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**NORMAL FORMS FOR CR SINGULAR CODIMENSION-TWO
LEVI-FLAT SUBMANIFOLDS**

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Real-analytic Levi-flat codimension-two CR singular submanifolds are a natural generalization to \mathbb{C}^m , $m > 2$, of Bishop surfaces in \mathbb{C}^2 . Such submanifolds, for example, arise as zero sets of mixed-holomorphic equations with one variable antiholomorphic. We classify the codimension-two Levi-flat CR singular quadrics, and we notice that new types of submanifolds arise in dimension three or higher. In fact, the nondegenerate submanifolds, i.e., higher order perturbations of $z_m = \bar{z}_1 z_2 + \bar{z}_1^2$, have no analogue in dimension two. We prove that the Levi foliation extends through the singularity in the real-analytic nondegenerate case. Furthermore, we prove that the quadric is a (convergent) normal form for a natural large class of such submanifolds, and we compute its automorphism group. In general, we find a formal normal form in \mathbb{C}^3 in the nondegenerate case that shows infinitely many formal invariants.

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1. Introduction

Let $M \subset \mathbb{C}^{n+1}$ be a real submanifold. A fundamental question in CR geometry is to classify M at a point up to local biholomorphic transformations. One approach is to find a normal form for M .

A real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ is Levi-flat if the Levi form vanishes identically. Roughly speaking, a Levi-flat submanifold is a family of complex submanifolds. Intuitively, a Levi-flat submanifold is as close to a complex submanifold as possible. In the real-analytic smooth hypersurface case, it is well known that M can locally be transformed into the real hyperplane given by

$$(1) \quad \operatorname{Im} z_1 = 0.$$

We therefore focus on higher codimension case, in particular on codimension two. A codimension-two submanifold is again given by a single equation, but in this case a complex valued equation. A new phenomenon that appears in codimension two is that M may no longer be a CR submanifold. Let $T_p^c M \subset T_p M$ be the largest subspace with $JT_p^c M = T_p^c M$, where J is the complex structure on \mathbb{C}^{n+1} . A submanifold is CR if $\dim T_p^c M$ is constant.

Real submanifolds of dimension $n + 1$ in \mathbb{C}^{n+1} with a nondegenerate complex tangent point has been studied extensively after the fundamental work of E. Bishop [1965]. In \mathbb{C}^2 , he studied the submanifolds

$$(2) \quad w = z\bar{z} + \gamma(z^2 + \bar{z}^2) + O(3),$$

where $\gamma \in [0, \infty]$ is called the Bishop invariant, with $\gamma = \infty$ interpreted as $w = z^2 + \bar{z}^2 + O(3)$. One of Bishop's motivations was to study the hull of holomorphy of the real submanifolds by attaching analytic discs. His work on the family of attached analytic discs has been refined by Kenig and Webster [1982; 1984], Huang and Krantz [1995], and Huang [1998]. The normal form theory for real submanifolds for Bishop surfaces or submanifolds was established by Moser and Webster [1983]; see also Moser [1985], Gong [1994a; 1994b; 2004], Huang and Yin [2009a], and Coffman [2010]. We mention that the Moser–Webster normal form does not deal with the case of vanishing Bishop invariant.

The formal normal form and its application to holomorphic classification for surfaces with vanishing Bishop invariant were achieved by Huang and Yin [2009a] by a completely different method. Real submanifolds with complex tangents have been studied in other situations. See, for example, [Lebl et al. 2014], where CR singular submanifolds that are images of CR manifolds were studied. Normal forms for the quadratic part of general codimension-two CR singular submanifolds in \mathbb{C}^3 was completely solved by Coffman [2009]. Huang and Yin [2009b] studied the normal form for codimension-two CR singular submanifolds of the form $w =$

$|z|^2 + O(3)$. Dolbeault, Tomassini and Zaitsev [2005; 2011] and Huang and Yin [2012] studied CR singular submanifolds of codimension-two that are boundaries of Levi-flat hypersurfaces. Burcea [2013] constructed the formal normal form for codimension-two CR singular submanifolds approximating a sphere. Coffman [2006] found an algebraic normal form for nondegenerate CR singular manifolds in high codimension and one-dimensional complex tangent.

To motivate our work, we observe that in Bishop's work, the real submanifolds are Levi-flat away from their CR singular sets. Our purpose is to understand such submanifolds in higher dimensional case with codimension being exactly two. Notice that the latter is the smallest codimension for CR singularity to be present in (smooth) submanifolds. Regarding CR singular Levi-flat real codimension-two submanifolds on \mathbb{C}^{n+1} as a natural generalization of Bishop surfaces to \mathbb{C}^{n+1} , we wish to find their normal forms. For singular Levi-flat hypersurfaces and related work on foliations with singularity, see [Bedford 1977; Burns and Gong 1999; Brunella 2007; Cerveau and Lins Neto 2011; Lebl 2013; Fernández-Pérez 2013].

Our techniques revolve around the study of the Levi map (the generalization of the Levi form to higher codimension submanifolds) of codimension-two submanifolds. Extending the CR structure through the singular point via Nash blowup and then extending the Levi map to this blowup has been studied previously by Garrity [2000].

A CR submanifold is *Levi-flat* if the Levi map vanishes identically. Locally, all CR real-analytic Levi-flat submanifolds of real codimension two can be, after holomorphic change of coordinates, written as

$$(3) \quad \operatorname{Im} z_1 = 0, \quad \operatorname{Im} z_2 = 0.$$

If a submanifold M is CR singular, denote by M_{CR} the set of points where M is CR. We say M is Levi-flat if M_{CR} is Levi-flat in the usual sense. A Levi-flat CR singular submanifold has no local biholomorphic invariants at the CR points, just as in the case of Bishop surfaces.

A real, real-analytic codimension-two submanifold that is CR singular at the origin can be written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$ as

$$(4) \quad w = \rho(z, \bar{z})$$

for ρ that is $O(2)$. We will be concerned with submanifolds where the quadratic part in ρ is nonzero in any holomorphic coordinates. We say that such submanifolds have a *nondegenerate complex tangent*. For example, the Bishop surfaces in \mathbb{C}^2 are precisely the CR singular submanifolds with nondegenerate complex tangent.

First, let us classify the quadratic parts of CR singular Levi-flats, and in the process completely classify the CR singular Levi-flat quadrics, that is, those where ρ is a quadratic.

Theorem 1.1. *Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a germ of a real-analytic real codimension-two submanifold, CR singular at the origin, written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ as*

$$(5) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

for quadratic A and B , where $A + B \neq 0$ (nondegenerate complex tangent). Suppose that M is Levi-flat (that is, M_{CR} is Levi-flat).

(i) *If M is a quadric, then M is locally biholomorphically equivalent to one and exactly one of the following:*

$$(A.1) \quad w = \bar{z}_1^2,$$

$$(A.2) \quad w = \bar{z}_1^2 + \bar{z}_2^2,$$

$$\vdots$$

$$(A.n) \quad w = \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2,$$

$$(B.\gamma) \quad w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0,$$

$$(C.0) \quad w = \bar{z}_1 z_2,$$

$$(C.1) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

(ii) *If M is real-analytic, then the quadric*

$$(6) \quad w = A(z, \bar{z}) + B(z, \bar{z})$$

is Levi-flat, and can be put via a biholomorphic transformation into exactly one of the forms above.

By part (ii), the quadratic part in (5) is an invariant of M at a point. We say the type of M at the origin is A.x, B. γ , or C.x depending on the type of the quadratic form. Following Bishop, we call types B. γ and A.1 Bishop-like, and we could think of $\gamma = \infty$ as A.1.

By type being *stable* we mean that the type does not change at all complex tangents in a neighborhood of the origin under any small (or higher order) perturbations that stay within the class of Levi-flat CR singular submanifolds. As a consequence of [Theorem 1.1](#) and because the rank is lower semicontinuous, we get that the only types that are stable are A.n and C.1, although A.n are degenerate because the form $A(z, \bar{z})$ is identically zero. See also [Proposition 15.1](#).

The quadrics A.k for $k \geq 2$ do not possess a nonsingular foliation extending the Levi foliation of M_{CR} through the origin. In fact, there is a singular complex subvariety of dimension one through the origin contained in M . See [Section 6](#).

In the sequel, when we wish to refer to the quadric of certain type we will use the notation $M_{C,1}$ to denote the quadric of type C.1.

The quadratic form $A(z, \bar{z})$ carries the “Levi map” of the submanifold. Type C.1 is the unique quadric that is stable and has nonzero A . Having nonzero A is also stable in a neighborhood of the origin under any small (or higher order) perturbations. Therefore, we say a type is *nondegenerate* if it is C.1 and we focus mostly on such submanifolds. First, we show that submanifolds of type C.x possess a nonsingular real-analytic foliation that extends the Levi foliation, due to the form $A(z, \bar{z})$:

Theorem 1.2. *Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular submanifold of type C.1 or C.0, that is, M is given by*

$$(7) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(3) \quad \text{or} \quad w = \bar{z}_1 z_2 + O(3).$$

Then there exists a nonsingular real-analytic foliation defined on M that extends the Levi foliation on M_{CR} ; and consequently, there exists a CR real-analytic mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n+1}$ such that F is a diffeomorphism onto $F(U) = M \cap U'$, for some neighborhood U' of zero.

Here the CR structure on $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ is induced from $\mathbb{C}^2 \times \mathbb{C}^{n-1}$. As a corollary of this theorem we obtain in [Section 8](#) using the results of [\[Lebl et al. 2014\]](#) that the CR singular set of any type C.1 submanifold is a Levi-flat submanifold of dimension $2n - 2$ and CR dimension $n - 2$.

The Levi foliation on a type C.x submanifold cannot extend to a whole neighborhood of M as a nonsingular holomorphic foliation. If it did, we could flatten the foliation and M would be a Cartesian product, in particular Bishop-like. Thus, the study of normal form theory for the special case when the foliation extends to a neighborhood is reduced to the case of Bishop surfaces, which have been studied extensively.

A codimension-two submanifold in \mathbb{C}^m can arise from

$$(8) \quad f(\bar{z}', z'') = 0$$

for a suitable holomorphic function f in m variables. The zero set admits two holomorphic foliations. We are interested in the case where one of foliations has leaves of maximum dimension $m - 2$, while the other has leaves of minimum dimension zero. Therefore, we will assume that $z' = z_1$ and $z'' = (z_2, \dots, z_m)$. Functions holomorphic in some variables and anti-holomorphic in other variables, such as (8), are often called *mixed-holomorphic* or mixed-analytic, and come up often in complex geometry, the simplest example being the standard inner product. An interesting feature of the mixed-holomorphic setting is that the equation can be complexified into \mathbb{C}^m , so the sets share some of the properties of complex varieties. However, they have a different automorphism group if we wish to classify them under biholomorphic transformations. Such mixed-analytic sets are automatically real codimension two, are Levi-flat or complex, and may have CR singularities. We study their normal form in [Section 9](#). See also [Theorem 1.3](#) below.

When a type C.1 CR singular submanifold has a defining equation that does not depend on $\bar{z}_2, \dots, \bar{z}_n$ we prove that it is automatically Levi-flat, and it is equivalent to $M_{C.1}$.

Theorem 1.3. *Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic submanifold given by*

$$(9) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \dots, z_n),$$

where r is $O(3)$. Then M is Levi-flat and at the origin and M is locally biholomorphically equivalent to the quadric $M_{C.1}$ submanifold

$$(10) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

The theorem is also true formally; given a formal submanifold of the form (9), it is formally equivalent to $M_{C.1}$.

A key idea in the proof of the convergence of the normalizing transformation is that the form $B(\bar{z}, \bar{z}) = \bar{z}_1^2$ induces a natural mixed-holomorphic involution on quadric $M_{C.1}$. This involution also plays a key role in computing the automorphism group of the quadric in [Theorem 12.4](#).

Finally, we also compute the automorphism group for the quadric $M_{C.1}$; see [Theorem 12.4](#). In particular we show that the automorphism group is infinite-dimensional.

Not every type C.1 Levi-flat submanifold is biholomorphically equivalent to the C.1 quadric. We will find a formal normal form for type C.1 Levi-flat submanifolds in \mathbb{C}^3 that shows infinitely many formal invariants. Let us give a simplified statement. For details, see [Theorem 14.3](#).

Theorem 1.4. *Let M be a real-analytic Levi-flat type C.1 submanifold in \mathbb{C}^3 . There exists a formal biholomorphic map transforming M into the image of*

$$(11) \quad \hat{\varphi}(z, \bar{z}, \xi) = (z + A(z, \xi, w)w\eta, \xi, w)$$

with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Here $A = 0$, or A satisfies certain normalizing conditions.

When $A \neq 0$, the formal automorphism group preserving the normal form is finite or one-dimensional.

We do not know if the formal normal form above can be achieved by convergent transformations, even if $A = 0$.

2. Invariants of codimension-two CR singular submanifolds

Before we impose the Levi-flat condition, let us find some invariants of codimension-two CR singular submanifolds in \mathbb{C}^{n+1} with CR singularity at zero. Such a submanifold, locally near the origin, can be put into the form

$$(12) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

where $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and A and B are quadratic forms. We think of A and B as matrices and z a column vector and write the forms as z^*Az and $z^*B\bar{z}$ respectively. The matrix B is not unique. Hence we make B symmetric to make the choice of the matrix B canonical. The following proposition is not difficult and well known. Since the details are important and will be used later, let us prove:

Proposition 2.1. *A biholomorphic transformation of (12) taking the origin to itself and preserving the form of (12) takes the matrices (A, B) to*

$$(13) \quad (\lambda T^*AT, \lambda T^*B\bar{T}),$$

for $T \in \text{GL}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}^*$. If $(F_1, \dots, F_n, G) = (F, G)$ is the transformation, then the linear part of G is $\lambda^{-1}w$ and the linear part of F restricted to z is Tz .

Let us emphasize that A is an arbitrary complex matrix and B is symmetric, but not necessarily Hermitian.

Proof. Let $(F_1, \dots, F_n, G) = (F, G)$ be a change of coordinates taking

$$(14) \quad w = \tilde{A}(z, \bar{z}) + \tilde{B}(\bar{z}, \bar{z}) + O(3) = \rho(z, \bar{z})$$

to

$$(15) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3).$$

Then

$$(16) \quad G(z, \rho(z, \bar{z})) \\ = A(F(z, \rho(z, \bar{z})), \bar{F}(\bar{z}, \bar{\rho}(\bar{z}, z))) + B(\bar{F}(\bar{z}, \bar{\rho}(\bar{z}, z)), \bar{F}(\bar{z}, \bar{\rho}(\bar{z}, z))) + O(3)$$

is true for all z . The right hand side has no linear terms, so the linear terms in G do not depend on z . That is, $G = \lambda^{-1}w + O(2)$, where λ is a nonzero scalar and the negative power is for convenience.

Let $T = [T_1, T_2]$ denote the matrix representing the linear terms of F . Here T_1 is an $n \times n$ matrix and T_2 is $n \times 1$. Since the linear terms in G do not depend on any z_j , T_1 is nonsingular. Then the quadratic terms in (16) are

$$(17) \quad \lambda^{-1}(\tilde{A}(z, \bar{z}) + \tilde{B}(\bar{z}, \bar{z})) = z^*T_1^*AT_1z + z^*T_1^*B\bar{T}_1\bar{z}.$$

In other words as matrices,

$$(18) \quad \tilde{A} = \lambda T_1^*AT_1 \quad \text{and} \quad \tilde{B} = \lambda T_1^*B\bar{T}_1. \quad \square$$

We will need to at times reduce to the three-dimensional case, and so we need:

Lemma 2.2. *Let $M \subset \mathbb{C}^{n+1}$, $n \geq 3$, be a real-analytic Levi-flat CR singular submanifold of the form*

$$(19) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

where A and B are quadratic. Let L be a nonsingular $(n-2) \times n$ matrix. If $A+B$ is not zero on the set $\{Lz=0\}$, then the submanifold

$$(20) \quad M_L = M \cap \{Lz=0\}$$

is a Levi-flat CR singular submanifold.

