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ON CERTAIN DUAL q-INTEGRAL EQUATIONS

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We consider three different systems of dual q-integral equations where the kernel is the third Jackson q-Bessel functions. We solve the first system by applying the multiplying factor method (ansatz solution) and the second by employing the fractional q-calculus, and we use the q-Mellin transform to reduce the third system to a Fredholm q-integral equation of the second kind. Examples are included.

1. Introduction

Dual integral equations arise in a natural way while solving certain mixed boundary value problems. See [Sneddon 1966; Sneddon and Lowengrub 1969; Titchmarsh 1986]. Many of the dual integral equations are of the form

$$\int_0^\infty w(u)A(u)K(u,x)\,du = \lambda(x), \quad 0 < x < a,$$
$$\int_0^\infty A(u)K(u,x)\,du = \mu(x), \quad a < x < \infty,$$

where $w(u)$ is the weight function, $K(x,u)$ is the kernel function. Several authors have described various methods to solve dual integral equations, especially when the kernel is a Bessel function. Busbridge [1938], Tranter [1951], Noble [1955; 1963], Sneddon [1960], Copson [1961], Peters [1961], Williams [1961], Erdélyi and Sneddon [1962], Nasim [1986] and others have described different methods to solve

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dual integral equations. An account of these methods is given in the introduction of [Erdélyi and Sneddon 1962] and, at greater length, in [Sneddon 1966, Chapter IV].

We now briefly mention three methods whose q-analogs will be treated in this work. The first approach, developed by Noble [1955] and Copson [1961], is the multiplying factor method. This approach involves the application of a certain multiplying factor, and after some manipulations we can solve the dual integral equations. The second approach uses fractional calculus to solve dual integral equations, and was developed by Erdélyi and Kober [1940] and Erdélyi [1951]. Their technique became a standard tool for solving dual integral equations. For example, see [Erdélyi and Sneddon 1962; Love 1963; Kesarwani 1967]. Finally, the third approach uses the Mellin transform to reduce the dual integral equations to a Fredholm equation of the second kind which can then be solved numerically. See [Williams 1961; Nasim 1986; Titchmarsh 1986].

In this paper, we are interested in solving dual q-integral equations when the kernel is the third Jackson q-Bessel function defined in (2-14) below and the q-integral is Jackson's q-integral. This paper is organized as follows. Section 2 includes the main notions and terminology from q-analysis which we need in our investigations. It also includes some q-integrals involving the third Jackson q-Bessel function. Section 3 includes the fractional q-integral operators and their calculus, which we need in our analysis. In Section 4, we apply the multiplying factor method to solve certain dual q-integral equations. In Section 5, we solve certain dual q-integral equations by using the fractional q-calculus method. In the last section, Section 6, we solve dual q-integral equations by using the q-Mellin transform introduced in [Fitouhi et al. 2006].

2. Preliminaries; *q*-notation

In the following, q is a positive number which is less than one. Let A_q, B_q, and $\mathbb{R}_{q,+}$ be the sets defined by

$$A_q := \{q^m : m \in \mathbb{N}_0\}, \quad B_q := \{q^{-m} : m \in \mathbb{N}\}, \quad \mathbb{R}_{q,+} := A_q \cup B_q, \tag{2-1}$$

where $\mathbb{N}_0 = \{0, 1, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$. We introduce some of the needed q-notation and results. The q-shifted factorial, see [Gasper and Rahman 2004], and the multiple q-shifted factorial are defined by

$$\begin{gathered}(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\\ (a_1, a_2, \dots, a_k; q)_n := \prod_{j=1}^{k}(a_j;q)_n.\end{gathered} \tag{2-2}$$

The limit $\lim_{n\to\infty}(a;q)_n$ exists and is denoted by $(a;q)_\infty$. For $\gamma\in\mathbb{C}$, $aq^\gamma\neq q^{-n}$, $n\in\mathbb{N}$, we define $(a;q)_\gamma$ to be

$$(a;q)_\gamma := \frac{(a;q)_\infty}{(aq^\gamma;q)_\infty}. \tag{2-3}$$

The q-hypergeometric series (or basic hypergeometric series) ${}_r\phi_s$ is defined by

$$ {}_r\phi_s(a_1,a_2,\dots,a_r;b_1,b_2,\dots,b_s;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,b_2,\dots,b_s;q)_n} z^n(-q^{(n-1)/2})^{n(s-r+1)}. \tag{2-4}$$

The series representation for ${}_r\phi_s$ converges absolutely for all $z\in\mathbb{C}$ if $r\le s$ and converges only for $|z|<1$ if $r=s+1$.

Lemma 2.1 [Koornwinder and Swarttouw 1992]. *If $|z|<1$, then for $m,n\in\mathbb{Z}$,*

$$\sum_{k=-\infty}^{\infty} z^{k+n}\frac{(q^{n+k+1};q)_\infty}{(q;q)_\infty}{}_1\phi_1(0;q^{n+k+1};q,z^2) \times z^{k+m}\frac{(q^{m+k+1};q)_\infty}{(q;q)_\infty}{}_1\phi_1(0;q^{m+k+1};q,z^2)=\delta_{nm}.$$

If $\mu\in\mathbb{R}$, a subset A of $\mathbb{R}$ is called a μ-geometric set if $\mu z\in A$ for all $z\in A$. Let f be a function, real- or complex-valued, defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(z) := \frac{f(z)-f(qz)}{z-qz},\quad z\in A\setminus\{0\}. \tag{2-5}$$

If $0\in A$, the q-derivative at zero is defined by

$$D_q f(0) := \lim_{n\to\infty}\frac{f(zq^n)-f(0)}{zq^n},\quad z\in A\setminus\{0\},$$

if the limit exists and does not depend on z. See [Annaby and Mansour 2012]. The nonsymmetric q-product rule is

$$D_q(fg)(x) = g(x)D_q f(x) + f(qx)D_q g(x). \tag{2-6}$$

A right inverse to D_q, the Jackson q-integration [Jackson 1910], is

$$\int_0^z f(t)\,d_qt := z(1-q)\sum_{n=0}^{\infty} q^n f(zq^n),\quad z\in A, \tag{2-7}$$

provided that the series converges, and

$$\int_a^b f(t)\,d_qt := \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt,\quad a,b\in A.$$

If A is q^{-1}-geometric, then the q-integration over $[z, \infty)$, $z \in A$, is defined by

$$\int_z^\infty f(t)\,d_qt := \sum_{n=1}^\infty zq^{-n}(1-q)f(zq^{-n}), \tag{2-8}$$

and defined on $(0, \infty)$ by

$$\int_0^\infty f(t)\,d_qt := \sum_{n=-\infty}^\infty q^n(1-q)f(q^n). \tag{2-9}$$

The q-integration by parts rule is

$$\int_0^a f(qt)D_qg(t)\,d_qt = f(a)g(a) - \lim_{n\to\infty} f(q^n)g(q^n) - \int_0^a D_qf(t)g(t)\,d_qt. \tag{2-10}$$

For $\eta \in \mathbb{C}$ and a function f defined on $\mathbb{R}_{q,+}$, we define the spaces

$$L_{q,\eta}(\mathbb{R}_{q,+}) := \left\{ f : \|f\|_{q,\eta} := \int_0^\infty |t^\eta f(t)|\,d_qt < \infty \right\},$$
$$L_{q,\eta}(A_q) := \left\{ f : \|f\|_{A_q,\eta} := \int_0^1 |t^\eta f(t)|\,d_qt < \infty \right\},$$
$$L_{q,\eta}(B_q) := \left\{ f : \|f\|_{B_q,\eta} := \int_1^\infty |t^\eta f(t)|\,d_qt < \infty \right\},$$

and

$$L_q(C) := L_{q,0}(C), \qquad C \in \{A_q, B_q, \mathbb{R}_{q,+}\}.$$

Clearly, $L_{q,\eta}(\mathbb{R}_{q,+}) = L_{q,\eta}(A_q) \cap L_{q,\eta}(B_q)$.

Lemma 2.2. *For $\alpha \in \mathbb{C}$, we have*

$$\sum_{k=0}^n q^{2k\alpha} \frac{(q^{2\alpha}; q^2)_{n-k}(q^{2k+2}; q^2)_{n-k}}{(q^2; q^2)_{n-k}} = 1. \tag{2-11}$$

Proof. The left-hand side of (2-11) is

$$\begin{aligned}(q^2; q^2)_n \sum_{k=0}^n \frac{q^{2k\alpha}}{(q^2; q^2)_k} \frac{(q^{2\alpha}; q^2)_{n-k}}{(q^2; q^2)_{n-k}} &= (q^{2\alpha}; q^2)_n \lim_{\varepsilon\to0} {}_2\phi_1(q^{-2n}, \varepsilon; q^{2-2n-2\alpha}; q^2, q^2) \\ &= (q^{2\alpha}; q^2)_n \lim_{\varepsilon\to0} \frac{(q^{2-2n-2\alpha}/\varepsilon; q^2)_n}{(q^{2-2n-2\alpha}; q^2)_n} \varepsilon^n = \lim_{\varepsilon\to0} (q^{2\alpha}\varepsilon; q^2)_n = 1;\end{aligned}$$

here we used [Gasper and Rahman 2004, (I.9), (II.6)]. This completes the proof. □

The q-gamma function [Jackson 1904; Gasper and Rahman 2004] is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z \in \mathbb{C},\ |q| < 1, \tag{2-12}$$

where we take the principal values of q^z and $(1-q)^{1-z}$. The q-binomial theorem (see [Andrews et al. 1999, p. 488]) takes the form

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{2-13}$$

The third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$ for $z \in \mathbb{C}$, (see [Jackson 1905; Ismail 2005]) is defined by

$$\begin{aligned} J_\nu(z;q) = J_\nu^{(3)}(z;q) &:= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu {}_1\phi_1(0;q^{\nu+1};q,qz^2) \\ &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2} z^{2n}}{(q;q)_n (q^{\nu+1};q)_n}, \end{aligned} \tag{2-14}$$

and satisfies

$$D_q[(\cdot)^{-\nu} J_\nu^{(3)}(\cdot\,;q^2)](z) = -\frac{q^{1-\nu} z^{-\nu}}{1-q} J_{\nu+1}^{(3)}(qz;q^2), \tag{2-15}$$

$$D_q[(\cdot)^{\nu} J_\nu^{(3)}(\cdot\,;q^2)](z) = \frac{z^{\nu}}{1-q} J_{\nu-1}^{(3)}(z;q^2); \tag{2-16}$$

see [Koornwinder and Swarttouw 1992; Swarttouw 1992]. The q-Bessel function $J_\nu(\cdot\,;q^2)$, $\nu > -1$, satisfies

$$|J_\nu(q^n;q^2)| \le \frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty} \begin{cases} q^{n\nu} & \text{if } n \ge 0, \\ q^{n^2-(\nu+1)n} & \text{if } n < 0. \end{cases} \tag{2-17}$$

See [Koelink 1994]. The following identity, which was introduced by Koornwinder and Swarttouw [1992], is useful in our investigations.

$$\frac{(q^{\alpha-t+1};q^2)_\infty}{(q^{\alpha+t+1};q^2)_\infty} = \sum_{-\infty}^{\infty} q^{n(t+1)} J_\alpha(q^n;q^2), \tag{2-18}$$

where t and α are complex numbers such that $\Re(t) > -\Re(\alpha) - 1$. We recall that the functions $\cos(z;q)$ and $\sin(z;q)$ are defined, for $z \in \mathbb{C}$, by

$$\begin{aligned} \cos(z;q) &:= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} (zq^{-1/2}(1-q))^{1/2} J_{-1/2}(z(1-q)/\sqrt{q}\,;q^2), \\ \sin(z;q) &:= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} (z(1-q))^{1/2} J_{1/2}(z(1-q);\ q^2). \end{aligned}$$

Proposition 2.3 [Koelink and Swarttouw 1994, p. 694]. *For* $\Re(\nu) > -1$, $x > 0$, *and* $a, b \in \mathbb{C} \setminus \{0\}$, *we have*

$$(2\text{-}19)\quad (a^2 - b^2) \int_0^x t J_\nu(aqt; q^2) J_\nu(bqt; q^2)\, d_q t$$
$$= (1-q) q^{\nu-1} x [a J_{\nu+1}(aqx; q^2) J_\nu(bx; q^2) - b J_\nu(ax; q^2) J_{\nu+1}(aqx; q^2)].$$

Koornwinder and Swarttouw [1992] introduced the following inverse pair of q-integral transforms under the side condition $f, g \in L_q^2(\mathbb{R}_{q,+})$:

$$(2\text{-}20)\quad g(\lambda) = \int_0^\infty f(x) J_\nu(\lambda x; q^2) x\, d_q x, \qquad f(x) = \int_0^\infty g(\lambda) J_\nu(\lambda x; q^2) \lambda\, d_q \lambda,$$

where $\lambda, x \in \mathbb{R}_{q,+}$. This pair of q-integral transforms is a q-analog of the Hankel transform pair

$$g(\lambda) = \int_0^\infty f(x) J_\nu(\lambda x) x\, dx, \qquad f(x) = \int_0^\infty g(\lambda) J_\nu(\lambda x) \lambda\, d\lambda.$$

The following result is a discrete q-analog of the Weber–Schafheitlin integral.

Proposition 2.4 [Koornwinder and Swarttouw 1992, p. 455–456]. *Let* α, β, *and* γ *be complex numbers and* $\xi, \rho \in \mathbb{R}_{q,+}$. *Then*

$$\frac{1}{1-q} \int_0^\infty t^{-\gamma} J_\alpha(\xi t; q^2) J_\beta(\rho t; q^2)\, d_q t$$
$$= \rho^\beta \xi^{(\gamma-\beta-1)} \frac{(q^{\alpha-\beta+\gamma+1}, q^{2\beta+2}; q^2)_\infty}{(q^{\alpha+\beta-\gamma+1}, q^2; q^2)_\infty}$$
$$\times {}_2\phi_1\left(q^{\beta-\alpha-\gamma+1}, q^{\beta+\alpha-\gamma+1}; q^{2\beta+2}; q^2, \frac{\rho^2}{\xi^2} q^{-\beta+\alpha+\gamma+1}\right)$$

if

$$\Re(-\beta+\alpha+\gamma+1) \geq 0,\ \rho < \xi \quad or \quad \Re(-\beta+\alpha+\gamma+1) > 0,\ \rho \leq \xi;$$

and

$$\frac{1}{1-q} \int_0^\infty t^{-\gamma} J_\alpha(\xi t; q^2) J_\beta(\rho t; q^2)\, d_q t$$
$$= \xi^\alpha \rho^{(\gamma-\alpha-1)} \frac{(q^{\beta-\alpha+\gamma+1}, q^{2\alpha+2}; q^2)_\infty}{(q^{\beta+\alpha-\gamma+1}, q^2; q^2)_\infty}$$
$$\times {}_2\phi_1\left(q^{\alpha-\beta-\gamma+1}, q^{\beta+\alpha-\gamma+1}; q^{2\alpha+2}; q^2, \frac{\xi^2}{\rho^2} q^{-\alpha+\beta+\gamma+1}\right)$$

if

$$\Re(-\alpha+\beta+\gamma+1) \geq 0,\ \xi < \rho \quad or \quad \Re(-\alpha+\beta+\gamma+1) > 0,\ \xi \leq \rho.$$

