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We prove that a harmonic map from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold must be holomorphic under an assumption on energy density. This can be considered as a complex analogue of the Liouville-type theorem for harmonic maps obtained by Sealey.

1. Introduction

The classical Liouville theorem says that a bounded harmonic function on $\mathbb{R}^n$ (or holomorphic function on $\mathbb{C}^n$) has to be constant. Sealey [1982] (see also [Xin 1996]) gave an analogue for harmonic maps. He proved that a harmonic map of finite energy from $\mathbb{R}^n$ ($n \geq 2$) to any Riemannian manifold must be a constant map. In this paper we consider the complex analogue of Sealey's result by asking: *Must a harmonic map with finite $\bar{\partial}$-energy from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold be holomorphic?*

On the other hand, from Siu and Yau's proof of the Frankel conjecture [1980] (the key is to prove that a stable harmonic map from S^2 to $\mathbb{CP}^n$ is holomorphic or conjugate holomorphic), we know that it is very important to study the holomorphicity of harmonic maps. So the above question is obviously interesting. We hope that it is true. But we do not know how to prove it. Our partial result can be stated as follows:

Theorem 1.1. *Let f be a harmonic map from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold. Let $e(f)$ be the energy density and $e''(f)$ the $\bar{\partial}$-energy density. If*

$$e(f)e''(f)(p) = O\left(\frac{1}{R^{4n+\alpha}}\right)^{1} \tag{1-1}$$

for some $\alpha > 0$, where R denotes the distance from the origin to p, then f is a holomorphic map.

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[1] The notation O means $e(f)e''(f)(p) \leq C/R^{4n+\alpha}$ for some $C > 0$ and sufficiently large R.

The condition (1-1) implies that the $\bar\partial$-energy is finite. Since

$$(e''(f))^2 \le e(f)e''(f) = O\left(\frac{1}{R^{4n+\alpha}}\right),$$

one has

$$e''(f) = O\left(\frac{1}{R^{2n+\alpha/2}}\right).$$

This leads to

$$\int_{\mathbb{C}^n} e''(f)\,dv < \infty.$$

Note that we do not have any curvature assumption for the target manifold.

We should mention some other related results on holomorphicity of harmonic maps. For instance, Dong [2013] established many holomorphicity results under the assumption of the target manifolds having strongly seminegative curvature. Xin [1985] obtained some holomorphicity results on harmonic maps from a complete Riemann surface into $\mathbb{CP}^n$.

If the target manifold is $\mathbb{C}^m$ (in this case every component of the map is a harmonic function), then the answer of above question is positive (see [Wan 2010]).

The main idea of the proof of Theorem 1.1 is to consider a one-parameter family of maps and study the $\bar\partial$-energy variation.

The rest of the paper is organized as follows. Section 2 contains some basic materials on harmonic maps. In Section 3, we study the first variation of the $\bar\partial$-energy. Theorem 1.1 is proved in Section 4.

2. Preliminaries

The materials in this section may be found in [Xin 1996].

2A. ***Basic concepts of harmonic maps.*** Let f be a smooth map between two Riemannian manifolds (M, g) and (N, h). We can define the energy density of f by

$$e(f) = \frac{1}{2}\operatorname{trace}|df|^2 = \frac{1}{2}\sum_{i=1}^{m}\langle f_* e_i, f_* e_i\rangle,$$

where $\{e_i\}$ $(i = 1, \dots, m = \dim M)$ is a local orthonormal frame field of M. The energy integral is defined by

$$E(f) = \int_M e(f)\,dv.$$

If we choose local coordinates $\{x^i\}$ and $\{y^\alpha\}$ in M and N, respectively, the energy density can be written as

$$e(f)(x) = \frac{1}{2} g^{ij}(x)\frac{\partial f^\alpha(x)}{\partial x^i}\frac{\partial f^\beta(x)}{\partial x^j} h_{\alpha\beta}(f(x)). \tag{2-1}$$

The tension field of f is

$$\tau(f) = (\nabla_{e_i} df)(e_i),$$

where ∇ is the induced connection on the pullback bundle $f^{-1}TN$ over M from those of M and N.