Proof. Clearly, if M_L is not contained in the CR singularity of M , then M_L is a Levi-flat CR singular submanifold. $M_{L'}$ is not contained in the CR singularity of M for a dense open subset of $(n-2) \times n$ matrices L' . If M_L is a subset of the CR singularity of M , pick a CR point p of M_L then pick a sequence L_n approaching L such that M_{L_n} are not contained in the CR singularity of M . As $A+B$ is not zero on the set $\{Lz=0\}$, then M_L is not a complex submanifold, and therefore a CR singular submanifold. As the Levi form of M_{L_n} vanishes at all CR points of M_{L_n} , the Levi form of M_L vanishes at p , so M_L is Levi-flat. \square

3. Levi-flat quadrics

Let us first focus on Levi-flat quadrics. We will prove later that the quadratic part of a Levi-flat submanifold is Levi-flat. Let M be defined in $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ by

$$(21) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}).$$

Being Levi-flat has several equivalent formulations. The main idea is that the $T^{(1,0)}M \times T^{(0,1)}M$ vector fields are completely integrable at CR points and we obtain a foliation of M at CR points by complex submanifolds of complex dimension $n-1$. An equivalent notion is that the Levi map is identically zero; see [Baouendi et al. 1999]. The Levi map for a CR submanifold defined by two real equations $\rho_1 = \rho_2 = 0$ (for ρ_1 and ρ_2 with linearly independent differentials) is the pair of Hermitian forms

$$(22) \quad i\partial\bar{\partial}\rho_1 \quad \text{and} \quad i\partial\bar{\partial}\rho_2,$$

applied to $T^{(1,0)}M$ vectors. The full quadratic forms $i\partial\bar{\partial}\rho_1$ and $i\partial\bar{\partial}\rho_2$ of course depend on the defining equations themselves and are therefore extrinsic information. It is important to note that for the Levi map we restrict it to $T^{(1,0)}M$ vectors. We can define these two forms $i\partial\bar{\partial}\rho_1$ and $i\partial\bar{\partial}\rho_2$ even at a CR singular point $p \in M$.

These forms are the complex Hessian matrices of the defining equations. For our quadric M , they are the real and imaginary parts of the $(n+1) \times (n+1)$ complex matrix

$$(23) \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

where the variables are ordered as (z_1, \dots, z_n, w) .

For M to be Levi-flat, the quadratic form defined by \tilde{A} has to be zero when restricted to the $(n-1)$ -dimensional space spanned by $T_p^{(1,0)}M$ for every $p \in M_{\text{CR}}$. In other words, for every $p \in M_{\text{CR}}$,

$$(24) \quad v^* \tilde{A} v = 0 \quad \text{for all } v \in T_p^{(1,0)}M.$$

The space $T_p^{(1,0)}M$ is of dimension $n-1$, and furthermore, the vector $\partial/\partial w$ is not in $T_p^{(1,0)}M$. Therefore, $z^*Az = 0$ for $z \in \mathbb{C}^n$ in a subspace of dimension $n-1$.

Before we proceed, let us note the following general fact about CR singular Levi-flat submanifolds:

Lemma 3.1. *Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a Levi-flat connected real-analytic real codimension-two submanifold, CR singular at the origin. Then there exists a germ of a complex analytic variety of complex dimension $n-1$ through the origin, contained in M .*

Proof. Through each point of M_{CR} there exists a germ of a complex variety of complex dimension $n-1$ contained in M . The set of CR points is dense in M . Take a sequence p_k of CR points converging to the origin and take complex varieties of dimension $n-1$, $W_k \subset M$ with $p_k \in W_k$. A theorem of Fornæss (see [Kohn 1979, Theorem 6.23] for a proof using the methods of Diederich and Fornæss [1978]) implies that there exists a variety through $W \subset M$ with $0 \in W$ and of complex dimension at least $n-1$. \square

Let us first concentrate on $n=2$, in which case $T^{(1,0)}M$ is one-dimensional at CR points. Write

$$(25) \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}.$$

Note that B is symmetric. A short computation shows that the vector field can be written as

$$(26) \quad \alpha \frac{\partial}{\partial w} + \beta_1 \frac{\partial}{\partial z_1} + \beta_2 \frac{\partial}{\partial z_2} = \alpha \frac{\partial}{\partial w} + \beta \frac{\partial}{\partial z},$$

where

$$(27) \quad \begin{aligned} \beta_1 &= \bar{a}_{21} \bar{z}_1 + \bar{a}_{22} \bar{z}_2 + 2\bar{b}_{12} z_1 + 2\bar{b}_{22} z_2, \\ \beta_2 &= -\bar{a}_{11} \bar{z}_1 - \bar{a}_{12} \bar{z}_2 - 2\bar{b}_{11} z_1 - 2\bar{b}_{12} z_2, \\ \alpha &= a_{11} \bar{z}_1 \beta_1 + a_{21} \bar{z}_2 \beta_1 + a_{12} \bar{z}_1 \beta_2 + a_{22} \bar{z}_2 \beta_2. \end{aligned}$$

M_{CR} is dense in M , since the CR singular set is defined by $\beta_1 = \beta_2 = 0$. Thus for M to be Levi-flat we need to check that the following product is identically zero:

$$(28) \quad [\beta^* \quad \bar{\alpha}] \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \beta^* A \beta.$$

If A is the zero matrix, then M is automatically Levi-flat. We diagonalize B via T into a diagonal matrix with ones and zeros on the diagonal. We obtain (recall $n = 2$) the submanifolds

$$(29) \quad w = \bar{z}_1^2 \quad \text{or} \quad w = \bar{z}_1 + \bar{z}_2^2.$$

The first submanifold is of the form $M \times \mathbb{C}$, where $M \subset \mathbb{C}^2$ is a Bishop surface.

Let us from now on suppose that $A \neq 0$.

As M is Levi-flat, then through each CR point $p = (z_p, w_p) \in M_{\text{CR}}$ we have a complex submanifold of dimension one in M . It is well known that this submanifold is contained in the Segre variety (see also [Section 4](#))

$$(30) \quad w = A(z, \bar{z}_p) + B(\bar{z}_p, \bar{z}_p), \quad \bar{w}_p = \bar{A}(\bar{z}_p, z) + \bar{B}(z, z).$$

By [Lemma 3.1](#) we obtain a complex variety $V \subset M$ of dimension one through the origin. Suppose without loss of generality that V is irreducible. V has to be contained in the Segre variety at the origin, in particular $w = 0$ on V . Therefore, to simplify notation, let us consider V to be subvariety of $\{w = 0\}$. Denote by \bar{V} the complex conjugate of V . Then as V is irreducible, $V \times \bar{V}$ is also irreducible (the smooth part of V is connected and so is the smooth part of $V \times \bar{V}$; see [[Whitney 1972](#)]). Hence, by complexifying, we have $A(z, \bar{\xi}) + B(\bar{\xi}, \bar{\xi}) = 0$ for all $z \in V$ and $\xi \in V$.

If $B \neq 0$, then setting $z = 0$, we have $B(\bar{\xi}, \bar{\xi}) = 0$ on V . As B is homogeneous and V is irreducible, V is a one-dimensional complex line. If $B = 0$, then $A(z, \bar{\zeta}) = 0$ for $z, \zeta \in V$ as mentioned above. We consider two cases. Suppose first that every $\sum_{j=1}^2 a_{ij} \bar{\zeta}_j$ is identically zero for all $\zeta \in V$ and $i = 1, 2$. Then V is contained in some complex line $\sum_{j=1}^2 \bar{a}_{ij} \zeta_j = 0$. Suppose now that $A(z, \bar{\zeta}_*)$ is not identically zero for some $\zeta_* \in V$. Then V is contained in the complex line $A(z, \bar{\zeta}_*) = 0$. This shows that V is a complex line.

Thus as $A(z, \bar{z}) + B(\bar{z}, \bar{z})$ is zero on a one-dimensional linear subspace, we make this subspace $\{z_1 = 0\}$ and so each monomial in $A(z, \bar{z}) + B(\bar{z}, \bar{z})$ is divisible by either z_1 or \bar{z}_1 . Therefore, A and B are matrices of the form

$$(31) \quad \begin{bmatrix} * & * \\ * & 0 \end{bmatrix},$$

that is, $a_{22} = 0$ and $b_{22} = 0$.

To normalize the pair (A, B) , we apply arbitrary invertible transformations $(T, \lambda) \in \text{GL}_n(\mathbb{C}) \times \mathbb{C}^*$ as

$$(32) \quad (A, B) \mapsto (\lambda T^* A T, \lambda T^* B \bar{T}).$$

Recall that we are assuming that $A \neq 0$. If $a_{21} = 0$ or $a_{12} = 0$, then A has rank one, and via a transformation T of the form

$$(33) \quad z'_1 = z_1, \quad z'_2 = z_2 + cz_1 \quad \text{or} \quad z'_2 = z_1, \quad z'_1 = z_2 + cz_1$$

and rescaling by nonzero λ , the matrix A can be put in the form

$$(34) \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The transformation T and λ must also be applied to B and this could possibly make $b_{22} \neq 0$. However, we will show that we actually have $b_{22} = 0$. Thus $B = 0$ on $z_1 = 0$ still holds true.

Let us first focus on

$$(35) \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We apply the $T^{(1,0)}$ vector field computed above. Only a_{11} is nonzero in A . Therefore $\beta^*A\beta$, which must be identically zero, is

$$(36) \quad 0 = \beta^*A\beta = \bar{\beta}_1\beta_1 = \overline{(2\bar{b}_{12}z_1 + 2\bar{b}_{22}z_2)}(2\bar{b}_{12}z_1 + 2\bar{b}_{22}z_2) \\ = 4(|b_{12}|^2z_1\bar{z}_1 + |b_{22}|^2z_2\bar{z}_2 + b_{12}\bar{b}_{22}\bar{z}_1z_2 + \bar{b}_{12}b_{22}z_1\bar{z}_2).$$

This polynomial must be identically zero and hence all coefficients must be identically zero. So $b_{12} = 0$ and $b_{22} = 0$. In other words, only b_{11} in B can be nonzero, in which case we make it nonnegative via a diagonal T to obtain the quadric

$$(37) \quad w = |z_1|^2 + \gamma\bar{z}_1^2, \quad \gamma \geq 0.$$

Next let us focus on

$$(38) \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

As above, we compute $\beta^*A\beta$:

$$(39) \quad 0 = \beta^*A\beta = \bar{\beta}_1\beta_2 = \overline{(2\bar{b}_{12}z_1 + 2\bar{b}_{22}z_2)}(-\bar{z}_2 - 2\bar{b}_{11}z_1 - 2\bar{b}_{12}z_2) \\ = -2b_{12}\bar{z}_1\bar{z}_2 - 2b_{22}\bar{b}_{11}z_1\bar{z}_2 - 4\bar{b}_{11}b_{12}z_1\bar{z}_1 - 4b_{12}\bar{b}_{12}\bar{z}_1z_2 \\ - 2b_{22}\bar{z}_2^2 - 4b_{22}\bar{b}_{12}z_2\bar{z}_2.$$

Again, as this polynomial must be identically zero, all coefficients must be zero. Hence $b_{12} = 0$ and $b_{22} = 0$. Again only b_{11} is left possibly nonzero.

Suppose that $b_{11} \neq 0$. Then let s be such that $b_{11}\bar{s}^2 = 1$, and let $\bar{t} = 1/\bar{s}$. The matrix $T = \begin{bmatrix} s & 0 \\ 0 & \bar{t} \end{bmatrix}$ is such that $T^*AT = A$ and $T^*B\bar{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If $b_{11} = 0$, we have

$B = 0$. Therefore we have obtained two distinct possibilities for B , and thus the two submanifolds

$$(40) \quad w = \bar{z}_1 z_2 \quad \text{or} \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

We emphasize that after A is normalized by a transformation of the form (33), only one coordinate change is needed to normalize b_{11} and this coordinate change preserves A . Both are required in a reduction proof for higher dimensions.

We have handled the rank-one case. Next we focus on the rank-two case, that is, $a_{21} \neq 0$ and $a_{12} \neq 0$ (recall $a_{22} = 0$). We normalize (rescale) A to have $a_{12} = 1$ and take

$$(41) \quad A = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix}.$$

Again, let us compute $\beta^* A \beta$. In the computation for the rank-two case, recall that we have not done any normalization other than rescaling, so we can safely still assume that $b_{22} = 0$,

$$(42) \quad \begin{aligned} 0 &= \beta^* A \beta = a_{11} \bar{\beta}_1 \beta_1 + \bar{\beta}_1 \beta_2 + a_{21} \beta_1 \bar{\beta}_2 \\ &= a_{11} \overline{(\bar{a}_{21} \bar{z}_1 + 2\bar{b}_{12} z_1)} (\bar{a}_{21} \bar{z}_1 + 2\bar{b}_{12} z_1) \\ &\quad + \overline{(\bar{a}_{21} \bar{z}_1 + 2\bar{b}_{12} z_1)} (-\bar{a}_{11} \bar{z}_1 - \bar{z}_2 - 2\bar{b}_{11} z_1 - 2\bar{b}_{12} z_2) \\ &\quad + a_{21} \overline{(-\bar{a}_{11} \bar{z}_1 - \bar{z}_2 - 2\bar{b}_{11} z_1 - 2\bar{b}_{12} z_2)} (\bar{a}_{21} \bar{z}_1 + 2\bar{b}_{12} z_1) \\ &= (-4|b_{12}|^2 - |a_{21}|^2) \bar{z}_1 z_2 + (\text{other terms}). \end{aligned}$$

All coefficients must be zero. So $a_{21} = 0$, and A would not be rank two.

Let us now focus on $n > 2$. First suppose that $A = 0$. Then as before, M is automatically Levi-flat and by diagonalizing B we obtain the n distinct submanifolds

$$(43) \quad \begin{aligned} w &= \bar{z}_1^2, \\ w &= \bar{z}_1^2 + \bar{z}_2^2, \\ &\vdots \\ w &= \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2. \end{aligned}$$

Thus suppose from now on that $A \neq 0$. As before, we have an irreducible $(n-1)$ -dimensional variety $V \subset M$ through the origin, such that $w = 0$ and $A(z, \bar{z}) + B(\bar{z}, \bar{z}) = 0$ on V .

We wish to show that $A(z, \bar{z}) + B(\bar{z}, \bar{z}) = 0$ on an $(n-1)$ -dimensional linear subspace. For any $\xi \in V$, we obtain $A(z, \bar{\xi}) + B(\bar{\xi}, \bar{\xi}) = 0$ for all $z \in V$. If V is contained in the kernel of the matrix A^* , then V is a linear subspace of dimension

$n - 1$. So suppose that $\bar{\xi}$ is not in the kernel of the matrix A^t . Then for a fixed $\bar{\xi}$, we obtain a linear equation $A(z, \bar{\xi}) + B(\bar{\xi}, \bar{\xi}) = 0$ for $z \in V$.

Therefore, as $A(z, \bar{z}) + B(\bar{z}, \bar{z})$ needs to be zero on an $(n - 1)$ -dimensional subspace, we can just make this $\{z_1 = 0\}$ and so each monomial is divisible by either z_1 or \bar{z}_1 . Therefore, A and B is of the form

$$(44) \quad \begin{bmatrix} * & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix},$$

that is, only first column and first row are nonzero. We normalize A via

$$(45) \quad (A, B) \mapsto (\lambda T^* A T, \lambda T^* B \bar{T}),$$

as before. We use column operations on all but the first column to make all but the first two columns have nonzero elements. Similarly we can do row operations on all but the first two rows to make all but first three rows nonzero. That is, A has the form

$$(46) \quad \begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By [Lemma 2.2](#), setting $z_3 = \cdots = z_n = 0$, we obtain a Levi-flat submanifold where the matrix corresponding to A is the principal 2×2 submatrix of A . This submatrix cannot be of rank two and hence either $a_{12} = 0$ or $a_{21} = 0$. If $a_{21} = 0$ and $a_{12} \neq 0$, then setting $z_2 = z_3$ and $z_4 = \cdots = z_n = 0$, we again must have a rank-one matrix and therefore $a_{31} = 0$.

Therefore, if $a_{12} \neq 0$ then all but a_{11} and a_{12} are zero. If $a_{12} = 0$, then via a further linear map not involving z_1 we can ensure that $a_{31} = 0$. In particular, A is of rank one and can only be nonzero in the principal 2×2 submatrix. At this point B is still of the form (44).

Via a linear change of coordinates in the first two variables, the principal 2×2 submatrix of A can be normalized into one of the two possible forms

$$(47) \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Recall that $A = 0$ was already handled.

Via the two-dimensional computation we obtain that $b_{22} = b_{12} = b_{21} = 0$. We use a linear map in z_1 and z_2 to normalize the principal 2×2 matrix of B , so that the submanifold restricted to (z_1, z_2, w) is in (B. γ), (C.0) or (C.1).

Finally we need to show that all entries of B other than b_{11} are zero. As we have done a linear change of coordinates in z_1 and z_2 , B may not be in the form (44), but we know $b_{jk} = 0$ as long as $j > 2$ and $k > 2$.

Now fix $k = 3, \dots, n$. Restrict to the submanifold given by $z_1 = \lambda z_2$ for $\lambda = 1$ or $\lambda = -1$, and $z_j = 0$ for all $j = 3, \dots, n$ except for $j = k$. In the variables (z_2, z_k, w) , we obtain a Levi-flat submanifold where the matrix corresponding to A is $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$. The matrix corresponding to B is

$$(48) \quad \begin{bmatrix} b_{11} & b_{1k} + \lambda b_{2k} \\ b_{1k} + \lambda b_{2k} & 0 \end{bmatrix}.$$

Via the two-dimensional calculation we have $b_{1k} + \lambda b_{2k} = 0$. As this is true for $\lambda = 1$ and $\lambda = -1$, we get that $b_{1k} = b_{2k} = 0$.