Lemma 2.1 gives us the orthogonality relation

$$\int_0^\infty t J_\alpha(\xi t; q^2) J_\alpha(\rho t; q^2)\, d_q t = \frac{1-q}{\xi^2}\delta_{\rho,\xi}, \quad \Re(\alpha) > -1, \tag{2-21}$$

where ρ, ξ are in $\mathbb{R}_{q,+}$ and $\delta_{\rho,\xi}$ is the Kronecker delta. The following is a q-analog of the Sonine–Schafheitlin integral. If in Proposition 2.4 we take $\gamma = 1$ and $\alpha = \beta$, we obtain

$$\int_0^\infty t^{-1} J_\alpha(\xi t; q^2) J_\alpha(\rho\, t; q^2)\, d_q t = \begin{cases} (1-q)/(1-q^{2\alpha})(\rho/\xi)^\alpha & \text{if } \rho < \xi, \\ (1-q)/(1-q^{2\alpha})(\xi/\rho)^\alpha & \text{if } \xi < \rho. \end{cases}$$

Corollary 2.5. *Let $\alpha, \beta \in \mathbb{C}$, $\rho, t \in \mathbb{R}_{q,+}$. If $\Re(\beta) > \Re(\alpha) > -1$, then*

$$\int_0^\infty t^{\alpha-\beta+1} J_\alpha(\xi\, t; q^2) J_\beta(\rho\, t; q^2)\, d_q t \tag{2-22}$$
$$= \begin{cases} 0 & \textit{if } \xi > \rho, \\ \dfrac{(1-q)(1-q^2)^{1-\beta+\alpha}}{\Gamma_{q^2}(\beta-\alpha)} \xi^\alpha \rho^{\beta-2\alpha-2} (q^2\xi^2/\rho^2; q^2)_{\beta-\alpha-1} & \textit{if } \xi \le \rho. \end{cases}$$

Proof. This follows from Proposition 2.4 by taking $\gamma = \beta - \alpha - 1$. □

Corollary 2.6. *Let m, n be nonnegative integers and $\nu > -n - m - k$. Then*

$$\int_0^\infty t^{-1} J_{\nu+2n+k}(t; q^2) J_{\nu+2m+k}(t; q^2)\, d_q t = \begin{cases} 0 & \textit{if } m \ne n, \\ \dfrac{1-q}{1-q^{2\nu+4n+2k}} & \textit{if } m = n. \end{cases} \tag{2-23}$$

Proof. This result follows by applying Proposition 2.4 with $\gamma = 1$, $\alpha = \nu + 2m + k$, $\beta = \nu + 2n + k$, and $\xi = \rho = 1$. □

The little q-Jacobi polynomial [Gasper and Rahman 2004, p. 27] is defined by

$$p_n(x; a, b \mid q) := {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx).$$

Corollary 2.7. *Let $\Re(\nu) > -1$, $\rho, t \in \mathbb{R}_{q,+}$. Then*

$$\int_0^\infty t^{1-k} J_{\nu+2m+k}(t; q^2) J_\nu(\rho\, t; q^2)\, d_q t \tag{2-24}$$
$$= \begin{cases} 0 & \rho > 1, \\ \dfrac{(1-q)\rho^\nu (q^{2m+2}\rho^2, q^{2m+2k}, q^{2\nu+2}; q^2)_\infty}{(q^{2m+2k}\rho^2, q^{2\nu+2m+2}, q^2; q^2)_\infty} p_m(q^{2m}\rho^2; q^{2k-2}, q^{2\nu} \mid q^2) & \rho < 1. \end{cases}$$

Proof. This result follows by applying Proposition 2.4 with $\gamma = k-1$, $\alpha = \nu+2m+k$, $\beta = \nu$, and $\xi = 1$, in addition to the transformation

$${}_2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1(c/a, c/b; c; q, abz/c). \qquad \square$$

Corollary 2.8. *Let* $\Re(\nu) > -1$, $\rho, t \in \mathbb{R}_{q,+}$. *Then*

$$J_{\nu+2m+k}(t;q^2) = \frac{(1-q)t^k(q^{2m+2k}, q^{2\nu+2};q^2)_\infty}{(q^{2\nu+2m+2};q^2;q^2)_\infty} \times \int_0^1 \rho^{\nu+1}(q^{2m+2}\rho^2;q^2)_{k-1} J_\nu(\rho t;q^2)\, p_m(q^{2m}\rho^2;q^{2k-2},q^{2\nu}\mid q^2)\, d_q\rho. \tag{2-25}$$

Proof. This follows from Corollary 2.7 and the q-Hankel transform pair (2-20). □

Proposition 2.9. *Let* x, u, *and* α *be complex numbers. If* $\Re(\gamma+\beta) > -1$ *and* $\Re(\beta) > -1$, *then*

$$\int_0^x t^\gamma (q^2t^2/x^2;q^2)_\alpha J_\beta(ut;q^2)\, d_qt = \frac{x^{\gamma+\beta+1}u^\beta(1-q)(q^{2\beta+2}, q^{2\alpha+\gamma+\beta+3};q^2)_\infty}{(q^{2\alpha+2}, q^{\gamma+\beta+1};q^2)_\infty} \times {}_2\phi_2(0, q^{\gamma+\beta+1}; q^{2\beta+2}, q^{2\alpha+\gamma+\beta+3}; q^2, q^2x^2u^2). \tag{2-26}$$

In particular, if $\gamma = \beta+1$, *then*

$$\int_0^x t^{\beta+1}(q^2t^2/x^2;q^2)_\alpha J_\beta(ut;q^2)\, d_qt = x^{\beta-\alpha+1}u^{-\alpha-1}(1-q)(1-q^2)^\alpha \Gamma_{q^2}(\alpha+1) J_{\alpha+\beta+1}(xu;q^2). \tag{2-27}$$

Proof. According to (2-14), we have

$$\int_0^x t^\gamma (q^2t^2/x^2;q^2)_\alpha J_\beta(ut;q^2)\, d_qt = \frac{(q^{2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)} u^{2k+\beta}}{(q^2;q^2)_k(q^{2\beta+2};q^2)_k} \int_0^x t^{\gamma+\beta+2k}(q^2t^2/x^2;q^2)_\alpha\, d_qt. \tag{2-28}$$

By using (2-3) and the q-binomial theorem with base q^2 instead of q on the inner series in (2-13), we obtain

$$\int_0^x t^{\gamma+\beta+2k}(q^2t^2/x^2;q^2)_\alpha\, d_qt = \frac{(1-q)x^{2k+\gamma+\beta+1}(q^2;q^2)_\infty(q^{2k+2\alpha+\gamma+\beta+3};q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty(q^{2k+\gamma+\beta+1};q^2)_\infty}. \tag{2-29}$$

Substituting (2-29) in (2-28) and using (2-4), the desired result follows. The particular case follows by direct substitution in (2-26) and the definition (2-14). □

Proposition 2.10. *Let ν and α be complex numbers such that $\Re(\nu) > -1$. For $x, u \in \mathbb{R}_{q,+}$,*

$$\int_x^\infty (x^2/t^2; q^2)_{\alpha-1} t^{2\alpha-\nu-1} J_\nu(tu; q^2)\, d_q t = x^{\alpha-\nu} u^{-\alpha} q^\alpha (1-q) \frac{(q^2; q^2)_\infty}{(q^{2\alpha}; q^2)_\infty} J_{\nu-\alpha}(xu/q; q^2). \tag{2-30}$$

Proof. Using the ${}_1\phi_1$ transformation (see [Gasper and Rahman 2004, p. 29])

$$(c; q)_\infty {}_1\phi_1(0; a; q, z) = (z; q)_\infty {}_1\phi_1(0; z; q, c)$$

and (2-14), one can verify that

$$J_\nu(z; q^2) = \frac{(q^2 z^2; q^2)_\infty}{(q^2; q^2)_\infty} z^\nu \sum_{j=0}^\infty (-1)^j q^{j(j+1)} \frac{q^{2\nu j}}{(q^2, q^2 z^2; q^2)_j}.$$

Hence,

$$\int_x^\infty (x^2/t^2; q^2)_{\alpha-1} t^{2\alpha-\nu-1} J_\nu(tu; q^2)\, d_q t = \frac{u^\nu}{(q^2; q^2)_\infty} \sum_{j=0}^\infty \frac{q^{j^2+j+2\nu j}}{(q^2; q^2)_j} \int_x^\infty (x^2/t^2; q^2)_{\alpha-1} t^{2\alpha-1} (q^{2j+2} t^2 u^2; q^2)_\infty\, d_q t. \tag{2-31}$$

Using Lemma 2.2, we can prove that

$$\int_x^\infty (x^2/t^2; q^2)_{\alpha-1} t^{2\alpha-1} (q^{2j+2} t^2 u^2; q^2)_\infty\, d_q t = (1-q) u^{-2\alpha} \frac{(q^2; q^2)_\infty}{(q^{2\alpha}; q^2)_\infty} q^{-2\alpha j+2\alpha} (x^2 u^2 q^{2j}; q^2)_\infty \tag{2-32}$$

for $x, t \in \mathbb{R}_{q,+}$. Substituting (2-32) into (2-31) yields the desired result. □

3. Fractional q-calculus

In this section, we introduce fractional q-integral operators and their properties which we need in our fractional q-calculus approach for solving certain dual q-integral equations. A comprehensive study of the fractional q-calculus and equations is in [Samko et al. 1987; Butzer and Westphal 2000; Annaby and Mansour 2012]. Al-Salam [1966] defined a two-parameter q-fractional operator by

$$K_q^{\eta,\alpha}\phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (x/t; q)_{\alpha-1} t^{-\eta-1} \phi(tq^{1-\alpha})\, d_q t,$$

where $\alpha \neq -1, -2, \ldots$. This is a q-analog of the Erdélyi and Sneddon fractional operator (see [Erdélyi 1951; Erdélyi and Sneddon 1962])

$$K^{\eta,\alpha} f(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\eta-1} f(t)\, dt.$$

The following operator is a slight modification of the operator $K_q^{\eta,\alpha}$, which we found very convenient in our analysis. This operator, denoted by $\mathcal{K}_q^{\eta,\alpha}$, is defined as

$$\mathcal{K}_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (x/t;q)_{\alpha-1} t^{-\eta-1} \phi(qt)\, d_q t, \tag{3-1}$$

where $\alpha \neq -1, -2, \ldots$. Using (2-8), the operator $\mathcal{K}_q^{\eta,\alpha}$ has the series representation

$$\mathcal{K}_q^{\eta,\alpha} \phi(x) = (1-q)^\alpha \sum_{n=0}^\infty q^{n\eta} \frac{(q^\alpha;q)_n}{(q;q)_n} \phi(xq^{-n}), \tag{3-2}$$

which is valid for all α.

Proposition 3.1. *Let $\eta \in \mathbb{C}$. If $\phi \in L_{q,-\eta-1}(B_q)$, then $\mathcal{K}_q^{\eta,\alpha}\phi(x)$ exists for all $x \in \mathbb{R}_{q,+}$ and belongs to $L_{q,\mu}(B_q)$ for any $\mu \in \mathbb{C}$ such that $\Re(\mu) < -\Re(\eta) - 1$.*

Proof. Let $m \in \mathbb{Z}$. Since

$$\mathcal{K}_q^{\eta,\alpha} \phi(q^m) = (1-q)^\alpha \sum_{n=0}^\infty q^{n\eta} \frac{(q^\alpha;q)_n}{(q;q)_n} \phi(q^{m-n}),$$

we obtain

$$|\mathcal{K}_q^{\eta,\alpha} \phi(q^m)| \le (1-q)^{\Re(\alpha)} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty q^{n\Re(\eta)} |\phi(q^{m-n})|. \tag{3-3}$$

Therefore, if $m > 0$, we obtain

$$\begin{aligned} &|\mathcal{K}_q^{\eta,\alpha} \phi(q^m)| \\ &\quad \le (1-q)^{\Re(\alpha)} q^{m\Re(\eta)} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \Bigl(\sum_{k=0}^m q^{-k\Re(\eta)} |\phi(q^k)| + \frac{1}{1-q} \|\phi\|_{B_q,-1-\eta} \Bigr). \end{aligned} \tag{3-4}$$

If $m < 0$, then

$$|\mathcal{K}_q^{\eta,\alpha} \phi(q^m)| \le (1-q)^{\Re(\alpha)-1} q^{m\Re(\eta)} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \|\phi\|_{B_q,-1-\eta}. \tag{3-5}$$

Consequently, $K_q^{\eta,\alpha}(x)$ exists for all $x \in \mathbb{R}_{q,+}$. If $\mu \in \mathbb{C}$ and $\Re(\mu) < -\Re(\eta) - 1$, then

$$\begin{aligned} &\int_1^\infty |t^\mu \mathcal{K}_q^{\eta,\alpha} \phi(t)|\, d_q t \\ &\qquad \le (1-q)^{\Re(\alpha)-1} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \|\phi\|_{B_q,-1-\eta} \sum_{j=0}^\infty q^{-j\Re(1+\mu+\eta)} < \infty, \end{aligned} \tag{3-6}$$

where $\Re(\mu) < -\Re(\eta) - 1$. Thus $\mathcal{K}_q^{\eta,\alpha}\phi \in L_{q,\mu}(B_q)$. □

Al-Salam [1966, p. 138–139] proved formally the semigroup identity

$$K_q^{\eta,\alpha} K_q^{\eta+\alpha,\beta}\phi(x) = K_q^{\eta,\alpha+\beta}\phi(x), \tag{3-7}$$

where η, α, and β are complex numbers and without imposing any conditions on the function ϕ. Using the same technique we can prove that the semigroup property

$$\mathcal{K}_q^{\eta,\alpha}\mathcal{K}_q^{\eta+\alpha,\beta}\phi(x) = \mathcal{K}_q^{\eta,\alpha+\beta}\phi(x) \tag{3-8}$$

holds for $x \in \mathbb{R}_{q,+}$ whenever $\phi \in L_{q,-\eta-\alpha-1}(B_q)$, η, α, β are complex numbers and $\Re(\alpha) < 0$. Therefore, if we take $\beta = -\alpha$ in (3-8) and note that $\mathcal{K}_q^{\eta,0}$ is the identity operator, we obtain

$$(\mathcal{K}_q^{\eta,\alpha})^{-1}\phi(x) = \mathcal{K}_q^{\eta+\alpha,-\alpha}\phi(x), \quad x \in \mathbb{R}_{q,+}, \tag{3-9}$$

for any $\phi \in L_{q,-\eta-\alpha-1}(B_q)$, and η, α are complex numbers.

Agarwal [1969] defined the two-parameter family

$$I_q^{\eta,\alpha}\phi(x) := \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} t^{\eta}\phi(t)\,d_qt, \quad \alpha \neq 0, -1, -2, \ldots. \tag{3-10}$$

of fractional q-integral operators, which can be written as

$$I_q^{\eta,\alpha}\phi(x) = (1-q)^{\alpha} \sum_{n=0}^{\infty} q^{(\eta+1)n} \frac{(q^{\alpha};q)_n}{(q;q)_n}\phi(xq^n), \tag{3-11}$$

which is valid for all α. The special case $I_q^{0,\alpha}$ is the q-analog of the Riemann–Liouville fractional operator introduced in [Al-Salam 1966] and is denoted by I_q^{α}. Hence,

$$I_q^{\alpha}\phi(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\phi(t)\,d_qt, \quad \alpha \neq 0, -1, -2, \ldots. \tag{3-12}$$

In [Annaby and Mansour 2012, p. 121], the authors solved the q-analog of the Abel integral equation on a continuous domain of the form $[0, a]$. In the following we state without proof a modified version of [ibid., Theorem 4.7] which holds when the domain of solution is discrete.