Definition 2.1. We say that f is a harmonic map if $\tau(f) = 0$.

From the variation point of view, a harmonic map can be seen as the critical point of the energy integral functional. Let f_t be a one-parameter family of maps. We can regard it as a smooth map $M \times (-\epsilon, \epsilon) \to N$. Let $f_0 = f$ and $(df_t/dt)|_{t=0} = v$. Then we have the first-variation formula (see [Xin 1996])

$$\frac{d}{dt} E(f_t)\big|_{t=0} = \int_M \operatorname{div} W \, dv - \int_M \langle v, \tau(f)\rangle \, dv, \tag{2-2}$$

where $W = \langle v, f_* e_j\rangle e_j$. If M is compact, then $\int_M \operatorname{div} W \, dv = 0$. We know that a harmonic map is the critical point of the energy functional.

2B. $\bar{\partial}$*-energy.* Let us consider the complex case. Let f be a smooth map from $\mathbb{C}^n$ to a Kähler manifold N. Let J be the standard complex structure of $\mathbb{C}^n$ and J' the complex structure of N. Let ω and ω^N be the corresponding Kähler forms of $\mathbb{C}^n$ and N (i.e., $\omega(\cdot\,,\cdot) = \langle J\cdot, \cdot\rangle$ and $\omega^N(\cdot\,,\cdot) = \langle J'\cdot, \cdot\rangle$). The $\bar{\partial}$-energy density is defined by

$$\begin{aligned} e''(f) = |\bar{\partial} f|^2 = |f_* J - J' f_*|^2 \\ &= \tfrac{1}{4}(|f_* e_i|^2 + |f_* J e_i|^2 - 2\langle J' f_* e_i, f_* J e_i\rangle) \\ &= \tfrac{1}{2}(e(f) - \langle f^* \omega^N, \omega^M\rangle), \end{aligned}$$

where $\{e_i, Je_i\}$ $(i = 1, \dots, n)$ is the Hermitian frame of $\mathbb{C}^n$ and $\langle f^*\omega^N, \omega\rangle$ denotes the induced norm. We say that f is holomorphic if $f_* J = J' f_*$. Obviously, f is holomorphic if and only if $|\bar{\partial} f|^2 \equiv 0$.

It is well known that a holomorphic map between two Kähler manifolds must be harmonic (see [Xin 1996]).

We denote the $\bar{\partial}$-energy by

$$E_{\bar{\partial}}(f) = \int_{\mathbb{C}^n} |\bar{\partial} f|^2 \, dv.$$

3. $\bar{\partial}$-energy variation

Let us consider the one-parameter family of maps $f_t(x) = f(tx) : \mathbb{C}^n \to N$, $t \in (1-\epsilon, 1+\epsilon)$ and $f_1 = f$. Let B_R denote the Euclidean ball in $\mathbb{C}^n$ of radius R around 0. We write

$$E(R, t) = \int_{B_R} |\bar{\partial} f_t|^2 \, dv.$$

Lemma 3.1. $E(R,t) = t^{2-2n}E(Rt,1)$.

Proof. Under the standard Hermitian metric of $\mathbb{C}^n$, $g^{ij} = \delta_{ij}$, from (2-1) we have

$$e(f_t)(x) = t^2 e(f)(tx).$$

By using the natural coordinates, it is easy to show that

$$\langle f_t^* \omega^N, \omega\rangle(x) = t^2 \langle f^* \omega^N, \omega\rangle(tx).$$

So we get

$$|\bar\partial f_t|^2(x) = t^2 |\bar\partial f|^2(tx).$$

It is easy to check that

$$\int_{B_R} |\bar\partial f_t|^2\, dv = t^{2-2n} \int_{B_{Rt}} |\bar\partial f|^2\, dv.$$

