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The Weil representation of the symplectic group associated to a finite abelian group of odd order is shown to have a multiplicity-free decomposition. When the abelian group is p-primary, the irreducible representations occurring in the Weil representation are parametrized by a partially ordered set which is independent of p. As p varies, the dimension of the irreducible representation corresponding to each parameter is shown to be a polynomial in p which is calculated explicitly. The commuting algebra of the Weil representation has a basis indexed by another partially ordered set which is independent of p. The expansions of the projection operators onto the irreducible invariant subspaces in terms of this basis are calculated. The coefficients are again polynomials in p. These results remain valid in the more general setting of finitely generated torsion modules over a Dedekind domain.

1. Introduction

1A. ***Overview.*** Heisenberg groups were introduced by Weyl [1949, Chapter 4] in his mathematical formulation of quantum kinematics. Best known among them are the Lie groups whose Lie algebras are spanned by position and momentum operators which satisfy Heisenberg's commutation relations. Weyl also considered Heisenberg groups which are finite modulo their centers, such as the Pauli group (generated by the Pauli matrices), which he used to characterize the kinematics of electron spin.

A fundamental property of Heisenberg groups, predicted by Weyl and proved by Stone [1930] and von Neumann [1931] for real Heisenberg groups is known as the Stone–von Neumann theorem. Mackey [1949] extended this theorem to locally compact Heisenberg groups (see Section 1C for the case that is pertinent to this paper, and [Prasad 2011] for a more detailed and general exposition). By considering Heisenberg groups associated to finite fields, local fields and adèles, Weil [1964] demonstrated the importance of Heisenberg groups in number theory.

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Weil exploited the Stone–von Neumann–Mackey theorem to construct a projective representation of a group of automorphisms of the Heisenberg group, now commonly known as the Weil representation. Along with parabolic induction and the technique of Deligne and Lusztig [1976] using l-adic cohomology, the Weil representation is one of the most important techniques for constructing representations of reductive groups over finite fields (see [Gérardin 1977; Srinivasan 1979]) or local fields (see [Gérardin 1975] and Mœglin, Vignéras and Waldspurger [Mœglin et al. 1987]).

Tanaka [1967a; 1967b] showed how the Weil representation can be used to construct all the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ for odd p by looking at Weil representations associated to the abelian groups $\mathbb{Z}/p^k\mathbb{Z} \oplus \mathbb{Z}/p^l\mathbb{Z}$ for $l \leq k$. However, most of the literature on Weil representations associated to finite abelian groups has focused on vector spaces over finite fields and on constructing representations of classical groups over finite fields.

The representation theory of groups over finite principal ideal local rings was initiated by Kloosterman [1946], who studied $\mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$. In contrast to general linear groups over finite fields, whose character theory was worked out by Green [1955], the representation theory for general linear groups over these rings is quite hard. It has been shown (by Aubert, Onn, Prasad and Stasinski [Aubert et al. 2010] and Singla [2010]) that this problem is intricately related to the problem of understanding the representations of automorphism groups of finitely generated torsion modules over discrete valuation rings. However, explicit constructions have been available either for a very small class of representations [Hill 1995a; 1995b] or for a very small class of groups [Onn 2008; Stasinski 2009; Singla 2010].

This article concerns the decomposition of the Weil representation of the full symplectic group associated to a finite abelian group of odd order (and more generally, a finite module of odd order over a Dedekind domain) into irreducible representations. When the module in question is elementary (e.g., $(\mathbb{Z}/p\mathbb{Z})^n$ for some odd prime p), it is well-known that the Weil representation, which may be realized on the space of functions on the abelian group, breaks up into two irreducible subspaces consisting of even and odd functions. Besides this, only the case where all the invariant factors are equal (e.g., $(\mathbb{Z}/p^k\mathbb{Z})^n$) has been understood completely (see [Prasad 1998, Theorem 2] for the case where k is even, and Cliff, McNeilly and Szechtman [Cliff et al. 2000] for the general case). A small part of the decomposition has been explained in the general case by Cliff, McNeilly and Szechtman [Cliff et al. 2003]. Maktouf and Torasso [2014; 2012] have shown that the restriction of the Weil representation of a symplectic group over a p-adic field to a maximal compact subgroup or to a maximal elliptic torus is multiplicity-free and have given an explicit description of the irreducible subrepresentations.