Theorem 3.2. *The q-Abel integral equation*

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\phi(t)\,d_qt = f(x) \quad (0 < \Re(\alpha) < 1,\ x \in A_q)$$

has a unique solution $\phi \in L_q(A_q)$, given by

$$\phi(x) = D_{q,x} I_q^{1-\alpha} f(x),$$

if and only if f *and* $I_q^{1-\alpha} f$ *are* $L_q(A_q)$ *functions with* $I_q^{1-\alpha} f(0) = 0$.

Proof. The proof is similar to the proof of [Annaby and Mansour 2012, Theorem 4.7] and is omitted. □

Proposition 3.3. *Let* η *and* α *be complex numbers. If* $\phi \in L_{q,\eta}(A_q)$, *then* $I_q^{\eta,\alpha}\phi(x)$ *exists for all* $x \in \mathbb{R}_{q,+}$ *and belongs to* $L_{q,\mu}(A_q)$ *for all* $\mu \in \mathbb{C}$ *such that* $\Re(\mu-\eta) > 0$.

Proof. Assume that $\phi \in L_{q,\eta}(A_q)$. Then

$$I_q^{\eta,\alpha}\phi(q^m) = (1-q)^{\alpha} q^{-(\eta+1)m} \sum_{n=m}^{\infty} q^{(\eta+1)n} \frac{(q^{\alpha};q)_{n-m}}{(q;q)_{n-m}} \phi(q^n).$$

Thus, if $m \geq 0$, we obtain

$$|I_q^{\eta,\alpha}\phi(q^m)| \leq (1-q)^{\Re(\alpha)} q^{-(\Re(\eta)+1)m} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \|\phi\|_{A_q,\eta};$$

and if $m < 0$, we obtain

$$|I_q^{\eta,\alpha}\phi(q^m)| \leq (1-q)^{\Re(\alpha)} q^{-(\Re(\eta)+1)m} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \times \left(\sum_{k=m}^{-1} q^{k(\Re(\eta)+1)} |\phi(q^k)| + \frac{1}{1-q} \|\phi\|_{A_q,\eta} \right) < \infty.$$

Moreover,

$$\int_0^1 |t^{\mu} I_q^{\eta,\alpha}\phi(t)| \leq (1-q)^{\Re(\alpha)-1} \frac{(-q^{\Re(\alpha)};q)_\infty}{(q;q)_\infty} \|\phi\|_{A_q,\eta} \sum_{j=0}^{\infty} q^{j\Re(\mu-\eta)} < \infty,$$

provided that $\Re(\mu-\eta) > 0$. This completes the proof. □

Proposition 3.4. *Let* η *and* α *be complex numbers, and* $\Re(\alpha) < 0$. *If* $\phi \in L_{q,\eta+\alpha}(A_q)$ *then*

$$(I_q^{\eta,\alpha})^{-1} = I_q^{\eta+\alpha,-\alpha}. \tag{3-13}$$

Proof. This follows by noting that

$$I_q^{\eta,\alpha} I_q^{\eta+\alpha,-\alpha}\phi(x) = \sum_{k=0}^{\infty} q^{(\eta+1)k} \frac{(q^{\alpha};q)_k}{(q;q)_k} \sum_{m=0}^{\infty} q^{(\eta+1+\alpha)m} \frac{(q^{-\alpha};q)_m}{(q;q)_m} \phi(q^{k+m}x). \tag{3-14}$$

Make the substitution $n = m + k$ on the inner series of (3-14). This gives

$$I_q^{\eta,\alpha} I_q^{\eta+\alpha,-\alpha}\phi(x) = \sum_{k=0}^{\infty} q^{-\alpha k} \frac{(q^{\alpha};q)_k}{(q;q)_k} \sum_{n=k}^{\infty} q^{(\eta+1+\alpha)n} \frac{(q^{-\alpha};q)_{n-k}}{(q;q)_{n-k}} \phi(q^n x). \tag{3-15}$$

If $\Re(\alpha) < 0$ and $\phi \in L_{q,\eta+\alpha}(A_q)$, then the double series in (3-15) is absolutely convergent and we can interchange the order of summation to obtain

$$I_q^{\eta,\alpha} I_q^{\eta+\alpha,-\alpha}\phi(x) = \sum_{n=0}^{\infty} q^{(\eta+1+\alpha)n}\phi(q^n x)\sum_{k=0}^{n} q^{-\alpha k}\frac{(q^{\alpha};q)_k}{(q;q)_k}\frac{(q^{-\alpha};q)_{n-k}}{(q;q)_{n-k}} = \sum_{n=0}^{\infty} q^{(\eta+1+2\alpha)n}\phi(q^n x)\frac{(q^{-\alpha};q)_n}{(q;q)_n}\frac{(q^{1-n};q)_n}{(q^{1+\alpha-n};q)_n} = \phi(x), \tag{3-16}$$

where we applied [Gasper and Rahman 2004, Equation (II.6)] to the inner series. □

A direct calculation gives

$$I_q^{\eta,\alpha} x^{\beta} f(x) = x^{\beta} I_q^{\eta+\beta,\alpha} f(x),$$

where η, α, β are complex numbers, and $f \in L_{q,\eta+\beta}(A_q)$. Agarwal [1969] also proved the following semigroup identity when η, λ, and μ are positive constants:

$$I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu}\phi(x) = I_q^{\eta,\mu+\lambda}\phi(x) = I_q^{\eta+\lambda,\mu} I_q^{\eta,\lambda}\phi(x) = I_q^{\eta,\mu} I_q^{\mu+\eta,\lambda}\phi(x) = I_q^{\eta+\mu,\lambda} I_q^{\eta,\mu}\phi(x). \tag{3-17}$$

But, using the same technique introduced in [ibid.], we can prove that

$$I_q^{\eta,\lambda} I_q^{\eta+\lambda,\mu}\phi(x) = I_q^{\eta,\mu+\lambda}\phi(x), \quad x \in \mathbb{R}_{q,+}, \tag{3-18}$$

holds for complex numbers η, λ, and μ whenever $\phi \in L_{q,\eta+\lambda}(A_q)$, and $\Re(\lambda) < 0$.

It should be mentioned here that in most of the proofs of the semigroup properties in [Agarwal 1969; Al-Salam 1966], the domain where the fractional integrals and the related properties hold is not determined precisely.

Let $S_q^{\eta,\alpha}$ be the operator defined by

$$S_q^{\eta,\alpha}\phi(x) := \frac{x^{-\alpha/2}}{(1-q)}\int_0^{\infty} y^{-\alpha/2} J_{2\eta+\alpha}(\sqrt{xy};q)\phi(y)\,d_q y = x^{-\alpha/2}\sum_{n=-\infty}^{\infty} q^{n(1-\alpha/2)} J_{2\eta+\alpha}(\sqrt{x}q^{n/2};q)\phi(q^n). \tag{3-19}$$

This operator is a q-analog of the modified Hankel transform operator introduced by Erdélyi and Kober [1940]; it is also a modification of the q-Hankel transform operator introduced in [Koornwinder and Swarttouw 1992] and defined by

$$\mathcal{H}_q^{\nu}(f)(x) = \int_0^{\infty} f(t) J_{\nu}(xt;q^2)\,(xt)^{1/2}\,d_q t.$$

Proposition 3.5. *Let η and α be complex numbers such that $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$ then $S_{q^2}^{\eta,\alpha}\phi(x)$ exists for all $x \in \mathbb{R}_{q^2,+}$ and belongs to $L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$.*

Proof. Let $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$. From (2-17), if $x \in \mathbb{R}_{q^2,+}$, there exists $M > 0$ such that

$$\left|\sum_{n=-\infty}^{\infty} q^{n(2-\alpha)} J_{2\eta+\alpha}(\sqrt{x}q^n; q^2)\phi(q^{2n})\right| \le M \sum_{n=-\infty}^{\infty} q^{2n(\Re(\eta)+1)}|\phi(q^{2n})| < \infty$$

because $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$. Thus, $S_{q^2}^{\eta,\alpha}(\phi)(x)$ exists for all $x \in \mathbb{R}_{q^2,+}$. Now we prove that $S_{q^2}^{\eta,\alpha} \in L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$. Indeed,

$$\begin{aligned} (3\text{-}20)\quad & \int_0^\infty t^{\eta+\alpha}|S_{q^2}^{\eta,\alpha}\phi(t)|\, d_{q^2}t \\ &= \frac{1}{1-q^2}\int_0^\infty |t^{\eta+\alpha/2}| \int_0^\infty |y^{-\alpha/2}|\,|J_{2\eta+\alpha}(\sqrt{yt}; q^2)|\,|\phi(y)|\, d_{q^2}y\, d_{q^2}t \\ &\le \frac{1}{1-q^2}\|\phi\|_\eta \int_0^\infty |t^{\eta+\alpha/2}| \sup_{y\in\mathbb{R}_{q^2,+}} |y^{-\eta-\alpha/2} J_{2\eta+\alpha}(\sqrt{ty}; q^2)|\, d_{q^2}t. \end{aligned}$$

Using the estimates (2-17), the q-integral on the third line of (3-20) is convergent when $\Re(2\eta+\alpha) > -1$, and the proposition follows. □

Proposition 3.6. *$S_{q^2}^{\eta,\alpha}$ defines a one-to-one linear operator from $L_{q^2,\eta}(\mathbb{R}_{q^2,+})$ into $L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$. Also,*

$$(S_{q^2}^{\eta,\alpha})^{-1} = S_{q^2}^{\eta+\alpha,-\alpha}. \tag{3-21}$$

Proof. Clearly $S_{q^2}^{\eta,\alpha}$ is linear. To prove that it is one-to-one, assume that there is a function $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$ such that $S_{q^2}^{\eta,\alpha}\phi(x) = 0$ for all $x \in \mathbb{R}_{q^2,+}$. Hence,

$$\sum_{n=-\infty}^{\infty} q^{n(2-\alpha)}\phi(q^{2n}) J_{2\eta+\alpha}(q^n\xi; q^2) = 0 \quad \text{for all } x \in \mathbb{R}_{q,+}. \tag{3-22}$$

Multiplying both sides of (3-22) by $\xi J_{2\eta+\alpha}(q^r\xi; q^2)$ $(r \in \mathbb{Z})$, calculating the q-integration for $\xi \in (0, \infty)$, then applying (2-21), we obtain

$$\sum_{n=-\infty}^{\infty} q^{n(2-\alpha)}\phi(q^{2n})q^{-n-r}\delta_{nr} = 0.$$

That is, $\phi(q^{2r}) = 0$ for all $r \in \mathbb{Z}$, and $S_{q^2}^{\eta,\alpha}$ is a one-to-one operator. Now we prove (3-21). From (3-19), for $j \in \mathbb{Z}$, we have

$$S_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,-\alpha}\phi(q^{2j}) = q^{-j\alpha} \sum_{n=-\infty}^{\infty} q^{n(2-\alpha)} J_{2\eta+\alpha}(q^{j+n}; q^2) S_{q^2}^{\eta,\alpha}\phi(q^{2n}).$$

Hence,

$$\begin{aligned} (3\text{-}23)\quad & S_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,-\alpha}\phi(q^{2j}) \\ &= q^{-j\alpha} \sum_{n=-\infty}^{\infty} q^{2n} J_{2\eta+\alpha}(q^{j+n}; q^2) \sum_{k=-\infty}^{\infty} q^{k(2+\alpha)} J_{2\eta+\alpha}(q^{k+n}; q^2)\phi(q^{2k}). \end{aligned}$$

Using (2-17) and $\phi \in L_{q,\eta}(\mathbb{R}_{q,+})$, we can prove that the series on the left-hand side of (3-23) is absolutely convergent. Consequently, we can interchange the order of summation to obtain

$$S_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,-\alpha} \phi(q^{2j}) = q^{-j\alpha} \sum_{k=-\infty}^{\infty} q^{k(2+\alpha)} \phi(q^{2k}) \sum_{n=-\infty}^{\infty} q^{2n} J_{2\eta+\alpha}(q^{j+n}; q^2) J_{2\eta+\alpha}(q^{k+n}; q^2).$$

Therefore, from (2-21),

$$S_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,-\alpha} \phi(q^{2j}) = q^{-j} q^{-j\alpha} \sum_{k=-\infty}^{\infty} q^{k(1+\alpha)} \phi(q^{2k}) \delta_{jk} = \phi(q^{2j}), \tag{3-24}$$

and the desired result follows. □

The next result gives another sufficient condition for the existence of $S_{q^2}^{\eta,\alpha}$.

Proposition 3.7. *Let η and α be complex numbers satisfying $\Re(2\eta+\alpha) > 0$. Let ϕ be a function defined on $\mathbb{R}_{q^2,+}$. If there exists $\mu \in \mathbb{C}$ such that*

$$\phi|_{A_{q^2}} \in L_{q^2,\eta}(A_{q^2}), \quad \phi|_{B_{q^2}} \in L_{q^2,\mu}(B_{q^2}),$$

then $S_{q^2}^{\eta,\alpha}\phi(x)$ exists for all $x \in \mathbb{R}_{q^2,+}$.

Proof. Let $x \in \mathbb{R}_{q^2,+}$. From (2-17), there exists $M > 0$ such that

$$|J_{2\eta+\alpha}(\sqrt{x}q^n; q^2)| \le M q^{n\Re(2\eta+\alpha)} \quad \text{for all } m \in \mathbb{N}_0.$$

Since $\phi|_{A_{q^2}} \in L_{q^2,\eta}(A_{q^2})$, then

$$\left|\sum_{n=0}^{\infty} q^{n(2-\alpha)} J_{2\eta+\alpha}(\sqrt{x}q^n; q^2)\phi(q^{2n})\right| \le M \sum_{n=0}^{\infty} q^{2n(\Re(\eta)+1)} |\phi(q^{2n})| < \infty. \tag{3-25}$$

From (2-17), there exists $K > 0$ such that

$$|J_{2\eta+\alpha}(\sqrt{x}q^n; q^2)| \le K q^{n^2-(\Re(2\eta+\alpha)+1)n} \quad \text{for all } n \in \mathbb{Z}^-.$$

Since $\phi|_{B_{q^2}} \in L_{q^2,\mu}(B_{q^2})$, then

$$\left|\sum_{n=1}^{\infty} q^{-n(2-\alpha)} J_{2\eta+\alpha}(\sqrt{x}q^{-n}; q^2)\phi(q^{-2n})\right| \le K \sum_{n=1}^{\infty} q^{-2n(-\Re(\eta+\alpha)+1/2)+n^2} |\phi(q^{-2n})| < \infty. \tag{3-26}$$

Combining (3-25) and (3-26), we can then conclude that $S_{q^2}^{\eta,\alpha}\phi(x)$ exists for all $x \in \mathbb{R}_{q^2,+}$. □

Proposition 3.8. *Let α, β, and λ be complex numbers such that $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$, then*

$$I_{q^2}^{\eta+\alpha,\beta} S_{q^2}^{\eta,\alpha}\phi(x) = (1-q^2)^\beta S_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x \in \mathbb{R}_{q^2,+}).$$

Proof. Let $x \in \mathbb{R}_{q^2,+}$ be fixed. Using definitions (3-10) and (3-19) we get

$$I_{q^2}^{\eta+\alpha,\beta} S_{q^2}^{\eta,\alpha}\phi(x) = \frac{x^{-\eta-\alpha-1}}{(1-q^2)\Gamma_{q^2}(\beta)} \times \int_0^x (q^2t/x;q^2)_{\beta-1} t^{\eta+\alpha/2} \int_0^\infty y^{-\alpha/2} J_{2\eta+\alpha}(\sqrt{ty};q^2)\phi(y)\,d_{q^2}y\,d_{q^2}t. \tag{3-27}$$