We have proved the following classification result. It is not difficult to see that the submanifolds in the list are biholomorphically inequivalent by Proposition 2.1. The ranks of A and B are invariants. It is obvious that the A matrix of B. γ and C.x submanifolds are inequivalent. Therefore, it is only necessary to directly check that B. γ are inequivalent for different $\gamma \geq 0$, which is easy.

Lemma 3.2. *If M defined in $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, $n \geq 1$, by*

$$(49) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$$

is Levi-flat, then M is biholomorphic to one and exactly one of the expressions (A.1)–(C.1) of Theorem 1.1.

The normalizing transformation used above is linear.

Lemma 3.3. *If M , defined by*

$$(50) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

is Levi-flat at all points where M is CR, then the quadric

$$(51) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$$

is also Levi-flat.

Proof. Write M as

$$(52) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + r(z, \bar{z}),$$

where r is $O(3)$.

Let A be the matrix giving the quadratic form $A(z, \bar{z})$ as before. The Levi map is given by taking the $n \times n$ matrix

$$(53) \quad L = L(p) = A + \left[\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \right]_{j,k}$$

and applying it to vectors in $\pi(T^{(1,0)}M)$, where π is the projection onto the $\{w=0\}$ plane. That is, we parametrize M by the $\{w=0\}$ plane, and work there as before.

Let

$$(54) \quad \begin{aligned} a_j &= -\bar{A}_{z_j} - \bar{B}_{z_j} - \bar{r}_{z_j}, \\ b &= \bar{A}_{z_1} + \bar{B}_{z_1} + \bar{r}_{z_1}, \\ c &= a_j(A_{z_1} + B_{z_1} + r_{z_1}) + b(A_{z_j} + B_{z_j} + r_{z_j}). \end{aligned}$$

Then for $j = 2, \dots, n$, we write the $T^{(1,0)}$ vector fields as

$$(55) \quad X_j = a_j \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial z_j} + c \frac{\partial}{\partial w}.$$

Hence

$$(56) \quad a_j \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial z_j}$$

are the vector fields in $\pi(T^{(1,0)}M)$.

Notice that a_j , b , and c vanish at the origin. Furthermore, if we take the linear terms of a_j , b , and the quadratic terms in c , that is,

$$(57) \quad \begin{aligned} \tilde{a}_j &= -\bar{A}_{z_j} - \bar{B}_{z_j}, \\ \tilde{b} &= \bar{A}_{z_1} + \bar{B}_{z_1}, \\ \tilde{c} &= \tilde{a}_j(A_{z_1} + B_{z_1}) + \tilde{b}(A_{z_j} + B_{z_j}), \end{aligned}$$

then away from the CR singular set of the quadric,

$$(58) \quad \tilde{X}_j = \tilde{a}_j \frac{\partial}{\partial z_1} + \tilde{b} \frac{\partial}{\partial z_j} + \tilde{c} \frac{\partial}{\partial w}$$

span the $T^{(1,0)}$ vector fields on the quadric $w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$.

Since M is Levi-flat, then we have that

$$(59) \quad \pi_*(X_j)^* L \pi_*(X_j) = 0.$$

The terms linear in z and \bar{z} respectively in the expression $\pi_*(X_j)^* L \pi_*(X_j)$ are

$$(60) \quad \pi_*(\tilde{X}_j)^* A \pi_*(\tilde{X}_j).$$

As this is identically zero, the quadric $w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$ is Levi-flat. \square

4. Quadratic Levi-flat submanifolds and their Segre varieties

A very useful invariant in CR geometry is the Segre variety. Suppose that a real-analytic variety $X \subset \mathbb{C}^N$ is defined by

$$(61) \quad \rho(z, \bar{z}) = 0,$$

where ρ is a real-analytic real vector-valued with $p \in X$. Suppose that ρ converges on some polydisc Δ centered at p . We complexify and treat z and \bar{z} as independent variables, and the power series of ρ at (p, \bar{p}) converges on $\Delta \times \Delta$. The Segre variety at p is then defined as the variety

$$(62) \quad Q_p = \{z \in \Delta : \rho(z, \bar{p}) = 0\}.$$

Of course the variety depends on the defining equation itself and the polydisc Δ . For ρ , it is useful to take the defining equation or equations that generate the ideal of the complexified X in $\mathbb{C}^N \times \mathbb{C}^N$ at p . If ρ is polynomial we take $\Delta = \mathbb{C}^N$.

It is well known that any irreducible complex variety that lies in X and goes through the point p also lies in Q_p . In the case of Levi-flat submanifolds, we generally get equality as germs. For example, for the CR Levi-flat submanifold M given by

$$(63) \quad \operatorname{Im} z_1 = 0, \quad \operatorname{Im} z_2 = 0,$$

the Segre variety Q_0 through the origin is precisely $\{z_1 = z_2 = 0\}$, which happens to be the unique complex variety in M through the origin.

Let us take the Levi-flat quadric

$$(64) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}).$$

As we want to take the generating equations in the complexified space we also need the conjugate

$$(65) \quad \bar{w} = \bar{A}(\bar{z}, z) + \bar{B}(z, z).$$

The Segre variety is then given by

$$(66) \quad w = 0, \quad \bar{B}(z, z) = 0.$$

Through any CR singular point of a real-analytic Levi-flat M , there is a complex variety of dimension $n - 1$ that is the limit of the leaves of the Levi foliation of M_{CR} , via [Lemma 3.1](#). Let us take all possible such limits, and call their union Q'_p . Notice that there could be other complex varieties in M through p of dimension $n - 1$. Note that $Q'_p \subset Q_p$.

Let us write down and classify the Segre varieties for all the quadric Levi-flat submanifolds in \mathbb{C}^{n+1} :

Type	Segre variety Q_0	Q_0 singular?	$\dim_{\mathbb{C}} Q_0$	$Q_0 \subset M?$	Q'_0
A.1	$w = 0, z_1^2 = 0$	no	$n - 1$	yes	Q_0
A.k	$w = 0$ $z_1^2 + \dots + z_k^2 = 0$	yes	$n - 1$	yes	Q_0
B.0	$w = 0$	no	n	no	$w = 0$ $z_1 = 0$
B. $\gamma, \gamma > 0$	$w = 0, z_1^2 = 0$	no	$n - 1$	yes	Q_0
C.0	$w = 0$	no	n	no	$w = 0$ $z_1 = 0$
C.1	$w = 0, z_1^2 = 0$	no	$n - 1$	yes	Q_0

The submanifold C.0 also contains the complex variety $\{w = 0, z_2 = 0\}$, but this variety is transversal to the leaves of the foliation, and so cannot be in Q'_0

Notice that in the cases A.k for all k , B. γ for $\gamma > 0$, and C.1, the variety Q_0 actually gives the complex variety Q'_0 contained in M through the origin. In these cases, the variety is nonsingular only in the set theoretic sense. Scheme-theoretically the variety is always at least a double line or double hyperplane in general.

5. The CR singularity of Levi-flat quadrics

Proposition 5.1. *Let $M \subset \mathbb{C}^{n+1}$ be given by*

$$(67) \quad w = \rho(z, \bar{z}),$$

where ρ is $O(2)$, and M is not a complex submanifold. Then the set S of CR singularities of M is given by

$$(68) \quad S = \{(z, w) : \bar{\partial}\rho = 0, w = \rho(z, \bar{z})\}.$$

Proof. This is well-known; we give a proof for convenience. In codimension two, a real submanifold is either CR singular, complex, or generic. A submanifold is generic if $\bar{\partial}$ of all the defining equations are pointwise linearly independent (see [Baouendi et al. 1999]). As M is not complex, to find the set of CR singularities, we find the set of points where M is not generic. We need both defining equations for M ,

$$(69) \quad w = \rho(z, \bar{z}) \quad \text{and} \quad \bar{w} = \rho(z, \bar{z}).$$

As the second equation always produces a $d\bar{w}$ while the first does not, the only way that the two can be linearly dependent is for the $\bar{\partial}$ of the first equation to be zero. In other words, $\bar{\partial}\rho = 0$. □

Let us compute and classify the CR singular sets for the CR singular Levi-flat quadrics:

Type	CR singularity S	$\dim_{\mathbb{R}} S$	CR structure of S
A. k	$z_1 = \cdots = z_k = 0, w = 0$	$2n - 2k$	complex
B.0	$z_1 = 0, w = 0$	$2n - 2$	complex
B. $\frac{1}{2}$	$z_1 + \bar{z}_1 = 0, w = 0$	$2n - 1$	Levi-flat
B. $\gamma, \gamma > 0, \gamma \neq \frac{1}{2}$	$z_1 = 0, w = 0$	$2n - 2$	complex
C.0	$z_2 = 0, w = 0$	$2n - 2$	complex
C.1	$z_2 + 2\bar{z}_1 = 0, w = -\frac{1}{4}z_2^2$	$2n - 2$	Levi-flat

By Levi-flat we mean that S is a Levi-flat CR submanifold in $\{w = 0\}$. There is a conjecture that a real subvariety that is Levi-flat at CR points has a stratification by Levi-flat CR submanifolds. This computation gives further evidence of this conjecture.

6. Levi foliations and images of generic Levi-flats

A CR Levi-flat submanifold $M \subset \mathbb{C}^n$ of codimension two has a certain canonical foliation defined on it with complex analytic leaves of real codimension two in M . The submanifold M is locally equivalent to $\mathbb{R}^2 \times \mathbb{C}^{n-2}$, defined by

$$(70) \quad \operatorname{Im} z_1 = 0, \quad \operatorname{Im} z_2 = 0.$$

The leaves of the foliation are the submanifolds given by fixing z_1 and z_2 at a real constant. By foliation we always mean the standard nonsingular foliation as locally comes up in the implicit function theorem. This foliation on M is called the *Levi foliation*. It is obvious that the Levi foliation on M extends to a neighborhood of M as a nonsingular holomorphic foliation. The same is not true in general for CR singular submanifolds. We say that a smooth holomorphic foliation \mathcal{L} defined in a neighborhood of M is an *extension* of the Levi foliation of M_{CR} , if \mathcal{L} and the Levi foliation have the same germs of leaves at each CR point of M . We also say that a smooth real-analytic foliation $\tilde{\mathcal{L}}$ on M is an extension of the Levi foliation on M_{CR} if $\tilde{\mathcal{L}}$ and the Levi foliation have the same germs of leaves at each CR point of M . In our situation (real-analytic), M_{CR} is a dense and open subset of M . This implies that the leaves of \mathcal{L} and $\tilde{\mathcal{L}}$ through a CR singular point are complex analytic submanifolds contained in M . The latter could lead to an obvious obstruction to extension. First let us see what happens if the foliation of M_{CR} is the restriction of a nonsingular holomorphic foliation of a whole neighborhood of M .

The Bishop-like quadrics, that is, A.1 and B. γ in \mathbb{C}^{n+1} , have a Levi foliation that extends as a holomorphic foliation to all of \mathbb{C}^{n+1} . That is because these submanifolds are of the form

$$(71) \quad N \times \mathbb{C}^{n-1}.$$

For submanifolds of the form (71) we can find normal forms using the well-developed theory of Bishop surfaces in \mathbb{C}^2 .

Proposition 6.1. *Suppose $M \subset \mathbb{C}^{n+1}$ is a real-analytic Levi-flat CR singular submanifold where the Levi foliation on M_{CR} extends near $p \in M$ to a nonsingular holomorphic foliation of a neighborhood of p in \mathbb{C}^{n+1} . Then at p , M is locally biholomorphically equivalent to a submanifold of the form*

$$(72) \quad N \times \mathbb{C}^{n-1},$$

where $N \subset \mathbb{C}^2$ is a CR singular submanifold of real dimension two. Therefore if M has a nondegenerate complex tangent, then it is Bishop-like, i.e., of type A.1 or B. γ .

Furthermore, two submanifolds of the form (72) are locally biholomorphically (resp. formally) equivalent if and only if the corresponding N s are locally biholomorphically (resp. formally) equivalent in \mathbb{C}^2 .

Proof. We flatten the holomorphic foliation near p so that in some polydisc Δ , the leaves of the foliation are given by $\{q\} \times \mathbb{C}^{n-1} \cap \Delta$ for $q \in \mathbb{C}^2$. Let us suppose that M is closed in Δ . At any CR point of M , the leaf of the Levi foliation agrees with that of the holomorphic foliation and therefore the leaf that lies in M agrees with a leaf of the form $\{q\} \times \mathbb{C}^{n-1}$ as a germ and so $\{q\} \times \mathbb{C}^{n-1} \cap \Delta \subset M$. As M_{CR} is dense in M , then M is a union of sets of the form $\{q\} \times \mathbb{C}^{n-1} \cap \Delta$ and the first part follows.

It is classical that every Bishop surface (two-dimensional real submanifold of \mathbb{C}^2 with a nondegenerate complex tangent) is equivalent to a submanifold whose quadratic part is of the form A.1 or B. γ .

Finally, the proof that two submanifolds of the form (72) are equivalent if and only if the N s are equivalent is straightforward. \square

Not every Bishop-like submanifold is a cross product as above. In fact the Bishop invariant may well change from point to point. See Section 15. In such cases the foliation does not extend to a nonsingular holomorphic foliation of a neighborhood.

Let us now focus on extending the Levi foliation to M , and not to a neighborhood of M . Let us prove a useful proposition about recognizing certain CR singular Levi-flats from the form of the defining equation. That is, if the r in the equation does not depend on \bar{z}_2 through \bar{z}_n .

Proposition 6.2. *Suppose that near the origin $M \subset \mathbb{C}^{n+1}$ is given by*

$$(73) \quad w = r(z_1, \bar{z}_1, z_2, z_3, \dots, z_n),$$

where r is $O(2)$ and $\partial r / \partial \bar{z}_1 \neq 0$. Then M is a CR singular Levi-flat submanifold and the Levi foliation of M_{CR} extends through the origin to a real-analytic foliation on M . Furthermore, there exists a real-analytic CR mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n+1}$, $F(0) = 0$, which is a diffeomorphism onto its image $F(U) \subset M$.

Near zero, M is the image of a CR mapping that is a diffeomorphism onto its image of the standard CR Levi-flat. The proposition also holds in two dimensions ($n = 1$), although in this case it is somewhat trivial.

Proof. As in [Lebl et al. 2014], let us define the mapping F by

$$(74) \quad (x, y, \xi) \mapsto (x + iy, \xi, r(x + iy, x - iy, \xi)),$$

where $\xi = (\xi_2, \dots, \xi_n) \in \mathbb{C}^{n-1}$. Near points where M is CR, this mapping is a CR diffeomorphism and hence M must be Levi-flat. Furthermore, since F is a diffeomorphism, it takes the Levi foliation on $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ to a foliation on M near zero. \square

Lemma 6.3. *Let $M \subset \mathbb{C}^{n+1}$ be a CR singular real-analytic Levi-flat submanifold of codimension two through the origin.*

Then M is a CR singular Levi-flat submanifold whose Levi foliation of M_{CR} extends through the origin to a nonsingular real-analytic foliation on M if and only if there exists a real-analytic CR mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n+1}$, $F(0) = 0$, which is a diffeomorphism onto its image $F(U) \subset M$.

Proof. One direction is easy and was used above. For the other direction, suppose that we have a foliation extending the Levi foliation through the origin. Let us consider M_{CR} an abstract CR manifold. That is a manifold M_{CR} together with the bundle $T^{(0,1)}M_{\text{CR}} \subset \mathbb{C} \otimes TM_{\text{CR}}$. The extended foliation on M gives a real-analytic subbundle $\mathcal{W} \subset TM$. Since we are extending the Levi foliation, when $p \in M_{\text{CR}}$, then $\mathcal{W}_p = T_p^c M$, where $T_p^c M = J(T_p^c M)$ is the complex tangent space and J is the complex structure on \mathbb{C}^{n+1} . Since M_{CR} is dense in M , then $J\mathcal{W} = \mathcal{W}$ on M .

