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We prove two main results: (1) For any integers $n \geq 1$ and $g \geq 2$, there is a closed 3-manifold M_g^n admitting a distance-n, genus-g Heegaard splitting, unless $(g, n) = (2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic unless $(g, n) = (3, 1)$. (2) For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many nonhomeomorphic closed 3-manifolds admitting distance-n, genus-g Heegaard splittings.

1. Introduction

Let S be a compact surface with $\chi(S) \leq -2$ but not a 4-punctured sphere. Harvey [1981] defined the curve complex $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices $x_0, x_1, \ldots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if they are represented by pairwise disjoint simple closed curves. For two vertices x and y of $\mathcal{C}(S)$, the distance of x and y, denoted by $d_{\mathcal{C}(S)}(x, y)$, is defined to be the minimal number of 1-simplexes in a simplicial path joining x to y. In other words, $d_{\mathcal{C}(S)}(x, y)$ is the smallest integer $n \geq 0$ such that there is a sequence of vertices $x_0 = x, \ldots, x_n = y$, such that x_{i-1} and x_i are represented by two disjoint essential simple closed curves on S for each $1 \leq i \leq n$. For two sets of vertices in $\mathcal{C}(S)$, say X and Y, $d_{\mathcal{C}(S)}(X, Y)$ is defined to be $\min\{d_{\mathcal{C}(S)}(x, y) \mid x \in X, y \in Y\}$. Now let S be a torus or a once-punctured torus. In this case, the curve complex $\mathcal{C}(S)$ is defined as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices $x_0, x_1, \ldots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if x_i and x_j are represented by two simple closed curves c_i and c_j on S, such that c_i intersects c_j in just one point for each $0 \leq i \neq j \leq k$.

Let M be a compact orientable 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$, where $\partial_+ V$ (resp. $\partial_+ W$) is the positive boundary of V (resp. W). Let $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) be the set

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of vertices in $\mathcal{C}(S)$ such that each element of $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) represents the boundary of an essential disk in V (resp. W). Then the distance of the Heegaard splitting $V \cup_S W$, denoted by $d_{\mathcal{C}(S)}(V, W)$, is defined to be $d_{\mathcal{C}(S)}(\mathcal{D}(V), \mathcal{D}(W))$; see [Hempel 2001].

It is well known that a 3-manifold admitting a high distance Heegaard splitting has good topological and geometric properties. For example, Hartshorn [2002] and Scharlemann [2006] showed that a 3-manifold admitting a high distance Heegaard splitting contains no essential surface with small Euler characteristic number; Scharlemann and Tomova [2006] showed that a high distance Heegaard splitting is the unique minimal Heegaard splitting up to isotopy. By Geometrization theorem and Hempel's work [2001] in Heegaard splittings of Seifert manifolds, a 3-manifold M admitting a distance at least three Heegaard splitting is hyperbolic. From this point of view, Heegaard distance is an active topic in Heegaard splitting. Here we give a brief survey on the existences of high distance Heegaard splittings. Hempel [ibid.] showed that for any integers $g \geq 2$, and $n \geq 2$, there is a 3-manifold that admits a distance at least n Heegaard splitting of genus g. Similar results were obtained using different methods in [Evans 2006; Campisi and Rathbun 2012]. Minsky, Moriah and Schleimer [Minsky et al. 2007] proved the same result for knot complements, and Li [2013] constructed the non-Haken manifolds admitting high distance Heegaard splittings. In general, generic Heegaard splittings have Heegaard distances at least n for any $n \geq 2$; see [Lustig and Moriah 2009; 2010; 2012]. By studying Dehn filling, Ma, Qiu and Zou announced that they had proved that distances of genus-two Heegaard splittings cover all nonnegative integers except one. Recently, Ido, Jang and Kobayashi [Ido et al. 2014] proved that, for any $n > 1$ and $g > 1$, there is a compact 3-manifold with two boundary components which admits a distance-n Heegaard splitting of genus g; Johnson informed us that he had proved that there is always a closed 3-manifold admitting a distance-n (≥ 5), genus-g Heegaard splitting and a genus larger strongly irreducible Heegaard splitting.

The main result of this paper is the following:

Theorem 1.1. *For any integers $n \geq 1$ and $g \geq 2$, there is a closed 3-manifold M_g^n which admits a distance-n Heegaard splitting of genus g unless $(g, n) = (2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic unless $(g, n) = (3, 1)$.*

Remark 1.2. (1) It is well known that there is no distance-one Heegaard splitting of genus two.

(2) Hempel [2001] showed that any Heegaard splitting of a Seifert 3-manifold has distance at most two. Now a natural question is: For any integer $g \geq 2$, is there a closed hyperbolic 3-manifold admitting a distance-2 Heegaard splitting of genus g?

When $g = 2$, Eudave-Muñoz [1999] proved that there is a hyperbolic (1, 1)-knot in 3-sphere, say K. In this case, the complement of K, say M_K, admits a distance-2 Heegaard splitting of genus two. By the main results in [Scharlemann 2006; Kobayashi and Qiu 2008; Agol 2010], there is an essential simple closed curve r on ∂M_K such that the manifold obtained by doing a Dehn filling on M_K along r, say M_K^r, is still hyperbolic. Hence M_K^r admits a distance-2 Heegaard splitting of genus two. Maybe the answer to this question has been well known for $g \geq 3$, but we find no published paper or book related to it.

(3) If M admits a distance-1 Heegaard splitting of genus three, then M contains an essential torus. Hence M is not hyperbolic.

(4) The proof of Theorem 1.1 implies the following fact: Let n be a positive integer, let $\{F_1, \ldots, F_n\}$ be a collection of closed orientable surfaces, and let I and $J = \{1, \ldots, n\} \backslash I$ be two subsets of $\{1, \ldots, n\}$. Then, for any integers

$$g \geq \max\left\{\sum_{i \in I} g(F_i),\ \sum_{j \in J} g(F_j)\right\}$$

and $m \geq 2$, there is a compact 3-manifold M admitting a distance-m Heegaard splitting of genus g, denoted by $M = V \cup_S W$, such that $F_i \subset \partial_- V$ for $i \in I$, $F_j \subset \partial_- W$ for $j \in J$. We omit the proof.

By the arguments in Theorem 1.1, we have:

Theorem 1.3. *For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many nonhomeomorphic closed 3-manifolds admitting distance-n Heegaard splittings of genus g.*

We organize this paper as follows. In Section 2, we introduce some results on curve complex. Then we will prove Theorem 1.1 for $n \neq 2$ in Section 3, for $n = 2$ in Section 5 and Theorem 1.3 in Section 4.

2. Preliminaries of curve complex

Let S be a compact surface of genus at least one and $\mathcal{C}(S)$ the curve complex of S. We say that a simple closed curve c in S is essential if c bounds no disk in S and is not parallel to ∂S. Hence each vertex of $\mathcal{C}(S)$ is represented by the isotopy class of an essential simple closed curve in S. For simplicity, we do not distinguish the essential simple closed curve c and its isotopy class c.

Lemma 2.1 [Minsky 1996; Masur and Minsky 1999; 2000]. *$\mathcal{C}(S)$ is connected, and the diameter of $\mathcal{C}(S)$ is infinite.*

We say that a collection $\mathcal{G} = \{a_0, a_1, \ldots, a_n\}$ is a geodesic in $\mathcal{C}(S)$ if $a_i \subset \mathcal{C}^0(S)$ and $d_{\mathcal{C}(S)}(a_i, a_j) = |i - j|$, for any $0 \leq i, j \leq n$. And the length of $\mathcal{G}$, denoted by $\mathcal{L}(\mathcal{G})$, is defined to be n. By the connectedness of $\mathcal{C}^1(S)$, there is always a shortest path in $\mathcal{C}^1(S)$ connecting any two vertices of $\mathcal{C}(S)$. For any two vertices α, β with $d_S(\alpha, \beta) = n$, we say that a geodesic $\mathcal{G}$ connects α, β if $\mathcal{G} = \{a_0 = \alpha, \dots, a_n = \beta\}$. Now for any two subsimplicial complex $X, Y \subset \mathcal{C}(S)$, we say that a geodesic $\mathcal{G}$ realizes the distance between X and Y if $\mathcal{G}$ connects a vertex $\alpha \in X$ and a vertex $\beta \in Y$ such that $\mathcal{L}(\mathcal{G}) = d_{\mathcal{C}(S)}(X, Y)$.

Let F be a compact surface of genus at least one with nonempty boundary. Similar to the definition of the curve complex $\mathcal{C}(F)$, we define the arc and curve complex $\mathcal{AC}(F)$ as follows. Each vertex of $\mathcal{AC}(F)$ is the isotopy class of an essential simple closed curve or an essential properly embedded arc in F, and a set of vertices forms a simplex of $\mathcal{AC}(F)$ if these vertices are represented by pairwise disjoint arcs or curves in F. For any two vertices which are realized by disjoint curves or arcs, we place an edge between them. All the vertices and edges form the 1-skeleton of $\mathcal{AC}(F)$, denoted by $\mathcal{AC}^1(F)$. For each edge, we assign it length one. Thus for any two vertices α and β in $\mathcal{AC}^1(F)$, the distance $d_{\mathcal{AC}(F)}(\alpha, \beta)$ is defined to be the minimal length of paths in $\mathcal{AC}^1(F)$ connecting α and β. Similarly, we can define the geodesic in $\mathcal{AC}(F)$.

