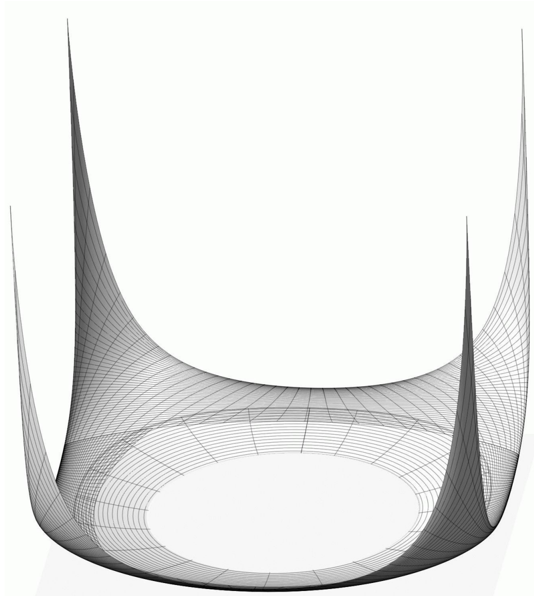

Figure 4. A 2-periodic isocurved surface obtained from Scherk's surface.

Extending this fundamental domain to the whole plane minus the lattice

$$\left\{\left(\tfrac{\pi}{2}+m\pi,\ \tfrac{\pi}{2}+n\pi\right)\in\mathbb{R}^2 \mid m,n\in\mathbb{Z}\right\}$$

in the obvious way, we obtain a 2-periodic surface, with singular curves that project into lines in the $x_3 = 0$ plane having the form $\{(m\pi/2, s) \mid m \in \mathbb{Z}, s \in \mathbb{R}\}$ or $\{(s, m\pi/2) \mid m \in \mathbb{Z}, s \in \mathbb{R}\}$ and, surprising, circular holes. Part of this surface is depicted in Figure 4.

3C. ***Hyperbolic isocurved surfaces.*** A slight variation of the geometric method used to construct elliptic isocurved surfaces from a minimal surface in $\mathbb{R}^3$, allows us to construct hyperbolic isocurved surfaces from timelike minimal surfaces in Minkowski space $\mathbb{L}^3$, that is, $\mathbb{R}^3$ with the Lorentzian metric $ds_L^2 = dx_1^2 + dx_2^2 - dx_3^2$.

For a given oriented timelike surface Σ immersed in $\mathbb{L}^3$ with unit normal N (in the Minkowski metric), we induce a congruence of h-geodesics, C_Σ in the following way. We define the center σ and the vertical plane containing the circle exactly as we did before. The only difference is the choice of the radius. Again, we consider the radius as the inverse of the size of the horizontal projection of N. But since now we are using the Minkowski metric, the size of this projection is bigger or equal to one. In this way, our radius function R will be smaller or equal to one.

Since the ideas and techniques are the same as the ones in Section 3A, we will limit ourselves to state the results without giving detailed proofs.

As before, we will search for a differentiable map $Y : \Sigma \to \mathbb{R}^3_+$ defined by

$$Y = \sigma + R(\cos\theta\, e_1 + \sin\theta\, e_3), \tag{3-19}$$

where σ, e_1, and R are defined, respectively, by (3-1), (3-2), and (3-3), where N is now a unit (spacelike) normal field to Σ with respect to the Minkowski metric, and $\theta : \Sigma \to \mathbb{R}$ is, as before, an unknown differentiable function such that, for any point $p \in \Sigma$ where Y is an immersion, the tangent plane of Y at $Y(p)$ is orthogonal to the h-geodesic of C_Σ associated to p. If Y satisfies the condition above, then we say that Y has the *orthogonal property*.

Our next theorem is analogous to Theorem 3.1.

Theorem 3.5. *Let Σ be an oriented surface in $\mathbb{L}^3$ with unit normal vector field N and such that $T_p\Sigma$ is not horizontal for every $p \in \Sigma$. The function θ that appears in the expression of the map Y given by* (3-19) *and such that Y has the orthogonal property exists if and only if Σ is either a minimal timelike surface or a vertical cylinder over a planar curve in the plane $x_3 = 0$.*

Proof. The proof follows exactly the same lines of the proof given for Theorem 3.1. The only difference is that if we write Σ locally as the graph of a function ψ, then the radius function is given by

$$R = \frac{\sqrt{|\nabla\psi|^2 - 1}}{|\nabla\psi|^2}.$$

The integrability condition associated to the existence of θ now becomes

$$(1 + \psi_v^2)\psi_{uu} - 2\psi_u\psi_v\psi_{uv} + (1 + \psi_v^2)\psi_{uu} = 0,$$

and this is the PDE associated to a minimal surface in $\mathbb{L}^3$. □

Now we state the theorem analogous to Theorem 3.3.

Theorem 3.6. *Let S be an immersed surface in $\mathbb{L}^3_+$. If S is orthogonal to the h-geodesics of the congruence C_Σ induced by an oriented immersed surface Σ, then S is an isocurved surface.*

Proof. The proof follows the lines of the proof of Theorem 3.3. We only note that in the present case, the relation between the radius function R and the (Euclidean) angle formed by N and the horizontal plane is

$$R = \sqrt{1 - \tan^2\theta}. \qquad \square$$

3D. ***Examples of hyperbolic isocurved surfaces.*** We now apply the geometric method to construct examples of hyperbolic isocurved surfaces. As in the elliptic case, the function β that appears in (3-6) has a geometric interpretation and we do not need minimal graphs to apply our method.