From (2-17), there exists $M > 0$ such that

$$|J_{2\eta+\alpha}(\sqrt{ty};q^2)| \le M(ty)^{\Re(\eta+\alpha/2)} \quad \text{for all } y, t \in \mathbb{R}_{q^2,+}.$$

Consequently,

$$\begin{aligned}\left|\frac{1}{\Gamma_q^2(\beta)}\int_0^x (q^2t/x;q^2)_{\beta-1} t^{\eta+\alpha/2} \int_0^\infty y^{-\alpha/2} J_{2\eta+\alpha}(\sqrt{ty};q^2)\phi(y)\,d_{q^2}y\,d_{q^2}t\right| \\ \le \frac{M}{|\Gamma_q^2(\beta)|}\|\phi\|_{q^2,\eta}\left|\int_0^x (q^2t/x;q^2)_{\beta-1} t^{2\eta+\alpha}\,d_{q^2}t\right| \\ = \frac{M}{|\Gamma_q^2(\beta)|}\|\phi\|_{q^2,\eta}|x^{2\eta+\alpha+1}B_q(\beta, 2\eta+\alpha+1)| < \infty,\end{aligned}$$

since $\Re(2\eta+\alpha)+1 > 0$. Hence, the series is absolutely convergent, and we can interchange the order of summation to obtain

$$I_{q^2}^{\eta+\alpha,\beta} S_{q^2}^{\eta,\alpha}\phi(x) = \frac{x^{-\eta-\alpha-1}}{\Gamma_{q^2}(\beta)(1-q^2)} \times \int_0^\infty y^{-\alpha/2}\phi(y) \int_0^x (q^2t/x;q^2)_{\beta-1} t^{\eta+\alpha/2} J_{2\eta+\alpha}(\sqrt{ty};q^2)\,d_{q^2}t\,d_{q^2}y. \tag{3-28}$$

Using

$$\int_0^x f(t)\,d_{q^2}t = \frac{1+q}{x}\int_0^x tf(t^2/x)\,d_qt \tag{3-29}$$

and (2-27), we obtain

$$\begin{aligned}\int_0^x (q^2t/x;q^2)_{\beta-1} t^{\eta+\alpha/2} J_{2\eta+\alpha}(\sqrt{ty};q^2)\,d_{q^2}t\,d_{q^2}y \\ = \frac{1+q}{x^{1+\eta+\alpha/2}}\int_0^x (q^2t^2/x^2;q^2)_{\beta-1} t^{2\eta+\alpha+1} J_{2\eta+\alpha}\left(t\sqrt{\frac{y}{x}};q^2\right)d_qt \\ = (1-q^2)^\beta y^{-\beta/2} x^{1+\eta+(\alpha-\beta)/2}\Gamma_{q^2}(\beta) J_{2\eta+\alpha+\beta}(\sqrt{xy};q^2).\end{aligned} \tag{3-30}$$

Substituting (3-30) into (3-28) yields the desired result and completes the proof. □

Proposition 3.9. *Let* α, β, *and* η *be complex numbers. If* $\phi \in L_{q^2,\eta+\alpha-\gamma}(\mathbb{R}_{q^2,+})$ *for some* $\gamma \in \mathbb{C}$, $\Re(\gamma) > \max\{0, \Re(\alpha)\}$, *then*

$$\mathcal{K}_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) = (1-q^2)^{\alpha} S_{q^2}^{\eta,\alpha+\beta}\phi(x) \tag{3-31}$$

for all $x \in \mathbb{R}_{q^2,+}$.

Proof. Using definitions (3-2), (3-19), (2-8), and (2-9) we get

$$\begin{aligned}\mathcal{K}_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) &= \frac{q^{-2\eta}x^{\eta}}{\Gamma_{q^2}(\alpha)}\int_x^{\infty}(x/t;q^2)_{\alpha-1}(S_{q^2}^{\eta+\alpha,\beta}\phi)(q^2t)\,d_{q^2}t\\ &= \frac{x^{\eta}}{\Gamma_{q^2}(\alpha)}\int_{q^2x}^{\infty}(q^2x/t;q^2)_{\alpha-1}(S_{q^2}^{\eta+\alpha,\beta}\phi)(t)\,d_{q^2}t\\ &= \frac{x^{\eta}}{(1-q^2)\Gamma_{q^2}(\alpha)}\int_{q^2x}^{\infty}\Big((q^2x/t;q^2)_{\alpha-1}t^{-\eta-1-\beta/2}\\ &\qquad\times\int_0^{\infty}y^{-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{yt};q^2)\phi(y)\,d_{q^2}y\Big)\,d_{q^2}t.\end{aligned} \tag{3-32}$$

Set $c := -1-\Re(2\eta+\beta+2\alpha-2\gamma)$. From (2-18), there exists $C > 0$ such that

$$|J_{2\eta+2\alpha+\beta}(q^r;q^2)| \le Cq^{-r(c+1)} \quad (r \in \mathbb{Z}). \tag{3-33}$$

Hence,

$$\begin{aligned}&\left|\int_{q^2x}^{\infty}(q^2x/t;q^2)_{\alpha-1}t^{-\eta-1-\beta/2}\int_0^{\infty}y^{-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{yt};q^2)\phi(y)\,d_{q^2}y\,d_{q^2}t\right|\\ &\qquad\le C\|\phi\|_{q^2,\eta+\alpha-\gamma}|(q^2x/t;q^2)_{\alpha-1}t^{\alpha-\gamma-1}\,d_{q^2}t| < \infty\end{aligned}$$

whenever $\Re(\gamma) > \Re(\alpha)$. Hence, the double q^2-integration in (3-32) is absolutely convergent, and we can interchange the order of q^2-integration to obtain

$$\begin{aligned}K_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) &= \frac{x^{\eta}}{\Gamma_{q^2}(\alpha)}\\ &\times\int_0^{\infty}y^{-\beta/2}\phi(y)\int_{q^2x}^{\infty}(q^2x/t;q^2)_{\alpha-1}t^{-\eta-1-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{yt};q^2)\,d_{q^2t}\,d_{q^2}y.\end{aligned}$$

Using

$$\int_a^{\infty}f(t)\,d_{q^2}t = \frac{1+q}{a}\int_a^{\infty}tf\Big(\frac{t^2}{a}\Big)\,d_qt$$

and Proposition 2.10, one finds that

$$\begin{aligned}\int_{q^2x}^{\infty}&(q^2x/t;q^2)_{\alpha-1}t^{-\eta-1-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{yt};q^2)\,d_{q^2}t\\&=(1+q)(q^2x)^{\eta+\beta/2}\int_{q^2x}^{\infty}(q^4x^2/t^2;q^2)_{\alpha-1}t^{-2\eta-1-\beta}J_{2\eta+2\alpha+\beta}\Big(\frac{t\sqrt{y}}{q\sqrt{x}};q^2\Big)\,d_{q^2}t\\&=(1+q)y^{-\alpha/2}x^{-\eta-(\alpha+\beta)/2}\frac{(q^2;q^2)_\infty}{(q^{2\alpha};q^2)_\infty}J_{2\eta+\alpha}(\sqrt{xy};q^2).\end{aligned}$$

Hence,

$$\begin{aligned}K_{q^2}^{\eta,\alpha}S_{q^2}^{\eta+\alpha,\beta}\phi(x)&=(1-q^2)^{\alpha-1}x^{-(\alpha+\beta)/2}\int_0^\infty y^{-(\alpha+\beta)/2}\phi(y)J_{2\eta+\alpha}(\sqrt{xy};q^2)\,d_{q^2}y\\&=(1-q^2)^{\alpha}S_{q^2}^{\eta,\alpha+\beta}(x).\end{aligned}$$

□

Proposition 3.10. *Let η, α, and β be complex numbers satisfying $\Re(\beta+\alpha)>0$ and $\Re(2\eta+\alpha)>-1$. If $\phi\in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$, then*

$$\mathcal{S}_{q^2}^{\eta+\alpha,\beta}S_{q^2}^{\eta,\alpha}\phi(x)=(1-q^2)^{-\beta-\alpha}I_{q^2}^{\eta,\alpha+\beta}\phi(x)\quad(x\in\mathbb{R}_{q^2,+}).$$

Proof. Let $x\in\mathbb{R}_{q^2,+}$ be fixed. Using definitions (3-10) and (3-19) we get

$$S_{q^2}^{\eta+\alpha,\beta}S_{q^2}^{\eta,\alpha}\phi(x)=\frac{x^{-\beta/2}}{(1-q^2)^2}\int_0^\infty\Big(t^{-(\alpha+\beta)/2}J_{2\eta+2\alpha+\beta}(\sqrt{tx};q^2)\times\int_0^\infty y^{-\alpha/2}J_{2\eta+\alpha}(\sqrt{ty};q^2)\phi(y)\,d_{q^2}y\Big)\,d_{q^2}t. \tag{3-34}$$

From (2-17), there exists $M>0$ such that

$$|J_{2\eta+\alpha}(\sqrt{ty};q^2)|\le M(ty)^{\Re(\eta+\alpha/2)}\quad\text{for all }\ y,\ t\in\mathbb{R}_{q^2,+}.$$

Consequently,

$$\begin{aligned}\Big|\int_0^\infty t^{-(\alpha+\beta)/2}J_{2\eta+2\alpha+\beta}(\sqrt{tx};q^2)\int_0^\infty y^{-\alpha/2}J_{2\eta+\alpha}(\sqrt{ty};q^2)\phi(y)\,d_{q^2}y\,d_{q^2}t\Big|\\\le M\|\phi\|_{q^2,\eta}\Big|\int_0^\infty t^{\eta-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{tx};q^2)\,d_{q^2}t\Big|<\infty.\end{aligned}$$

Hence, the series is absolutely convergent, and we can interchange the order of summation to obtain

$$\mathcal{S}_{q^2}^{\eta+\alpha,\beta}S_{q^2}^{\eta,\alpha}\phi(x)=\frac{x^{-\beta/2}(1+q)}{(1-q^2)^2}\int_0^\infty\Big(y^{-\alpha/2}\phi(y)\times\int_0^\infty t^{1-\beta-\alpha}J_{2\eta+2\alpha+\beta}(\sqrt{x}t;q^2)J_{2\eta+\alpha}(\sqrt{y}t;q^2)\,d_qt\Big)\,d_{q^2}y.$$

Therefore, applying Corollary 2.5 with $\Re(\beta+\alpha)>0$ and $\Re(2\eta+\alpha)>-1$, we obtain

$$\begin{aligned}\mathcal{S}_{q^2}^{\eta+\alpha,\beta} S_{q^2}^{\eta,\alpha}\phi(x) &= \frac{(1-q^2)^{-\beta-\alpha}x^{-\eta-1}}{\Gamma_{q^2}(\beta+\alpha)}\int_0^x y^{\eta}(q^2y/x;q^2)_{\alpha+\beta-1}\phi(y)\,d_{q^2}y \\ &= (1-q^2)^{-\beta-\alpha} I_{q^2}^{\eta,\alpha+\beta}\phi(x). \qquad\square\end{aligned}$$

Proposition 3.11. *Let η, α, and β be complex numbers satisfying $\Re(\beta+\alpha)>0$ and $\Re(2\eta+\alpha)>-1$. If $\phi\in L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$, then*

$$\mathcal{S}_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) = (1-q^2)^{-\beta-\alpha}\mathcal{K}_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x\in\mathbb{R}_{q^2,+}).$$

Proof. Let $x\in\mathbb{R}_{q^2,+}$ be fixed. Using definitions (3-10) and (3-19) we get

$$\text{(3-35)}\quad S_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) = \frac{x^{-\alpha/2}}{(1-q^2)^2}\int_0^\infty\Big(t^{-(\alpha+\beta)/2}J_{2\eta+\alpha}(\sqrt{tx};q^2) \times\int_0^\infty y^{-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{ty};q^2)\phi(y)\,d_{q^2 y}\Big)\,d_{q^2}t.$$

From (2-17), there exists $M>0$ such that

$$|J_{2\eta+2\alpha+\beta}(\sqrt{ty};q^2)|\le M(ty)^{\Re(\eta+\alpha+\beta/2)} \quad \text{for all } y,t\in\mathbb{R}_{q^2,+}.$$

Consequently,

$$\begin{aligned}&\left|\int_0^\infty t^{-(\alpha+\beta)/2}J_{2\eta+\alpha}(\sqrt{tx};q^2)\int_0^\infty y^{-\beta/2}J_{2\eta+2\alpha+\beta}(\sqrt{ty};q^2)\phi(y)\,d_{q^2}y\,d_{q^2}t\right| \\ &\qquad\le M\|\phi\|_{q^2,\eta+\alpha}\left|\int_0^\infty t^{\eta+\alpha/2}J_{2\eta+\alpha}(\sqrt{tx};q^2)\,d_{q^2}t\right|<\infty.\end{aligned}$$

Hence, the series is absolutely convergent, and we can interchange the order of summation to obtain

$$\mathcal{S}_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) = \frac{x^{-\alpha/2}(1+q)}{(1-q^2)^2}\int_0^\infty\Big(y^{-\beta/2}\phi(y) \times\int_0^\infty t^{1-\beta-\alpha}J_{2\eta+2\alpha+\beta}(\sqrt{y}t;q^2)J_{2\eta+\alpha}(\sqrt{x}t;q^2)\,d_qt\Big)\,d_{q^2}y.$$

Therefore, applying Corollary 2.5 with $\Re(\beta+\alpha)>0$ and $\Re(2\eta+\alpha)>-1$, we obtain

$$\begin{aligned}\mathcal{S}_{q^2}^{\eta,\alpha} S_{q^2}^{\eta+\alpha,\beta}\phi(x) &= \frac{(1-q^2)^{-\beta-\alpha}q^{-\eta}x^{\eta}}{\Gamma_{q^2}(\beta+\alpha)}\int_x^\infty y^{-\eta-1}(q^2x/y;q^2)_{\alpha+\beta-1}\phi(q^2y)\,d_{q^2}y \\ &= (1-q^2)^{-\beta-\alpha}\mathcal{K}_{q^2}^{\eta,\alpha+\beta}\phi(x). \qquad\square\end{aligned}$$

4. The multiplying factor method

Titchmarsh [1986, p. 334–339] solved the dual integral equations

$$\int_0^\infty \xi^{2\alpha}\psi(\xi)J_\nu(\rho\xi)\,d\xi = f(\rho),\ (0<\rho<1),$$
$$\int_0^\infty \psi(\xi)J_\nu(\rho\xi)\,d\xi = 0,\ (1<\rho), \tag{4-1}$$

for $\alpha > 0$ by using difficult analysis involving Mellin transforms. His techniques were extended to the case $\alpha > -1$ by Busbridge [1938]. Sneddon [1960] used the Abel integral formula to solve the system in the case when $\nu = 0$ and $\alpha = \pm\frac{1}{2}$. The technique was generalized by Copson [1961] for any α satisfying $-1<\alpha<1$, $\alpha\neq 0$, and $\nu > -1$. In this section, we introduce and solve a q-analog of Copson's result.

Theorem 4.1. *Let α and ν be complex numbers such that $\Re(\nu) > -1$ and let f be a function defined on A_q. Consider the dual q-integral equations*

$$\int_0^\infty \xi^{2\alpha}\psi(\xi)J_\nu(\rho\xi;q^2)d_q\xi = f(\rho), \quad \rho\in A_q, \tag{4-2}$$

$$\int_0^\infty \psi(\xi)J_\nu(\rho\xi;q^2)d_q\xi = 0, \quad \rho\in B_q. \tag{4-3}$$

We consider three cases:

Case I. *If $\alpha = 0$, $\Re(\nu) > -1$, then the q-integral equations* (4-2)–(4-3) (*which are now not dual*) *have a solution of the form*

$$\psi(\xi) = \frac{\xi}{1-q}\int_0^1 t^2 J_\nu(\xi t;q^2)f(t)\,d_qt, \tag{4-4}$$

provided that $t^{\nu+1}f(t)\in L_{q^2}(A_{q^2})$.