Define the real-analytic subbundle $\mathcal{V} \subset \mathbb{C} \otimes TM$ as

$$(75) \quad \mathcal{V}_p = \{X + iJ(X) : X \in \mathcal{W}_p\}.$$

At CR points $\mathcal{V}_p = T_p^{(0,1)}M$ (see, for example, [Baouendi et al. 1999, p. 8]). Then we can find vector fields X^1, \dots, X^{n-1} in \mathcal{W} such that

$$(76) \quad X^1, J(X^1), X^2, J(X^2), \dots, X^{n-1}, J(X^{n-1})$$

is a basis of \mathcal{W} near the origin. Then the basis for \mathcal{V} is given by

$$(77) \quad X^1 + iJ(X^1), X^2 + iJ(X^2), \dots, X^{n-1} + iJ(X^{n-1}).$$

As the subbundle is integrable, we obtain that (M, \mathcal{V}) gives an abstract CR manifold, which at CR points agrees with M_{CR} . This manifold is Levi-flat as it is Levi-flat on a dense open set. As it is real-analytic, it is embeddable; and hence there exists a real-analytic CR diffeomorphism from a neighborhood of $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ to a neighborhood of zero in M (as an abstract CR manifold). This is our mapping F . \square

The quadrics A.k, $k \geq 2$, defined by

$$(78) \quad w = \bar{z}_1^2 + \cdots + \bar{z}_k^2,$$

contain the singular variety defined by $w = 0, z_1^2 + \cdots + z_k^2 = 0$, and hence the Levi foliation cannot extend to a nonsingular foliation of the submanifold. The quadric A.1 does admit a holomorphic foliation, but other type A.1 submanifolds do not in general. For example, the submanifold

$$(79) \quad w = \bar{z}_1^2 + \bar{z}_2^3$$

is of type A.1 and the unique complex variety through the origin is $0 = z_1^2 + z_2^3$, which is singular. Therefore the foliation cannot extend to M .

7. Extending the Levi foliation of type C.x submanifolds

Let us prove [Theorem 1.2](#), that is, let us start with a type C.0 or C.1 submanifold and show that the Levi foliation must extend real-analytically to all of M . Equivalently, we show that the real-analytic bundle $T^{(1,0)}M_{\text{CR}}$ extends to a real-analytic subbundle of $\mathbb{C} \otimes TM$. Taking real parts we obtain an involutive subbundle of TM extending $T^c M_{\text{CR}} = \text{Re}(T^{(1,0)}M_{\text{CR}})$.

Proof of Theorem 1.2. Let M be the submanifold given by

$$(80) \quad w = \bar{z}_1 z_2 + \epsilon \bar{z}_1^2 + r(z, \bar{z}),$$

where $\epsilon = 0, 1$. Let us treat the z variables as the parameters on M . Let π be the projection onto the $\{w = 0\}$ plane, which is tangent to M at zero as a real $2n$ -dimensional hyperplane. We will look at all the vector fields on this plane $\{w = 0\}$. All vectors in $\pi(T^{(1,0)}M)$ can be written in terms of $\partial/\partial z_j$ for $j = 1, \dots, n$.

The Levi map is given by taking the $n \times n$ matrix

$$(81) \quad L = L(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \left[\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \right]_{j,k} (p)$$

to vectors $v \in \pi(T^{(1,0)}M)$ (π is the projection) as v^*Lv . The excessive term in L vanishes at zero.

Notice that for $p \in M_{\text{CR}}$, $\pi(T_p^{(1,0)}M)$ is $(n-1)$ -dimensional. As M is Levi-flat, then v^*Lv vanishes for $v \in \pi(T_p^{(1,0)}M)$. Write the vector $v = (v_1, \dots, v_n)^t$. The zero set of the function

$$(82) \quad (z, v) \in \mathbb{C}^n \times \mathbb{C}^n \xrightarrow{\varphi} v^*L(z, \bar{z})v$$

is a variety V of real codimension two at the origin of $\mathbb{C}^n \times \mathbb{C}^n$ because of the form of L . That is, at $z = 0$, the only vectors v such that $v^*Lv = 0$ are those where $v_1 = 0$ or $v_2 = 0$. So the codimension is at least two. And we know that v^*Lv vanishes for vectors in $\pi(T_p^{(1,0)}M)$ for $p \in M$ near zero, which is real codimension two at each z corresponding to a CR point. Therefore, $V \cap (\pi(M_{\text{CR}}) \times \mathbb{C}^n)$ has a connected component that is equal to a connected component of the real-analytic subbundle $\pi(T^{(1,0)}M_{\text{CR}})$. We will verify that the latter is connected.

We show below that this subbundle extends past the CR singularity. The key point is to show that the restriction of $\pi(T^{(1,0)}(M_{\text{CR}}))$ extends to a smooth real-analytic submanifold of $T^{(1,0)}\mathbb{C}^n$. Write

$$(83) \quad \varphi(z, v) = v_1 \bar{v}_2 + \sum a_{jk}(z) v_j \bar{v}_k,$$

where $a_{jk}(0) = 0$.

By Proposition 5.1, $\pi(M \setminus M_{\text{CR}})$ is contained in

$$(84) \quad z_2 + 2\epsilon \bar{z}_1 + r \bar{z}_1 = 0.$$

Thus M_{CR} is connected. Assume that $v \cdot (\partial/\partial z) \in T_p^{(1,0)}M$ at a CR point p . Then

$$(85) \quad (z_2 + 2\epsilon \bar{z}_1 + r \bar{z}_1) \bar{v}_1 + \sum_{j>1} r \bar{z}_j \bar{v}_j = 0.$$

When p is in the open set $U_\delta \subset \pi(M_{\text{CR}})$ defined by $|z_2 + 2\epsilon \bar{z}_1| > \frac{1}{2}|z|$ and $0 < |z| < \delta$, v is contained in

$$(86) \quad V_C: |v_1| \leq |v|/C.$$

When δ is sufficiently small, $\varphi(z, v) = 0$ admits a unique solution

$$(87) \quad v_1 = f(z, v_3, \dots, v_n), \quad v_2 = 1$$

by imposing $v \in V_C$. Note that f is given by convergent power series. For $|z| < \delta$, define

$$(88) \quad w_j = (w_{j1}(z), \dots, w_{jn}(z)) \in V_C, \quad j = 2, \dots, n$$

such that $\varphi(z, w_j(z)) = 0$ and

$$(89) \quad w_{j2} = 1, \quad w_{jk} = \delta_{jk}, \quad j \geq 2, k > 2.$$

To see why we can do so, fix $p \in U_\delta$. First we can find a vector w_2 in $E_p = \pi(T_p^{(1,0)}M_{\text{CR}})$ such that $v_2 = 1$. Otherwise, $E_p \subset V_C$ cannot have dimension $n-1$. Let E'_p be the vector subspace of E_p with $v_2 = 0$. Then E'_p has rank $n-2$ and remains in the cone V_C . Then E'_p has an element w_2 with v_2 component being one. Repeating this, we find w_2, \dots, w_n in E_p such that the v_j component of w_i is zero

for $2 < j < i$. Using linear combinations, we find a unique basis $\{w_2, \dots, w_n\}$ of E_p that satisfies condition (89).

Assume that C is sufficiently large. By the above uniqueness assertion on $\varphi(z, v) = 0$, we conclude that when $p \in U_\delta$, $\{w_2(p), \dots, w_n(p)\}$ is a basis of $\pi(T_p^{(1,0)}M_{\text{CR}})$. Also it is real analytic at $p = 0$. Define

$$(90) \quad \omega_j(z) = w_j(z) \cdot \frac{\partial}{\partial z}, \quad |z| < \delta.$$

We lift the functions ω_j via π to a subbundle of $\mathbb{C} \otimes TM$; let us call them $\tilde{\omega}_j$. Then consider the vector fields $w_j^* = 2 \operatorname{Re} \tilde{\omega}_j = \tilde{\omega}_j + \overline{\tilde{\omega}_j}$ and $w_{n+j}^* = \operatorname{Im} \tilde{\omega}_j$ for $j = 2, \dots, n$. Above CR points over U_δ , \tilde{w}_j is in $TM_{\text{CR}} \otimes \mathbb{C}$ and so tangent to M . We thus obtain a $(2n - 2)$ -dimensional real-analytic subbundle of TM that agrees with the real-analytic real subbundle of TM_{CR} induced by the Levi foliation above U_δ . Since M_{CR} and the subbundles are real analytic and M_{CR} is connected, they agree over M_{CR} .

The real-analytic distribution spanned by $\{\omega_j^*\}$ has constant rank $2n - 2$ everywhere and is involutive on an open subset of M_{CR} and hence everywhere. \square

8. CR singular set of type C.x submanifolds

Let $M \subset \mathbb{C}^{n+1}$ be a codimension-two Levi-flat CR singular submanifold that is an image of $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ via a real-analytic CR map, and let $S \subset M$ be the CR singular set of M . In [Lebl et al. 2014] it was proved that near a generic point of S , exactly one of the following is true:

- (i) S is a Levi-flat submanifold of dimension $2n - 2$ and CR dimension $n - 2$.
- (ii) S is a complex submanifold of complex dimension $n - 1$ (real dimension $2n - 2$).
- (iii) S is a Levi-flat submanifold of dimension $2n - 1$ and CR dimension $n - 1$.

We only have the above classification for a generic point of S , and S need not be a CR submanifold everywhere. See [Lebl et al. 2014] for examples.

If M is a Levi-flat CR singular submanifold and the Levi foliation of M_{CR} extends to M , then by Lemma 6.3 at a generic point S has to be of one of the above types. A corollary of Theorem 1.2 is:

Corollary 8.1. *Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular type C.1 or type C.0 submanifold. Let $S \subset M$ denote the CR singular set. Then near the origin S is a submanifold of dimension $2n - 2$, and at a generic point, S is either CR Levi-flat of dimension $2n - 2$ (CR dimension $n - 2$) or a complex submanifold of complex dimension $n - 1$.*

Furthermore, if M is of type C.1, then at the origin S is a CR Levi-flat submanifold of dimension $2n - 2$ (CR dimension $n - 2$).

Proof. Let M be given by

$$(91) \quad w = \bar{z}_1 z_2 + \epsilon \bar{z}_1^2 + r(z, \bar{z}),$$

where r is $O(3)$ and $\epsilon = 0$ or $\epsilon = 1$.

By [Proposition 5.1](#) the CR singular set is exactly where

$$(92) \quad z_2 + \epsilon 2\bar{z}_1 + r_{\bar{z}_1}(z, \bar{z}) = 0 \quad \text{and} \quad r_{\bar{z}_j}(z, \bar{z}) = 0 \quad \text{for all } j = 2, \dots, n.$$

By considering the real and imaginary parts of the first equation and applying the implicit function theorem, the set $\tilde{S} = \{z : z_2 + \epsilon 2\bar{z}_1 + r_{\bar{z}_1}(z, \bar{z}) = 0\}$ is a real submanifold of real dimension $2n - 2$ (real codimension two in M). Now $S \subset \tilde{S}$, but as we saw above that S is of dimension at least $2n - 2$. Therefore $S = \tilde{S}$ near the origin. The conclusion of the first part then follows from the classification above.

The stronger conclusion for type C.1 submanifolds follows by noticing that when $\epsilon = 1$, the submanifold

$$(93) \quad z_2 + 2\bar{z}_1 + r_{\bar{z}_1}(z, \bar{z}) = 0$$

is CR and not complex at the origin. □

9. Mixed-holomorphic submanifolds

Let us study sets in \mathbb{C}^m defined by

$$(94) \quad f(\bar{z}_1, z_2, \dots, z_m) = 0,$$

for a single holomorphic function f of m variables.

Such sets have much in common with complex varieties, since they are in fact complex varieties when \bar{z}_1 is treated as a complex variable. The distinction is that the automorphism group is different since we are interested in automorphisms that are holomorphic, not mixed-holomorphic.

Proposition 9.1. *If $M \subset \mathbb{C}^m$ is a submanifold with a defining equation of the form (94), where f is a holomorphic function that is not identically zero, then M is a real codimension-two set and M is either a complex submanifold or a Levi-flat submanifold, possibly CR singular. Furthermore, if M is CR singular at $p \in M$, and has a nondegenerate complex tangent at p , then M has type A.k, C.0, or C.1 at p .*

Proof. Since the zero set of f is a complex variety in the $(\bar{z}_1, z_2, \dots, z_m)$ space, we get automatically that it is real codimension two. We also have that as it is a submanifold, then it can be written as a graph of one variable over the rest.

Let $m = n + 1$ for convenience and suppose that $M \subset \mathbb{C}^{n+1}$ is a submanifold through the origin. By factorization for germs of holomorphic functions and by the

smoothness assumption on M we may assume that $df(0) \neq 0$. Call the variables (z_1, \dots, z_n, w) and write M as a graph. One possibility is that we write M as

$$(95) \quad \bar{w} = \rho(z_1, \dots, z_n),$$

where $\rho(0) = 0$ and ρ has no linear terms. M is complex if $\rho \equiv 0$. Otherwise M is CR singular and we rewrite it as

$$(96) \quad w = \bar{\rho}(\bar{z}_1, \dots, \bar{z}_n).$$

We notice that the matrix representing the Levi-map must be identically zero, so we must get Levi-flat. If there are any quadratic terms, we obtain a type A.k submanifold.

Alternatively, M can be written as

$$(97) \quad w = \rho(\bar{z}_1, z_2, \dots, z_n),$$

with $\rho(0) = 0$. If ρ does not depend on \bar{z}_1 then M is complex. Assume that ρ depends on \bar{z}_1 . If ρ has linear terms in \bar{z}_1 , then M is CR. Otherwise it is a CR singular submanifold, and near a nonCR singular point it is a generic codimension-two submanifold. The CR singular set of M is defined by $\partial\rho/\partial\bar{z}_1 = 0$.

Suppose that M is CR singular. That M is Levi-flat follows from [Proposition 6.2](#). We can therefore normalize the quadratic term, after linear terms in z_2, \dots, z_n are absorbed into w .

If not all quadratic terms are zero, we must have an A.k, C.0, or C.1 type submanifold. \square

Let us now study normal forms for such sets in \mathbb{C}^2 and \mathbb{C}^m , $m \geq 3$. First in two variables we can completely answer the question. This result is surely well known and classical.

Proposition 9.2. *If $M \subset \mathbb{C}^2$ is a submanifold with a defining equation of the form (94), then it is locally biholomorphically equivalent to a submanifold in coordinates $(z, w) \in \mathbb{C}^2$ of the form*

$$(98) \quad w = \bar{z}^d$$

for $d = 0, 1, 2, 3, \dots$, where d is a local biholomorphic invariant of M . If $d = 0$, M is complex; if $d = 1$, it is a CR totally-real submanifold; and if $d \geq 2$, then M is CR singular.

Proof. Write the submanifold as a graph of one variable over the other. Without loss of generality and after possibly taking a conjugate of the equation, we have

$$(99) \quad w = f(\bar{z})$$

for some holomorphic function f . Assume $f(0) = 0$. If f is identically zero, then $d = 0$ and we are finished. If f is not identically zero, we apply a holomorphic change of coordinates in z , and the rest follows easily. \square

In three or more variables, if $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a submanifold through the origin, then when the quadratic part is nonzero we have seen above that it can be a type A. k , C.0, or C.1 submanifold. If the submanifold is the nondegenerate type C.1 submanifold, then we will show in the next section that M is biholomorphically equivalent to the quadric $M_{C.1}$.

Before we move to type C.1, let us quickly consider the mixed-holomorphic submanifolds of type A. n . The submanifolds of type A. n in \mathbb{C}^{n+1} can in some sense be considered nondegenerate when talking about mixed-holomorphic submanifolds.

Proposition 9.3. *If $M \subset \mathbb{C}^{n+1}$ is a submanifold of type A. n at the origin of the form*

$$(100) \quad w = \bar{z}_1^2 + \cdots + \bar{z}_n^2 + r(\bar{z}),$$

where $r \in O(3)$. Then M is locally near the origin biholomorphically equivalent to the A. n quadric

$$(101) \quad w = \bar{z}_1^2 + \cdots + \bar{z}_n^2.$$

Proof. The complex Morse lemma (see, e.g., [Ebeling 2007, Proposition 3.15]) states that there is a local change of coordinates near the origin in just the z variables such that

$$(102) \quad z_1^2 + \cdots + z_n^2 + \bar{r}(z)$$

is equivalent to $z_1^2 + \cdots + z_n^2$. \square

It is not difficult to see that the normal form for mixed-holomorphic submanifolds in \mathbb{C}^{n+1} of type A. k , $k < n$, is equivalent to a local normal form for a holomorphic function in n variables. Therefore, for example, the submanifold $w = \bar{z}_1^2 + \bar{z}_2^3$ is of type A.1 and is not equivalent to any quadric.

10. Formal normal form for certain type C.1 submanifolds I

In this section we prove the formal normal form in [Theorem 1.3](#). That is, we prove that if $M \subset \mathbb{C}^{n+1}$ is defined by

$$(103) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \dots, z_n),$$

where r is $O(3)$, then M is Levi-flat and formally equivalent to

$$(104) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

That M is Levi-flat follows from [Proposition 6.2](#).

Lemma 10.1. *If $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is given by*

$$(105) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \dots, z_n),$$

where r is an $O(3)$ formal power series, then M is formally equivalent to $M_{C.1}$ given by

$$(106) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

In fact, the normalizing transformation can be of the form

$$(107) \quad (z, w) = (z_1, \dots, z_n, w) \mapsto (z_1, f(z, w), z_3, \dots, z_n, g(z, w)),$$

where f and g are formal power series.