Thus we obtain the lemma. □

We now prove the following variation formula for $\bar\partial$-energy:

Lemma 3.2. $\displaystyle \frac{\partial E(R,t)}{\partial t}\bigg|_{t=1} = \frac{R}{2}\int_{\partial B_R}\left(\left|f_*\frac{\partial}{\partial r}\right|^2 - \left\langle J' f_*\frac{\partial}{\partial r},\ f_* J\frac{\partial}{\partial r}\right\rangle\right) dv.$

Proof. Let $\{e_1, \dots, e_{2n} = \partial/\partial r\}$ be a local orthonormal frame field, where $\partial/\partial r$ denotes the unit radial vector field. By the definition of $f_t(x)$, it is easy to see that the variation vector field of f_t at $t = 1$ is

$$v = \frac{df_t}{dt}\bigg|_{t=1} = r f_* \frac{\partial}{\partial r}.$$

The proof is separated into two steps.

Step 1. From (2-2), we have

$$\begin{aligned} \frac{d}{dt}\int_{B_R} e(f_t)\, dv\bigg|_{t=1} &= \int_{B_R} \operatorname{div}\langle v, f_* e_j\rangle e_j\, dv - \int_{B_R} \langle v, \tau(f)\rangle\, dv \\ &= \int_{\partial B_R}\left\langle v, f_*\frac{\partial}{\partial r}\right\rangle dv = R\int_{\partial B_R}\left|f_*\frac{\partial}{\partial r}\right|^2 dv. \end{aligned}$$

Since f is harmonic, we know that the tension field $\tau(f)$ is 0, and the second equality follows from the divergence theorem.

Step 2. On the other hand, from [Xin 1996] we know that $(d/dt)f_t^*\omega^N = d\theta_t$, where $\theta_t = f_t^* i(f_{t*}\partial/\partial t)\omega^N$. Since $(df_t/dt)|_{t=1} = rf_*\partial/\partial r$, we get $\theta_1 = \theta = rf^* i(f_*\partial/\partial r)\omega^N$. Then

$$\begin{aligned}
\frac{d}{dt}\int_{B_R}\langle f_t^*\omega^N,\omega\rangle\,dv\Big|_{t=1} &= \int_{B_R}\langle d\theta,\omega\rangle\,dv\\
&= \int_{B_R} d(\theta\wedge *\omega) + \int_{B_R}\langle\theta,\delta\omega\rangle\,dv\\
&= \int_{\partial B_R}\theta\wedge *\omega - \int_{B_R}\langle\theta, *d\omega^{n-1}\rangle\,dv\\
&= \int_{\partial B_R}\theta\wedge *\omega\\
&= -\int_{\partial B_R}\theta(e_i)\omega\Big(e_i,\frac{\partial}{\partial r}\Big)\,dv\\
&= -R\int_{\partial B_R}\omega^N\Big(f_*\frac{\partial}{\partial r}, f_*e_i\Big)\omega\Big(e_i,\frac{\partial}{\partial r}\Big)\,dv\\
&= -R\int_{\partial B_R}\Big\langle J' f_*\frac{\partial}{\partial r}, f_*e_i\Big\rangle\Big\langle Je_i,\frac{\partial}{\partial r}\Big\rangle\,dv\\
&= R\int_{\partial B_R}\Big\langle J' f_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r}\Big\rangle\,dv.
\end{aligned}$$

Noting that $\langle d\theta,\omega\rangle\,dv = d\theta\wedge *\omega$, the second equality follows from the differential rules, where δ and $*$ are the codifferential and star operators. By Stokes' theorem and the definition of δ, the third equality holds. The fifth equality follows from direct computation. Since we may choose $e_1 = J\,\partial/\partial r$, the last equality holds.