In this paper, we describe all the invariant subspaces for the Weil representation for all finite modules of odd order over a Dedekind domain. To be specific, it is shown that the Weil representation has a multiplicity-free decomposition (Theorem 5.5). When the underlying finitely generated torsion module is primary of type λ for some partition λ (see (4.1) and (10.1)), the irreducible components are parametrized by elements of a partially ordered set which depends only on λ, and not on the underlying ring. As the local ring varies, for a fixed element of this partially ordered set, the dimension of the corresponding representation is shown to be a polynomial in the order of its residue field whose coefficients do not depend on the ring (Theorem 9.3). These polynomials are computed explicitly (Theorem 9.12). The centralizer algebra of the Weil representation also has a combinatorial basis indexed by a partially ordered set which depends only on λ and not on the underlying ring. The projection operators onto the irreducible invariant subspaces, when expressed in terms of this basis, are also shown to have coefficients which are polynomials in the order of the residue field whose coefficients also do not depend on the ring (Theorem 9.17), and these polynomials are computed explicitly (Theorems 9.18 and 9.19). Thus the decomposition of the Weil representation into irreducible invariant subspaces is, despite its apparent complexity, combinatorial in nature.

The results in this paper could serve as a starting point from which more subtle constructions involving the Weil representation (such as Howe duality) which have worked so well in the case of classical groups over finite fields can be extended to groups of automorphisms of finitely generated torsion modules over a discrete valuation ring.

It is worth noting that every Heisenberg group that is finite modulo its center is isomorphic to one of the groups considered here (for the precise statement, see Prasad, Shapiro and Vemuri [Prasad et al. 2010], particularly Section 3 and Corollary 5.7). For example, the seemingly different Heisenberg groups used by Tanaka [1967a] to construct representations in the principal series and cuspidal series of finite SL_2 are isomorphic. The difference lies in the realization of the special linear group as a group of automorphisms. The decomposition of any Weil representation associated to a finite abelian group will therefore always be a refinement of one of the decompositions described in this paper.

In order to concentrate on the important ideas without being distracted by technicalities, the main body of this paper uses the setting of finite abelian groups. Section 10 explains how to carry over the results to finitely generated torsion modules over discrete valuation rings and even more generally, finite modules over Dedekind domains.

To obtain our results, we use the combinatorial theory of orbits in finite abelian groups developed in [Dutta and Prasad 2011] (the relevant part is recalled in Section 4A), well-known basic facts about Heisenberg groups and Weil representations which are recalled in Section 1C (of which simple proofs can be found in

[Prasad 2009]), and the standard combinatorial theory of partially ordered sets, as set out in Chapter 3 of the book [Stanley 1997].

1B. ***Structure of the paper.*** In Section 1C, we recall the definition of the Heisenberg group and its Schrödinger representation. Following Weil [1964], we deduce the existence of the Weil representation from the irreducibility and uniqueness of the Schrödinger representation. Section 1D contains a precise formulation of our main problem — the decomposition of the Weil representation associated to a finite abelian group into irreducible summands.

In Section 2, we use the primary decomposition of finite abelian groups to reduce the main problem to the case of primary finite abelian groups. In Section 3 we explain the relationship between the multiplicities of the summands in the decomposition of the Weil representation and the number of orbits for the action of the symplectic group on the quotient of the Heisenberg group by its center.

In Section 4A, we recall the combinatorial theory of orbits and characteristic subgroups in a finite abelian group developed in [Dutta and Prasad 2011]. An important order-reversing involution on the lattice of characteristic subgroups is introduced in Section 4B. The theory from that paper is extended to symplectic orbits on the quotient of the Heisenberg group modulo its center in Section 4C.