Proof. Suppose that the normalization was done to degree $d - 1$, then suppose that

$$(108) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r_1(z_1, \bar{z}_1, z_2, \dots, z_n) + r_2(z_1, \bar{z}_1, z_2, \dots, z_n),$$

where r_1 is degree- d homogeneous and r_2 is $O(d + 1)$. Write

$$(109) \quad r_1(z_1, \bar{z}_1, z_2, \dots, z_n) = \sum_{j=0}^k \sum_{|\alpha|+j=d} c_{j,\alpha} \bar{z}_1^j z^\alpha,$$

where k is the highest power of \bar{z}_1 in r_1 , and α is a multiindex.

If k is even, then we use the transformation that replaces w with

$$(110) \quad w + \sum_{|\alpha|+k=d} c_{j,\alpha} w^{k/2} z^\alpha.$$

Let us look at the degree- d terms in

$$(111) \quad (\bar{z}_1 z_2 + \bar{z}_1^2) + \sum_{|\alpha|+k=d} c_{j,\alpha} (\bar{z}_1 z_2 + \bar{z}_1^2)^{k/2} z^\alpha \\ = \bar{z}_1 z_2 + \bar{z}_1^2 + r_1(z_1, \bar{z}_1, z_2, \dots, z_n).$$

We need not include r_2 as the terms have degree $d + 1$ or higher. After canceling out the new terms on the left, we notice that the formal transformation removed all the terms in r_1 with a power \bar{z}_1^k and replaced them with terms that have a smaller power of \bar{z}_1 .

Next suppose that k is odd. We use the transformation that replaces z_2 with

$$(112) \quad z_2 - \sum_{|\alpha|+k=d} c_{j,\alpha} w^{(k-1)/2} z^\alpha.$$

Let us look at the degree- d terms in

$$(113) \quad \bar{z}_1 z_2 + \bar{z}_1^2 \\ = \bar{z}_1 \left(z_2 - \sum_{|\alpha|+k=d} c_{j,\alpha} w^{(k-1)/2} z^\alpha \right) + \bar{z}_1^2 + r_1(z_1, \bar{z}_1, z_2, \dots, z_n).$$

Again we need not include r_2 as the terms have degree $d+1$ or higher, and we need not add the new terms to z_2 in the argument list for r_1 since all those terms would be of higher degree. Again we notice that the formal transformation removed all the terms in r_1 with a power \bar{z}_1^k and replaced them with terms that have a smaller power of \bar{z}_1 .

The procedure above does not change the form of the submanifold, but it lowers the degree of \bar{z}_1 by one. Since we can assume that all terms in r_1 depend on \bar{z}_1 , we are finished with degree- d terms after k iterations of the above procedure. \square

11. Convergence of normalization for certain type C.1 submanifolds

A key point in the computation below is the following natural involution for the quadric $M_{C.1}$. Notice that the map

$$(114) \quad (z_1, z_2, \dots, z_n, w) \mapsto (-\bar{z}_2 - z_1, z_2, \dots, z_n, w)$$

takes $M_{C.1}$ to itself. The involution simply replaces the \bar{z}_1 in the equation with $-z_2 - \bar{z}_1$. The way this involution is defined is by noticing that the equation $w = \bar{z}_1 z_2 + \bar{z}_1^2$ has generically two solutions for \bar{z}_1 keeping z_2 and w fixed. In the same way we could define an involution on all type C.1 submanifolds of the form $w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}_1, z_2, \dots, z_n)$, although we will not require this construction.

We prove convergence via:

Lemma 11.1. *Let m_1, \dots, m_N be positive integers. Suppose $T(z)$ is a formal power series in $z \in \mathbb{C}^N$. Suppose $T(t^{m_1} v_1, \dots, t^{m_N} v_N)$ is a convergent power series in $t \in \mathbb{C}$ for all $v \in \mathbb{C}^N$. Then T is convergent.*

The proof is a standard application of the Baire category theorem and the Cauchy inequality. See [Baouendi et al. 1999, p. 153, Theorem 5.5.30], where all m_j are one. For $m_j > 1$ we first change variables by setting $v_j = w_j^{m_j}$ and apply the lemma with $m_j = 1$.

The following lemma finishes the proof of [Theorem 1.3](#). By absorbing any holomorphic terms into w , we assume that $r(z_1, 0, z_2, \dots, z_n) \equiv 0$. In [Lemma 10.1](#) we have also constructed a formal transformation that only changes the z_2 and w coordinates, so it is enough to prove convergence in this case. Key points of this proof are that the right hand side of the defining equation for $M_{C.1}$ is homogeneous, and that we have a natural involution on $M_{C.1}$.

Lemma 11.2. *If $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is given by*

$$(115) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \dots, z_n),$$

where r is $O(3)$ and convergent, and $r(z_1, 0, z_2, \dots, z_n) \equiv 0$. Suppose that two formal power series $f(z, w)$ and $g(z, w)$ satisfy

$$(116) \quad \begin{aligned} g(z, \bar{z}_1 z_2 + \bar{z}_1^2) \\ = \bar{z}_1 f(z, \bar{z}_1 z_2 + \bar{z}_1^2) + \bar{z}_1^2 + r(z_1, \bar{z}_1, f(z, \bar{z}_1 z_2 + \bar{z}_1^2), z_3, \dots, z_n). \end{aligned}$$

Then f and g are convergent.

Proof. Equation (116) is true formally, treating z_1 and \bar{z}_1 as independent variables. Notice that (116) has one equation for two unknown functions.

We now use the involution on $M_{C,1}$ to create a system that we can solve uniquely. We replace \bar{z}_1 with $-z_2 - \bar{z}_1$. We leave z_1 untouched (treating as an independent variable). We obtain an identity in formal power series:

$$(117) \quad \begin{aligned} g(z, \bar{z}_1 z_2 + \bar{z}_1^2) = (-z_2 - \bar{z}_1) f(z, \bar{z}_1 z_2 + \bar{z}_1^2) + (-z_2 - \bar{z}_1)^2 \\ + r(z_1, (-z_2 - \bar{z}_1), f(z, \bar{z}_1 z_2 + \bar{z}_1^2), z_3, \dots, z_n). \end{aligned}$$

The formal series $\xi = f(z, \bar{z}_1 z_2 + \bar{z}_1^2)$ and $\omega = g(z, \bar{z}_1 z_2 + \bar{z}_1^2)$ are solutions of the system

$$(118) \quad \omega = \bar{z}_1 \xi + \bar{z}_1^2 + r(z_1, \bar{z}_1, \xi, z_3, \dots, z_n),$$

$$(119) \quad \omega = (-z_2 - \bar{z}_1) \xi + (-z_2 - \bar{z}_1)^2 + r(z_1, (-z_2 - \bar{z}_1), \xi, z_3, \dots, z_n).$$

We next replace z_j with $t z_j$ and \bar{z}_1 with $t \bar{z}_1$ for $t \in \mathbb{C}$. Because $\bar{z}_1 z_2 + \bar{z}_1^2$ is homogeneous of degree two, we obtain that the formal series in t , given by $\xi(t) = f(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2))$, $\omega(t) = g(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2))$ for every $(z_1, \bar{z}_1, z_2, \dots, z_n) \in \mathbb{C}^{n+1}$, are solutions of the system

$$(120) \quad \omega = t \bar{z}_1 \xi + t^2 \bar{z}_1^2 + r(tz_1, t \bar{z}_1, \xi, tz_3, \dots, tz_n),$$

$$(121) \quad \omega = t(-z_2 - \bar{z}_1) \xi + t^2(-z_2 - \bar{z}_1)^2 + r(tz_1, t(-z_2 - \bar{z}_1), \xi, tz_3, \dots, tz_n).$$

We eliminate ω to obtain an equation for ξ ,

$$(122) \quad \begin{aligned} t(2\bar{z}_1 + z_2)(\xi - tz_2) \\ = r(tz_1, t(-z_2 - \bar{z}_1), \xi, tz_3, \dots, tz_n) - r(tz_1, t \bar{z}_1, \xi, tz_3, \dots, tz_n). \end{aligned}$$

We now treat ξ as a variable and we have a holomorphic (convergent) equation. The right hand side must be divisible by $t(2\bar{z}_1 + z_2)$: It is divisible by t since r is divisible by \bar{z}_1 . It is also divisible by $2\bar{z}_1 + z_2$ as setting $z_2 = -2\bar{z}_1$ makes the right hand side vanish. Therefore,

$$(123) \quad \xi - tz_2 = \frac{r(tz_1, t(-z_2 - \bar{z}_1), \xi, tz_3, \dots, tz_n) - r(tz_1, t \bar{z}_1, \xi, tz_3, \dots, tz_n)}{t(2\bar{z}_1 + z_2)},$$

where the right hand side is a holomorphic function (that is, a convergent power series) in $z_1, \bar{z}_1, z_2, \dots, z_n, t, \xi$. For any fixed $z_1, \bar{z}_1, z_2, \dots, z_n$, we solve for ξ in terms of t via the implicit function theorem, and we obtain that ξ is a holomorphic function of t . The power series of ξ is given by $\xi(t) = f(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2))$.

Let $v \in \mathbb{C}^{n+1}$ be any nonzero vector. Via a proper choice of $z_1, \bar{z}_1, z_2, \dots, z_n$ (still treating \bar{z}_1 and z_1 as independent variables) we write $v = (z, \bar{z}_1 z_2 + \bar{z}_1^2)$. We apply the above argument to $\xi(t) = f(tv_1, \dots, tv_n, t^2 v_{n+1})$, and $\xi(t)$ converges as a series in t . As we get convergence for every $v \in \mathbb{C}^{n+1}$, we obtain that f converges by Lemma 11.1. Then via (120) we obtain that $g(tv_1, \dots, tv_n, t^2 v_{n+1})$ converges as a series in t for all v , and hence g converges. \square

12. Automorphism group of the C.1 quadric

With the normal form achieved in previous sections, let us study the automorphism group of the C.1 quadric in this section. We will again use the mixed-holomorphic involution that is obtained from the quadric.

We study the local automorphism group at the origin. That is the set of germs at the origin of biholomorphic transformations taking M to M and fixing the origin.

First we look at the linear parts of automorphisms. We already know that the linear term of the last component only depends on w . For $M_{C.1}$ we can say more about the first two components.

Proposition 12.1. *Let $(F, G) = (F_1, \dots, F_n, G)$ be a formal invertible or biholomorphic automorphism of $M_{C.1} \subset \mathbb{C}^{n+1}$, that is the submanifold of the form*

$$(124) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

Then $F_1(z, w) = az_1 + \alpha w + O(2)$, $F_2(z, w) = \bar{a}z_2 + \beta w + O(2)$, and $G(z, w) = \bar{a}^2 w + O(2)$, where $a \neq 0$.

Proof. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be such that

$$F_1(z, w) = a \cdot z + \alpha w + O(2) \quad \text{and} \quad F_2(z, w) = b \cdot z + \beta w + O(2).$$

Then from Proposition 2.1 we have

$$(125) \quad \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda [a^* \quad b^* \quad \cdots] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ \vdots \end{bmatrix}.$$

Therefore $\lambda \bar{a}_1 b_2 = 1$, and $\bar{a}_j b_k = 0$ for all $(j, k) \neq (1, 2)$. Similarly,

$$(126) \quad \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda [a^* \quad b^* \quad \cdots] \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{b} \\ \vdots \end{bmatrix}.$$

Therefore $\lambda \bar{a}_1^2 = 1$, and $\bar{a}_j \bar{a}_k = 0$ for all $(j, k) \neq (1, 1)$. Putting these two together we obtain that $a_j = 0$ for all $j \neq 1$, and as $a_1 \neq 0$ we get $b_j = 0$ for all $j \neq 2$. As λ is the reciprocal of the coefficient of w in G , we are finished. \square

Lemma 12.2. *Let $M_{C,1} \subset \mathbb{C}^3$ be given by*

$$(127) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

Suppose that a local biholomorphism (resp. formal automorphism) (F_1, F_2, G) transforms $M_{C,1}$ into $M_{C,1}$. Then F_1 depends only on z_1 , and F_2 and G depend only on z_2 and w .

Proof. Let us define a $(1, 0)$ tangent vector field on M by

$$(128) \quad Z = \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial w}.$$

Write $F = (F_1, F_2, G)$. F must take Z into a multiple of itself when restricted to $M_{C,1}$. That is, on $M_{C,1}$ we have

$$(129) \quad \frac{\partial F_1}{\partial z_2} + \bar{z}_1 \frac{\partial F_1}{\partial w} = 0,$$

$$(130) \quad \frac{\partial F_2}{\partial z_2} + \bar{z}_1 \frac{\partial F_2}{\partial w} = \lambda,$$

$$(131) \quad \frac{\partial G}{\partial z_2} + \bar{z}_1 \frac{\partial G}{\partial w} = \lambda \bar{F}_1(\bar{z}, \bar{w}),$$

for some function λ . Let us take the first equation and plug in the defining equation for M_1 ,

$$(132) \quad \frac{\partial F_1}{\partial z_2}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + \bar{z}_1 \frac{\partial F_1}{\partial w}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = 0.$$

This is true for all $z \in \mathbb{C}^2$, and so we may treat z_1 and \bar{z}_1 as independent variables.

We have an involution on $M_{C,1}$ that takes \bar{z}_1 to $-z_2 - \bar{z}_1$. Therefore we also have

$$(133) \quad \frac{\partial F_1}{\partial z_2}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (-z_2 - \bar{z}_1) \frac{\partial F_1}{\partial w}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = 0.$$

This means that $\partial F_1 / \partial w$ and therefore $\partial F_1 / \partial z_2$ must be identically zero. That is, F_1 only depends on z_1 .

We have that the following must hold for all z :

$$(134) \quad G(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) F_2(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{F}_1(\bar{z}_1))^2.$$

Again we treat z_1 and \bar{z}_1 as independent variables. Differentiate with respect to z_1 :

$$(135) \quad \frac{\partial G}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) \frac{\partial F_2}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2).$$

We plug in the involution again to obtain

$$(136) \quad \frac{\partial G}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(-z_2 - \bar{z}_1) \frac{\partial F_2}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2).$$

Therefore as F_1 is not identically zero, then as before both $\partial F_2 / \partial z_1$ and $\partial G / \partial z_1$ must be identically zero. \square

Lemma 12.3. *Take $M_{C,1} \subset \mathbb{C}^3$ given by*

$$(137) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2,$$

and let (F_1, F_2, G) be a local automorphism at the origin. Then F_1 uniquely determines F_2 and G . Furthermore, given any invertible function of one variable F_1 with $F_1(0) = 0$, there exist unique F_2 and G that complete an automorphism and they are determined by

$$(138) \quad \begin{aligned} F_2(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) &= \bar{F}_1(\bar{z}_1) + \bar{F}_1(-\bar{z}_1 - z_2), \\ G(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) &= -\bar{F}_1(\bar{z}_1) \bar{F}_1(-\bar{z}_1 - z_2). \end{aligned}$$

We should note that the lemma also works formally. Given any formal F_1 , there exist unique formal F_2 and G satisfying the above property.

Proof. By Lemma 12.2, F_1 depends only on z_1 and F_2 and G depend only on z_2 and w . We write the automorphism as a composition of the two mappings $(F_1(z_1), z_2, w)$ and $(z_1, F_2(z_2, w), G(z_2, w))$.

We plug the transformation into the defining equation for $M_{C,1}$ and obtain

$$(139) \quad G(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) F_2(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{F}_1(\bar{z}_1))^2.$$

We use the involution $(z_1, z_2) \mapsto (-\bar{z}_1 - z_2, z_2)$ which preserves $M_{C,1}$ and obtain a second equation

$$(140) \quad G(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(-\bar{z}_1 - z_2) F_2(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{F}_1(-\bar{z}_1 - z_2))^2.$$

We eliminate G and solve for F_2 :

$$(141) \quad \begin{aligned} F_2(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) &= \frac{(\bar{F}_1(-\bar{z}_1 - z_2))^2 - (\bar{F}_1(\bar{z}_1))^2}{\bar{F}_1(\bar{z}_1) - \bar{F}_1(-\bar{z}_1 - z_2)} \\ &= \bar{F}_1(\bar{z}_1) + \bar{F}_1(-\bar{z}_1 - z_2). \end{aligned}$$

Next we note that trivially, F_2 is unique if it exists: its difference vanishes on $M_{C,1}$.

If we suppose that F_1 is convergent, then just as before, substituting z_2 with tz_2 and \bar{z}_1 with $t\bar{z}_1$, we are restricting to curves (tz_2, t^2w) for all (z_2, w) . The series is convergent in t for every fixed z_2 and w . Therefore if F_2 exists and F_1 is convergent, then F_2 is convergent by [Lemma 11.1](#).