When F is a subsurface of S, we say that F is essential in S if the induced map of the inclusion from $\pi_1(F)$ to $\pi_1(S)$ is injective. Furthermore, we say that F is a proper essential subsurface of S if F is essential in S and at least one boundary component of F is essential in S. For more details, see [Masur and Minsky 2000].

If F is an essential subsurface of S, there is some connection between $\mathcal{AC}(F)$ and $\mathcal{C}(S)$. For any $\alpha \in \mathcal{C}^0(S)$, there is an essential simple closed curve α_{geo} representing α such that the geometric intersection number $i(\alpha_{\text{geo}}, \partial F)$ is minimal. Hence each component of $\alpha_{\text{geo}} \cap F$ is essential in F. Now for $\alpha \in \mathcal{C}(S)$, let $\kappa_F(\alpha)$ be the collection of isotopy classes of the essential components of $\alpha_{\text{geo}} \cap F$.

For any $\gamma \in \mathcal{C}(F)$, we define the set $\sigma_F(\gamma)$ as follows: $\gamma' \in \sigma_F(\gamma)$ if and only if γ' is the essential boundary component of a closed regular neighborhood of $\gamma \cup \partial F$. Set $\sigma_F(\varnothing) = \varnothing$. Now let $\pi_F = \sigma_F \circ \kappa_F$. Then the map π_F links $\mathcal{C}(F)$ and $\mathcal{C}(S)$, which is the subsurface projection map in [ibid.].

We say $\alpha \in \mathcal{C}^0(S)$ cuts F if $\pi_F(\alpha) \neq \varnothing$. If $\alpha, \beta \in \mathcal{C}^0(S)$ both cut F, we denote $d_{\mathcal{C}(F)}(\alpha, \beta) = \text{diam}_{\mathcal{C}(F)}(\pi_F(\alpha), \pi_F(\beta))$. And if $d_{\mathcal{C}(S)}(\alpha, \beta) = 1$, then

$$d_{\mathcal{AC}(F)}(\alpha, \beta) \leq 1,$$
$$d_{\mathcal{C}(F)}(\alpha, \beta) \leq 2,$$

observed by H. Masur and Y. N. Minsky. When the two vertices α and β have distance k in $\mathcal{C}(S)$, we have a direct consequence of the above observation:

Lemma 2.2. *Let F and S be as above, $\mathcal{G} = \{\alpha_0, \dots, \alpha_k\}$ be a geodesic in $\mathcal{C}(S)$ such that α_i cuts F for each $0 \leq i \leq k$. Then $d_{\mathcal{C}(F)}(\alpha_0, \alpha_k) \leq 2k$.*

Moreover, Masur and Minsky [ibid.] proved:

Lemma 2.3 (bounded geodesic image theorem). *Let F be an essential proper subsurface of S, and let γ be a geodesic segment in $\mathcal{C}(S)$, so that $\pi_F(v) \neq \varnothing$ for every vertex v of γ. Then there is a constant $\mathcal{M}$ depending only on S so that* $\operatorname{diam}_{\mathcal{C}(F)}(\pi_F(\gamma)) \leq \mathcal{M}$.

When S is closed with $g(S) \geq 2$, there is always a compact 3-manifold M with S as its compressible boundary. Let $\mathcal{D}(M, S)$, called the disk complex for S, be the subset of vertices of $\mathcal{C}(S)$, where each element bounds a disk in M. For an essential simple closed curve on S, say c, we say that it is disk-busting if $S-c$ is incompressible in M.

Now let's consider the subsurface projection of disk complex. The following disk image theorem is proved by Li [2012], Masur and Schleimer [2013] independently.

For any I-bundle J over a bounded compact surface P, $\partial J = \partial_v J \cup \partial_h J$, where the vertical boundary $\partial_v J$ is the I-bundle related to ∂P, and the horizontal boundary $\partial_h J$ is the portion of ∂J transverse to the I-fibers.

Lemma 2.4. *Let M be a compact orientable and irreducible* 3*-manifold. S is a boundary component of M. Suppose $\partial M - S$ is incompressible. Let $\mathcal{D}$ be the disk complex of S, and let $F \subset S$ be an essential subsurface. Assume each component of ∂F is disk-busting. Then either*

(1) *M is an I-bundle over some compact surface, F is a horizontal boundary of the I-bundle and the vertical boundary of this I-bundle is a single annulus. Or,*

(2) *The image of this complex, $\kappa_F(\mathcal{D})$, lies in a ball of radius three in $\mathcal{AC}(F)$. In particular, $\kappa_F(\mathcal{D})$ has diameter six in $\mathcal{AC}(F)$. Moreover, $\pi_F(\mathcal{D})$ has diameter at most twelve in $\mathcal{C}(F)$.*

Hempel introduced a full simplex X on S which is a dimension $3g(S)-4$ simplex in $\mathcal{C}(S)$. Then after attaching 2-handles and 3-handles along the vertices of X on the same side of S, there is a handlebody H_X with $\partial H_X = S$.

Lemma 2.5 [Hempel 2001]. *Let S be a closed, orientable surface of genus at least two. For any positive number d and any full simplex X of $\mathcal{C}(S)$, there is another full simplex Y of $\mathcal{C}(S)$ such that $d_{\mathcal{C}(S)}(\mathcal{D}(H_X), \mathcal{D}(H_Y)) \geq d$.*

Through subsurface projection, the bounded geodesic image theorem links the geodesic in the curve complex of the entire surface to the curve complex of a proper subsurface. Since the diameter of the curve complex is infinite, we can construct a geodesic of any given length in the curve complex. Furthermore, we require that the constructed geodesic satisfies that both the first and last vertices are represented by separating essential simple closed curves.

We organize our results:

Lemma 2.6. *Let g, n, m, s, t be integers such that $g, m, n \geq 2$, $1 \leq t, s \leq g-1$. Let S_g be a closed surface of genus g. Then there are two essential separating curves α*

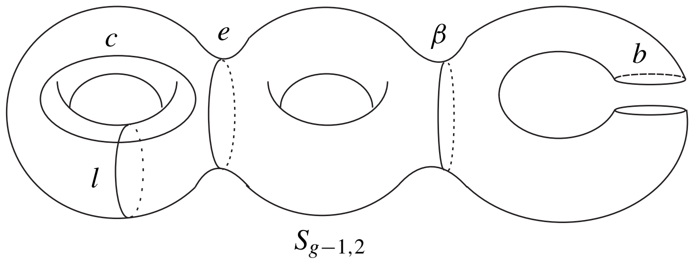

Figure 1. Self-banding.

and β *in* S_g *such that* $d_{\mathcal{C}(S_g)}(\alpha, \beta) = n$; *one component of* $S_g - \alpha$ *has genus* t; *one component of* $S_g - \beta$ *has genus* s. *Furthermore*, *there is a geodesic*

$$\mathcal{G} = \{a_0 = \alpha, a_1, \dots, a_{n-1}, a_n = \beta\}$$

in $\mathcal{C}(S_g)$ *such that*

(1) a_i *is nonseparating in* S_g *for* $1 \le i \le n-1$, *and*

(2) $m\mathcal{M} + 2 \le d_{\mathcal{C}(S^{a_i})}(a_{i-1}, a_{i+1}) \le m\mathcal{M} + 6$, *where* S^{a_i} *is the surface* $S - N(a_i)$ *for* $1 \le i \le n-1$ *and* $\mathcal{M}$ *is the constant in Lemma 2.3.*

Proof. Let α be an essential separating curve in S such that one component of $S_g - \alpha$, say S_1, has genus t.

Suppose first that $n = 2$. Let b be a nonseparating curve in S_g which is disjoint from α. Let S^b be the surface $S_g - N(b)$, where $N(b)$ is an open regular neighborhood of b in S_g. Then S^b is a genus-$(g-1)$ surface with two boundary components. Furthermore, α is an essential separating simple closed curve in S^b.

By Lemma 2.1, $\mathcal{C}^1(S^b)$ is connected and its diameter is infinite. Hence there is an essential simple closed curve c in S^b with $d_{\mathcal{C}(S^b)}(\alpha, c) = m\mathcal{M} + 4$. Note that $g - 1 \ge 1$. If c is separating in S^b, then there is a nonseparating essential simple closed curve c^* in S^b such that $c \cap c^* = \varnothing$. Hence $d_{\mathcal{C}(S^b)}(c, c^*) = 1$, and

$$m\mathcal{M} + 3 \le d_{\mathcal{C}(S^b)}(\alpha, c^*) \le m\mathcal{M} + 5.$$

So there is a nonseparating essential simple closed curve c in S^b such that

$$m\mathcal{M} + 3 \le d_{\mathcal{C}(S^b)}(\alpha, c) \le m\mathcal{M} + 5.$$

Let l be a nonseparating simple closed curve in S^b such that l intersects c in one point, and let e be the boundary of the closed regular neighborhood of $c \cup l$ in S^b. Then e bounds a once-punctured torus T containing l and c. Since $s \le g - 1$, there is an essential separating simple closed curve β in S^b such that β bounds a once-punctured surface of genus s containing T as a subsurface, see Figure 1.