Section 5 contains the first major theorem of this article, namely that the decomposition of the Weil representation associated to a finite abelian group into simple representations is multiplicity-free (Theorem 5.5). This is achieved by computing the structure constants of its endomorphism algebra to show that this algebra is commutative (Lemma 5.3). It follows that the set of invariant subspaces of the Weil representation, partially ordered by inclusion, forms a Boolean lattice (Corollary 5.6).

The task of describing the irreducible components of the Weil representation is carried out using combinatorial analysis in Sections 6–9. In Section 6 two elementary types of invariant subspaces of the Weil representation are identified. The first type are the subspaces of $L^2(A)$ consisting of even and odd functions; the second type are associated to so-called small order ideals (these subspaces are far from being mutually disjoint and irreducible). In Section 7, we describe a tensor product decomposition of the invariant subspaces associated to small order ideals. In Section 8, we refine the invariant subspaces of Section 6 to construct a family of invariant subspaces, which as a poset under inclusion is described in terms of a combinatorially defined poset Q_λ. In Section 9 the irreducible subrepresentations of the Weil representations are extracted from the invariant subspaces of Section 8 (Theorem 9.3). The rest of Section 9 is devoted to the explicit computation of the dimensions of these subrepresentations as well as formulae for the orthogonal projections onto them in terms of a natural basis for the endomorphism algebra of the Weil representation.

Finally, in Section 10 we explain how to extend the ideas of this paper to analyze the Weil representation associated to any finite module over a Dedekind domain.

1C. *Basic definitions.* Let A be a finite abelian group of odd order. Let $\hat{A}$ denote the Pontryagin dual of A. This is the group of all homomorphisms $A \to U(1)$, where $U(1)$ denotes the group of unit complex numbers. Let $K = A \times \hat{A}$. For each $k = (x, \chi) \in K$, the unitary operator on $L^2(A)$ defined by

$$W_k f(u) = \chi(u - x/2) f(u - x) \quad \text{for all } f \in L^2(A),\ u \in A$$

is called a Weyl operator. These operators satisfy

$$W_k W_l = c(k, l) W_{k+l} \quad \text{for all } k, l \in K,$$

where, if $k = (x, \chi)$ and $l = (y, \lambda)$, then

$$c(k, l) = \chi(y/2)\lambda(x/2)^{-1}.$$

Observe that $c(k, l)$ is bimultiplicative; for example, $c(k, l + l') = c(k, l)c(k, l')$ for all $k, l, l' \in K$.

The subgroup

$$H = \{cW_k \mid c \in U(1), k \in K\}$$

of the group of unitary operators on $L^2(A)$ is called the Heisenberg group associated to A. This group is known to physicists as a generalized Pauli group or a Weyl–Heisenberg group. As defined here, it comes with a unitary representation on $L^2(A)$, called the Schrödinger representation. Mackey's generalization [1949, Theorem 1] of the Stone–von Neumann theorem applies:

Theorem 1.1. *The Schrödinger representation of H is irreducible. Let $U(\mathcal{H})$ be the group of unitary operators on a Hilbert space $\mathcal{H}$, and let $\rho : H \to U(\mathcal{H})$ be an irreducible unitary representation of H such that $\rho(cW_0) = c\,\mathrm{Id}_{\mathcal{H}}$ for every $c \in U(1)$. Then there exists, up to scaling, a unique isometry $W : L^2(A) \to \mathcal{H}$ such that*

$$W W_k = \rho(W_k) W \quad \textit{for all } k \in K.$$

If g is an automorphism of K such that

$$c(gk, gl) = c(k, l) \quad \text{for all } k, l \in K, \tag{1.2}$$

then $\rho_g : H \to U(L^2(A))$ defined by

$$\rho_g(cW_k) = cW_{g(k)} \quad \text{for all } c \in U(1),\ k \in K$$

is an irreducible unitary representation of H on $L^2(A)$ such that $\rho(cW_0) = c\,\mathrm{Id}_{L^2(A)}$. By Theorem 1.1, there exists a unitary operator W_g on $L^2(A)$ such that

$$W_g W_k = W_{g(k)} W_g \quad \text{for all } k \in K.$$

Writing W_g^* for the adjoint of the unitary operator W_g, we have:

$$W_g W_k W_g^* = W_{g(k)} \quad \text{for all } k \in K. \tag{1.3}$$

If g_1 and g_2 are two such automorphisms, both $W_{g_1 g_2}$ and $W_{g_1} W_{g_2}$ intertwine the Schrödinger representation with $\rho_{g_1 g_2}$, and hence must differ by a unitary scalar:

$$W_{g_1} W_{g_2} = c(g_1, g_2) W_{g_1 g_2} \quad \text{for some } c(g_1, g_2) \in U(1).$$