Case II. *If $0<\Re(\alpha)<1$ and $\Re(\nu+\alpha)>0$, the dual q-integral equations* (4-2)–(4-3) *have a solution of the form*

$$\psi(\xi) = \frac{\xi^{1-\alpha}}{(1-q)^2(1-q^2)^\alpha}\int_0^1 t^{1-\nu-\alpha}J_{\nu+\alpha}(\xi t;q^2)\, I_{q^2}^{\alpha}[(\,\cdot\,)^{\nu/2}f(\sqrt{\cdot}\,)](t^2)\,d_qt, \tag{4-5}$$

provided that

$$I_{q^2}^{\alpha}[(\,\cdot\,)^{\nu/2}f(\sqrt{\cdot}\,)](\rho^2) \text{ and } I_{q^2}^{\alpha-1}[(\,\cdot\,)^{\nu/2}f(\sqrt{\cdot}\,)](\rho^2) \text{ are } L_{q^2}(A_{q^2}) \text{ functions.} \tag{4-6}$$

Case III. *If $-1<\Re(\alpha)<0$ and $\Re(\nu+\alpha)>-1$, the dual q-integral equations* (4-2)–(4-3) *have a solution of the form*

$$\psi(\xi) = \frac{\xi^{1-\alpha}}{(1-q)(1-q^2)^\alpha}\int_0^1 t^{1-\nu-\alpha}J_{\nu+\alpha}(\xi t;q^2)I_{q^2}^{\alpha}[(\,\cdot\,)^{\nu/2}f(\sqrt{\cdot}\,)](t^2)\,d_qt, \tag{4-7}$$

provided that

(4-8) $I_{q^2}^{\alpha}[(\cdot)^{\nu/2} f(\sqrt{\cdot})](\rho^2)$ *and* $I_{q^2}^{\alpha+1}[(\cdot)^{\nu/2} f(\sqrt{\cdot})](\rho^2)$ *are* $L_{q^2}(A_{q^2})$ *functions.*

Proof. Let $\phi \in L_{q,\nu+\alpha-2}(A_q)$ be a function which shall be defined later on. Define the function ψ on $\mathbb{R}_{q,+}$ by

$$\psi(\xi) = \xi^{1-\alpha} \int_0^1 \phi(t) J_{\nu+\alpha}(\xi t; q^2)\, d_q t. \tag{4-9}$$

Hence ψ is a well-defined function. We now prove that ψ satisfies (4-3). Indeed,

$$\int_0^\infty \psi(\xi) J_\nu(\rho\,\xi; q^2)\, d_q\xi = \int_0^\infty \xi^{1-\alpha} J_\nu(\rho\xi; q^2) \int_0^1 \phi(t) J_{\nu+\alpha}(\xi t; q^2)\, d_q t\, d_q\xi. \tag{4-10}$$

From (2-17), there exists $M > 0$ such that

$$|J_{\nu+\alpha}(\xi\, t; q^2)| \le M |(\xi\, t)^{\nu+\alpha}|$$

for all ξ, t in $\mathbb{R}_{q,+}$. Also, $\phi \in L_{q,\nu+\alpha-2}(A_q)$ implies that $\phi \in L_{q,\nu+\alpha}(A_q)$. Consequently,

$$\left| \int_0^\infty \psi(\xi) J_\nu(\rho\,\xi; q^2)\, d_q\xi \right| \le M \|\phi\|_{A_q,\nu+\alpha} \int_0^\infty |\xi^{1+\nu} J_\nu(\rho\,\xi; q^2)|\, d_q\xi < \infty$$

whenever $\Re(\nu) > -1$, where we applied again (2-17). Therefore, the double q-integration in (4-10) is absolutely convergent and we can interchange the order of q-integration to obtain

$$\int_0^\infty \psi(\xi) J_\nu(\rho\,\xi; q^2)\, d_q\xi = \int_0^1 \phi(t) \int_0^\infty \xi^{1-\alpha} J_\nu(\rho\xi; q^2) J_{\nu+\alpha}(\xi t; q^2)\, d_q\xi\, d_q\, t. \tag{4-11}$$

Thus, by replacing α, β by ν, $\nu+\alpha$, respectively, and applying Corollary 2.5, the q-integration in (4-11) vanishes, and this proves (4-3). In the following we prove (4-2). We distinguish among three cases.

Case I. $\alpha = 0$ and $\Re(\nu) > -1$. By (4-11), (2-21), Equation (4-2) is reduced to

$$\frac{1-q}{\rho^2} \int_0^1 \phi(t) \delta_{\rho,t}\, d_q t = f(\rho), \tag{4-12}$$

that is,

$$\phi(\rho) = \frac{\rho^2 f(\rho)}{1-q}. \tag{4-13}$$

Substituting (4-13) in (4-9), we obtain (4-4).

Case II. $0 < \Re(\alpha) < 1$ and $\Re(\nu+\alpha) > 0$. From (2-15), we obtain

$$\psi(\xi) = -q^{\nu+\alpha-1}(1-q)\xi^{-\alpha}\int_0^1 \phi(t)t^{\nu+\alpha-1}D_{q,t}[t^{1-\nu-\alpha}J_{\nu+\alpha-1}(q^{-1}\xi t;q^2)]\,d_qt.$$

Applying the q-integration by parts rule (2-10), we obtain

$$\begin{aligned}\psi(\xi) = (1-q)\xi^{-\alpha}\int_0^1 D_{q,t}[t^{\nu+\alpha-1}\phi(t)]t^{1-\nu-\alpha}J_{\nu+\alpha-1}(\xi t;q^2)\,d_qt \\ -(1-q)q^{\nu+\alpha-1}\xi^{-\alpha}\phi(1)J_{\nu+\alpha-1}(q^{-1}\xi;q^2) \\ +(1-q)q^{\nu+\alpha-1}\xi^{-\alpha}\lim_{n\to\infty}\phi(q^n)J_{\nu+\alpha-1}(q^{n-1}\xi;q^2).\end{aligned}$$

Since $\phi \in L_{q,\nu+\alpha-2}(A_q)$, we have

$$\lim_{n\to\infty} q^{n(\nu+\alpha-1)}\phi(q^n) = 0.$$

Therefore,

$$\begin{aligned}\psi(\xi) = (1-q)\xi^{-\alpha}\int_0^1 D_{q,t}[t^{\nu+\alpha-1}\phi(t)]t^{1-\nu-\alpha}J_{\nu+\alpha-1}(\xi t;q^2)\,d_qt \\ -(1-q)q^{\nu+\alpha-1}\xi^{-\alpha}\phi(1)J_{\nu+\alpha-1}(q^{-1}\xi;q^2).\end{aligned}\tag{4-14}$$

Substituting (4-14) into (4-2), we obtain

$$\begin{aligned}f(\rho) = -(1-q)q^{\nu+\alpha-1}\phi(1)\int_0^\infty \xi^\alpha J_{\nu+\alpha-1}(q^{-1}\xi;q^2)J_\nu(\rho\xi;q^2)\,d_q\xi \\ +(1-q)\int_0^\infty \xi^\alpha\int_0^1 \varphi(t)t^{1-\nu-\alpha}J_{\nu+\alpha-1}(\xi t;q^2)J_\nu(\rho\xi;q^2)\,d_qt\,d_q\xi,\end{aligned}\tag{4-15}$$

where for the convenience of the reader, we set

$$\varphi(t) := D_{q,t}[t^{\nu+\alpha-1}\phi(t)] \quad (t\in A_q).\tag{4-16}$$

Since $\rho \in A_q$, then from Corollary 2.5, the first q-integral in (4-15) vanishes. As for the second double q-integral, the conditions on ϕ imply that $\varphi \in L_q(A_q)$. Therefore, under the conditions on ν and α, the double q-integration is absolutely convergent; and we can interchange the order of q-integration to obtain

$$f(\rho) = (1-q)\int_0^1 \varphi(t)t^{1-\nu-\alpha}\int_0^\infty \xi^\alpha J_{\nu+\alpha-1}(\xi t;q^2)J_\nu(\rho\xi;q^2)\,d_q\xi\,d_qt.$$

But under the conditions that $\Re(\alpha) < 1$ and $\Re(\nu+\alpha) > 0$, we have

$$\int_0^\infty \xi^\alpha J_{\nu+\alpha-1}(t\xi;q^2)J_\nu(\rho\xi;q^2)\,d_q\xi = \begin{cases}\dfrac{(1-q)(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}t^{\nu+\alpha-1}\rho^{-\nu-2\alpha}(q^2t^2/\rho^2;q^2)_{-\alpha} & \text{if } \rho\ge t,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$f(\rho) = \frac{(1-q)^2(1-q^2)^{\alpha}}{\Gamma_{q^2}(1-\alpha)} \rho^{-\nu-2\alpha} \int_0^{\rho} (q^2t^2/\rho^2; q^2)_{-\alpha} \varphi(t)\, d_q t$$
$$= \frac{(1-q)^2(1-q^2)^{\alpha} \rho^{-(\nu+2\alpha)}}{(1+q)\Gamma_{q^2}(1-\alpha)} \int_0^{\rho^2} (q^2t/\rho^2; q^2)_{-\alpha} \frac{\varphi(\sqrt{t})}{\sqrt{t}}\, d_{q^2} t. \tag{4-17}$$

Now, we apply the q-Abel integral (Theorem 3.2) under the conditions in (4-6). This gives

$$\frac{\varphi(\rho)}{\rho} = \frac{(1+q)}{(1-q)^2(1-q^2)^{\alpha}} (D_{q^2} I^{\alpha}_{q^2} g)(\rho^2), \quad g(\rho^2) := \rho^{\nu} f(\rho),\ \ \rho \in A_q. \tag{4-18}$$

From (4-16) and the fact that $\phi \in L_{q,\nu+\alpha-1}(A_q)$, we obtain

$$\phi(t) = t^{-\nu+\alpha-1} \int_0^t \varphi(\rho)\, d_q \rho.$$

Thus, from (4-18),

$$\phi(t) = \frac{(1+q)t^{1-\nu-\alpha}}{(1-q)^2(1-q^2)^{\alpha}} \int_0^t \rho (D_{q^2} I^{\alpha}_{q^2} g)(\rho^2)\, d_q \rho$$
$$= \frac{t^{1-\nu-\alpha}}{(1-q)^2(1-q^2)^{\alpha}} \int_0^{t^2} (D_{q^2} I^{\alpha}_{q^2} g)(\rho)\, d_{q^2} \rho$$
$$= \frac{t^{1-\nu-\alpha}}{(1-q)^2(1-q^2)^{\alpha}} (I^{\alpha}_{q^2} g)(t^2). \tag{4-19}$$

Hence, $\psi(\cdot)$ is given by (4-9).

Case III. $-1 < \Re(\alpha) < 0$ and $\Re(\nu+\alpha) > -1$. We substitute (4-9) in (4-2) to obtain

$$f(\rho) = \int_0^{\infty} \xi^{\alpha+1} \int_0^1 \phi(t) J_{\nu+\alpha}(\xi t; q^2) J_{\nu}(\rho\xi; q^2)\, d_q t\, d_q \xi. \tag{4-20}$$

If $\phi \in L_{q,\nu+\alpha}(A_q)$ and $\Re(\nu+\alpha) > -1$, the double q-integration in (4-20) is absolutely convergent. Therefore, we can interchange the order of q-integration to obtain

$$f(\rho) = \int_0^1 \phi(t) \int_0^{\infty} \xi^{\alpha+1} J_{\nu+\alpha}(\xi t; q^2) J_{\nu}(\rho\xi; q^2)\, d_q \xi\, d_q t.$$

Since $\Re(\alpha) < 0$, then $\Re(\nu) > \Re(\nu+\alpha) > -1$. Thus applying Corollary 2.5 yields

$$f(\rho) = \frac{\rho^{-\nu-2\alpha-2}(1-q)}{\Gamma_{q^2}(-\alpha)(1-q^2)^{-\alpha}} \int_0^{\rho^2} \left(\frac{q^2 t}{\rho^2}; q^2\right)_{-\alpha-1} t^{(\nu+\alpha-1)/2} \phi(\sqrt{t})\, d_{q^2} t.$$

Since f satisfies (4-8), we can apply the q-Abel integral (Theorem 3.2) to obtain

$$\begin{aligned}\phi(\rho)&=\frac{\rho^{1-\nu-\alpha}}{(1-q)(1-q^2)^{\alpha}}D_{q^2,\rho^2}I_{q^2}^{\alpha+1}[(\cdot)^{\nu/2}f(\sqrt{\cdot})](\rho^2)\\&=\frac{\rho^{1-\nu-\alpha}}{(1-q)(1-q^2)^{\alpha}}I_{q^2}^{\alpha}[(\cdot)^{\nu/2}f(\sqrt{\cdot})](\rho^2),\end{aligned}\tag{4-21}$$

where $\rho\in A_q$. Thus, substituting (4-21) into (4-9) gives (4-7) and completes the proof of the theorem. □

Example 4.2. If $\nu=\frac{1}{2}$ and $\alpha=\frac{1}{4}$, then the solution of the system

$$\begin{aligned}&\frac{\rho^{-1/2}\sqrt{1-q^2}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty\psi(\xi)\sin\Big(\frac{\rho\xi}{1-q};q^2\Big)\,d_q\xi=f(\rho),\qquad\rho\in A_q,\\&\frac{\rho^{-1/2}\sqrt{1-q^2}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty\xi^{-\frac{1}{2}}\psi(\xi)\sin\Big(\frac{\rho\xi}{1-q};q^2\Big)\,d_q\xi=0,\qquad\rho\in B_q,\end{aligned}$$

is

$$\psi(\xi)=\frac{\xi^{3/4}}{(1-q)^2(1-q^2)^{1/4}}\int_0^1t^{1/4}J_{3/4}(\xi t;q^2)I_{q^2}^{1/4}[(\cdot)^{-1/4}f(\sqrt{\cdot})](t^2)\,d_qt,$$

where $I_{q^2}^{1/4}[(\cdot)^{1/4}f(\sqrt{\cdot})](t^2)$ and $I_{q^2}^{-3/4}[(\cdot)^{1/4}f(\sqrt{\cdot})](t^2)$ are $L_{q^2}(A_{q^2})$ functions.

In particular, if $f(t)=t^{3/2}$, then

$$\psi(\xi)=\frac{\xi^{3/4}\Gamma_{q^2}(3/2)}{(1-q)^2(1-q^2)^{1/4}\Gamma_{q^2}(7/4)}\int_0^1t^{7/4}J_{3/4}(\xi t;q^2)\,d_qt,$$

and by (2-16), we obtain

$$\psi(\xi)=\frac{\Gamma_{q^2}(3/2)}{(1-q)(1-q^2)^{1/4}\Gamma_{q^2}(7/4)}\xi^{-1/4}J_{7/4}(\xi;q^2).$$

Example 4.3. If $\upsilon=2$ and $\alpha=-\frac{1}{2}$, then the solution of the system

$$\begin{aligned}&\int_0^\infty\xi^{-1}\psi(\xi)J_2(\rho\xi;q^2)\,d_q\xi=f(\rho),\qquad\rho\in A_q,\\&\int_0^\infty\psi(\xi)J_2(\rho\xi;q^2)d_q\xi=0,\qquad\rho\in B_q,\end{aligned}$$

is

$$\psi(\xi)=\frac{\xi^{3/2}(1-q^2)^{1/2}}{(1-q)}\int_0^1t^{-1/2}J_{3/2}(\xi t;q^2)I_{q^2}^{-1/2}[(\cdot)f(\sqrt{\cdot})](t^2)\,d_qt,$$

where $I_{q^2}^{-1/2}[(\cdot)f(\sqrt{\cdot})](t^2)$ and $I_{q^2}^{1/2}[(\cdot)f(\sqrt{\cdot})](t^2)$ are $L_{q^2}(A_{q^2})$ functions.