So β is also separating in S_g. Now we prove that

$$d_{\mathcal{C}(S_g)}(\alpha, \beta) = 2 \quad \text{and} \quad d_{\mathcal{C}(S_g)}(\alpha, c) = 2.$$

Since $\alpha \cap b = \varnothing$, $\beta \cap b = \varnothing$ and $c \cap b = \varnothing$, $d_{\mathcal{C}(S_g)}(\alpha, \beta) \le 2$ and $d_{\mathcal{C}(S_g)}(\alpha, c) \le 2$. Since $c \cap \beta = \varnothing$, by the assumption on $d_{\mathcal{C}(S^b)}(\alpha, c)$,

$$m\mathcal{M} + 2 \le d_{\mathcal{C}(S^b)}(\alpha, \beta) \le m\mathcal{M} + 6.$$

So $d_{\mathcal{C}(S_g)}(\beta, \alpha) = 2$. For if $d_{\mathcal{C}(S_g)}(\alpha, \beta) \le 1$, then, by Lemma 2.3, $d_{\mathcal{C}(S^b)}(\alpha, \beta) \le \mathcal{M}$, a contradiction. Similarly, $d_{\mathcal{C}(S_g)}(\alpha, c) = 2$. And

$$\mathcal{G} = \{a_0 = \alpha, a_1 = b, a_2 = \beta\} \quad \text{and} \quad \mathcal{G}^* = \{a_0 = \alpha, a_1 = b, a_2 = c\}$$

are two geodesics of $\mathcal{C}(S_g)$. Furthermore, $\mathcal{G}$ satisfies the conclusion of Lemma 2.6.

Now we prove this lemma by induction on n.

Assumption. Let $k \ge 2$. Suppose that there are two essential separating simple closed curves α and β, and a nonseparating simple closed curve c in S_g such that

$$d_{\mathcal{C}(S_g)}(\alpha, \beta) = k,$$
$$d_{\mathcal{C}(S_g)}(\alpha, c) = k,$$

and one component of $S_g - \alpha$ has genus t while one component of $S_g - \beta$ has genus s. Furthermore, there is a geodesic $\mathcal{G}^* = \{\alpha, a_1, \dots, a_{k-1}, a_k = c\}$ where a_i is nonseparating in S_g for each $1 \le i \le k$, satisfying

$$m\mathcal{M} + 3 \le d_{\mathcal{C}(S^{a_i})}(a_{i-1}, a_{i+1}) \le m\mathcal{M} + 5 \quad \text{for any } 1 \le i \le k-2,$$
$$m\mathcal{M} + 3 \le d_{\mathcal{C}(S^{a_{k-1}})}(a_{k-2}, c) \le m\mathcal{M} + 5,$$

and a geodesic $\mathcal{G} = \{\alpha = a_0, a_1, \dots, a_{k-1}, \beta\}$ satisfying the conclusions (1) and (2) of Lemma 2.6.

Let S^c be the surface $S_g - N(c)$, where $N(c)$ is an open regular neighborhood of c in S_g. Since c is nonseparating in S_g, S^c is a genus-$(g-1)$ surface with two boundary components. Since $\mathcal{G}^* = \{\alpha, a_1, \dots, a_{k-1}, c\}$ is also a geodesic connecting α to c, a_{k-1} is an essential nonseparating simple closed curve in S^c. By the above argument, there is an essential nonseparating curve h and an essential separating curve e in S^c such that

(1) e bounds an once-punctured torus T^* containing h;

(2) $m\mathcal{M} + 3 \le d_{\mathcal{C}(S^c)}(h, a_{k-1}) \le m\mathcal{M} + 5$;

(3) $m\mathcal{M} + 2 \le d_{\mathcal{C}(S^c)}(e, a_{k-1}) \le m\mathcal{M} + 6$.

And there is also an essential separating simple closed curve γ which bounds a genus-s subsurface of S^c containing T^* as a subsurface, while γ is also separating

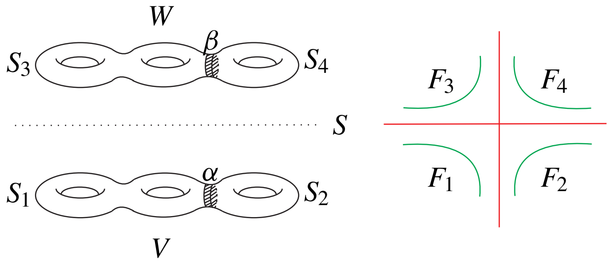

Figure 2. Heegaard splitting I.

in S_g. Since h is disjoint from γ,

$$m\mathcal{M}+2 \le d_{\mathcal{C}(S^c)}(\gamma, a_{k-1}) \le m\mathcal{M}+6.$$

Now we prove that $d_{\mathcal{C}(S_g)}(\alpha, h) = k+1$, $d_{\mathcal{C}(S_g)}(\alpha, \gamma) = k+1$.

Suppose, on the contrary, that $d_{\mathcal{C}(S_g)}(\alpha, h) = x \le k$. Then there exists a geodesic $\mathcal{G}_1 = \{\alpha = b_0, \dots, b_x = h\}$. Note that each of α and h is not isotopic to c and the length is less than or equal to k. Since $d_{\mathcal{C}(S_g)}(\alpha, c) = k$, b_j is not isotopic to c for $1 \le j \le x-1$. This means b_j cuts S^c for each $0 \le j \le x$. By Lemma 2.3, $d_{\mathcal{C}(S^c)}(\alpha, h) \le \mathcal{M}$. Since $d_{\mathcal{C}(S_g)}(\alpha, c) = k$, a_j is not isotopic to c for $0 \le j \le k-1$. By using Lemma 2.3 again, $d_{S^c}(\alpha, a_{k-1}) \le \mathcal{M}$. Then $d_{\mathcal{C}(S^c)}(a_{k-1}, h) \le 2\mathcal{M}$. It contradicts the choice of h.

Now $\mathcal{G}' = \{a_0 = \alpha, a_1, \dots, a_{k-1}, c, \gamma\}$ and $\mathcal{G}'' = \{a_0 = \alpha, a_1, \dots, a_{k-1}, c, h\}$ are two geodesics satisfying the conclusion. □

3. Proof of Theorem 1.1 ($n \neq 2$)

In this section, we will prove:

Proposition 3.1. *For any positive integers $n \neq 2$ and $g \ge 2$, there is a closed 3-manifold which admits a distance-n Heegaard splitting of genus g unless $(g, n) = (2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic unless $(g, n) = (3, 1)$.*

Proof. We first suppose that $n \ge 3$.

Let S be a closed surface of genus g. By Lemma 2.6, there are two separating essential simple closed curves α and β such that $d_{\mathcal{C}(S)}(\alpha, \beta) = n$ for $n \ge 3$. Let V be the compression body obtained by attaching a 2-handle to $S \times [0, 1]$ along $\alpha \times \{1\}$, and let W be the compression body obtained by attaching a 2-handle to $S \times [-1, 0]$ along $\beta \times \{-1\}$. Then $V \cup_S W$ is a Heegaard splitting where S is the surface $S \times \{0\}$; see Figure 2.

Since V contains only one essential disk B with $\partial B = \alpha$ up to isotopy and W contains only one essential disk D with $\partial D = \beta$ up to isotopy, $d_{\mathcal{C}(S)}(V, W) = n$.

Let F_1 and F_2 be the components of $\partial_- V$, and S_1 and S_2 the two components of $S - \alpha$. Similarly, let F_3 and F_4 be the components of $\partial_- W$, and S_3 and S_4 the

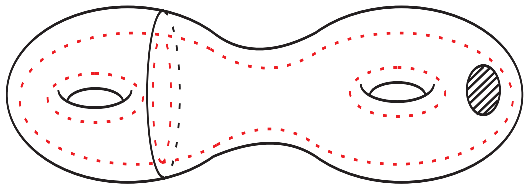

Figure 3. A spanning annulus.

two components of $S-\beta$. Now B cuts V into two manifolds $F_1 \times I$ and $F_2 \times I$, and D cuts W into two manifolds $F_3 \times I$ and $F_4 \times I$; see Figure 2. By Lemma 2.6, we assume that S_3 is a once-punctured torus.

We first consider the compression body V. We assume that $F_i = F_i \times \{0\}$, $S_i \cup B = F_i \times \{1\}$ for $1 \le i \le 2$. Let $f_{F_i} : S_i \cup B \to F_i$ be the natural homeomorphism such that $f_{F_i}(x \times \{1\}) = x \times \{0\}$ for $i = 1, 2$. And f_{F_i} is well defined. Then, for any two essential simple closed curves $\zeta, \theta \subset S_i \cup B$,

$$d_{\mathcal{C}(F_i)}(f_{F_i}(\zeta), f(\theta)) = d_{\mathcal{C}(S_i \cup B)}(\zeta, \theta) \quad \text{for } i = 1, 2;$$

see Figure 3. Hence f_{F_i} induces an isomorphism from $\mathcal{C}(S_i \cup B)$ to $\mathcal{C}(F_i)$, for any $i = 1, 2$. Denote the isomorphism by f_{F_i} too. Note that the shaded disk in Figure 3 is B.