Let $\mathrm{Sp}(K)$ be the group of all automorphisms g of K which satisfy (1.2). We have shown that $g \mapsto W_g$ is a projective representation of $\mathrm{Sp}(K)$ on $L^2(A)$. This representation is known as the Weil representation.

Remark 1.4. The operators W_g, for $g \in \mathrm{Sp}(K)$ can be normalized in such a way that $c(g_1, g_2) = 1$ for all g_1, g_2 (see Remark 6.6). Thus the Weil representation can be taken to be an ordinary representation of $\mathrm{Sp}(K)$.

Remark 1.5. The overlap of notation between the Weyl operators and the Weil representation is suggested by (1.3), which implies that they can be combined to construct a representation of $H \rtimes \mathrm{Sp}(K)$. The operators in this representation are precisely the unitary operators which normalize H. The resulting group is sometimes known as a Clifford group or a Jacobi group. It plays a prominent role in the stabilizer formalism for quantum error-correcting codes (see Chapter X of [Nielsen and Chuang 2000]).

1D. ***Formulation of the problem.*** We investigate the decomposition

$$L^2(A) = \bigoplus_{\pi \in \mathrm{Sp}(K)^\wedge} m_\pi \mathcal{H}_\pi \tag{1.6}$$

into irreducible representations. Here $\mathrm{Sp}(K)^\wedge$ denotes the set of equivalence classes of irreducible unitary representations of $\mathrm{Sp}(K)$ and, for each $\pi : \mathrm{Sp}(K) \to U(\mathcal{H}_\pi)$ in $\mathrm{Sp}(K)^\wedge$, m_π denotes the multiplicity of π in the Weil representation. Although the Weil representation is defined only up to multiplication by a scalar representation, the multiplicities and dimensions of the irreducible representations occurring in the decomposition are invariant under such twists (see Remark 2.2). As explained in Section 1A, the outcome of this paper is an understanding of this decomposition.

2. Product decompositions

We shall recall and apply a well-known observation on Weil representations associated to a product of abelian groups (see [Gérardin 1977, Corollary 2.5]).

2A. ***Projective equivalence.*** Since Weil representations are defined only up to scalar factors, we use a definition of equivalence of representations that is weaker than unitary equivalence:

Definition 2.1 (Projective equivalence). Let G be a group and $\rho_i : G \to U(\mathcal{H}_i)$ for $i = 1, 2$ be two unitary representations of G. We say that ρ_1 and ρ_2 are projectively equivalent if there exists a homomorphism $\chi : G \to U(1)$ such that ρ_2 is unitarily equivalent to $\rho_1 \otimes \chi$.

Remark 2.2. If, as a representation of G,

$$\mathcal{H}_i = \bigoplus_{\pi \in \hat{G}} m_\pi^{(i)} \mathcal{H}_\pi$$

is the decomposition of $\mathcal{H}_i$ into irreducibles for representations as in Definition 2.1, then $m^{(2)}_{\pi\otimes\chi} = m^{(1)}_\pi$, so there is a bijection between the sets of irreducible representations of G that appear in $\mathcal{H}_1$ and $\mathcal{H}_2$ which preserves multiplicities and dimensions.

2B. ***Tensor product decomposition.*** If A admits a product decomposition $A = A' \times A''$, then

$$L^2(A) = L^2(A') \otimes L^2(A''). \tag{2.3}$$

Let $K' = A' \times (A')^\wedge$ and $K'' = A'' \times (A'')^\wedge$. Thus $K = K' \times K''$. Let S' and S'' be subgroups of $\mathrm{Sp}(K')$ and $\mathrm{Sp}(K'')$ respectively. Then $S = S' \times S''$ is a subgroup of $\mathrm{Sp}(K)$.