In particular, if we take $f(t) = t^{-1}$ then

$$\psi(\xi) = \frac{\xi^{3/2}(1-q^2)^{1/2}\Gamma_{q^2}(3/2)}{(1-q)} \int_0^1 t^{-1/2} J_{3/2}(\xi t; q^2)\, d_q t,$$

and by applying (2-15), we can see that

$$\psi(\xi) = -\frac{q(1-q^2)\Gamma_{q^2}(3/2)}{\Gamma_{q^2}(1/2)} \sin\left(\frac{q^{-1}\xi}{1-q}; q\right) - \xi.$$

5. The fractional q-calculus approach

In this section, we solve certain dual q-integral equations by using the fractional q-calculus approach. Peters [1961] solved the dual integral equations

$$\int_0^\infty \xi^{-2\alpha}\Psi(\xi) J_\mu(2\rho\xi) d\xi = F(\rho) \quad (0 < \rho < 1),$$
$$\int_0^\infty \xi^{-2\beta}\Psi(\xi) J_\nu(2\rho\xi) d\xi = G(\rho) \quad (\rho > 1),$$

by using fractional calculus. Here we give a q-type analog of Peters' problem.

Theorem 5.1. *Let α, β, μ, and ν be complex numbers and let*

$$\lambda := \tfrac{1}{2}(\mu+\nu) - (\alpha-\beta) > -1.$$

Assume that

$$\Re(\mu) > -1, \quad \Re(\nu) > -1, \quad \Re(\lambda) > -1, \quad \text{and} \quad \Re(\lambda-\mu-2\alpha) > 0.$$

Let $f \in L_{q^2,\mu/2+\alpha}(A_{q^2})$ and $g \in L_{q^2,-\mu/2+\alpha-1}(B_{q^2})$. Then the dual q^2-integral equations

(5-1)
$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho) J_\mu(\sqrt{\rho\xi}; q^2) d_{q^2}\rho = f(\xi) \quad (\xi \in A_{q^2}),$$
$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}\psi(\rho) J_\nu(\sqrt{\rho\xi}; q^2) d_{q^2}\rho = g(\xi) \quad (\xi \in B_{q^2}),$$

have the solution

$$\psi(\xi) = (1-q^2)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda(\sqrt{\rho\xi}; q^2) I_{q^2}^{\mu/2+\alpha,\lambda-\mu} f(\rho)\, d_{q^2}\rho$$
$$+ (1-q^2)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda(\sqrt{\rho\xi}; q^2)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda} g(\rho)\, d_{q^2}\rho,$$

in $L_{q^2,\mu/2-\alpha}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\nu/2-\beta}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\nu/2-\beta-\gamma}(\mathbb{R}_{q^2,+})$, for γ satisfying

(5-2)
$$1 + \Re(\nu) > \Re(\gamma) > \max\{0, \Re(\nu-\lambda)\}.$$

Proof. We shall extend the domain of the functions f and g in (5-1) to be $\mathbb{R}_{q^2,+}$ by introducing the functions f_1 and g_1, where

$$f_1 \equiv f \text{ on } A_{q^2} \quad \text{and} \quad f_1 \equiv 0 \text{ on } B_{q^2},$$
$$g_1 \equiv 0 \text{ on } A_{q^2} \quad \text{and} \quad g_1 \equiv g \text{ on } B_{q^2},$$

respectively. From (3-19), (5-1) can be written as

$$\begin{aligned}(1-q^2)S_{q^2}^{\mu/2-\alpha,2\alpha}\psi(\xi) &= f(\xi) \quad (\xi \in A_{q^2}),\\ (1-q^2)S_{q^2}^{\nu/2-\beta,2\beta}\psi(\xi) &= g(\xi) \quad (\xi \in B_{q^2}).\end{aligned} \tag{5-3}$$

Under the conditions on the functions ψ, f, and g, we can apply Propositions 3.8 and 3.9 to obtain

$$\frac{I_{q^2}^{\mu/2+\alpha,\lambda-\mu} f(\xi)}{(1-q^2)^{\lambda-\mu+1}} = \frac{I_{q^2}^{\mu/2+\alpha,\lambda-\mu} S_{q^2}^{\mu/2-\alpha,2\alpha}\psi(\xi)}{(1-q^2)^{\lambda-\mu+1}} = S_{q^2}^{\mu/2-\alpha,\lambda-\mu+2\alpha}\psi(\xi)$$

and

$$\frac{\mathcal{K}_{q^2}^{\mu/2-\alpha,\nu-\lambda} g(\xi)}{(1-q^2)^{\nu-\lambda+1}} = \frac{\mathcal{K}_{q^2}^{\mu/2-\alpha,\nu-\lambda} S_{q^2}^{\nu/2-\beta,2\beta}\psi(\xi)}{(1-q^2)^{\nu-\lambda+1}} = S_{q^2}^{\mu/2-\alpha,\lambda-\mu+2\alpha}\psi(\xi).$$

Thus, the last two identities can be described by

$$S_{q^2}^{\mu/2-\alpha,\lambda-\mu+2\alpha}\psi(\xi) = h(\xi) \quad (\xi \in R_{q^2,+}), \tag{5-4}$$

where h is the function defined by

$$h(\xi) := \begin{cases}(1-q^2)^{\mu-\lambda-1} I_{q^2}^{\mu/2+\alpha,\lambda-\mu} f(\xi) & \text{if } \xi \in A_{q^2},\\ (1-q^2)^{\lambda-\nu-1}\mathcal{K}_{q^2}^{\mu/2-\alpha,\nu-\lambda} g(\xi) & \text{if } \xi \in B_{q^2}.\end{cases}$$

Thus, applying the inversion formula in Proposition 3.6 yields

$$\psi(\xi) = S_{q^2}^{-\mu/2+\alpha+\lambda,-\lambda+\mu-2\alpha} h(\xi).$$

In other words,

$$\begin{aligned}\psi(\xi) &= (1-q^2)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda(\sqrt{\rho\xi};q^2) I_{q^2}^{\mu/2+\alpha,\lambda-\mu} f(\rho)\,d_{q^2}\rho\\ &\quad + (1-q^2)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda(\sqrt{\rho\xi};q^2)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda} g(\rho)\,d_{q^2}\rho.\end{aligned} \tag{5-5}$$

From (2-18), the q^2-integral on $[1,\infty)$ in (5-5) is absolutely convergent, and from Proposition 3.3, one can verify that if $\Re(\lambda-\mu-2\alpha) > 0$ then the q^2-integral on $[0, 1]$ in (5-5) is absolutely convergent. Hence, the function ψ is well defined. Also, using (2-18),

$$\psi \in L_{q^2,\mu/2-\alpha}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\nu/2-\beta}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\nu/2-\beta-\gamma}(\mathbb{R}_{q^2,+})$$

when γ satisfies the condition in (5-2). □

Example 5.2. If $\nu=\frac{1}{2}$, $\mu=-\frac{1}{2}$, $\alpha=0$, and $\beta=\frac{1}{2}$, then $\lambda=\frac{1}{2}$ and the solution of the system

$$\frac{\sqrt{1-q^2}\rho^{-1/4}}{\Gamma_{q^2}(1/2)}\int_0^\infty \xi^{-1/4}\psi(\xi)\cos\left(\frac{\sqrt{q\rho\xi}}{1-q};q\right)d_{q^2}\xi=f(\rho),\quad \rho\in A_{q^2},$$
$$\frac{\sqrt{1-q^2}\rho^{-3/4}}{\Gamma_{q^2}(1/2)}\int_0^\infty \xi^{-3/4}\psi(\xi)\sin\left(\frac{\sqrt{\rho\xi}}{1-q};q\right)d_{q^2}\xi=g(\rho),\quad \rho\in B_{q^2},$$

is

$$\psi(\xi)=\frac{\xi^{1/4}}{(1-q^2)^{3/2}\Gamma_{q^2}(1/2)}\int_0^1 \rho^{-1/4}\sin\left(\frac{\sqrt{\rho\xi}}{1-q};q\right)I_{q^2}^{-1/4,1}f(\rho)\,d_{q^2}\rho + \frac{\xi^{1/4}}{(1-q^2)^{3/2}\Gamma_{q^2}(1/2)}\int_1^\infty \rho^{-1/4}\sin\left(\frac{\sqrt{\rho\xi}}{1-q};q\right)g(\rho)\,d_{q^2}\rho.$$

The solution is in $L_{q^2,-1/4}(\mathbb{R}_{q^2,+})\cap L_{q^2,-1/4-\gamma}(\mathbb{R}_{q^2,+})$, where $0<\Re(\gamma)<\frac{3}{2}$.

In particular, if $f(\rho)=g(\rho)=\rho^{-1/4}$, then

$$\psi(\xi)=\frac{\xi^{1/4}}{(1-q)\sqrt{1-q^2}\Gamma_{q^2}(3/2)}\int_0^1 \sin\left(\frac{\sqrt{\xi}\rho}{1-q};q\right)d_q\rho + \frac{\xi^{1/4}}{(1-q)\sqrt{1-q^2}\Gamma_{q^2}(1/2)}\int_1^\infty \sin\left(\frac{\sqrt{\xi}\rho}{1-q};q\right)d_q\rho;$$

and by applying (2-15) and Proposition 2.10, we obtain

$$\psi(\xi)=\frac{\xi^{-1/4}}{\sqrt{1-q^2}}\left[\frac{q^{3/2}}{\Gamma_{q^2}(1/2)}-\frac{1}{\Gamma_{q^2}(3/2)}\right]\cos\left(\frac{\sqrt{\xi}}{(1-q)\sqrt{q}};q\right)+\frac{\xi^{-1/4}}{\sqrt{1-q^2}\Gamma_{q^2}(1/2)}.$$

6. q-Mellin transform method

Nasim [1986] showed that the dual integral equations

$$\int_0^\infty t^{-2\alpha}J_\nu(xt)[1+w(t)]\phi(t)\,dt=f(x),\quad 0<x<1,$$
$$\int_0^\infty t^{-2\beta}J_\mu(xt)\phi(t)\,dt=g(x),\quad 1<x<\infty,$$

where w is an arbitrary weight function, can be reduced to a single (rather complicated) Fredholm equation of the second kind by using the Mellin transform. In this section we give a q-analog of Nasim's problem where we employ the q-Mellin transform to reduce certain dual q-integral equations into a Fredholm q-integral equation of the second kind.

First, we write down some definitions and results which we use later on. Fitouhi et al. [2006] defined a q-analog of the Mellin transform through the identity

$$\mathcal{M}_q(f)(s) := \int_0^\infty t^{s-1} f(t)\, d_q t.$$

Let α and β be real numbers and $\mathcal{M}_q^{\alpha,\beta}$ be the space of all functions defined on $\mathbb{R}_{q,+}$ such that

$$f|_{A_q} \in L_{q,\alpha-1}(A_q) \quad \text{and} \quad f|_{B_q} \in L_{q,\beta-1}(B_q).$$

The next lemma includes a sufficient condition for the existence of the q-Mellin transform which is slightly different from the one introduced in [Fitouhi et al. 2006].

Lemma 6.1. *Let α, β be real numbers such that $\alpha \le \beta$. If $f \in \mathcal{M}_q^{\alpha,\beta}$, then the q-Mellin transform of the function f exists on the strip $\alpha \le \Re(s) \le \beta$.*

Proof. Assume that $f \in \mathcal{M}_q^{\alpha,\beta}$. Then, using that

$$\mathcal{M}_q f(s) = \int_0^1 t^{s-1} f(t)\, d_q t + \int_1^\infty t^{s-1} f(t)\, d_q t,$$

and the inequalities

$$|t^s| \le t^\alpha \quad \text{for } \Re(s) \ge \alpha \text{ and all } t \in [0, 1],$$

and

$$|t^s| \le t^\beta \quad \text{for } \Re(s) \le \beta \text{ and all } t \in [1, \infty),$$

we obtain

$$|\mathcal{M}_q f(s)| \le \int_0^1 t^{\alpha-1} |f(t)|\, d_q t + \int_1^\infty t^{\beta-1} |f(t)|\, d_q t < \infty,$$

and the desired result follows. □

By $(\alpha_{q,f}, \beta_{q,f})$ we mean the fundamental (largest) strip on which the q-Mellin transform exists for $s \in \mathbb{C}$ such that $\alpha_{q,f} < \Re(s) < \beta_{q,f}$.

Lemma 6.2. *Let α, β be real numbers such that $\alpha < \beta$. If $f \in \mathcal{M}_q^{\alpha,\beta}$, then $\mathcal{M}_q f(s)$ is an analytic function on the simply connected domain defined by the strip $(\alpha_{q,f}, \beta_{q,f})$.*

Proof. If we set $F_n(s) := (1-q)\sum_{-n}^{n} q^{ks} f(q^k)$, we can verify that:

(1) $F_n(s)$ is an entire function for each $n \in \mathbb{N}_0$;

(2) $F_n(s)$ tends uniformly to $\mathcal{M}_q f(s)$ as n tends to ∞ for $\Re(s) \in (\alpha_{q,f}, \beta_{q,f})$.

Hence, $\mathcal{M}_q f(s)$ is an analytic function in the domain defined by the strip $\Re(s) \in (\alpha_{q,f}, \beta_{q,f})$. □

A direct consequence of the previous lemma is that

$$\int_C x^{-s}\mathcal{M}_q f(s)\,ds = (1-q)\sum_{-\infty}^{\infty} f(q^k)\int_C x^{-s}q^{ks}\,ds$$

for any contour that lies in the interior of the domain defined by the strip $\Re(s) \in (\alpha_{q,f}, \beta_{q,f})$. Fitouhi et al. [2006] chose C to be the contour that connects the points $c - i\pi/\log(q)$ and $c + i\pi/\log(q)$, where $c \in (\alpha_{q,f}, \beta_{q,f})$, to introduce and prove the q-Mellin inversion formula

$$f(x) = \frac{\log(q)}{2i\pi(1-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \mathcal{M}_q(f)(s)x^{-s}\,ds, \quad x \in \mathbb{R}_{q,+}. \tag{6-1}$$

In addition, they introduced a q-Parseval's formula for the Mellin transform, under suitable conditions on the functions f and g,

$$\int_0^{\infty} f(xt)g(t)\,d_qt = \frac{\log(q)}{2i\pi(1-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \mathcal{M}_q(f)(s)\mathcal{M}_q(g)(1-s)x^{-s}\,ds, \tag{6-2}$$

where c is in the fundamental strip of defining f, g. They also proved:

Theorem 6.3. *Let K and g be a pair of functions defined on $\mathbb{R}_{q,+}$ such that the strip $I_{K;g} = (\beta_{q,K}, \alpha_{q,K}) \cap (1-\beta_{q,g}, 1-\alpha_{q,g})$ is not empty. If*

$$f(x) = \int_0^{\infty} g(t)K(xt)\,d_qt,$$

then

$$\mathcal{M}_q(f)(s) = \mathcal{M}_q(K)(s)\mathcal{M}_q(g)(1-s), \quad s \in I_{K;g}.$$

The q-Mellin transform of the third Jackson q-Bessel has been calculated in [Fitouhi et al. 2006] by using the identity of Koornwinder and Swarttouw [1992, p. 449]:

$$\frac{(t^{-1}z;q)_\infty}{(tz;q)_\infty} = \sum_{-\infty}^{\infty} t^n z^n \frac{(z^2;q)_\infty}{(q;q)_\infty}\,{}_1\phi_1(0;z^2;q,q^{n+1}),$$

where t and z are complex numbers such that $0 < |t| < |z|^{-1}$. Using the same technique, we can prove that

$$\begin{aligned}\mathcal{M}_q(z^\beta J_\alpha(z;q^2)(s)) \\ = (1-q)(1-q^2)^{s+\beta-1}\frac{\Gamma_{q^2}(\frac{1}{2}(\alpha+s+\beta))}{\Gamma_{q^2}(\frac{1}{2}(\alpha-s-\beta+2))}, \quad \Re(s) > -\Re(\alpha+\beta).\end{aligned} \tag{6-3}$$

The following two lemmas are needed in the sequel.