Now we need to show the existence of the formal solution F_2 . Notice that the right hand side of (141) is invariant under the involution. It suffices to show that any power series in \bar{z}_1, z_2 that is invariant under the involution is a formal power series in z_2 and $\bar{z}_1z_2 + \bar{z}_1^2$. Let us treat $\xi = \bar{z}_1$ as an independent variable. The original involution becomes a holomorphic involution in ξ, z_2 :

$$(142) \quad \tau : \xi \rightarrow -\xi - z_2, \quad z_2 \rightarrow z_2.$$

By a theorem of Noether we obtain a set of generators for the ring of invariants by applying the averaging operation $R(f) = \frac{1}{2}(f + f \circ \tau)$ to all monomials in ξ and z_2 of degree two or lower. By direct calculation it is not difficult to see that $\xi, \xi z_2 + \xi^2$ generate the ring of invariants. Therefore any invariant power series in z_2, ξ is a power series in $\xi, \xi z_2 + \xi^2$. This shows the existence of F_2 . The existence of G follows in the same way.

The equation for $G(z_2, \bar{z}_1z_2 + \bar{z}_1^2) = -\bar{F}_1(\bar{z}_1)\bar{F}_1(-\bar{z}_1 - z_2)$ is obtained by plugging in the equation for F_2 . Its existence, uniqueness, and convergence in the case where F_1 converges, follow exactly the same as for F_2 . \square

Theorem 12.4. *If $M \subset \mathbb{C}^{n+1}$, $n \geq 2$ is given by*

$$(143) \quad w = \bar{z}_1z_2 + \bar{z}_1^2,$$

and $(F_1, F_2, \dots, F_n, G)$ is a local automorphism at the origin, then F_1 depends only on z_1 , F_2 and G depend only on z_2 and w , and F_1 completely determines F_2 and G via (138). The mapping $(z_1, z_2, F_3, \dots, F_n)$ has rank n at the origin.

Furthermore, given any invertible function F_1 of one variable with $F_1(0) = 0$, and arbitrary holomorphic functions F_3, \dots, F_n such that $F_j(0) = 0$, and $(z_1, z_2, F_3, \dots, F_n)$ has rank n at the origin, then there exist unique F_2 and G so that (F_1, \dots, F_n, G) is an automorphism.

Proof. Let (F_1, \dots, F_n, G) be an automorphism. Then we have

$$(144) \quad G(z_1, \dots, z_n, w) \\ = \bar{F}_1(\bar{z}_1, \dots, \bar{z}_n, \bar{w})F_2(z_1, \dots, z_n, w) + (\bar{F}_1(\bar{z}_1, \dots, \bar{z}_n, \bar{w}))^2.$$

[Proposition 12.1](#) says that the linear terms in G only depend on w , the linear terms of F_1 depend only on z_1 and w , and the linear terms of F_2 only depend on z_2 and w .

Let us embed $M_{C.1} \subset \mathbb{C}^3$ into M via setting $z_3 = \alpha_3 z_2, \dots, z_n = \alpha_n z_2$, for arbitrary $\alpha_3, \dots, \alpha_n$. Then we obtain

$$(145) \quad G(z_1, z_2, \alpha_3 z_2, \dots, \alpha_n z_2, w) \\ = \bar{F}_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \dots, \bar{\alpha}_n \bar{z}_2, \bar{w}) F_2(z_1, z_2, \alpha_3 z_2, \dots, \alpha_n z_2, w) \\ + (\bar{F}_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \dots, \bar{\alpha}_n \bar{z}_2, \bar{w}))^2.$$

By noting what the linear terms are, we notice that the above is the equation for an automorphism of $M_{C.1}$. Therefore by [Lemma 12.2](#) we have

$$(146) \quad \frac{\partial F_1}{\partial w} = 0, \quad \frac{\partial F_2}{\partial z_1} = 0, \quad \text{and} \quad \frac{\partial G}{\partial z_1} = 0,$$

as that is true for all $\alpha_3, \dots, \alpha_n$. Plugging in the defining equation for $M_{C.1}$, we obtain an equation that holds for all z and we can treat z and \bar{z} independently. We plug in $z = 0$ to obtain

$$(147) \quad 0 = \bar{F}_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \dots, \bar{\alpha}_n \bar{z}_2, 0) F_2(0, \dots, 0, \bar{z}_1^2) \\ + (\bar{F}_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \dots, \bar{\alpha}_n \bar{z}_2, 0))^2.$$

Differentiating with respect to $\bar{\alpha}_j$ we obtain $\partial F_1 / \partial z_j = 0$, for $j = 3, \dots, n$. We set $\bar{\alpha}_j = 0$ in the equation, differentiate with respect to \bar{z}_2 and obtain that $\partial F_1 / \partial z_2 = 0$. In other words, F_1 is a function of z_1 only. We rewrite (145) by writing F_1 as a function of z_1 only and F_2 and G as functions of z_2, \dots, z_n, w , and we plug in $w = \bar{z}_1 z_2 + \bar{z}_1^2$ to obtain

$$(148) \quad G(z_2, \alpha_3 z_2, \dots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2) \\ = \bar{F}_1(\bar{z}_1) F_2(z_2, \alpha_3 z_2, \dots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{F}_1(\bar{z}_1))^2.$$

By [Lemma 12.3](#), we know that F_1 uniquely determines $F_2(z_2, \alpha_3 z_2, \dots, \alpha_n z_2, w)$ and $G(z_2, \alpha_3 z_2, \dots, \alpha_n z_2, w)$. These two functions therefore do not depend on $\alpha_3, \dots, \alpha_n$, and in turn F_2 and G do not depend on z_3, \dots, z_n as claimed. Furthermore F_1 does uniquely determine F_2 and G .

Finally since the mapping is a biholomorphism, and from what we know about the linear parts of F_1, F_2 , and G , it is clear that $(z_1, z_2, F_3, \dots, F_n)$ has rank n .

The other direction follows by applying [Lemma 12.3](#). We start with F_1 , determine F_2 and G as in three dimensions. Then add F_3, \dots, F_n and the rank condition guarantees an automorphism. \square

13. Normal form for certain type C.1 submanifolds II

The goal of this section is to find the normal form for Levi-flat submanifolds $M \subset \mathbb{C}^{n+1}$ given by

$$(149) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + \operatorname{Re} f(z),$$

for a holomorphic $f(z)$ of order $O(3)$.

Since $f(z)$ can be absorbed into w via a holomorphic transformation, the goal is really to prove:

Theorem 13.1. *Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic Levi-flat given by*

$$(150) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}),$$

where r is $O(3)$. Then M can be put into the $M_{C,1}$ normal form

$$(151) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2,$$

by a convergent normalizing transformation.

Furthermore, if r is a polynomial and the coefficient of \bar{z}_1^3 in r is zero, then there exists an invertible polynomial mapping taking $M_{C,1}$ to M .

In [Theorem 1.3](#), we have already shown that a submanifold of the form

$$(152) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}_1)$$

is necessarily Levi-flat and has the normal form $M_{C,1}$. The first part of [Theorem 13.1](#) will follow once we prove:

Lemma 13.2. *If $M \subset \mathbb{C}^{n+1}$ is given by*

$$(153) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}),$$

where r is $O(3)$ and M is Levi-flat, then r depends only on \bar{z}_1 .

Proof. First let us assume that $n = 2$. For $p \in M_{CR}$, $T_p^{(1,0)}M$ is one dimensional. The Levi map is the matrix

$$(154) \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

applied to the $T^{(1,0)}M$ vectors. As M is Levi-flat, then the Levi map has to vanish. The only vectors v for which $v^*Lv = 0$, are the ones without $\partial/\partial z_1$ component or $\partial/\partial z_2$ component, that is, vectors of the form

$$(155) \quad a \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial w} \quad \text{or} \quad a \frac{\partial}{\partial z_2} + b \frac{\partial}{\partial w}.$$

We apply these vectors to the defining equation and its conjugate and we obtain in the first case the equations

$$(156) \quad b = 0, \quad a \left(\bar{z}_2 + 2z_1 + \frac{\partial \bar{r}}{\partial z_1} \right) = 0.$$

This cannot be satisfied identically on M since this is supposed to be true for all z , but a cannot be identically zero and the second factor in the second equation has only one nonholomorphic term, which is \bar{z}_2 .

Let us try the second form and we obtain the equations

$$(157) \quad b = a\bar{z}_1, \quad a \left(\frac{\partial \bar{r}}{\partial z_2} \right) = 0.$$

Again a cannot be identically zero, and hence the second factor of the second equation $\partial \bar{r} / \partial z_2$ must be identically zero, which is possible only if r depends only on \bar{z}_1 .

Finally, it is possible to pick $b = \bar{z}_1$ and $a = 1$, to obtain a $T^{(1,0)}$ vector field

$$(158) \quad \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial w},$$

and therefore these submanifolds are necessarily Levi-flat.

Next suppose that $n > 2$. Notice that replacing z_k with $\lambda_k \xi$ for $k \geq 2$ and then fixing λ_k for $k \geq 2$, we get

$$(159) \quad w = \bar{z}_1 \lambda_2 \xi + \bar{z}_1^2 + r(\bar{z}_1, \bar{\lambda}_2 \bar{\xi}, \dots, \bar{\lambda}_n \bar{\xi}).$$

By [Lemma 2.2](#), we obtain a Levi-flat submanifold in $(z_1, \xi, w) \in \mathbb{C}^3$, and hence can apply the above reasoning to obtain that $r(\bar{z}_1, \bar{\lambda}_2 \bar{\xi}, \dots, \bar{\lambda}_n \bar{\xi})$ does not depend on $\bar{\xi}$. As this was true for any λ_k , we have that r can only depend on \bar{z}_1 . \square

It is left to prove the claim about the polynomial normalizing transformation:

Lemma 13.3. *Suppose that $M \subset \mathbb{C}^{n+1}$ is given by*

$$(160) \quad w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}_1),$$

where r is a polynomial that vanishes to the fourth order. Then there exists an invertible polynomial mapping taking $M_{C.1}$ to M .

Proof. We will take a transformation of the form

$$(161) \quad (z_1, z_2, w) \mapsto (z_1, z_2 + f(z_2, w), w + g(z_2, w)).$$

We are therefore trying to find polynomials f and g that satisfy

$$(162) \quad \bar{z}_1 z_2 + \bar{z}_1^2 + g(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{z}_1 (z_2 + f(z_2, \bar{z}_1 z_2 + \bar{z}_1^2)) + \bar{z}_1^2 + r(\bar{z}_1).$$

If we simplify, we obtain

$$(163) \quad g(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) - \bar{z}_1 f(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = r(\bar{z}_1).$$

Consider the involution $S: (\bar{z}_1, z_2) \rightarrow (-\bar{z}_1 - z_2, z_2)$. Its invariant polynomials $u(\bar{z}_1, z_2)$ are precisely the polynomials in $z_2, z_2 \bar{z}_1 + \bar{z}_1^2$. The polynomial $r(\bar{z}_1)$ can

be uniquely written as

$$(164) \quad r^+(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{z}_1 + \frac{1}{2} z_2) r^-(z_2, \bar{z}_1 z_2 + \bar{z}_1^2)$$

in two polynomials r^\pm . Taking $f = -r^-$ and $g = r^+ + \frac{1}{2} z_2 r^-$, we find the desired solutions. \square

14. Normal form for general type C.1 submanifolds

In this section we show that generically a Levi-flat type C.1 submanifold is not formally equivalent to the quadric $M_{C.1}$ submanifold. In fact, we find a formal normal form that shows infinitely many invariants. There are obviously infinitely many invariants if we do not impose the Levi-flat condition. The trick therefore is, how to impose the Levi-flat condition and still obtain a formal normal form.

Let $M \subset \mathbb{C}^3$ be a real-analytic Levi-flat type C.1 submanifold through the origin. We know that M is an image of $\mathbb{R}^2 \times \mathbb{C}$ under a real-analytic CR map that is a diffeomorphism onto its target; see [Theorem 1.2](#). After a linear change of coordinates we assume that the mapping is

$$(165) \quad (x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \mapsto (x + iy + a(x, y, \xi), \xi + b(x, y, \xi), (x - iy)\xi + (x - iy)^2 + r(x, y, \xi)),$$

where a, b are $O(2)$ and r is $O(3)$. As the mapping is a CR mapping and a local diffeomorphism, then given any such a, b , and r , the image is necessarily Levi-flat at CR points. Therefore the set of all these mappings gives us all type C.1 Levi-flat submanifolds.

We precompose with an automorphism of $\mathbb{R}^2 \times \mathbb{C}$ to make $b = 0$. We cannot similarly remove a as any automorphism must have real-valued first two components (the new x and the new y), and hence those components can only depend on x and y but not on ξ . So if a depends on ξ , we cannot remove it by precomposing.

Next we notice that we can treat M as an abstract CR manifold. Suppose we have two equivalent submanifolds M_1 and M_2 , with F being the biholomorphic map taking M_1 to M_2 . If M_j is the image of a map φ_j , then note that φ_2^{-1} is CR on $(M_2)_{\text{CR}}$. Therefore, $G = \varphi_2^{-1} \circ F \circ \varphi_1$ is CR on $(F \circ \varphi_1)^{-1}((M_2)_{\text{CR}})$, which is dense in a neighborhood of the origin of $\mathbb{R}^2 \times \mathbb{C}$ (the CR singularity of M_2 is a thin set, and we pull it back by two real-analytic diffeomorphisms). A real-analytic diffeomorphism that is CR on a dense set is a CR mapping. The same argument works for the inverse of G , and therefore we have a CR diffeomorphism of $\mathbb{R}^2 \times \mathbb{C}$. We conclude:

Proposition 14.1. *If $M_j \subset \mathbb{C}^3$, $j = 1, 2$ are given by the maps φ_j*

$$(166) \quad (x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \xrightarrow{\varphi_j} (x + iy + a_j(x, y, \xi), \xi + b_j(x, y, \xi), \\ (x - iy)\xi + (x - iy)^2 + r_j(x, y, \xi)),$$

and M_1 and M_2 are locally biholomorphically (resp. formally) equivalent at zero, then there exist local biholomorphisms (resp. formal equivalences) F and G at zero, with $F(M_1) = M_2$, $G(\mathbb{R}^2 \times \mathbb{C}) = \mathbb{R}^2 \times \mathbb{C}$ as germs (resp. formally) and

$$(167) \quad \varphi_2 = F \circ \varphi_1 \circ G.$$

In other words, the proposition states that if we find a normal form for the mapping, we find a normal form for the submanifolds. Let us prove that the proposition also works formally.

Proof. We have to prove that G restricted to $\mathbb{R}^2 \times \mathbb{C}$ is CR, that is, $\partial G / \partial \bar{\xi} = 0$. Let us consider

$$(168) \quad \varphi_2 \circ G = F \circ \varphi_1.$$

The right hand side does not depend on $\bar{\xi}$ and thus the left hand side does not either. Write $G = (G^1, G^2, G^3)$. Let us write $b = b_2$ and $r = r_2$ for simplicity. Taking derivative of $\varphi_2 \circ G$ with respect to $\bar{\xi}$ we get

$$(169) \quad \begin{aligned} G_{\bar{\xi}}^1 + iG_{\bar{\xi}}^2 + a_x(G)G_{\bar{\xi}}^1 + a_y(G)G_{\bar{\xi}}^2 + a_{\bar{\xi}}(G)G_{\bar{\xi}}^3 &= 0, \\ G_{\bar{\xi}}^3 + b_x(G)G_{\bar{\xi}}^1 + b_y(G)G_{\bar{\xi}}^2 + b_{\bar{\xi}}(G)G_{\bar{\xi}}^3 &= 0, \\ (G_{\bar{\xi}}^1 - iG_{\bar{\xi}}^2)G^3 + (G^1 - iG^2)G_{\bar{\xi}}^3 + 2(G^1 - iG^2)(G_{\bar{\xi}}^1 - iG_{\bar{\xi}}^2) \\ &\quad + r_x(G)G_{\bar{\xi}}^1 + r_y(G)G_{\bar{\xi}}^2 + r_{\bar{\xi}}(G)G_{\bar{\xi}}^3 = 0. \end{aligned}$$

Suppose that the homogeneous parts of $G_{\bar{\xi}}^j$ are zero for all degrees up to $d - 1$. If we look at the degree- d homogeneous parts of the first two equations above we immediately note that it must be that $G_{\bar{\xi}}^1 + iG_{\bar{\xi}}^2 = 0$ and $G_{\bar{\xi}}^3 = 0$ in degree d . We then look at the degree- $(d + 1)$ part of the third equation. Recall that $[\cdot]_d$ is the degree- d part of an expression. We get

$$(170) \quad [G_{\bar{\xi}}^1 - iG_{\bar{\xi}}^2]_d [G^3 + 2G^1 - i2G^2]_1 = 0.$$

As G is an automorphism we cannot have the linear terms be linearly dependent and hence $G_{\bar{\xi}}^1 = G_{\bar{\xi}}^2 = 0$ in degree d . We finish by induction on d . \square

Using the proposition we can restate the result of [Theorem 1.3](#) by parametrization:

Corollary 14.2. *A real-analytic Levi-flat type $C.1$ submanifold $M \subset \mathbb{C}^3$ is biholomorphically equivalent to the quadric $M_{C.1}$ if and only if the mapping giving M is*

equivalent to a mapping of the form

$$(171) \quad (x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \mapsto (x + iy, \xi, (x - iy)\xi + (x - iy)^2 + r(x, y, \xi)).$$

That is, M is equivalent to $M_{C.1}$ if and only if we can get rid of the $a(x, y, \xi)$ via pre and post composing with automorphisms. The proof of the corollary follows as a submanifold that is realized by this map must be of the form $w = \bar{z}_1 z_2 + \bar{z}_1^2 + \rho(z_1, \bar{z}_1, z_2)$ and we apply [Theorem 1.3](#).