Theorem 2.4. *The Weil representation of S on $L^2(A)$ is projectively equivalent to the tensor product of the Weil representation of S' on $L^2(A')$ and the Weil representation of S'' on $L^2(A'')$.*

Proof. By (1.3), the Weil representations of S' and S'' satisfy

$$W_{g'} W_{k'} W_{g'}^* = W_{g'(k')} \quad \text{and} \quad W_{g''} W_{k''} W_{g''}^* = W_{g''(k'')}$$

for all $g' \in S'$, $g'' \in S''$, $k' \in K'$ and $k'' \in K''$, whence

$$(W_{g'} \otimes W_{g''})(W_{k'} \otimes W_{k''})(W_{g'} \otimes W_{g''})^* = W_{g'(k')} \otimes W_{g''(k'')}.$$

Since $W_{k'} \otimes W_{k''}$ coincides with $W_{(k',k'')}$ under the isomorphism (2.3), $W_{g'} \otimes W_{g''}$ satisfies the defining identity (1.3) for the Weil representation of S on $L^2(A)$. □

2C. ***Primary decomposition.*** A finite abelian group has a primary decomposition

$$A = \prod_{p \text{ prime}} A_p,$$

where A_p is the subgroup of elements of A annihilated by some power of p. Writing K_p for $A_p \times (A_p)^\wedge$,

$$K = \prod_p K_p \quad \text{and} \quad \mathrm{Sp}(K) = \prod_p \mathrm{Sp}(K_p).$$

Theorem 2.4, when applied to the primary decomposition, gives:

Corollary 2.5. *The Weil representation of* $\mathrm{Sp}(K)$ *on* $L^2(A)$ *is projectively equivalent to the tensor product over those primes* p *for which* $A_p \neq 0$ *of the Weil representations of* $\mathrm{Sp}(K_p)$ *on* $L^2(A_p)$.

In view of Corollary 2.5, it suffices to consider the case where A is a finite abelian p-group for some odd prime p.

3. Multiplicities and orbits

We now recall the relation between the decomposition of the Weil representation and orbits in K [Prasad 2009].

3A. *An orthonormal basis.*

Lemma 3.1. *The set* $\{W_k \mid k \in K\}$ *of Weyl operators is an orthonormal basis of* $\mathrm{End}_{\mathbb{C}}\, L^2(A)$.

Proof. For each $k \in K$ and $T \in \mathrm{End}_{\mathbb{C}}\, L^2(A)$, let

$$\tau(k)T = W_k T W_k^*.$$

Then $k \mapsto \tau(k)$ is a unitary representation of K on $\mathrm{End}_{\mathbb{C}}\, L^2(A)$. If $k = (x, \chi)$ and $l = (y, \lambda)$ are two elements of K, then

$$\begin{aligned}
\tau(k)W_l &= W_k W_l W_k^* \\
&= W_k W_l (W_l W_k)^* W_l \\
&= c(k,l) W_{k+l} c(l,k)^{-1} W_{l+k}^* W_l \\
&= \chi(y)\lambda(x)^{-1} W_l.
\end{aligned}$$

Thus the W_l are eigenvectors for the action of K with distinct eigencharacters. Therefore they form an orthonormal set of operators. Since $|K| = |A|^2 = \dim \mathrm{End}_{\mathbb{C}}\, L^2(A)$, this orthonormal set is a basis. □

3B. *Endomorphisms.*

By Lemma 3.1, every $T \in \mathrm{End}_{\mathbb{C}}\, L^2(A)$ has a unique expansion

$$T = \sum_{k \in K} T_k W_k, \quad \text{with each } T_k \in \mathbb{C}. \tag{3.2}$$

Theorem 3.3. *For every subgroup* S *of* $\mathrm{Sp}(K)$,

$$\mathrm{End}_S\, L^2(A) = \{T \in \mathrm{End}_{\mathbb{C}}\, L^2(A) \mid T_k = T_{g(k)} \textit{ for all } g \in S,\ k \in K\}.$$