Combining Steps 1 and 2, we obtain

$$\frac{d}{dt}\int_{B_R}|\bar\partial f_t|^2\,dv\Big|_{t=1} = \frac{R}{2}\int_{\partial B_R}\Big(\Big|f_*\frac{\partial}{\partial r}\Big|^2 - \Big\langle J' f_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r}\Big\rangle\Big)\,dv. \qquad \square$$

Remark 3.3. If M is a compact manifold, $\int_M\langle f^*\omega^N,\omega^M\rangle\,dv$ is a homotopy invariant. This was observed first by Lichnerowicz [1970].

4. Proof of Theorem 1.1

We use a similar trick to [Sealey 1982].

By Lemma 3.1, we obtain

$$\frac{\partial E(R,t)}{\partial t}\Big|_{t=1} = (2-2n)E(R,1) + R\frac{\partial E(R,1)}{\partial R}.$$

On the other hand, from Lemma 3.2 and the condition (1-1), one has

$$\begin{aligned}\frac{\partial E(R,t)}{\partial t}\Big|_{t=1} &= \frac{R}{2}\int_{\partial B_R}\Big(\Big|f_*\frac{\partial}{\partial r}\Big|^2 - \Big\langle J' f_*\frac{\partial}{\partial r}, f_* J\frac{\partial}{\partial r}\Big\rangle\Big)\,dv\\ &\geq \frac{R}{2}\int_{\partial B_R}\Big(\Big|f_*\frac{\partial}{\partial r}\Big|^2 - \Big|f_*\frac{\partial}{\partial r}\Big|\Big|f_* J\frac{\partial}{\partial r}\Big|\Big)\,dv\\ &\geq -\frac{R}{2}\int_{\partial B_R}\Big|f_*\frac{\partial}{\partial r}\Big|\Big|\Big|f_*\frac{\partial}{\partial r}\Big| - \Big|f_* J\frac{\partial}{\partial r}\Big|\Big|\,dv\\ &\geq -\frac{R}{2}R^{2n-1}\frac{1}{R^{2n+\alpha/2}}C\\ &= -\frac{C}{2}R^{-\alpha/2},\end{aligned}$$

where C is a positive constant. Hence for any $\epsilon > 0$, there exists an R_0 such that

$$\frac{\partial E(R,t)}{\partial t}\bigg|_{t=1} \geq -\epsilon$$

for all $R \geq R_0$. Therefore

$$R\frac{\partial E(R,1)}{\partial R} \geq -\epsilon + (2n-2)E(R,1)$$

for $R \geq R_0$.

If $E(\infty,1) = \int_{\mathbb{C}^n} |\bar{\partial} f|^2\,dv = E > 0$, then there exists an R_1 such that for all $R \geq R_1$ we have $E(R,1) \geq E_0 > 0$. Since $n \geq 2$, we can choose a sufficiently small ϵ such that

$$R\frac{\partial E(R,1)}{\partial R} \geq A = -\epsilon + (2n-2)E_0 > 0$$

when $R \geq R_2 = \max(R_0, R_1)$. Then

$$E(\infty,1) = \int_{\mathbb{C}^n} |\bar{\partial} f|^2\,dv \geq \int_{R_2}^{\infty} \frac{A}{R}\,dR = \infty.$$

This is a contradiction. Therefore $\int_{\mathbb{C}^n} |\bar{\partial} f|^2\,dv = 0$. Hence f is a holomorphic map.

Remark 4.1. Compared with the real case [Sealey 1982], Lemma 3.2 has the term $\langle J' f_*(\partial/\partial r), f_* J(\partial/\partial r)\rangle$. We need to use condition (1-1) to control it.

Remark 4.2. If we consider the ∂-energy density $e'(f) = |\partial f|^2 = |f_* J + J' f_*|^2$, the corresponding result of Theorem 1.1 also holds; i.e., if the condition (1-1) is replaced by $e(f)e'(f)(p) = O(1/R^{4n+\alpha})$, then the conclusion is that f is a conjugate holomorphic map ($|\partial f|^2 \equiv 0$).

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