Let $\iota : S_i \to S_i \cup B$ be the inclusion map for $i = 1, 2$. Note that ∂S_i contains only one component. If c is an essential simple closed curve in S_i, $\iota(c)$ is also essential in $S_i \cup B$. So, for any two essential simple closed curves $\zeta, \theta \subset S_i$,

$$d_{\mathcal{C}(S_i \cup B)}(\iota(\zeta), \iota(\theta)) \le d_{S_i}(\zeta, \theta) \quad \text{for } i = 1, 2.$$

Hence ι induces a distance nonincreasing map from $\mathcal{C}(S_i)$ to $\mathcal{C}(S_i \cup B)$, for any $i = 1, 2$. Denote the inclusion map by ι too. Then we define

$$\psi_{F_i} = f_{F_i} \circ \iota \circ \pi_{S_i}.$$

Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n \ge 2$, $\alpha \cap \beta \ne \varnothing$. By the argument in Section 2,

$$\operatorname{diam}_{\mathcal{C}(S_i)}(\pi_{S_i}(\beta)) \le 2.$$

Hence,

$$\operatorname{diam}_{\mathcal{C}(F_i)}(\psi_{F_i}(\beta)) \le 2.$$

We start to attach a handlebody to V along F_1. Then we have two cases:

(a) F_1 is a torus. By Lemma 2.1, there is an essential simple closed curve r in F_1 such that

$$d_{\mathcal{C}(F_1)}(\psi_{F_1}(\beta), r) \ge \mathcal{M} + 1. \tag{1}$$

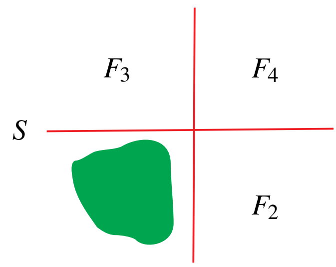

Figure 4. Heegaard splitting II.

Let J_r be a solid torus such that $\partial J_r = F_1$, and r bounds an essential disk in J_r. In this case, J_r contains only one essential disk up to isotopy. Let V_{F_1} be the manifold $V \cup J_r$.

(b) $g(F_1) \geq 2$. By Lemma 2.5, there is a full simplex X of $\mathcal{C}(F_1)$ such that

$$d_{\mathcal{C}(F_1)}(\mathcal{D}(H_X), \psi_{F_1}(\beta)) \geq \mathcal{M}+1,$$

where H_X is the handlebody obtained by attaching 2-handles to F_1 along X then 3-handles to cap off the possible 2-spheres. In this case, we denote the manifold $V \cup H_X$ by V_{F_1}.

In a word, V_{F_1} is a compression body with only one negative boundary component F_2, where $\partial_+ V_{F_1} = \partial_+ W$; see Figure 4. Hence $V_{F_1} \cup_S W$ is a Heegaard splitting.

Claim 3.2. *The Heegaard distance $d_{\mathcal{C}(S)}(V_{F_1}, W)$ is n.*

Proof. Suppose, otherwise, that $d_{\mathcal{C}(S)}(V_{F_1}, W) = k < n$. Since W contains only one essential disk D up to isotopy where $\partial D = \beta$, there is an essential disk B_1 in V_{F_1} such that $d_{\mathcal{C}(S)}(\partial B_1, \beta) = k \leq n-1$, i.e, there is a geodesic $\mathcal{G} = \{a_0 = \beta, \dots, a_k = \partial B_1\}$, where $k \leq n-1$. □

Claim 3.3. *$a_j \cap S_1 \neq \varnothing$, for any $0 \leq j \leq k$.*

Proof. Suppose that $a_j \cap S_1 = \varnothing$ for some $0 \leq j \leq k$. If $a_k \cap S_1 = \varnothing$, then $B_1 \subset F_2 \times I$ and B_1 is inessential in V_{F_1}. So $j \neq k$. Since $a_0 = \beta$, $j \neq 0$. Hence there is a geodesic $\mathcal{G}^* = \{\beta = a_0, \dots, a_j, \alpha\}$. It means that $d_{\mathcal{C}(S)}(\alpha, \beta) \leq k < n$, a contradiction. □

By Lemma 2.3, $d_{\mathcal{C}(S_1 \cup B)}(\partial B_1, \beta) \leq \mathcal{M}$ and $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\partial B_1), \psi_{F_1}(\beta)) \leq \mathcal{M}$. Depending on the intersection between B_1 and B, there are two cases:

(a) $B_1 \cap B = \varnothing$. Since B_1 is not isotopic to B, $\psi_{F_1}(\partial B_1)$ bounds an essential disk in H_X or J_r depending on $g(F_1)$, where H_X and J_r are constructed as above. Then

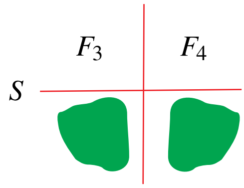

Figure 5. Heegaard splitting III.

by Lemma 2.3,

$$d_{\mathcal{C}(F_1)}(\psi_{F_1}(\partial B_1), \psi_{F_1}(\beta)) \leq \mathcal{M},$$
$$d_{\mathcal{C}(F_1)}(r, \psi_{F_1}(\beta)) \leq \mathcal{M} \quad \text{if } g(F_1) = 1,$$
$$d_{\mathcal{C}(F_1)}(\mathcal{D}(H_X), \psi_{F_1}(\beta)) \leq \mathcal{M} \quad \text{if } g(F_1) \geq 2.$$

It contradicts the choice of X or r.

(b) $B_1 \cap B \neq \varnothing$. Let a be an outermost arc of $B_1 \cap B$ on B_1. It means that a, together with a subarc $\gamma \subset \partial B_1$, bounds a disk B_γ such that $B_\gamma \cap B = a$. Since B cuts V_{F_1} into a handlebody H which contains F_1 and an I-bundle $F_2 \times I$, $B_\gamma \subset H$. Hence a curve in $\psi_{F_1}(\partial B_1)$ bounds an essential disk in H_X or J_r. By the argument in (a), it is impossible.

Now V_{F_1} is a compression body which has only one minus boundary component F_2. Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n \geq 3$, $\beta \cap S_2 \neq \varnothing$. By Lemmas 2.1 and 2.5, there is always a simplex Y on F_2 such that $d_{\mathcal{C}(F_2)}(\mathcal{D}(H_Y), \psi_{F_2}(\beta)) \geq \mathcal{M} + 1$, where H_Y is the handlebody or the solid torus obtained by attaching 2-handles to F_2 along Y and 3-handles to cap off the possible 2-spheres. Let V_{F_1,F_2} be the manifold obtained by attaching H_Y to V_{F_1} along F_2; see Figure 5. Then V_{F_1,F_2} is a handlebody where $\partial_+ V_{F_1,F_2} = \partial_+ W$. Hence $V_{F_1,F_2} \cup_S W$ is also a Heegaard splitting.

Claim 3.4. *The Heegaard distance $d_{\mathcal{C}(S)}(V_{F_1,F_2}, W)$ is n.*

Proof. Suppose, on the contrary, that $d_{\mathcal{C}(S)}(V_{F_1,F_2}, W) = k < n$. Since W contains only one essential disk D up to isotopy such that $\partial D = \beta$, there is an essential disk B_2 in V_{F_1,F_2} such that $d_{\mathcal{C}(S)}(\partial B_2, \beta) = k$, i.e., there is a geodesic $\mathcal{G} = \{a_0 = \beta, \ldots, a_k = \partial B_2\}$, where $k \leq n-1$. By the definition of Heegaard distance, $a_j \cap \partial S_2 \neq \varnothing$ for $0 \leq j \leq k-1$ when $k \geq 1$.

Note that $\partial B = \alpha$. Depending on the way of intersection between B_2 and B, there are two cases:

(a) $B_2 \cap B = \varnothing$. Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n > k$, B_2 is not isotopic to B. By the proof of Claim 3.2, ∂B_2 does not lie in S_1. Hence $\partial B_2 \subset S_2$. It implies that $\psi_{F_2}(\partial B_2)$ bounds an essential disk in H_Y. By Lemma 2.3, $d_{\mathcal{C}(S_2)}(\partial B_2, \beta) \leq \mathcal{M}$. Hence

$$d_{\mathcal{C}(F_2)}(\psi_{F_2}(\partial B_2), \psi_{F_2}(\beta)) \leq \mathcal{M}, \quad d_{\mathcal{C}(F_2)}(\mathcal{D}(H_Y), \psi_{F_2}(\beta)) \leq \mathcal{M}.$$

It contradicts the choice of Y.

(b) $B_2 \cap B \neq \varnothing$. Let a^* be an outermost arc of $B_2 \cap B$ on B_2. This means that a^*, together with a subarc $\gamma^* \subset \partial B_2$, bounds a disk B_{γ^*} such that $B_{\gamma^*} \cap B = a^*$. By the proof of Claim 3.2, $\gamma^* \subset S_2$. Thus $\psi_{F_2}(\partial B_2)$ bounds an essential disk in H_Y. By the same argument in Claim 3.2 again, it is impossible. □

Until now, we get a distance-n genus-g Heegaard splitting $V_{F_1,F_2} \cup_S W$. In this case, V_{F_1,F_2} is a handlebody, and W contains only one essential disk D such that $\partial D = \beta$. Furthermore, we cut S along β into two components S_3 and S_4, and cut W along D into two manifolds $F_3 \times I$ and $F_4 \times I$ such that $F_i = F_i \times \{0\}$, and $S_i \cup D = F_i \times \{1\}$ for $i = 3, 4$. Now the shaded disk in Figure 3 is D. Let $f_{F_i} : S_i \cup D \to F_i$ be the natural homeomorphism such that $f_{F_i}(x \times \{1\}) = x \times \{0\}$ for $i = 3, 4$. Then, for any two essential simple closed curves $\zeta, \theta \subset S_i \cup D$,

$$d_{\mathcal{C}(F_i)}(f_{F_i}(\zeta), f_{F_i}(\theta)) = d_{\mathcal{C}(S_i \cup D)}(\zeta, \theta) \quad \text{for } i = 3, 4;$$

see Figure 3. Hence f_{F_i} induces an isomorphism from $\mathcal{C}(S_i \cup D)$ to $\mathcal{C}(F_i)$, for any $i = 3, 4$. Denote the isomorphism by f_{F_i} too.