Proof. Note that $T \in \mathrm{End}_S\, L^2(A)$ if and only if $W_g T W_g^* = T$ for all $g \in S$. Expanding T as in (3.2) and using the defining identity (1.3) for W_g gives the theorem. □

Now suppose that as a representation of S, $L^2(A)$ has the decomposition

$$L^2(A) = \bigoplus_{\pi \in \hat{S}} m_{\pi,S} \mathcal{H}_\pi .$$

Then, together with Schur's lemma, Theorem 3.3 implies:

Corollary 3.4. *If $S \backslash K$ denotes the set of S-orbits in K,*

$$\sum_{\pi \in \hat{S}} m_{\pi,S}^2 = |S \backslash K|.$$

4. Orbits and characteristic subgroups

We first recall the theory of orbits (under the full automorphism group) and characteristic subgroups in a finite abelian group from [Dutta and Prasad 2011]. We then see how it applies to $\mathrm{Sp}(K)$-orbits in K.

4A. ***Orbits.*** Every finite abelian p-group is isomorphic to

$$A = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_l}\mathbb{Z} \tag{4.1}$$

for a unique sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of positive integers (in other words, a partition). Henceforth, we assume that A is of the above form. For each partition λ, let

$$P_\lambda = \big\{(v, k) \mid k \in \{\lambda_1, \ldots, \lambda_l\},\ 0 \leq v < k\big\}.$$

Say that $(v, k) \geq (v', k')$ if and only if $v' \geq v$ and $k' - v' \leq k - v$. This relation is a partial order on P_λ. For $x \in \mathbb{Z}/p^k\mathbb{Z}$, let

$$v(x) = \max\{0 \leq v \leq k \mid x \in p^v\mathbb{Z}/p^k\mathbb{Z}\}.$$

For $a = (a_1, \ldots, a_l) \in A$, let $I(a)$ be the order ideal in P_λ generated by $(v(a_i), \lambda_i)$ with $a_i \neq 0$ in $\mathbb{Z}/p^{\lambda_i}\mathbb{Z}$.

Example 4.2. When $\lambda = (5, 4, 4, 1)$ and $a = (p^4, p^2, p^3, 0)$, the Hasse diagram of P_λ is shown on the left hand side of Figure 1. The ideal $I(a)$ is represented by black dots on the right hand side of Figure 1. Note that the elements of P_λ are arranged in such a way that k is constant along verticals and decreases from left to right.

Theorem 4.3 [Dutta and Prasad 2011, Theorem 4.1]. *For $a, b \in A$, the element b is the image of a under an endomorphism of A if and only if $I(b) \subset I(a)$.*

Given $x = (v, k) \in P_\lambda$, let $e(x)$ denote the element in A all of whose entries are zero except for the leftmost entry with $\lambda_i = k$, which is p^v. For an order ideal I in P_λ, denote by $\max I$ the set of maximal elements in I, and let

$$a(I) = \sum_{x \in \max I} e(x).$$

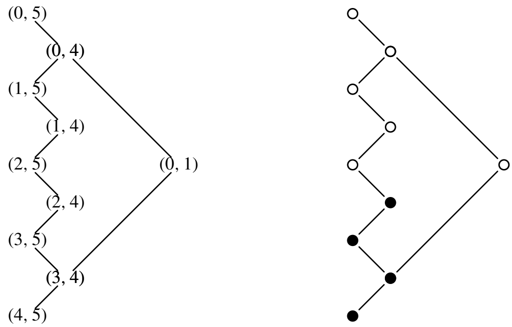


Figure 1. Left: the poset $P_{(5,4,4,1)}$. Right: the order ideal $I(p^4, p^2, p^3, 0)$.

Let G denote the group of all automorphisms of A.

Theorem 4.4 [Dutta and Prasad 2011, Theorem 5.4]. *The map $I \mapsto a(I)$ gives rise to a bijection from the set of order ideals in P_λ to the set of G-orbits in A.*

The elements $a(I)$, as I varies over the order ideals in P_λ, can be taken as representatives of the orbits. The inverse of the function of Theorem 4.4 is given by $a \mapsto I(a)$.