Lemma 6.4. *If* $\Re(s) < \nu + 2\alpha$ *and*

$$h_1^* := \mathcal{M}_q^{-1}\left(\frac{\Gamma_{q^2}(\frac{1}{2}\nu - \frac{1}{2}s + \alpha)}{\Gamma_{q^2}(\frac{1}{2}\mu - \frac{1}{2}s + \beta + 1)}\right),$$

then

$$h_1^*(x) = \frac{(1+q)x^{-\nu-2\alpha}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu-\alpha+\beta+1)}(q^2/x^2; q^2)_{-\nu/2+\mu/2-\alpha+\beta} \quad (x \in \mathbb{R}_{q,+}). \tag{6-4}$$

Proof. Applying (6-1) we obtain

$$\begin{aligned} h_1^*(x) &= \frac{\log(q)}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \mathcal{M}_q(h_1^*)(s)x^{-s}\, ds \\ &= \frac{\log(q)}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} x^{-s} \frac{\Gamma_{q^2}(\frac{1}{2}\nu - \frac{1}{2}s + \alpha)}{\Gamma_{q^2}(\frac{1}{2}\mu - \frac{1}{2}s + \beta + 1)}\, ds \\ &= \frac{\log(q)(1-q^2)^{\mu/2-\nu/2-\alpha+\beta+1}}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} x^{-s} \frac{(q^{\mu-s+2\beta+2}; q^2)_\infty}{(q^{\nu-s+2\alpha}; q^2)_\infty}\, ds. \end{aligned}$$

Applying the q-binomial theorem (2-13) with $z = q^{\nu-s+2\alpha}$ and for $\Re(s) < \nu + 2\alpha$, we obtain

$$h_1^*(x) = \frac{\log(q)(1-q^2)^{\mu/2-\nu/2-\alpha+\beta+1}}{2i\pi(1-q)} \sum_{n=0}^{\infty} \frac{(q^{\mu-\nu-2\alpha+2\beta+2}; q^2)_n}{(q^2; q^2)_n} q^{n(\nu+2\alpha)} \times \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} (q^n x)^{-s}\, ds.$$

Therefore, if $x := q^m$ $(m \in \mathbb{Z})$,

$$\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} q^{-s(n+m)}\, ds = \begin{cases} 2i\pi/\log(q) & \text{if } m = -n, \\ 0 & \text{if } m \neq -n. \end{cases}$$

Hence, $h_1^*(q^m) = 0$ for $m \in \mathbb{N}$; and for $m \in -\mathbb{N}_0$,

$$\begin{aligned} h_1^*(q^m) &= \frac{q^{-m(\nu+2\alpha)}(1-q^2)^{\mu/2-\nu/2-\alpha+\beta+1}}{1-q} \frac{(q^{\mu-\nu-2\alpha+2\beta+2}; q^2)_{-m}}{(q^2; q^2)_{-m}} \\ &= \frac{q^{-m(\nu+2\alpha)}(1+q)}{\Gamma_{q^2}(\frac{1}{2}\mu - \frac{1}{2}\nu - \alpha + \beta + 1)}(q^{2-2m}; q^2)_{\mu/2-\nu/2-\alpha+\beta}. \end{aligned}$$

Thus $h_1^*(x)$ is given by (6-4) for all $x \in \mathbb{R}_{q,+}$. □

Lemma 6.5. *If* $\Re(s) > -\nu + 2\alpha + 2$ *and*

$$h_2^* := \mathcal{M}_q^{-1}\left(\frac{\Gamma_{q^2}(\frac{1}{2}\nu + \frac{1}{2}s - \alpha - 1)}{\Gamma_{q^2}(\frac{1}{2}\mu + \frac{1}{2}s - \beta)}\right),$$

then, for $x \in \mathbb{R}_{q,+}$,

$$h_2^*(x) = \frac{(1+q)x^{\nu-2\alpha-2}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu+\alpha-\beta+1)}(q^2x^2;q^2)_{-\nu/2+\mu/2+\alpha-\beta}. \tag{6-5}$$

Proof. The proof is similar to the proof of Lemma 6.4 and is omitted. □

In the following, we use the q-Mellin transform to reduce the dual q-integral equations

$$\begin{aligned} \int_0^\infty u^{-2\alpha}\psi(u)[1+w(u)]J_\nu(u\rho;q^2)\,d_qu &= f(\rho), \quad \rho \in A_q, \\ \int_0^\infty u^{-2\beta}\psi(u)J_\mu(u\rho;q^2)\,d_qu &= g(\rho), \quad \rho \in B_q, \end{aligned} \tag{6-6}$$

to a Fredholm q-integral equation of the second kind. Before we start our mission, we set the following notation. Let $\{H_1, H_1^*\}$ and $\{H_2, H_2^*\}$ be the pair of functions defined by

$$H_1(s) := \mathcal{M}_q(u^{-2\alpha}J_\nu(u;q^2))(s) = (1-q)(1-q^2)^{s-2\alpha-1}\frac{\Gamma_{q^2}(\frac{1}{2}\nu+\frac{1}{2}s-\alpha)}{\Gamma_{q^2}(\frac{1}{2}\nu-\frac{1}{2}s+\alpha+1)},$$

$$H_1^*(s) := (1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\frac{\Gamma_{q^2}(\frac{1}{2}\nu-\frac{1}{2}s+\alpha+1)}{\Gamma_{q^2}(\frac{1}{2}\mu-\frac{1}{2}s+\beta+1)},$$

where $2\alpha-\nu < \Re(s) < 2+2\alpha+\nu$ and

$$H_2(s) := \mathcal{M}_q(u^{-2\beta}J_\mu(u;q^2))(s) = (1-q)(1-q^2)^{s-2\beta-1}\frac{\Gamma_{q^2}(\frac{1}{2}\mu+\frac{1}{2}s-\beta)}{\Gamma_{q^2}(\frac{1}{2}\mu-\frac{1}{2}s+\beta+1)},$$

$$H_2^*(s) := (1-q^2)^{\nu/2-\mu/2-\alpha+\beta}\frac{\Gamma_{q^2}(\frac{1}{2}\nu+\frac{1}{2}s-\alpha)}{\Gamma_{q^2}(\frac{1}{2}\mu+\frac{1}{2}s-\beta)},$$

where $\Re(s) > \max\{2\beta-\mu,\ 2\alpha-\nu\}$. It is worth noting that for

$$\max\{2\alpha-\nu, 2\beta-\mu\} < \Re(s) < 2+2\alpha+\nu,$$

we have

$$H_1(s)H_1^*(s) = H_2(s)H_2^*(s) = K(s), \tag{6-7}$$

where $K(s)$ is the function defined for $\Re(s) > 2\alpha-\nu$ by

$$K(s) := \mathcal{M}_q(k(u))(s), \quad k(u) := u^{\nu/2-\mu/2-\alpha-\beta}J_{\nu/2+\mu/2-\alpha+\beta}(u;q^2).$$

Theorem 6.6. *Let* α, β, γ, *and* ν *be real parameters satisfying*

$$\nu > 0 \quad \textit{and} \quad \max\{2\alpha-\nu, 2\beta-\mu\} < 2+2\alpha+\nu.$$

Let f and g be functions defined on A_q and B_q, respectively. Let f_1 and g_1 be the extensions of f and g defined on $\mathbb{R}_{q,+}$ by

$$f_1 \equiv (\cdot)^{-2\alpha} f(\cdot) \text{ on } A_q, \quad f_1 \equiv 0 \text{ on } B_q,$$
$$g_1 \equiv 0 \text{ on } A_q, \quad g_1 \equiv (\cdot)^{-2\beta} g(\cdot) \text{ on } B_q.$$

Assume that the q-Mellin transforms of the functions ψ, ψw, f_1, and g_1 exist on Ω_1, $\Omega_1 \neq \phi$, where

$$\Omega_1 := (\alpha_{q,f_1}, \beta_{q,f_1}) \cap (\alpha_{q,g_1}, \beta_{q,g_1}) \cap (1-\beta_{q,\psi}, 1-\alpha_{q,\psi}) \cap (1-\beta_{q,w\psi}, 1-\alpha_{q,w\psi}).$$

Set

$$\eta := -\tfrac{1}{4}\nu + \tfrac{3}{4}\mu + \tfrac{1}{2}\alpha + \tfrac{3}{2}\beta + 1, \quad \lambda := \nu - \mu - 2\alpha - 2\beta - 2.$$

Then the system (6-6) can be reduced to a Fredholm q-integral equation of the form

$$\psi(t) = A(t) - t^{\mu/2-\nu/2+\alpha+\beta+1} \int_0^\infty u^{\nu/2-\mu/2-\alpha-\beta} \psi(u) w(u) L(t,u)\, d_q u,$$

where

$$\begin{aligned} A(t) &= (1-q^2)^{\nu/2-\mu/2+\alpha-\beta+1} \\ &\quad \times S_q^{\eta,\lambda}\big(H(1-\rho)\rho^{-\mu-2\beta-2} I_{q^2}^{-\nu/2+\mu/2-\alpha+\beta}[(\cdot)^{\nu/2} f(\sqrt{\cdot})](\rho^2)\big) \\ &\quad + (1-q^2)^{\nu/2-\mu/2+\alpha-\beta+1} \\ &\quad \times S_q^{\eta,\lambda}\big(H(\rho-q^{-1})\rho^{-2\beta} \mathcal{K}_{q^2}^{\nu/2-\alpha+\beta,-\nu/2+\mu/2+\alpha-\beta}[g(q^{-1}\sqrt{\cdot})](\rho^2)\big); \end{aligned}$$

$H(x-a)$ is the Heaviside function defined to be 1 if $x \geq 0$ and 0 otherwise; and

$$\begin{aligned} L(t,u) = \frac{(1-q)q^{\nu/2+\mu/2-\alpha+\beta}}{t^2-u^2} &\times \big[t J_{\nu/2+\mu/2-\alpha+\beta+1}(t;q^2) J_{\nu/2+\mu/2-\alpha+\beta}(q^{-1}u;q^2) \\ &\quad - u J_{\nu/2+\mu/2-\alpha+\beta+1}(u;q^2) J_{\nu/2+\mu/2-\alpha+\beta}(q^{-1}t;q^2)\big]. \end{aligned}$$

Proof. Assume that Ψ, Φ, F_1, and G_1 are the q-Mellin transform of the functions ψ, ψw, f_1, and g_1, respectively, on Ω_1. Let

$$\Omega_2 := \{s \in \mathbb{C} : \Re s \in \Omega_1 \text{ and } \Re s > \max\{2\alpha-\nu, 2\beta-\nu\}\}.$$

Applying Theorem 6.3 gives

$$H_1(s)[\Psi(1-s) + \Phi(1-s)] = F_1(s), \quad H_2(s)\Psi(1-s) = G_1(s). \tag{6-8}$$

Multiplying the first equation in (6-8) by $H_1^*(s)$ and the second by $H_2^*(s)$ yields

$$\begin{aligned} K(s)\Psi(1-s) &= H_1^*(s)[F_1(s) - H_1(s)\Phi(1-s)], \\ K(s)\Psi(1-s) &= H_2^*(s)G_1(s), \end{aligned} \tag{6-9}$$

for $\Re(s) \in \Omega := \Omega_2 \cap \{s \in \mathbb{C} : \max\{2\alpha - \nu, 2\beta - \mu\} < \Re(s) < 2 + 2\alpha + \nu\}$, where we used (6-7).

In the following we calculate the value of the q-integral

$$\int_0^\infty k(\rho t)\psi(t)\, d_q t \quad (\rho \in \mathbb{R}_{q,+}). \tag{6-10}$$

We distinguish two cases:

Case 1. $\rho \in A_q$. In this case, from (6-2) and (6-9), we obtain

$$\begin{aligned} \int_0^\infty & k(\rho t)\psi(t)\, d_q t \\ &= \frac{\log(q)}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} K(s)\Psi(1-s)\rho^{-s}\, ds, \\ &= \frac{\log(q)}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} H_1^*(s)[F_1(s) - H_1(s)\Phi(1-s)]\rho^{-s}\, ds. \end{aligned} \tag{6-11}$$

Substituting

$$\rho^{-s} = \frac{1-q}{1-q^{\nu-s+2\alpha}} \rho^{-\nu-2\alpha+1} D_{q,\rho}\rho^{\nu-s+2\alpha}$$

into the third line of (6-11), we obtain

$$\begin{aligned} \int_0^\infty & k(\rho t)\psi(t)\, d_q t \\ &= \frac{\log(q)\rho^{-\nu-2\alpha+1}}{2i\pi(1-q^2)^{-\nu/2+\mu/2-\alpha+\beta+1}} D_{q,\rho} \\ &\quad \times \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} [F_1(s) - H_1(s)\Phi(1-s)]\rho^{\nu-s+2\alpha} \frac{\Gamma_{q^2}(\frac{1}{2}\nu - \frac{1}{2}s + \alpha)}{\Gamma_{q^2}(\frac{1}{2}\mu - \frac{1}{2}s + \beta + 1)}\, ds \\ &= \frac{\log(q)\rho^{-\nu-2\alpha+1}}{2i\pi(1-q^2)^{-\nu/2+\mu/2-\alpha+\beta+1}} D_{q,\rho}\rho^{\nu+2\alpha} \\ &\qquad \times \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} [F_1(s) - L(s)]\mathcal{H}_1^*(s)\rho^{-s}\, ds \\ &= \frac{(1-q)\rho^{-\nu-2\alpha+1}}{(1-q^2)^{-\nu/2+\mu/2-\alpha+\beta+1}} D_{q,\rho}\rho^{\nu+2\alpha} \int_0^\infty [f_1(t) - l(t)] h_1^*\Big(\frac{\rho}{t}\Big)\frac{1}{t}\, d_q t, \end{aligned} \tag{6-12}$$

where $\mathcal{H}_1^*(s) := \dfrac{\Gamma_{q^2}(\frac{1}{2}\nu - \frac{1}{2}s + \alpha)}{\Gamma_{q^2}(\frac{1}{2}\mu - \frac{1}{2}s + \beta + 1)}$, $h_1^*(x)$ is given by Lemma 6.4, and

$$l(\rho) = \mathcal{M}_q^{-1}(H_1(s)\Phi(1-s)).$$

On the other hand, using the q-Mellin transform inversion formula and the q-Parseval relation in (6-2), we obtain

$$\begin{aligned} l(\rho) &= \frac{\log(q)}{2i\pi(1-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} H_1(s)\Phi(1-s)\rho^{-s}\, ds \\ &= \rho^{-2\alpha} \int_0^\infty u^{-2\alpha}\psi(u)w(u)J_\nu(u\rho; q^2)\, d_qu. \end{aligned}$$