Let $\iota : S_i \to S_i \cup D$ be the inclusion map for $i = 3, 4$. Note that ∂S_i contains only one component. If c is an essential simple closed curve in S_i, $\iota(c)$ is also essential in $S_i \cup D$. Now, for any two essential simple closed curves $\zeta, \theta \subset S_i$,

$$d_{\mathcal{C}(S_i \cup D)}(\iota(\zeta), \iota(\theta)) \leq d_{S_i}(\zeta, \theta) \quad \text{for } i = 3, 4.$$

Hence ι induces a distance nonincreasing map from $\mathcal{C}(S_i)$ to $\mathcal{C}(S_i \cup D)$, for any $i = 3, 4$. Denote the inclusion map by ι too. Then we define

$$\psi_{F_i} = f_{F_i} \circ \iota \circ \pi_{S_i}.$$

Since $V_{F_1,F_2} \cup_S W$ is a distance-n (≥ 3) Heegaard splitting of genus g, and W contains only one essential disk D up to isotopy, S_3 and S_4 are incompressible in V_{F_1,F_2}. Hence $\beta = \partial S_3 = \partial S_4$ is disk-busting in V_{F_1,F_2}. Since the Heegaard distance $n \geq 3$ and $g(S_3) = 1$, V_{F_1,F_2} is not an I-bundle over some compact surface with S_i a horizontal boundary of the I-bundle while the vertical boundary of this I-bundle a single annulus for $i = 3, 4$. By Lemma 2.4, $\operatorname{diam}_{S_i}(\mathcal{D}(V_{F_1,F_2})) \leq 12$ for $i = 3, 4$. Hence $\operatorname{diam}_{F_i}(\psi_{F_i}(\mathcal{D}(V_{F_1,F_2}))) \leq 12$.

Since F_3 is a torus and $\operatorname{diam}_{F_3}(\psi_{F_3}(\mathcal{D}(V_{F_1,F_2}))) \leq 12$, by Lemma 2.1, there is an essential simple closed curve δ in F_3 such that $d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1,F_2})), \delta) \geq \mathcal{M} + 1$. Let W_{F_3} be the manifold obtained attaching a solid J_δ to W along F_3 so that δ bounds a disk in J_δ. Then W_{F_3} is a compression body.

Since $\operatorname{diam}_{F_4}(\psi_{F_4}(\mathcal{D}(V_{F_1,F_2}))) \leq 12$, by Lemmas 2.1 and 2.5, there is a simplex Z of $\mathcal{C}(F_4)$ such that

$$d_{\mathcal{C}(F_4)}(\mathcal{D}(H_Z), \psi_{F_4}(\mathcal{D}(V_{F_1,F_2}))) \geq \mathcal{M} + 1,$$

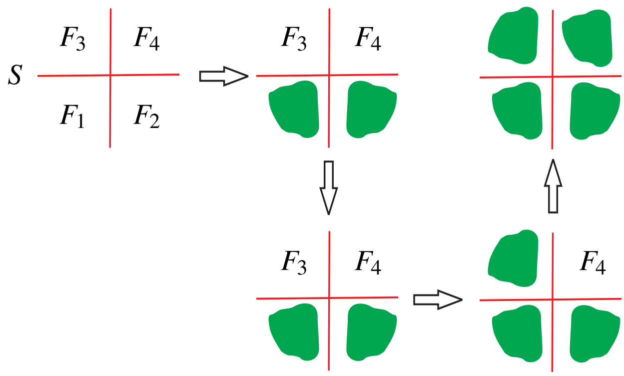

Figure 6. Heegaard splitting IV.

where H_Z is the handlebody or the solid torus obtained by attaching 2-handles to F_4 along Z then 3-handles to cap off the possible 2-spheres. In this case, let W_{F_3,F_4} be the handlebody $W_{F_3} \cup H_Z$ where $\partial_+ W_{F_3,F_4} = \partial_+ V_{F_1,F_2}$. Now $V_{F_1,F_2} \cup_S W_{F_3,F_4}$ is a Heegaard splitting of a closed 3-manifold; see Figure 6.

Claim 3.5. *The Heegaard distance $d_{\mathcal{C}(S)}(V_{F_1,F_2}, W_{F_3,F_4})$ is n.*

Proof. Let D be the essential disk in W_{F_3,F_4} bounded by β. Suppose, on the contrary, that the Heegaard distance is $k < n$. Then there is a geodesic

$$\mathcal{G} = \{a_0 = \partial B_1, \dots, a_k = \partial D_1\},$$

where $k \le n-1$, B_1 is an essential disk in V_{F_1,F_2}, and D_1 is an essential disk in W_{F_3,F_3}. $\alpha_i \cap \beta \ne \varnothing$, for any $0 \le i \le k-1$. If not, the distance of $V_{F_1,F_2} \cup_S W$ would be at most $k < n$. Similarly, D_1 is not isotopic to D.

Then we have two cases:

(a) $D_1 \cap D = \varnothing$. Then ∂D_1 lies in one of S_3 and S_4. We assume that ∂D_1 lies in S_3. The other case is similar. Hence $\psi_{F_3}(\partial D_1) = \delta$. By Lemma 2.3, $\operatorname{diam}_{S_3}(\mathcal{D}(\mathcal{G})) \le \mathcal{M}$. Since $\pi_{S_3}(\partial B_1) \in \pi_{S_3}(\mathcal{D}(V_{F_1,F_2}))$, we have

$$d_{\mathcal{C}(S_3)}(\pi_{S_3}(\mathcal{D}(V_{F_1,F_2})), \partial D_1) \le \mathcal{M}.$$

Hence,

$$d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1,F_2})), \psi_{F_3}(\partial D_1) = \delta) \le \mathcal{M},$$

a contradiction.

(b) $D_1 \cap D \ne \varnothing$. Let c be an outermost arc of $D_1 \cap D$ on D_1. This means that c, together with a subarc $\delta^* \subset \partial D_1$, bounds a disk D_c such that $D_c \cap D = c$. We assume that $\partial D_c \subset S_4$. The other case is similar. By Lemma 2.3, $\operatorname{diam}_{S_4}(\mathcal{G}) \le \mathcal{M}$. Hence

$$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\mathcal{D}(V_{F_1,F_2})), \psi_{F_4}(\partial D_1)) \le \mathcal{M}.$$

Note that $\psi_{F_4}(\partial B_1) \in \mathcal{D}(H_Z)$. Then by the same argument in (a), it is impossible. □

Now we prove the proposition for $n = 1$. It is known that if a Heegaard splitting has distance 1, there are on the Heegaard surface two disjoint nonisotopic essential simple closed curves that bound essential disks in different compression bodies. That is to say, a distance-1 Heegaard splitting is always weakly reducible. For a reducible Heegaard splitting, since there is an essential simple closed curve in the Heegaard surface bounding essential disks in both of these two compression bodies, it has distance zero. Hence it is only needed to prove the proposition for weakly reducible and irreducible Heegaard splittings.

Let M_1 and M_2 be two 3-manifolds with homeomorphic connected boundary. For any homeomorphism f from ∂M_1 to ∂M_2, let M_f be the manifold obtained by gluing M_1 and M_2 along f. Suppose M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ for $i = 1, 2$. In this case, M_f has a natural Heegaard splitting called the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$. The following facts are well known:

(1) If the gluing map f is complicated enough, then the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is unstabilized; see [Lackenby 2004; Bachman et al. 2006; Li 2010].

(2) If both $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ have high distance, then the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is unstabilized and irreducible; see [Kobayashi and Qiu 2008; Yang and Lei 2009].

Now let $M_i = V_i \cup_{S_i} W_i$ be a Heegaard splitting of genus two such that ∂M_i is a torus, and $d(S_i) > 8$ for $i = 1, 2$, then, by the main result in [Kobayashi and Qiu 2008], the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$, say $V \cup_S W$, is unstabilized.

Suppose that $g \geq 4$. By the above argument, there exist a Heegaard splitting $M_1 = V_1 \cup_{S_1} W_1$ of genus $g-1$ such that $g(\partial M_1) = 2$ and $d(S_1) \geq 2g$, and a Heegaard splitting $V_2 \cup_{S_2} W_2$ of genus three such that $g(\partial M_2) = 2$ and $d(S_2) \geq 2g$. Hence both M_1 and M_2 are hyperbolic. By the main result in [ibid.], the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$, say $M = V \cup_S W$, is unstabilized and weakly reducible. Furthermore, $g(S) = g$. By Thurston's theorem, both M_1 and M_2 have hyperbolic structures with totally geodesic boundaries. Hence M is hyperbolic. □

Remark 3.6. The strongly irreducible Heegaard splitting $V \cup_S W$ where both V and W contain only one essential separating disk up to isotopy independently is always a minimal Heegaard splitting of $M = V \cup_S W$. Li [2010] defined a subcomplex $\mathcal{U}(F_1)$, for $F_1 \subset \partial_- V$ and proved that for any handlebody H attached to M along F_1, if $d_{\mathcal{C}(F_1)}(\mathcal{U}(F_1), \mathcal{D}(H))$ is larger than a constant $\mathcal{K}$ which depends on M and H, then the new generated Heegaard splitting $V_{F_1} \cup_S W$ is still the minimal Heegaard splitting of $M^{F_1} = V_{F_1} \cup_S W$. Similar to the other boundaries of M. Now in our construction of distance-n (≥ 2) strongly irreducible Heegaard

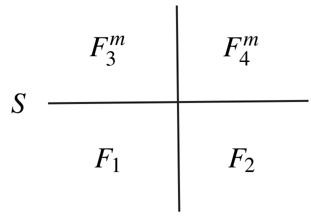

Figure 7. Heegaard splitting V.

splitting (for $n = 2$, see Section 5), we can choose a full simplex X in F_1 such that $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\mathcal{D}(W)), \mathcal{D}(H_X))$ is large enough and $d_{\mathcal{C}(F_1)}(\mathcal{U}(F_1), \mathcal{D}(H_X))$ is larger than $\mathcal{K}$. Then the new Heegaard splitting $V_{F_1} \cup_S W$ is still the minimal Heegaard splitting of $M^{F_1} = V_{F_1} \cup_S W$ and has the same distance.