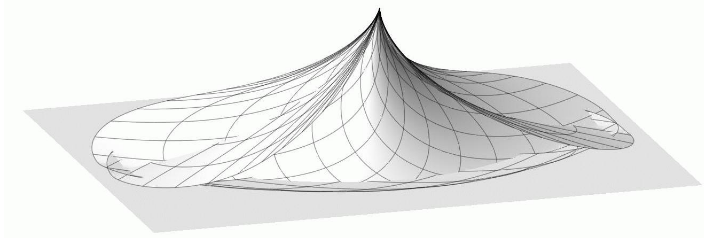

Figure 5. An Enneper-type hyperbolic isocurved surface.

For a simply connected minimal timelike surface Σ in $\mathbb{L}^3$, there is a notion of conjugate surface Σ^* and also a sort of Weierstrass representation; see [Milnor 1990]. It turns out that β in (3-6) is now given by $\beta = -x_3$, i.e., the negative of the height function of Σ^*.

We now use some known examples of timelike minimal surfaces to construct hyperbolic isocurved surfaces.

Example (Enneper-type). Following [Inoguchi and Toda 2004], we consider the Enneper-type timelike minimal surface given by

$$X(x, y) = A(x) + B(y),$$

where

$$A(x) = \frac{1}{2}\left(x^2, nx - \frac{x^3}{3}, x + \frac{x^3}{3}\right) \quad \text{and} \quad B(y) = \frac{1}{2}\left(-y^2, y - \frac{y^3}{3}, -y - \frac{y^3}{3}\right).$$

A piece of this surface appears in Figure 5.

Example. Our last example is induced from a timelike minimal surface that is the graph of the function

$$\psi(x, y) = \frac{y}{\tanh x},$$

that appears in [Milnor 1990]. The conjugate height function ψ^* in this case can be found by direct integration and is given by

$$\psi^* = -\sqrt{y^2 - 1 + \cosh^2 x}.$$

Our method yields the isocurved immersion that is illustrated in Figure 6. The actual expression of the immersion is rather complicated and we will omit it.

4. Final remarks

4A. ***Parabolic isocurved surfaces.*** We have shown how to locally construct examples of elliptic and hyperbolic isocurved surfaces from minimal surfaces in $\mathbb{R}^3$ and $\mathbb{L}^3$. As far as parabolic isocurved surfaces go, we have not found a

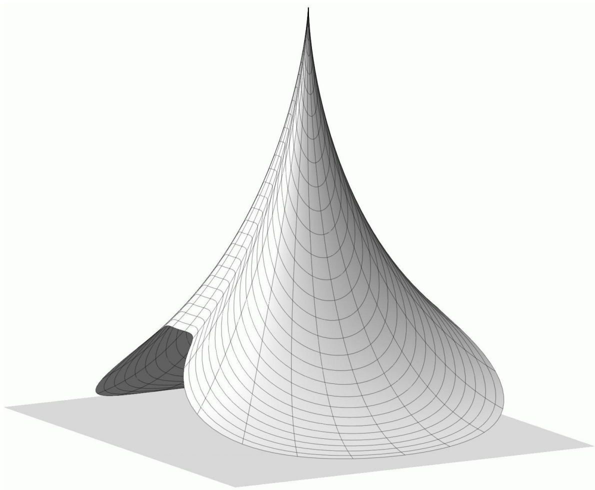

Figure 6. Hyperbolic isocurved surface.

geometric way to generate them. However, we note that there is an interesting family of examples of parabolic isocurved surfaces that was presented by Robert Bryant in his answer to a MathOverflow question posed by the second author; see http://mathoverflow.net/a/108813/53193. Bryant's examples have the form

$$X(s,t) = (a(s) + \cos s\,(t - \tanh t),\, b(s) + \sin s\,(t - \tanh t),\, \operatorname{sech} t)),$$

where $a(s)$ and $b(s)$ are functions such that

$$a'(s)\cos s + b'(s)\sin s = 0.$$

The simplest choice, $a(s) = b(s) = 0$, yields a surface of revolution with respect to the x_3-axis that has a tractrix as the profile curve.

It is also worth mentioning that there are smooth isocurved surfaces that change their type (from elliptic to hyperbolic) and have a smooth curve where the surface is parabolic. The simplest example of such surface is provided by the right circular cones with vertical axis.

4B. ***Anti-isocurved surfaces.*** In our study of hyperbolic isocurved surfaces, we have used timelike minimal surfaces in $\mathbb{L}^3$. One could ask what happens if instead of a timelike surface we start with a spacelike surface in $\mathbb{L}^3$ with vanishing mean curvature (these are known as maximal surfaces). The answer is that the associated congruence of h-geodesics admits orthogonal surfaces and it turns out that for these surfaces $K_h = -K_e$, a class of surfaces that could be called *anti-isocurved surfaces*.

In this context, we recall that there are smooth surfaces in $\mathbb{L}^3$ such as the graph of the function

$$\varphi(x, y) = \ln \frac{\cosh x}{\cosh y},$$

that have zero mean curvature and change their type (i.e., from spacelike to timelike), and each element of the induced family of orthogonal surfaces will be divided into a region where it is isocurved and another region where it is anti-isocurved.

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