On simplifying (6-12), using (6-4), we obtain, for $\rho \in A_q$,

$$\begin{aligned} \int_0^\infty k(\rho t)\psi(t)\, d_qt &= \frac{(1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\rho^{-\nu-2\alpha+1}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu-\alpha+\beta+1)} D_{q,\rho} \\ &\quad \times \int_0^\rho t^{\nu+2\alpha-1}[\rho^{-2\alpha}f(t)-l(t)](q^2t^2/\rho^2; q^2)_{-\nu/2+\mu/2-\alpha+\beta}\, d_qt. \end{aligned} \tag{6-13}$$

Hence,

$$\begin{aligned} &\int_0^\infty k(\rho t)\psi(t)\, d_qt \\ &= \frac{(1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\rho^{-\nu-2\alpha+1}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu-\alpha+\beta+1)} D_{q,\rho}\Bigg[\int_0^\rho t^{\nu-1}f(t)(q^2t^2/\rho^2; q^2)_{-\nu/2+\mu/2-\alpha+\beta}\, d_qt \\ &\qquad -\int_0^\rho t^{\nu-1}(q^2t^2/\rho^2; q^2)_{-\nu/2+\mu/2-\alpha+\beta}\int_0^\infty u^{-2\alpha}\psi(u)w(u)J_\nu(ut; q^2)\, d_qu\, d_qt\Bigg]. \end{aligned}$$

Since $\nu > 0$, we can apply Proposition 2.9 to obtain

$$\begin{aligned} &\int_0^\rho t^{\nu-1}(q^2t^2/\rho^2; q^2)_{-\nu/2+\mu/2-\alpha+\beta}J_\nu(ut; q^2)\, d_qt \\ &\qquad = \frac{\rho^{2\nu}u^\nu(1-q)(q^{\mu+\nu-2\alpha+2\beta+2}; q^2)_\infty}{(1-q^{2\nu})(q^{\mu-\nu-2\alpha+2\beta+2}; q^2)_\infty} \\ &\qquad\quad \times {}_2\phi_2(0, q^{2\nu}; q^{2\nu+2}, q^{\nu+\mu-2\alpha+2\beta+2}; q^2, q^2\rho^2u^2). \end{aligned}$$

Therefore,

$$\int_0^\infty k(\rho t)\psi(t)\, d_qt = m_1(\rho) = I_1 - I_2, \tag{6-14}$$

where

$$I_1 := \frac{(1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\rho^{-\nu-2\alpha+1}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu-\alpha+\beta+1)} D_{q,\rho}\int_0^\rho t^{\nu-1}\left(\frac{q^2t^2}{\rho^2}; q^2\right)_{-\nu/2+\mu/2-\alpha+\beta} f(t)\, d_qt$$

and

$$I_2 := \frac{(1-q)(q^{\nu+\mu-2\alpha+2\beta+2};q^2)_\infty}{(1-q^{2\nu})(q^2;q^2)_\infty}\rho^{1-\nu-2\alpha}$$
$$\times D_{q,\rho}\rho^{2\nu}\int_0^\infty u^{\nu-2\alpha}\psi(u)w(u)\,{}_2\phi_2(0,q^{2\nu};q^{2\nu+2},q^{\nu+\mu-2\alpha+2\beta+2};q^2,q^2\rho^2u^2)\,d_qu.$$

Using (2-14) and (2-12), we obtain

$$D_{q,\rho}\rho^{2\nu}{}_2\phi_2(0,q^{2\nu};q^{2\nu+2},q^{\nu+\mu-2\alpha+2\beta+2};q^2,q^2\rho^2u^2)$$
$$= \frac{(q^2;q^2)_\infty(1-q^{2\nu})\rho^{3\nu/2-\mu/2+\alpha-\beta-1}u^{-\nu/2-\mu/2+\alpha-\beta}}{(q^\nu+\mu-2\alpha+2\beta+2;q^2)_\infty(1-q)}J_{\nu/2+\mu/2-\alpha+\beta}(\rho u;q^2).$$

Thus,

$$I_2 = \rho^{\nu/2-\mu/2-\alpha-\beta}\int_0^\infty u^{\nu/2-\mu/2-\alpha-\beta}\psi(u)w(u)J_{\nu/2+\mu/2-\alpha+\beta}(\rho u;q^2)\,d_qu.$$

Applying [Annaby and Mansour 2012, Lemma 1.12], one can verify that

$$D_{q,\rho}\int_0^\rho t^{\nu-1}\Big(\frac{q^2t^2}{\rho^2};q^2\Big)_{-\nu/2+\mu/2-\alpha+\beta}f(t)\,d_qt = \int_0^\rho t^{\nu-1}D_{q,\rho}\Big(\frac{q^2t^2}{\rho^2};q^2\Big)_{-\nu/2+\mu/2-\alpha+\beta}f(t)\,d_qt. \tag{6-15}$$

Since $D_{q,\rho}(c/\rho^2;q^2)_\alpha = \dfrac{q^{-2}c}{\rho^3}\dfrac{1-q^{2\alpha}}{1-q}(c/\rho^2;q^2)_{\alpha-1}$ and

$$\int_0^\rho f(t)\,d_qt = \frac{1}{1+q}\int_0^{\rho^2}\frac{f(\sqrt{t})}{\sqrt{t}}\,d_qt,$$

we obtain

$$D_{q,\rho}\int_0^\rho t^{\nu-1}\Big(\frac{q^2t^2}{\rho^2};q^2\Big)_{-\nu/2+\mu/2-\alpha+\beta}f(\sqrt{t})\,d_qt = \frac{1-q^{-\nu+\mu-2\alpha+2\beta}}{(1-q^2)\rho^3}\int_0^{\rho^2}t^{\nu/2}\Big(\frac{q^2t}{\rho^2};q^2\Big)_{-\nu/2+\mu/2-\alpha+\beta-1}f(\sqrt{t})\,d_{q^2}t. \tag{6-16}$$

Hence, from (3-12),

$$I_1 = (1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\rho^{-\mu-2\beta-2}I_{q^2}^{-\nu/2+\mu/2-\alpha+\beta}[(\cdot)^{\nu/2}f(\sqrt{\cdot})](\rho^2).$$

The q-integral in (6-14) is now

$$m_1(\rho) = (1-q^2)^{\nu/2-\mu/2+\alpha-\beta}\rho^{-\mu-2\beta-2}I_{q^2}^{-\nu/2+\mu/2-\alpha+\beta}[(\cdot)^{\nu/2}f(\sqrt{\cdot})](\rho^2) - \rho^{\nu/2-\mu/2-\alpha-\beta}\int_0^\infty u^{\nu/2-\mu/2-\alpha-\beta}\psi(u)w(u)J_{\nu/2+\mu/2-\alpha+\beta}(\rho u;q^2)\,d_qu. \tag{6-17}$$

Case 2. $\rho \in B_q$. From (6-2) and (6-9), we obtain

$$\int_0^\infty k(\rho t)\psi(t)\,d_qt = \frac{\log(q)}{2i\pi(1-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} K(s)\Psi(1-s)\rho^{-s}\,ds = \frac{\log(q)}{2i\pi(1-q)}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} H_2^*(s)G_1(s)\rho^{-s}\,ds. \tag{6-18}$$

Substituting

$$\rho^{-s} = \rho^{\nu-2\alpha-1}\frac{1-q}{1-q^{-s-\nu+2\alpha+2}}D_{q,\rho}\rho^{-s-\nu+2\alpha+2}$$

in the second line of (6-18), we obtain

$$\begin{aligned}\int_0^\infty k(\rho t)\psi(t)\,d_qt &= m_2(\rho)\\ &= -\frac{\log(q)q^{\nu-2\alpha-2}\rho^{\nu-2\alpha-1}}{2i\pi(1-q^2)^{-\nu/2+\mu/2+\alpha-\beta+1}}\\ &\quad\times D_{q,\rho}\rho^{-\nu+2\alpha+2}\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} G_1(s)\mathcal{H}_2^*(s)q^s\rho^{-s}\,ds\\ &= -(1-q)(1-q^2)^{\nu/2-\mu/2-\alpha+\beta-1}q^{\nu-2\alpha-2}\rho^{\nu-2\alpha-1}\\ &\quad\times D_{q,\rho}\rho^{-\nu+2\alpha+2}\int_0^\infty g_1(t)h_2^*\Big(\frac{\rho}{qt}\Big)\frac{1}{t}\,d_qt,\end{aligned} \tag{6-19}$$

where $\mathcal{H}_2^*(s) := \Gamma_{q^2}(\frac{1}{2}\nu+\frac{1}{2}s-\alpha-1)/\Gamma_{q^2}(\frac{1}{2}\mu+\frac{1}{2}s-\beta)$ and $h_2^*(\rho)$ is given by Lemma 6.5. On simplifying (6-19), using (6-5), we get

$$m_2(\rho) = -\frac{(1-q^2)^{\nu/2-\mu/2-\alpha+\beta}\rho^{\nu-2\alpha-1}}{\Gamma_{q^2}(-\frac{1}{2}\nu+\frac{1}{2}\mu+\alpha-\beta+1)} \times D_{q,\rho}\int_\rho^\infty \frac{g(u)}{u^{\nu-2\alpha+2\beta-1}}\Big(\frac{\rho^2}{u^2};q^2\Big)_{-\nu/2+\mu/2+\alpha-\beta}\,d_qu.$$

In similar steps as in simplifying I_1 and by (3-1), we obtain

$$m_2(\rho) = -(1-q^2)^{\nu/2-\mu/2-\alpha+\beta}\rho^{-2\beta}\mathcal{K}_{q^2}^{\nu/2-\alpha+\beta,-\nu/2+\mu/2+\alpha-\beta}[g(q^{-1}\sqrt{\cdot})](\rho^2). \tag{6-20}$$

If we set $m(\rho)$ to be

$$m(\rho) = \begin{cases} m_1(\rho) & \text{if } \rho\in A_q,\\ m_2(\rho) & \text{if } \rho\in B_q,\end{cases}$$

then combining (6-17) and (6-20) gives

$$\rho^{\nu/2-\mu/2-\alpha-\beta}\int_0^\infty t^{\nu/2-\mu/2-\alpha-\beta}J_{\nu/2+\mu/2-\alpha+\beta}(\rho t;q^2)\psi(t)\,d_qt = m(\rho). \tag{6-21}$$

Hence, from (2-20) and (6-21), we obtain

$$\psi(t) = t^{-\nu/2+\mu/2+\alpha+\beta+1} \times \int_0^\infty \rho^{-\nu/2+\mu/2+\alpha+\beta+1} J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) m(\rho)\, d_q\rho. \tag{6-22}$$

Substituting for $m(\rho)$ in (6-22), we obtain

$$\psi(t) = t^{-\nu/2+\mu/2+\alpha+\beta+1} \int_0^1 \rho^{-\nu/2+\mu/2+\alpha+\beta+1} J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) m_1(\rho)\, d_q\rho + t^{-\nu/2+\mu/2+\alpha+\beta+1} \int_1^\infty \rho^{-\nu/2+\mu/2+\alpha+\beta+1} J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) m_2(\rho)\, d_q\rho.$$

Returning to (6-17) and (6-20), we obtain

$$\psi(t) = N_1 + N_2 - N_3,$$

where

$$N_1 := (1-q^2)^{\nu/2-\mu/2+\alpha-\beta} t^{-\nu/2+\mu/2+\alpha+\beta+1} \times \int_0^1 \rho^{-\nu/2-\mu/2+\alpha-\beta-1} J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) \times I_{q^2}^{-\nu/2+\mu/2-\alpha+\beta}[(\,\cdot\,)^{\nu/2} f(\sqrt{\cdot}\,)](\rho^2)\, d_q\rho,$$

$$N_2 := -(1-q^2)^{\nu/2-\mu/2-\alpha+\beta} t^{-\nu/2+\mu/2+\alpha+\beta+1} \times \int_1^\infty \rho^{-\nu/2+\mu/2+\alpha-\beta+1} J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) \times \mathcal{K}_{q^2}^{\nu/2-\alpha+\beta,-\nu/2+\mu/2+\alpha-\beta}[g(q^{-1}\sqrt{\cdot}\,)](\rho^2)\, d_q\rho,$$

$$N_3 := t^{-\nu/2+\mu/2+\alpha+\beta+1} \times \int_0^1 \left(\rho J_{\nu/2+\mu/2-\alpha+\beta}(\rho t; q^2) \int_0^\infty u^{\nu/2-\mu/2-\alpha-\beta} \psi(u) w(u) \times J_{\nu/2+\mu/2-\alpha+\beta}(\rho u; q^2)\, d_q u \right) d_q\rho.$$

Assuming that the weight function w satisfies $\psi w \in L_q^1(\mathbb{R}_{q,+})$, then the double q-integrals defining N_3 is absolutely convergent, and therefore we can interchange the order of q-integration to obtain

$$\psi(t) = A(t) - t^{\mu/2-\nu/2+\alpha+\beta+1} \int_0^\infty u^{\nu/2-\mu/2-\alpha-\beta} \psi(u) w(u) L(t,u)\, d_q u, \tag{6-23}$$

where $L(t,u)$ comes from (2-19). Thus, the solution of the single integral equation in (6-23) gives us the value of the unknown function $\psi(t)$, which is the solution of the dual q-integral equation in (6-6), as well. □

In particular, if we replace α by $-\alpha$ in (6-6), and set $w = g \equiv 0$, $\beta = 0$, and $\nu = \mu$, we obtain the dual q-integral equations (4-2)–(4-3), and from Theorem 6.6, its solution is given by

$$\psi(t) = \frac{t^{1-\alpha}}{(1-q^2)^\alpha} \int_0^1 \rho^{1-\nu-\alpha} J_{\nu+\alpha}(\rho t; q^2) I_{q^2}^{\alpha}[(\cdot)^{\nu/2} f(\sqrt{\cdot})](\rho^2)\, d_q\rho. \tag{6-24}$$

This coincides with the result of Section 4.

Example 6.7. If $\nu = \frac{1}{2}$, $\mu = -\frac{1}{2}$, $\alpha = \beta = -\frac{1}{4}$, then the solution of the system

$$\frac{\sqrt{1-q^2}\rho^{-1/2}}{\Gamma_{q^2}(1/2)} \int_0^\infty \phi(u)[1+w(u)] \sin\Big(\frac{u\rho}{1-q}; q\Big)\, d_q u = f(\rho), \quad \rho \in A_q,$$
$$\frac{\sqrt{1-q^2}\rho^{-1/2}}{\Gamma_{q^2}(1/2)} \int_0^\infty \phi(u) \cos\Big(\frac{u\rho\sqrt{q}}{1-q}; q\Big)\, d_q u = g(\rho), \quad \rho \in B_q,$$

takes the form

$$\psi(t) = A(t) - \frac{1}{(1-q)} \int_0^\infty u\psi(u) w(u) L(t,u)\, d_q u,$$

where

$$L(t,u) = \frac{1-q}{t^2-u^2}[t J_1(t;q^2) J_0(q^{-1}u;q^2) - u J_1(u;q^2) J_0(q^{-1}t;q^2)]$$

and

$$A(t) = (1-q^2)^{1/2}\Big[\int_0^1 \rho^{-1} J_0(\rho t; q^2) I_{q^2}^{-1/2}[(\cdot)^{1/4} f(\sqrt{\cdot})](\rho^2)\, d_q\rho - \int_1^\infty J_0(\rho t; q^2) \mathcal{K}_{q^2}^{1/4,-1/2}[g(q^{-1}\sqrt{\cdot})](\rho^2)\, d_q\rho\Big].$$

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