4. Proof of Theorem 1.3

Proposition 4.1. *For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many nonhomeomorphic closed 3-manifolds which admit distance-n, genus-g Heegaard splittings.*

Proof. Let S_g be a closed surface of genus g. By Lemma 2.6, for each $m \geq 2$, there is a geodesic $\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \ldots, a_{n-1}^m, a_n^m = \beta^m\}$ in $\mathcal{C}(S_g)$ such that

(1) a_i^m is nonseparating in S_g for $1 \leq i \leq n-1$, α and β^m are two essential separating simple closed curves on S_g,

(2) $m\mathcal{M} + 2 \leq d_{\mathcal{C}(S^{a_i^m})}(a_{i-1}^m, a_{i+1}^m) \leq m\mathcal{M} + 6$, where S^{a_i} is the surface $S - N(a_i)$ for $1 \leq i \leq n-1$, and

(3) one component of $S_g - \beta^m$ has genus one.

Without loss of generality, we assume that $\mathcal{M} \geq 6$. Let M_m be the manifold obtained by attaching two 2-handles to $S_g \times [-1, 1]$ along $\alpha \times \{-1\}$ and $\beta^m \times \{1\}$. We also use S_g representing the surface $S_g \times \{0\}$. Now M_m has a Heegaard splitting as $V_m \cup_{S_g} W_m$, where V_m is the compression body obtained by attaching a 2-handle to $S \times [-1, 0]$ along $\alpha \times \{-1\}$, and W_m is the manifold obtained by attaching a 2-handle to $S \times [0, 1]$ along $\beta^m \times \{1\}$. Then $\partial_- V_m$ contains two components F_1 and F_2, and $\partial_- W_m$ contains two components F_3^m and F_4^m; see Figure 7.

By the proof of Theorem 1.1 ($n \neq 2$), there is a closed 3-manifold M_m^* which admits a distance-n Heegaard splitting $V_m^* \cup_{S_g} W_m^*$, where V_m^* is obtained by attaching handlebodies H_{X_1} and H_{X_2} to V_m along F_1 and F_2, and W_m^* is obtained by attaching handlebodies H_{Y_1} and H_{Y_2} to W_m along F_3^m and F_4^m such that

$$d_{\mathcal{C}(F_i)}(\psi_{F_i}(\beta^m), \mathcal{D}(H_{X_i})) \geq \mathcal{M} + 15 \quad \text{for } i = 1, 2,$$

$$d_{\mathcal{C}(F_i)}(\psi_{F_i}(\alpha), \mathcal{D}(H_{Y_i})) \geq \mathcal{M} + 15 \quad \text{for } i = 3, 4.$$

Replace M_m^*, V_m^* and W_m^* by M_m, V_m and W_m. Now

$$\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \ldots, a_{n-1}^m, a_n^m = \beta^m\}$$

is a geodesic of $\mathcal{C}(S_g)$ realizing the distance of $M_m = V_m \cup_{S_g} W_m$.

Claim 4.2. *Let*

$$\mathcal{G} = \{b_0, \ldots, b_n\}$$

be a geodesic of $\mathcal{C}(S_g)$ *realizing the distance of* $V_m \cup_{S_g} W_m$. *Then*

$$b_i = a_i^m$$

for any $1 \le i \le n-1$.

Proof. Let S_1 and S_2 be the two components of $S_g - \alpha$. We assume that b_0 bounds a disk B_0 in V_m, and b_n bounds a disk D_n in W_m. We first prove that α (resp. β^m) is disjoint from b_1 (resp. b_{n-1}).

Let B be the essential disk bounded by α in V_m. Suppose, on the contrary, that $\alpha \cap b_1 \neq \varnothing$. Hence b_0 is not isotopic to $a_0^m = \alpha$. Then there are two cases:

(a) $B_0 \cap B \neq \varnothing$. Let a be an outermost arc of $B_0 \cap B$ on B_0. It means that a, together with a subarc of $\gamma \subset \partial B_0$, bounds a disk B_γ such that $B_\gamma \cap B = a$. We assume that $\gamma \subset S_1$. The other case is similar. By the argument in Section 3, $\psi_{F_1}(\partial B_0)$ bounds an essential disk in H_{X_1}. But with $b_1 \cap \partial S_1 \neq \varnothing$, it implies that $d_{\mathcal{C}(S_1)}(b_0, b_n) \le \mathcal{M}$. Hence $d_{\mathcal{C}(F_1)}(\psi_{F_1}(b_n), \mathcal{D}(H_{X_1})) \le \mathcal{M}$.

(b) $B_0 \cap B = \varnothing$. Since $b_1 \cap \alpha \neq \varnothing$, B_0 is not isotopic to B. Then ∂B_0 is essential in S_1 or S_2. We assume that $\partial B_0 \subset S_1$. The other case is similar. Hence by the arguments in the previous case, $d_{\mathcal{C}(F_1)}(\psi_{F_1}(b_n), \mathcal{D}(H_{X_1})) \le \mathcal{M}$.

However, since the Heegaard distance is at least four and $\alpha = \partial S_1 = \partial S_2$ bounds an essential disk in V^m, the curve α is disk-busting for W^m and W^m can not be the I-bundle over S_1 or S_2. Then by Lemma 2.4,

$$\operatorname{diam}_{\mathcal{C}(S_1)}(\mathcal{D}(W^m)) \le 12$$

and

$$\operatorname{diam}_{\mathcal{C}(S_2)}(\mathcal{D}(W^m)) \le 12.$$

Hence $\operatorname{diam}_{\mathcal{C}(F_1)}(\psi_{F_1}(\mathcal{D}(W^m))) \le 12$ and $\operatorname{diam}_{\mathcal{C}(F_2)}(\psi_{F_2}(\mathcal{D}(W^m))) \le 12$. Together with (a) and (b), by the triangle inequality, we have

$$d_{\mathcal{C}(F_1)}(\psi_{F_1}(\beta^m), \mathcal{D}(H_{X_1})) \le \mathcal{M} + 12.$$

It contradicts the choice of X_1 in F_1.

Let $\mathcal{G}^* = \{\alpha = a_0^m, b_1, \ldots, b_{n-1}, a_n^m\}$ be a new geodesic realizing the distance of $V_m \cup_{S_g} W_m$. Now we prove that b_1 is isotopic to a_1^m. The other case is similar.

Suppose, otherwise, that b_1 is not isotopic to a_1^m. Note that b_i is not isotopic to a_1^m. Otherwise, the distance of $V_m \cup_{S_g} W_m$ would be at most $n-1$. Let $S^{a_1^m}$ be the surface $S_g - N(a_1^m)$, where $N(a_1^m)$ is an open regular neighborhood of a_1^m on S_g. By Lemma 2.3,

$$d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_n^m)) \leq \mathcal{M}.$$

Now let's consider the shorter geodesic

$$\mathcal{G}^{**} = \{a_2^m, \dots, a_{n-1}^m, a_n^m = \beta^m\},$$

which is a subgeodesic of

$$\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \dots, a_{n-1}^m, a_n^m = \beta^m\}.$$

By the definition of geodesic in the curve complex, a_i^m is not isotopic to a_1^m for any $i \geq 2$. By Lemma 2.3 again,

$$d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_2^m), \pi_{S^{a_1^m}}(a_n^m)) \leq \mathcal{M}.$$

Hence

$$d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_2^m)) \leq 2\mathcal{M}.$$

This contradicts our assumption on $m\mathcal{M} \leq d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_2^m))$ and $m \geq 2$. Hence b_1 is isotopic to a_1^m. ☐

Replace $M_m = V_m \cup_{S_g} W_m$ by $M_m = V_m \cup_{S_g^m} W_m$.