We have seen that the involution τ on M , in particular when M is the quadric, is useful to compute the automorphism group and to construct Levi-flat submanifolds of type C.1. We will also need to deal with power series in z, \bar{z}, ξ . Thus we extend τ , which is originally defined on \mathbb{C}^2 , as follows:

$$(172) \quad \sigma(z, \bar{z}, \xi) = (z, -\bar{z} - \xi, \xi).$$

Here z, \bar{z}, ξ are treated as independent variables. Note that $z, \xi, w = \bar{z}\xi + \bar{z}^2$ are invariant by σ , while $\eta = \bar{z} + \frac{1}{2}\xi$ is skew invariant by σ . A power series in z, \bar{z}, ξ that is invariant by σ is precisely a power series in z, ξ, w . In general, a power series u in z, \bar{z}, ξ admits a unique decomposition

$$(173) \quad u(z, \bar{z}, \xi) = u^+(z, \xi, w) + \eta u^-(z, \xi, w).$$

First we introduce degree for power series $u(z, \bar{z}, \xi)$ and weights for power series $v(z, \xi, w)$. As usual we assign degree $i + j + k$ to the monomial $z^i \bar{z}^j \xi^k$. We assign weight $i + j + 2k$ to the monomial $z^i \xi^j w^k$. For simplicity, we will call them weight in both situations. Let us also set

$$(174) \quad \begin{aligned} [u]_d(z, \bar{z}, \xi) &= \sum_{i+j+k=d} u_{ijk} z^i \bar{z}^j \xi^k, \\ [v]_d(z, \xi, w) &= \sum_{i+j+2k=d} v_{ijk} z^i \xi^j w^k. \end{aligned}$$

Set $[u]_i^j = [u]_i + \dots + [u]_j$ and $[v]_i^j = [v]_i + \dots + [v]_j$ for $i \leq j$.

Theorem 14.3. *Let M be a real-analytic Levi-flat type C.1 submanifold in \mathbb{C}^3 . There exists a formal biholomorphic map transforming M into the image of*

$$(175) \quad \hat{\varphi}(z, \bar{z}, \xi) = (z + A(z, \xi, w)\eta, \xi, w)$$

with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Suppose further that $A \not\equiv 0$. Fix i_*, j_*, k_* such that j_* is the largest integer satisfying $A_{i_* j_* k_*} \neq 0$ and $i_* + j_* + 2k_* = s$. Then we can achieve

$$(176) \quad A_{i_*(j_*+n)k_*} = 0, \quad n = 1, 2, \dots$$

Furthermore, the power series A is uniquely determined up to the transformation

$$(177) \quad A(z, \xi, w) \rightarrow \bar{c}^3 A(cz, \bar{c}\xi, \bar{c}^2 w), \quad c \in \mathbb{C} \setminus \{0\}.$$

In the above normal form with $A \neq 0$, the group of formal biholomorphisms that preserve the normal form consists of dilations

$$(178) \quad (z, \xi, w) \rightarrow (\nu z, \bar{\nu}\xi, \bar{\nu}^2 w),$$

satisfying $\bar{\nu}^3 A(\nu z, \bar{\nu}\xi, \bar{\nu}^2 w) = A(z, \xi, w)$.

Proof. It will be convenient to write the CR diffeomorphism G of $\mathbb{R}^2 \times \mathbb{C}$ as (G_1, G_2) where G_1 is complex-valued and depends on z, \bar{z} , while G_2 depends on z, \bar{z}, ξ . Let M be the image of a mapping φ defined by

$$(179) \quad (z, \bar{z}, \xi) \xrightarrow{\varphi} (z + a(z, \bar{z}, \xi), \xi, \bar{z}\xi + \bar{z}^2 + r(z, \bar{z}, \xi))$$

with $a = O(2), r = O(3)$. We want to find a formal biholomorphic map F of \mathbb{C}^3 and a formal CR diffeomorphism G of $\mathbb{R}^2 \times \mathbb{C}$ such that

$$(180) \quad F\hat{\varphi}G^{-1} = \varphi$$

with $\hat{\varphi}$ in the normal form.

To simplify the computation, we will first achieve a preliminary normal form where $r = 0$ and the function a is skew-invariant by σ . For the preliminary normal form we will only apply F and G that are tangent to the identity. We will then use the general F and G to obtain the final normal form.

Let us assume that F and G are tangent to the identity. Let $M = F(\hat{\varphi}(\mathbb{R}^2 \times \mathbb{C}))$ where $\hat{\varphi}$ is determined by \hat{a}, \hat{r} . We write

$$(181) \quad F = I + (f_1, f_2, f_3), \quad G = I + (g_1, g_2).$$

The ξ components in $\varphi G = F\hat{\varphi}$ give us

$$(182) \quad g_2(z, \bar{z}, \xi) = f_2(z + \hat{a}(z, \bar{z}, \xi), \xi, \bar{z}\xi + \bar{z}^2 + \hat{r}(z, \bar{z}, \xi)).$$

Thus, we are allowed to define g_2 by the above identity for any choice of $f_2 = O(2)$. Eliminating g_2 in other components of $\varphi G = F\hat{\varphi}$, we obtain

$$(183) \quad f_1 \circ \hat{\varphi} - g_1 = a \circ G - \hat{a},$$

$$(184) \quad f_3 \circ \hat{\varphi} - \bar{z}f_2 \circ \hat{\varphi} = r \circ G - \hat{r} + 2\eta\tilde{g}_1 + \tilde{g}_1 f_2 \circ \hat{\varphi} + \tilde{g}_1^2,$$

where $\tilde{g}_1(z, \bar{z}) = \bar{g}_1(\bar{z}, z)$ and

$$(185) \quad (a \circ G)(z, \bar{z}, \xi) := a(G_1(z, \bar{z}), \bar{G}_1(\bar{z}, z), G_2(z, \bar{z}, \xi)).$$

Each power series $r(z, \bar{z}, \xi)$ admits a unique decomposition

$$(186) \quad r(z, \bar{z}, \xi) = r^+(z, \xi, w) + \eta r^-(z, \xi, w),$$

where both r^\pm are invariant by σ . Note that $r(z, \bar{z}, \xi)$ is a power series in z, ξ , and w , if and only if it is invariant by σ , i.e., if $r^- = 0$. We write

$$(187) \quad r^+ = wt(k) \quad \text{or} \quad wt(r^+) \geq k,$$

if $r_{abc}^+ = 0$ for $a + b + 2c < k$. Define $r^- = wt(k)$ analogously and write $\eta r^- = wt(k)$ if $r^- = wt(k - 1)$. We write $r = wt(k)$ if $(r^+, \eta r^-) = wt(k)$. Note that

$$(188) \quad r = O(k) \Rightarrow r = wt(k), \quad wt(rs) \geq wt(r) + wt(s).$$

The power series in z and \bar{z} play a special role in describing normal forms. Let us define T^\pm via

$$(189) \quad u(z, \bar{z}) = (T^+u)(z, \xi, w) + (T^-u)(z, \xi, w)\eta.$$

Let S_k^+ (resp. S_k^-) be spanned by monomials in z, \bar{z}, ξ which have weight k and are invariant (resp. skew-invariant) by σ . Then the range of ηT^- in S_k^- is a linear subspace R_k . We decompose

$$(190) \quad S_k^- = R_k \oplus (S_k^- \ominus R_k).$$

The decomposition is of course not unique. We will take

$$(191) \quad S_k^- \ominus R_k = \bigoplus_{\substack{a+b+2c=k-1 \\ c>0}} \mathbb{C}z^a \xi^b w^c \eta.$$

Here we have used $\eta = \bar{z} + \frac{1}{2}\xi$, $\eta^2 = w + \frac{1}{4}\xi^2$, and

$$(192) \quad T^+u(z, \xi, w) = \sum_{i,j \geq 0} \sum_{0 \leq \alpha \leq j/2} u_{ij} \binom{j}{2\alpha} z^i (w + \frac{1}{4}\xi^2)^\alpha (-\frac{1}{2}\xi)^{j-2\alpha},$$

$$(193) \quad T^-u(z, \xi, w) = \sum_{\substack{i \geq 0 \\ j > 0}} \sum_{0 \leq \alpha < j/2} u_{ij} \binom{j}{2\alpha + 1} z^i (w + \frac{1}{4}\xi^2)^\alpha (-\frac{1}{2}\xi)^{j-2\alpha-1}.$$

In particular, we have

$$(194) \quad T^-u(z, \xi, 0) = \sum_{\substack{i \geq 0 \\ j > 0}} (-1)^{j-1} u_{ij} z^i \xi^{j-1}.$$

This shows that

$$(195) \quad T^-u(z, \xi, 0) = \frac{1}{-\xi}(u(z, -\xi) - u(z, 0)).$$

We are ready to show that under the condition that $g_1(z, \bar{z})$ has no pure holomorphic terms, there exists a unique (F, G) which is tangent to the identity such that $\hat{r} = 0$,

$$(196) \quad \hat{a} \in \mathcal{N} := \bigoplus \mathcal{N}_k, \quad \mathcal{N}_k := S_k^- \ominus R_k.$$

We start with terms of weight two in (183) and (184) to get

$$(197) \quad [f_1]_2 - [g_1]_2 = [a]_2 - \eta[\hat{a}^-]_1,$$

$$(198) \quad [f_3]_2 = 0.$$

Note that $f_j^- = 0$. The first identity implies that

$$(199) \quad [f_1]_2 - [T^+g_1]_2 = [a^+]_2, \quad [T^-g_1]_1 = [\hat{a}^-]_1 - [a^-]_1.$$

The first equation is solvable with kernel defined by

$$(200) \quad [f_1]_k - [T^+g_1]_k = 0,$$

for $k = 2$. This shows that $[g_1]_2$ is still arbitrary and we use it to achieve

$$(201) \quad \eta[\hat{a}^-]_1 \in S_2^- \ominus R_2 = \{0\}.$$

Then the kernel space is defined by (200) and

$$(202) \quad [g_1(z, \bar{z}) - g_1(z, 0)]_k = 0$$

with $k = 2$. In particular, under the restriction

$$(203) \quad [g_1(z, 0)]_k = 0,$$

for $k = 2$, we have achieved $\hat{a}^- \in \mathcal{N}_2$ by unique $[f_1]_2, [g_1]_2, [f_2]_1, [f_3]_2$. By induction, we verify that if (203) holds for all k , we determine uniquely $[f_1]_k$ and $[g_1]_k$ by normalizing $[\hat{a}]_k \in \mathcal{N}_k$. We then determine $[f_2]_k$ and $[f_3]_{k+1}$ uniquely to normalize $[\hat{r}]_{k+1} = 0$. For details, let us find formulae for the solutions. We rewrite (183) as

$$(204) \quad T^-g_1 = -(a \circ G - \hat{a} - f_1 \circ \hat{\varphi})^-,$$

$$(205) \quad (f_1 \circ \hat{\varphi})^+ = (a \circ G - \hat{a})^+ + T^+g_1.$$

Using (194), we can solve

$$(206) \quad (-1)^{j-1}g_{1,ij} = -((a \circ G)^-)^{-}_{i(j-1)0}, \quad j \geq 1, i + j = k.$$

Then we have

$$(207) \quad (\hat{a}^-)_{ij0} = 0, \quad i + j = k - 1;$$

$$(208) \quad (\hat{a}^-)_{ijm} = ((a \circ G - f_1 \circ \hat{\varphi} + g_1)^-)_{ijm}, \quad m \geq 1, i + j + m = k - 1.$$

Note that $-[g_1]_k(z, -\bar{z}) = \bar{z}[(a \circ G - \hat{a})^-]_{k-1}(z, \bar{z}, 0)$. We obtain

$$(209) \quad [g_1]_k(z, \bar{z}) = \bar{z}[(a \circ G - \hat{a})^-]_{k-1}(z, -\bar{z}, 0).$$

Having determined $[g_1]_k$, we take

$$(210) \quad [f_1]_k = [(a \circ G - \hat{a} + g_1)^+]_k.$$

We then solve (184) by taking

$$(211) \quad [f_2]_k = [E^-]_k, \quad [f_3]_{k+1} = [(E - \frac{1}{2}\xi f_2)^+]_{k+1},$$

$$(212) \quad E := r \circ G - \hat{r} + 2\eta\tilde{g}_1 + \tilde{g}_1 f_2 \circ \hat{\varphi} + \tilde{g}_1^2.$$

We have achieved the preliminary normalization.

Assume now that

$$(213) \quad \varphi(z, \bar{z}, \xi) = (z + a^-(z, \xi, w)\eta, \xi, w),$$

$$(214) \quad \hat{\varphi}(z, \bar{z}, \xi) = (z + \hat{a}^-(z, \xi, w)\eta, \xi, w)$$

are in the preliminary normal form, i.e.,

$$(215) \quad w \mid a^-(z, \xi, w), \quad w \mid \hat{a}^-(z, \xi, w).$$

Let us assume that

$$(216) \quad a^-(z, \xi, w) = wt(s), \quad [a^-]_s \neq 0, \quad \hat{a}^-(z, \xi, w) = wt(s).$$

We assume that $\varphi G = F\hat{\varphi}$ with

$$(217) \quad F(z, \xi, w) = I + (f_1, f_2, f_3),$$

$$(218) \quad G(z, \bar{z}, \xi) = (z + g_1(z, \bar{z}), \xi + g_2(z, \bar{z}, \xi)).$$

Here f_i, g_j start with terms of weight and order at least two. In particular, we have

$$(219) \quad f_i = wt(N), \quad g_i = wt(N), \quad i = 1, 2;$$

$$(220) \quad f_3 = wt(N'), \quad N' \geq N \geq 2.$$

Set $(P, Q, R) := \varphi G$. Using $N \geq 2, s \geq 2$, and Taylor's theorem, we obtain

$$(221) \quad P = z + g_1(z, \bar{z}) + a^-(z, \xi, w)\eta + a^-(z, \xi, w)(\bar{g}_1(\bar{z}, z) + \frac{1}{2}g_2(z, \bar{z}, \xi)) \\ + \eta \nabla a^-(z, \xi, w)(g_1(z, \bar{z}), g_2(z, \bar{z}, \xi), (\xi + 2\bar{z})\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi)) \\ + wt(s + N + 1),$$

$$(222) \quad Q = \xi + g_2(z, \bar{z}, \xi),$$

$$(223) \quad R = w + (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) + wt(2N).$$

We also have $(P, Q, R) = F\hat{\varphi}$. Thus

$$(224) \quad P = z + \hat{a}^-(z, \xi, w)\eta + f_1(z, \xi, w) + \partial_z f_1(z, \xi, w)\hat{a}^-(z, \xi, w)\eta \\ + wt(N + s + 1),$$

$$(225) \quad Q = \xi + f_2(z, \xi, w) + \partial_z f_2(z, \xi, w)\hat{a}^-(z, \xi, w)\eta + wt(N + s + 1),$$

$$(226) \quad R = w + f_3(z, \xi, w) + \partial_z f_3(z, \xi, w)\hat{a}^-(z, \xi, w)\eta + wt(N' + s + 1).$$

We will use the above six identities for P, Q, R in two ways. First we use their lower order terms to get

$$(227) \quad f_1(z, \xi, w) = g_1(z, \bar{z}) + (a^-(z, \xi, w) - \hat{a}^-(z, \xi, w))\eta + wt(N + s),$$

$$(228) \quad f_2(z, \xi, w) = g_2(z, \bar{z}, \xi) + wt(N + s),$$

$$(229) \quad f_3(z, \xi, w) = (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) + wt(2N) + wt(N' + s).$$

Hence, we can take $N' = N + 1$. By (227) and the preliminary normalization, we first know that

$$(230) \quad \hat{a} = a + wt(N + s - 1),$$

$$(231) \quad f_1(z, \xi, w) = b(z) + wt(N + s), \quad g_1(z, \bar{z}) = b(z) + wt(N + s).$$

We compose (229) by σ and then take the difference of the two equations to get

$$(232) \quad f_2(z, \xi, w) = -\bar{b}(\bar{z}) - \bar{b}(-\bar{z} - \xi) + wt(2N - 1) + wt(N + s),$$

$$(233) \quad f_3(z, \xi, w) = -\bar{z}\bar{b}(-\bar{z} - \xi) + (\bar{z} + \xi)\bar{b}(\bar{z}) + wt(2N) + wt(N + s + 1).$$

Here we have used $N' = N + 1$. Let $b(z) = b_N z^N + wt(N + 1)$. Therefore, we have

$$(234) \quad g_2(z, \bar{z}, \xi) = -\bar{b}_N(\bar{z}^N + (-\bar{z} - \xi)^N) + wt(N + 1),$$

$$(235) \quad \bar{g}_1(\bar{z}, z) + \frac{1}{2}g_2(z, \bar{z}, \xi) = \eta \bar{b}_N \sum \bar{z}^i (-\bar{z} - \xi)^{N-1-i} + wt(N + 1),$$

$$(236) \quad (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) = \bar{b}_N(\bar{z}^{N-1} + (-\bar{z} - \xi)^{N-1})w + wt(N + 2).$$

Next, we use the two formulae for P and (231) to get the identity in higher weight,

$$(237) \quad \hat{a}^- = a^- + g_1^- + Lb_N + wt(N + s), \quad f_1 - g_1^+ = wt(N + s + 1).$$

Here we have used $f_1^- = 0$ and

$$(238) \quad Lb_N(z, \xi, w) := -Nb_N z^{N-1} [a^-]_s(z, \xi, w) - [a^-]_s(z, \xi, w) \bar{b}_N \sum \bar{z}^i (-\bar{z} - \xi)^{N-1-i} + \nabla[a^-]_s \cdot (b_N z^N, -\bar{b}_N (\bar{z}^N + (-\bar{z} - \xi)^N), \bar{b}_N w (\bar{z}^{N-1} + (-\bar{z} - \xi)^{N-1})).$$

Recall that $w \mid a^-$ and $w \mid \hat{a}^-$. We also have that $w \mid Lb_N(z, \xi, w)$ and Lb_N is homogenous in weighted variables and of weight $N + s - 1$. This shows that $[g_1^-(z, \xi, 0)]_{N+s-1} = 0$. By (194), we get

$$(239) \quad [g_1(z, \bar{z})]_{N+s} = [g_1(z, 0)]_{N+s}, \quad [\hat{a}^-]_{s+N-1} = [a^-]_{s+N-1} + Lb_N.$$

Let us make some observations. First, Lb_N depends only on b_N and it does not depend on coefficients of $b(z)$ of degree larger than N . We observe that the first identity says that all coefficients of $[g_1]_{N+s}$ must be zero, except that the coefficient $g_{1,(N+s)0}$ is arbitrary. On the other hand Lb_N , which has weight $N + s - 1$, depends only on $g_{1,N0}$, while $N + s - 1 > N$. Let us assume for the moment that we have $Lb_N \neq 0$ for all $b_N \neq 0$. We will then choose a suitable complement subspace \mathcal{N}_{N+s-1}^* in the space of weighted homogenous polynomials in z, ξ, w of weight $N + s - 1$ for Lb_N . Then $\hat{a}^- \in w \sum_{N>1} \mathcal{N}_{N+s-1}^*$ will be the required normal form. The normal form will be obtained by the following procedures: Assume that φ is not formally equivalent to the quadratic mapping in the preliminary normalization. We first achieve the preliminary normal form by a mapping $F^0 = I + (f_1^0, f_2^0, f_3^0)$ and $G^0 = I + (g_1^0, g_2^0)$ which are tangent to the identity. We can make F^0 and G^0 to be unique by requiring $f_1^1(z, 0) = 0$. Then a is normalized such that $\hat{a} = \hat{a}^- \eta$ with $[\hat{a}^-]_s$ being nonzero homogenous part of the lowest weight. We may assume that $[a]_{s+1} = [\hat{a}]_{s+1}$. Inductively, we choose $f_{1,N00}^1$ ($N = 2, 3, \dots$) to achieve $[\hat{a}^-]_{N+s-1} \in w \mathcal{N}_{N+s-1}^*$. In this step for a given N , we determine mappings $F^1 = I + (f_1^1, f_2^1, f_3^1)$ and $G^1 = I + (g_1^1, g_2^1)$ by requiring that $f_1^1(z, \xi, w)$ contains only one term ξ^N , while $f_1^1, f_2^1, g_1^1, g_2^1$ have weight at most N and f_3^1 has weight at most $N + 1$. In the process, we also show that $[f_1^1(z, \xi, w)]_2^{N+s}$ depends only on z , if we do not want to impose the restriction on f_1^1 . Moreover, the coefficient of ξ^{N+s-1} of f_1^1 can still be arbitrarily chosen without changing the normalization achieved for $[\hat{a}^-]_{N+s-1}$ via $[f_1^1]_N$. However, by achieving $[\hat{a}^-]_{N+s-1} \in w \mathcal{N}_{N+s-1}^*$ via F^1 and G^1 , we may destroy the preliminary normalization achieved via F_0 and G_0 . We will then restore the preliminary normalization via $F^2 = I + (f_1^2, f_2^2, f_3^2)$, $G^2 = I + (g_1^2, g_2^2)$ satisfying $g_1^2(z, 0) = 0$. This amounts to determining $g_1^2 = g_1$ and $f_1^2 = f_1$ via (204) and (205) for which the terms of weight at most $N + s$ have been determined by (237), and then $f_2^2 = f_2$, $f_3^2 = f_3$, $g_2^2 = g_2$ are determined by (211), (212), and (182), respectively. This allows us to repeat the procedure to achieve the normalization in any higher weight.

We will then remove the restriction that the normalizing mappings must be tangent to the identity. This will alter the normal form only by suitable linear dilations.

Suppose that $b_N \neq 0$. Let us verify that

$$(240) \quad Lb_N \neq 0.$$

We will also identify one of nonzero coefficients to describe the normalizing condition on \hat{a} . We write the two invariant polynomials

$$(241) \quad \bar{z}^N + (-\bar{z} - \xi)^N = \lambda_N \xi^N + \sum_{j < N} p_{ijk} z^i \xi^j w^k,$$

$$(242) \quad \sum_i \bar{z}^i (-\bar{z} - \xi)^{N-1-i} = \lambda'_{N-1} \xi^{N-1} + \sum_{j < N-1} q_{ijk} z^i \xi^j w^k.$$

If we plug in $w = \bar{z}^2 + \bar{z}\xi$, we obtain two polynomial identities in the variables z, \bar{z} , and ξ :

$$(243) \quad \bar{z}^N + (-\bar{z} - \xi)^N = \lambda_N \xi^N + \sum_{j < N} p_{ijk} z^i \xi^j (\bar{z}^2 + \bar{z}\xi)^k,$$

$$(244) \quad \sum_i \bar{z}^i (-\bar{z} - \xi)^{N-1-i} = \lambda'_{N-1} \xi^{N-1} + \sum_{j < N-1} q_{ijk} z^i \xi^j (\bar{z}^2 + \bar{z}\xi)^k.$$

If we set $\bar{z} = z = 0$, we obtain that

$$(245) \quad \lambda_N = \lambda'_N = (-1)^N.$$

Recall that j_* is the largest integer such that $(a^-)_{i_* j_* k_*} \neq 0$ and $i_* + j_* + 2k_* = s$. Since $w \mid [a^-]_s$, then $k_* > 0$. We obtain

$$(246) \quad (Lb_N)_{i_*(j_*+N-1)k_*} = (a^-)_{i_* j_* k_*} \bar{b}_N (-\lambda'_{N-1} - j_* \lambda_{N-1} + k_* \lambda_N) \neq 0.$$

Therefore, we can achieve

$$(247) \quad (\hat{a}^-)_{i_*(j_*+n)k_*} = 0, \quad n = 1, 2, \dots$$

This determines uniquely all b_2, b_3, \dots

We now remove the restriction that F and G are tangent to the identity. Suppose that both φ and $\hat{\varphi}$ are in the normal form. Suppose that $F\varphi = \hat{\varphi}G$. Then looking at the quadratic terms, we know that the linear parts of F and G must be dilations. In fact, the linear part of F must be the linear automorphism of the quadric. Thus the linear parts of F and G have the forms

$$(248) \quad G': (z, \xi) = (\nu z, \bar{\nu} \xi), \quad F'(z, \xi, w) = (\nu z, \bar{\nu} \xi, \bar{\nu}^2 w).$$

Then $(F')^{-1} \hat{\varphi} G'$ is still in the normal form. Since $(F')^{-1} F$ is holomorphic and $(G')^{-1} G$ is CR, by the uniqueness of the normalization, we know that $F' = F$ and

$G' = G$. Therefore, F and G change the normal form a^- as follows,

$$(249) \quad a^-(z, \xi, w) = \bar{v}\hat{a}^-(vz, \bar{v}\xi, \bar{v}^2w), \quad v \in \mathbb{C} \setminus \{0\}.$$

When $[\hat{a}^-]_s = [a^-]_s \neq 0$, we see that $|v| = 1$. Therefore, the formal automorphism group is discrete or one-dimensional. \square

Coffman [2006] used an analogous method of even/odd function decomposition to obtain a quadratic normal form for non Levi-flat real-analytic m -submanifolds in \mathbb{C}^n with a CR singularity satisfying certain nondegeneracy conditions, provided $\frac{2}{3}(n+1) \leq m < n$. He was able to achieve the convergent normalization by a rapid iteration method. Using the above decomposition of invariant and skew-invariant functions of the involution σ , one might achieve a convergent solution for approximate equations when M is formally equivalent to the quadric. However, when the iteration is employed, each new CR mapping $\hat{\varphi}$ might only be defined on a domain that is proportional to that of the previous φ by a *constant* factor. This is significantly different from the situations of [Moser 1985; Coffman 2006; 2010], where rapid iteration methods are applicable. Therefore, even if M is formally equivalent to the quadric, we do not know if they are holomorphically equivalent.

15. Instability of Bishop-like submanifolds

Let us now discuss stability of Levi-flat submanifolds under small perturbations that keep the submanifolds Levi-flat, in particular, we discuss which quadratic invariants are stable when moving from point to point on the submanifold. The only stable submanifolds are A. n and C.1. The Bishop-like submanifolds (or even just the Bishop invariant) are not stable under perturbation, which we show by constructing examples.

Proposition 15.1. *Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a connected real-analytic real codimension-two submanifold that has a nondegenerate CR singular at the origin. M can be written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ as*

$$(250) \quad w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

for quadratic A and B . In a neighborhood of the origin all complex tangents of M are nondegenerate, while ranks of A and B are upper semicontinuous. Suppose that M is Levi-flat (that is, M_{CR} is Levi-flat). The CR singular set of M that is not of type $B.\frac{1}{2}$ at the origin is a real-analytic subset of M of codimension at least two, while the CR singular set of M that is of type $B.\frac{1}{2}$ at the origin has codimension at least one. A. n has an isolated CR singular point at the origin and so does C.1 in \mathbb{C}^3 . Let $S_0 \subset M$ be the set of CR singular points. There is a neighborhood U of the origin such that for $S = S_0 \cap U$ we have:

- (i) If M is of type A. k for $k \geq 2$ at the origin, then it is of type A. j at each point of S for some $j \geq k$.

- (ii) If M is of type C.1 at the origin, then it is of type C.1 on S . If M is of type C.0 at the origin, then it is of type C.0 or C.1 on S .
- (iii) There exists an M that is of type B. γ at one point and of C.1 at CR singular points arbitrarily near. Similarly there exists an M of type A.1 at $p \in M$ that is either of type C.1, or B. γ , at points arbitrarily near p . There also exists an M of type B. γ at every point but where γ varies from point to point.

Proof. First we show that the rank of A and the rank of B are lower semicontinuous on S_0 , without imposing the Levi-flatness condition. Similarly the real dimension of the range of $A(z, \bar{z})$ is lower semicontinuous on S_0 . Write M as

$$(251) \quad w = \rho(z, \bar{z}),$$

where ρ vanishes to second order at zero. If we move to a different point of S_0 via an affine map $(z, w) \mapsto (Z + z_0, W + w_0)$. Then we have

$$(252) \quad W + w_0 = \rho(Z + z_0, \bar{Z} + \bar{z}_0).$$

We compute the Taylor coefficients

$$(253) \quad W = \frac{\partial \rho}{\partial z}(z_0, \bar{z}_0) \cdot Z + \frac{\partial \rho}{\partial \bar{z}}(z_0, \bar{z}_0) \cdot \bar{Z} + Z^* \left[\frac{\partial^2 \rho}{\partial z \partial \bar{z}}(z_0, \bar{z}_0) \right] Z \\ + \frac{1}{2} Z^t \left[\frac{\partial^2 \rho}{\partial z \partial z}(z_0, \bar{z}_0) \right] Z + \frac{1}{2} Z^* \left[\frac{\partial^2 \rho}{\partial \bar{z} \partial \bar{z}}(z_0, \bar{z}_0) \right] \bar{Z} + O(3).$$

The holomorphic terms can be absorbed into W . If $(\partial \rho / \partial \bar{z})(z_0, \bar{z}_0) \cdot \bar{Z}$ is nonzero, then this complex defining function has a linear term in W and a linear term in \bar{Z} and the submanifold is CR at this point. Therefore the set of complex tangents of M is defined by

$$(254) \quad \frac{\partial \rho}{\partial \bar{z}} = 0,$$

and each complex tangent point is nondegenerate. At a complex tangent point at the origin, A is given by $[(\partial^2 \rho / \partial z \partial \bar{z})(z_0, \bar{z}_0)]$ and B is given by $\frac{1}{2} [(\partial^2 \rho / \partial \bar{z} \partial \bar{z})(z_0, \bar{z}_0)]$. These matrices change continuously as we move along S . We first conclude that all CR singular points of M in a neighborhood of the origin are nondegenerate. Further holomorphic transformations act on A and B using [Proposition 2.1](#). Thus the ranks of A and B and the real dimension of the range of $A(z, \bar{z})$ are lower semicontinuous on S_0 as claimed. Furthermore as M is real-analytic, the points where the rank drops lie on a real-analytic subvariety of S_0 , or in other words a thin set. Let U be a small enough neighborhood of the origin so that $S = S_0 \cap U$ is connected.

Imposing the condition that M is Levi-flat, we apply [Theorem 1.1](#). By a simple computation, unless M is of type B. $\frac{1}{2}$, the set of complex tangents of M has codimension at least two; and $A.n$ has an isolated CR singular point and so does

C.1 in \mathbb{C}^3 . Item (i) follows as A.k are the only types where the rank of B is greater than one, and the theorem says M must be one of these types. For (ii), note that since A is of rank one when M is C.x at a point, M cannot be of type A.k nearby. If M is of type C.1 at a point then the range of A must be of real dimension two in a neighborhood, and hence on this neighborhood M cannot be of type B. γ .

Examples proving (iii) are given below. \square

Example 15.2. Define M via

$$(255) \quad w = |z_1|^2 + \gamma \bar{z}_1^2 + \bar{z}_1 z_2 z_3.$$

It is Levi-flat by [Proposition 6.2](#). At the origin M is of type B. γ , but at a point where $z_1 = z_2 = 0$ and $z_3 \neq 0$, the submanifold is CR singular and it is of type C.1.

Example 15.3. Similarly we obtain a CR singular Levi-flat M via

$$(256) \quad w = \bar{z}_1^2 + \bar{z}_1 z_2 z_3;$$

this M is A.1 at the origin, but C.1 at nearby CR singular points.

Example 15.4. If we define M via

$$(257) \quad w = \gamma \bar{z}_1^2 + |z_1|^2 z_2,$$

then M is a CR singular Levi-flat type A.1 submanifold at the origin, but type B. γ at points where $z_1 = 0$ but $z_2 \neq 0$.

Example 15.5. The Bishop invariant can vary from point to point. Define M via

$$(258) \quad w = |z_1|^2 + \bar{z}_1^2(\gamma_1(1 - z_2) + \gamma_2 z_2),$$

where $\gamma_1, \gamma_2 \geq 0$. It is not hard to see that M is Levi-flat. Again it is an image of $\mathbb{C}^2 \times \mathbb{R}^2$ in a similar way as above.

At the origin, the submanifold is Bishop-like with Bishop invariant γ_1 . When $z_1 = 0$ and $z_2 = 1$, the Bishop invariant is γ_2 . In fact when $z_1 = 0$, the Bishop invariant at that point is $|\gamma_1(1 - z_2) + \gamma_2 z_2|$.

[Proposition 6.2](#) says that this submanifold possesses a real-analytic foliation extending the Levi foliation through the singular points. [Proposition 6.1](#) says that if a foliation on M extends to a (nonsingular) holomorphic foliation, then the submanifold would be a simple product of a Bishop submanifold and \mathbb{C} . Therefore, if $\gamma_1 \neq \gamma_2$ then the Levi foliation on M cannot extend to a holomorphic foliation of a neighborhood of M .

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