The following claim reveals the connection between geodesics in the curve complex and closed 3-manifolds:

Claim 4.3. *For any t, s such that $2 \leq t \neq s \in N$, either*

1. *$M_t = V_t \cup_{S_g^t} W_t$ and $M_s = V_s \cup_{S_g^s} W_s$ are two different 3-manifolds up to homeomorphism, or,*
2. *M_t is homeomorphic to M_s, but $V_t \cup_{S_g^t} W_t$ and $V_s \cup_{S_g^s} W_s$ are two different Heegaard splittings of M_t up to homeomorphic equivalence.*

Proof. Suppose that M_t is homeomorphic to M_s for some $t, s \in N$ where $2 \leq t, s$ and $t \neq s$. If (2) fails, then $V_t \cup_{S_g^t} W_t$ and $V_s \cup_{S_g^s} W_s$ are homeomorphic. It means that there is a homeomorphism f from M_t to M_s such that $f((S_g^t; V_t, W_t)) = (S_g^s; V_s, W_s)$. We assume that $f(V_t) = V_s$ and $f(W_t) = W_s$. The other case is similar. It is well known that f induces an isomorphism from $\mathcal{C}(S_g^t)$ to $\mathcal{C}(S_g^s)$, still denoted by f. Then for the geodesic

$$\mathcal{G}^t = \{\alpha = a_0^t, a_1^t, \dots, a_{n-1}^t, a_n^t = \beta^t\}$$

which realizes the distance of $V_t \cup_{S_g^t} W_t$, $f(\mathcal{G})$ is also a geodesic in $\mathcal{C}(S_g^s)$ realizing the distance of $V_s \cup_{S_g^s} W_s$. By Claim 4.2, $f(a_j^t)$ is isotopic to a_j^s for $1 \leq j \leq n-1$.

Since $f(a_2^t)$ is isotopic to a_2^s, we can perform an isotopy on S_g^s such that the composition of f with the isotopy gives an homeomorphism $f^\star$ from S_g^t to S_g^t and $f^\star(a_2^t) = a_2^s$, $f^\star(V_t) = V_s$, $f^\star(W_t) = W_s$. It's also true that $f^\star$ induces an automorphism from $\mathcal{C}(S_g^t)$ to $\mathcal{C}(S_g^s)$, denoted by $f^\star$ too. Thus $f^\star(\mathcal{G}^t)$ is also a geodesic realizing the distance of $V_s \cup_{S_g^s} W_s$. By Claim 4.2 again, for any $1 \le j \le n-1$, $f^\star(a_j^t)$ is still isotopic to a_j^s. Hence $f^\star(a_1^t)$ (resp. $f^\star(a_3^t)$) is isotopic to a_1^s (resp. a_3^t).

Let $S^{a_2^t}$ be the surface $S_g^t - N(a_2^t)$, where $N(a_2^t)$ is an open regular neighborhood of a_2^t on S_g^t, and let $S^{a_2^s}$ be the surface of $S_g^s - N(a_2^s)$. Then $f^\star(S^{a_2^t}) = S^{a_2^s}$ and $f^\star|_{S^{a_2^t}}$ is a homeomorphism. Hence $f^\star$ also induces an isomorphism from $\mathcal{C}(S^{a_2^t})$ to $\mathcal{C}(S^{a_2^s})$, still denoted by $f^\star$. Now we also assume $a_1^t \cap a_2^t = \varnothing$ and $a_3^t \cap a_2^t = \varnothing$. Thus $f^\star(a_1^t) \cap (f^\star(a_2^t) = a_2^s) = \varnothing$ and $f^\star(a_3^t) \cap (f^\star(a_2^t) = a_2^s) = \varnothing$. Then $d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) = d_{\mathcal{C}(S^{a_2^s})}(f^\star(a_1^t), f^\star(a_3^t))$. On the other hand, $f^\star(a_1^t)$ (resp. $f^\star(a_3^s)$) must be isotopic to a_1^s (resp. a_3^s) in $S^{a_2^s}$. For if not, then after removing possible bigon capped by them, they bound no annuli in $S^{a_2^s}$, and thus they bound no annuli and bigon in S_g^s. By bigon criterion [Farb and Margalit 2012, Proposition 1.7], they realize the geometry intersection number. Since they are isotopic in S_g^s, they must bound an annulus in S_g^s. So

$$d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) = d_{\mathcal{C}(S^{a_2^s})}(f^\star(a_1^t), f^\star(a_3^t)),$$

$$d_{\mathcal{C}(S^{a_2^s})}(f^\star(a_1^t), f^\star(a_3^t)) = d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s).$$

It means that

$$d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) = d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s).$$

However, by the assumption,

$$t\mathcal{M} + 2 \le d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) \le t\mathcal{M} + 6,$$

$$s\mathcal{M} + 2 \le d_{\mathcal{C}(S^{a_2^s})}(a_1^t, a_3^t) \le s\mathcal{M} + 6,$$

$$\mathcal{M} \ge 6,$$

we have

$$d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) \ne d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s),$$

a contradiction. □

The Waldhausen conjecture proved by Johanson [1990; 1995] and Li [2006; 2007] implies that, for any positive integer g, an atoroidal closed 3-manifold M admits only finitely many Heegaard splittings of genus g up to homeomorphism. Since M_t admits a Heegaard splitting with distance at least four, it is atoroidal for any $t \ge 2$; see [Hartshorn 2002; Scharlemann 2006]. Now Theorem 1.3 immediately follows from Claim 4.3 and the Waldhausen conjecture. □

5. Proof of Theorem 1.1 ($n = 2$)

We rewrite the second part of Theorem 1.1:

Proposition 5.1. *For any integer $g \geq 2$, there is a closed hyperbolic 3-manifold which admits a distance-2 Heegaard splitting of genus g.*

Proof. By Remark 1.2(2), there is a hyperbolic closed 3-manifold which admits a distance-2 Heegaard splitting of genus two. So we only need to prove it for $g \geq 3$.

Assumption 1. Let S be a closed surface of genus g. By Lemma 2.6, there are two separating essential simple closed curves α and γ such that

(1) $d_{\mathcal{C}(S)}(\alpha, \gamma) = 2$,

(2) one component of $S - \alpha$, say S_1, has genus one while the component of $S - \alpha$, say S_2, has genus $g - 1$,

(3) one component of $S - \gamma$, say S_3, has genus one, while the component of $S - \gamma$, say S_4, has genus $g - 1$,

(4) there is a nonseparating slope β on S such that α and γ are disjoint from β, and $d_{\mathcal{C}(S^\beta)}(\alpha, \gamma) > 4$, where S^β is the surface $S - \eta(\beta)$, and

(5) $\beta \subset S_2 \cap S_4$.

Let V be the compression body obtained by attaching a 2-handle to $S \times [0, 1]$ along a separating curve $\alpha \times \{1\}$, and let W be the compression body obtained by attaching a 2-handle to $S \times [-1, 0]$ along a separating curve $\gamma \times \{-1\}$. Denote $S \times \{0\}$ by S too. Then $V \cup_S W$ is a Heegaard splitting. Since V contains only one essential disk B with $\partial B = \alpha$ up to isotopy, and W contains only one essential disk D with $\partial D = \gamma$ up to isotopy, $d_{\mathcal{C}(S)}(V, W) = 2$.

Let F_1 and F_2 be the components of $\partial_- V$, such that F_i is homeomorphic to $S_i \cup B$ for $i = 1, 2$. Similarly, let F_3 and F_4 be the components of $\partial_- W$ such that F_i is homeomorphic to $S_i \cup D$ for $i = 3, 4$. Then both S_1 and S_3 are once-punctured tori, and F_1 and F_3 are two tori; see Figure 2. Furthermore, both F_3 and F_4 have genus at least two. Now B cuts V into two manifolds $F_1 \times I$ and $F_2 \times I$, and D cuts W into two manifolds $F_3 \times I$ and $F_4 \times I$.

Since $d_{\mathcal{C}(S)}(V, W) = 2$, $\gamma \cap S_i \neq \varnothing$ for $i = 1, 2$, and $\alpha \cap S_i \neq \varnothing$ for $i = 3, 4$. Hence $\psi_{F_i}(\gamma) \neq \varnothing$ for $i = 1, 2$, and $\psi_{F_i}(\alpha) \neq \varnothing$ for $i = 3, 4$, where ψ is defined in Section 3.

Assumption 2. (1) Let δ be an essential simple closed curve on the torus F_1 such that $d_{\mathcal{C}(F_2)}(\psi_{F_2}(\gamma), \delta) \geq 5$.

(2) Let X be a full complex of $\mathcal{C}(F_2)$ such that $d_{\mathcal{C}(F_2)}(\psi_{F_2}(\gamma), \mathcal{D}(H_X)) \geq 24$, where H_X is the handlebody obtained by attaching 2-handles to F_2 along the vertices of X then 3-handles to cap off the spherical boundary components.

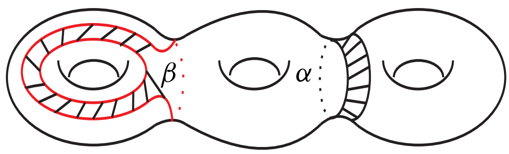

Figure 8. Essential annulus.

Let $V_{F_2} = V \cup H_X$, and let V_{F_1,F_2} be the handlebody obtained by doing a surgery on V_{F_2} along the slope δ on F_1. By Assumption 1, $g(S_3) = 1$, $g(S_4) \geq 2$, V_{F_1,F_2} is not an I-bundle over S_i for $i = 3, 4$. By Lemma 2.4, $\operatorname{diam}_{\mathcal{C}(S_i)}(\pi_{S_i}(\mathcal{D}(V_{F_1,F_2}))) \leq 12$ for $i = 3, 4$.

Assumption 3. (1) Let r be an essential simple closed curve on the torus F_3 such that $d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1,F_2})), r) \geq 24$.

(2) Let Y be a full complex of $\mathcal{C}(F_4)$ such that $d_{\mathcal{C}(F_4)}(\psi_{F_4}(\mathcal{D}(V_{F_1,F_2})), \mathcal{D}(H_Y)) \geq 24$, where H_Y is the handlebody obtained by attaching 2-handles to F_4 along the vertices of Y then 3-handles to cap off the spherical boundary components.

Let $W_{F_4} = W \cup H_Y$, and let W_{F_3,F_4} be the handlebody obtained by doing a surgery on W_{F_4} along the slope r on F_3. Now both $M^* = V_{F_2} \cup_S W_{F_4}$ and $V_{F_1,F_2} \cup_S W_{F_3,F_4}$ are Heegaard splittings. Furthermore, we can prove that these two Heegaard splittings have distance two by arguments in the proof of Proposition 3.1.

Now we consider $M^* = V_{F_2} \cup_S W_{F_4}$. Note that M^* has only two toroidal boundary components. Since the distance of $V_{F_2} \cup_S W_{F_4}$ is two, M^* is irreducible and ∂-irreducible.

Claim 5.2. *M^* is atoroidal.*

Proof. Suppose, on the contrary, that M^* contains an essential torus T. Since the distance of $V_{F_2} \cup_S W_{F_4}$ is two, $V_{F_2} \cup_S W_{F_4}$ is strongly irreducible. By Schultens' lemma [Schultens 1993], we may assume that each component of $T \cap S$ is essential on both T and S. Hence each component of $T \cap V_{F_2}$ and $T \cap W_{F_4}$ is an incompressible annulus in V_{F_2} or W_{F_4}.

Let A_0 be one component of $T \cap V_{F_2}$. We first prove that there is one component of ∂A_0, say a_0, not isotopic to β.

Now V_{F_2} contains a ∂-compressing disk B^* of A_0. Note that A_0 has a ∂-compression disk B^* in V_{F_2}. By doing a surgery on A_0 along B^*, we get a disk B_0 in V_{F_2}. Since A_0 is essential, B_0 is essential. Suppose that the two components of ∂A_0 are isotopic to β. Since β is nonseparating on S, ∂B_0 bounds a once-punctured torus containing β; see Figure 8.

By Assumption 1, $\beta \subset S_2$. Since S_2 has genus $g - 1 \geq 2$, ∂B_0 is not isotopic to $\alpha = \partial S_2$. By standard outermost disk argument, $\psi_{F_2}(\partial B_0)$ bounds an essential disk in H_X. Therefore $d_{\mathcal{C}(F_2)}(\mathcal{D}(H_X), \psi_{F_2}(\beta)) \leq 1$. Since $\gamma \cap \beta = \varnothing$,

$d_{\mathcal{C}(F_2)}(\psi_{F_2}(\beta), \psi_{F_2}(\gamma)) \leq 2$. Hence $d_{\mathcal{C}(F_2)}(\mathcal{D}(H_X), \psi_{F_2}(\gamma)) \leq 3$. It contradicts Assumption 2.

Let A_1 be a component of $T \cap W_{F_4}$ which is incident to A_0 at a_0. This means that a_0 is one component of ∂A_1. We consider two cases:

Case 1. $a_0 \cap \alpha = \varnothing$ and $a_0 \cap \gamma = \varnothing$.

Recall the definition of the surface S^β. Since a_0 is not isotopic to β, $a_0 \cap S^\beta \neq \varnothing$. Since $\alpha, \gamma \subset S^\beta$,

$$d_{\mathcal{C}(S_\beta)}(\pi_{S^\beta}(a_0), \alpha) \leq 1,$$
$$d_{\mathcal{C}(S^\beta)}(\gamma, \pi_{S^\beta}(a_0)) \leq 1.$$

Hence $d_{\mathcal{C}(S^\beta)}(\alpha, \gamma) \leq 2$. This contradicts Assumption 1.

Case 2. $a_0 \cap (\alpha \cup \gamma) \neq \varnothing$.

We assume that $a_0 \cap \alpha \neq \varnothing$. By the above argument, B_0 is an essential disk in V_{F_2} such that ∂B_0 is disjoint from a_0. Furthermore, ∂B_0 is not isotopic to α. Since B cuts V_{F_2} into $F_1 \times I$ and a handlebody H such that $S_2 \cup B = \partial H$, $\partial B_0 \cap S_2 \neq \varnothing$. Furthermore, all outermost disks of $B_0 \cap B$ on B_0 lie in H. Hence a curve in $\pi_{S_2}(\partial B_0)$ bounds an essential disk in H. This means a curve in $\psi_{F_2}(\partial B_0)$ bounds an essential disk in H_X.

If $a_0 \cap \gamma = \varnothing$, then

$$d_{\mathcal{C}(F_2)}(\psi_{F_2}(\partial B_0), \psi_{F_2}(\gamma)) \leq d_{\mathcal{C}(F_2)}(\psi_{F_2}(\partial B_0), \psi_{F_2}((a_0)) + d_{\mathcal{C}(F_2)}(\psi_{F_2}(a_0), \psi_{F_2}(\gamma)) \leq 4.$$

It contradicts Assumption 2. Hence $a_0 \cap \gamma \neq \varnothing$, and $\psi_{F_4}(a_0) \neq \varnothing$.

Since A_1 is an essential annulus in W_{F_4}, there is an essential disk D_0 obtained by doing boundary compression on A_1 in W_{F_4}. Furthermore $\partial D_0 \cap a_0 = \varnothing$. Since D cuts W_{F_4} into $F_3 \times I$ and a handlebody H^* containing H_Y, all outermost disks of $D_0 \cap D$ in D_0 lie in H^*. Hence $\psi_{F_4}(\partial D_0)$ bounds an essential disk in H_Y. Hence a curve in $\pi_{S_4}(\partial D_0) \neq \varnothing$. Since $\partial D_0 \cap a_0 = \varnothing$, by Lemma 2.2, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial D_0), \pi_{S_4}(a_0)) \leq 2$. According to the definition of ψ_{F_4}, $d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(a_0)) \leq 2$.

Recall that the essential disk B_0 is obtained by doing a boundary compression on A_0 in V_{F_2}. Since the distance of $V_{F_2} \cup_S W_{F_4}$ is two, $\partial B_0 \cap \gamma \neq \varnothing$. Since $g(S_3) = 1$ and $g(S_4) \geq 2$, V_{F_2} is not an I-bundle over S_4. By Lemma 2.4, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial B_0), \pi_{S_4}(\alpha)) \leq 12$. Hence

$$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(\alpha)) \leq 12.$$

Since $\partial B_0 \cap a_0 = \varnothing$,

$$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(a_0)) \leq 2.$$

The above inequalities implies that

$$\begin{aligned} d_{\mathcal{C}(F_4)}&(\psi_{F_4}(\partial D_0), \psi_{F_4}(\alpha)) \\ &\le d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(a_0)) + d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(a_0)) \\ &\qquad + d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(\alpha)) \\ &\le 16. \end{aligned}$$

It contradicts Assumption 3. □

Claim 5.3. *M^* is anannular.*

Proof. Since the distance of $M^* = V_{F_2} \cup_S W_{F_4}$ is two, $M^* = V_{F_2} \cup_S W_{F_4}$ is strongly irreducible and boundary irreducible. Suppose, on the contrary, that M^* contains an essential annulus A. Then there are two cases:

(a) ∂A lies in the same boundary component of M^*. Without assumption, we assume that $\partial A \subset F_2$. Hence the boundary of closed regular neighborhood of $F_2 \cup A$ consists of three tori, denoted by F_2, T_1 and T_2. By Claim 5.2, both T_1 and T_2 are inessential in M^*. Since the boundary of M^* is not connected, one of T_1 and T_2, says T_1, is compressible and the other one is boundary parallel. This means that M^* is a Seifert manifold, whose orbifold is an annulus with at most one cone point. By [Moriah and Schultens 1998], each irreducible Heegaard splitting of M^* is vertical or horizontal. Hence each irreducible Heegaard splitting of M^* has genus two. So each genus at least three Heegaard splitting of M^* is stabilized and reducible. A contradiction.

(b) ∂A lies in different boundary components of M^*. Then the boundary of $A \cup \partial M^*$ consists of three tori, denoted by T, F_2 and F_4. By Claim 5.2, T is inessential in M^*. It is not hard to see that T is not boundary parallel to F_2 or F_4. Then T is compressible in M^*. So M^* is a Seifert manifold, whose orbifold is an annulus with at most one cone point. By [ibid.] again, each irreducible Heegaard splitting of M^* is vertical or horizontal. Hence each irreducible Heegaard splitting of M^* has genus two. So each genus at least three Heegaard splitting of M^* is stabilized and reducible. A contradiction. □

Now M^* is a hyperbolic 3-manifold, $M^* = V_{F_2} \cup_S W_{F_4}$ is a distance-2 Heegaard splitting of genus g. Furthermore, M^* contains two toral boundary components F_1 and F_3. By the main results in [Agol 2010; Lackenby and Meyerhoff 2013], there are at most ten slopes δ on F_1 such that the manifold $M^*(\delta)$ obtained by doing Dehn filling on M^* along δ is nonhyperbolic. By Assumption 2, there are infinitely many slopes δ so that $M^*(\delta)$ has a distance-2 Heegaard splitting of genus g. Hence there is at least one slope δ on F_1 such that $M^*(\delta)$ is hyperbolic and $M^*(\delta)$ admits a distance-2 Heegaard splitting of genus g. Similarly, by Assumption 3, there is a hyperbolic closed manifold which admits a distance-2 Heegaard splitting of genus g. □

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