4B. ***Characteristic subgroups.*** For an order ideal $I \subset P_\lambda$,

$$A_I = \{a \in A \mid I(a) \subset I\}$$

is a characteristic subgroup of A of order $p^{[I]}$, where $[I]$ denotes the number of elements in I, counted with multiplicity (the multiplicity of (v, k) is the number of times that k occurs in the partition λ; see [ibid., Theorem 7.3]). Every characteristic subgroup of A is of the form A_I for some order ideal $I \subset P_\lambda$. In fact, $I \mapsto A_I$ is an isomorphism of the lattice of order ideals in P_λ onto the lattice of characteristic subgroups of A. Thus, the lattice of characteristic subgroups of A is a finite distributive lattice [Stanley 1997, Section 3.4]. If B is any group isomorphic to A, and $\phi : A \to B$ is an isomorphism, then since A_I is characteristic, the image $B_I = \phi(A_I)$ does not depend on the choice of ϕ. Consequently, it makes sense to talk of the subgroup $\hat{A}_I$ of $\hat{A}$, which is the image of A_I under any isomorphism $A \to \hat{A}$.

For each order ideal $I \subset P_\lambda$, its annihilator

$$A_I^\perp := \{\chi \in \hat{A} \mid \chi(a) = 1 \text{ for all } a \in A_I\}$$

is a characteristic subgroup of $\hat{A}$. Therefore, there exists an order ideal $I^\perp \subset P_\lambda$ such that $A_I^\perp = \hat{A}_{I^\perp}$. Clearly, $I \mapsto I^\perp$ is an order-reversing involution of the set of order ideals in P_λ. The Hasse diagram of P_λ has a horizontal axis of symmetry. $I^\perp$ can be visualized as the complement of the reflection of I about this axis of I (see Figure 2).

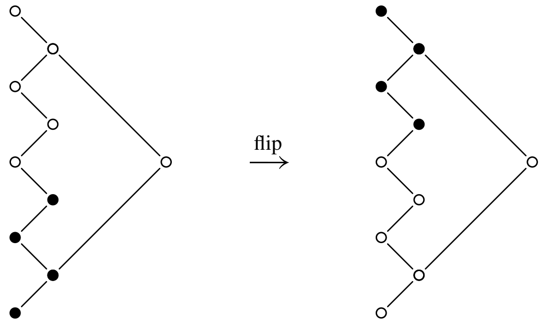


Figure 2. The involution on order ideals. Left: I (black dots). Right: $I^{\perp}$ (white dots).

4C. *Symplectic orbits.*

Theorem 4.5. *The map $I \mapsto (a(I), 0)$ (here 0 denotes the identity element of $\hat{A}$) gives rise to a bijection from the set of order ideals in P_λ to the set of $\mathrm{Sp}(K)$-orbits in K.*

Proof. We first show that each $\mathrm{Sp}(K)$-orbit in K intersects $A \times \{0\}$. Let $e_1, \dots, e_l$ denote the generators of A, so e_i is the element whose i-th coordinate is 1 and all other coordinates are 0. Each element $a \in A$ has an expansion

$$a = a_1 e_1 + \dots + a_l e_l \quad \text{with } 0 \le a_i < p^{\lambda_i} \text{ for each } i \in \{1, \dots, l\}. \tag{4.6}$$

Let ϵ_j denote the unique element of $\hat{A}$ for which

$$\epsilon_j(e_k) = e^{2\pi i \delta_{jk} p^{-\lambda_j}}.$$

Then each element $\alpha \in \hat{A}$ has an expansion

$$\alpha = \alpha_1 \epsilon_1 + \dots + \alpha_l \epsilon_l \quad \text{with } 0 \le \alpha_i < p^{\lambda_i} \text{ for each } i \in \{1, \dots, l\}. \tag{4.7}$$

Let $k = (a, \alpha) \in K$, with a and α as in (4.6) and (4.7), respectively. The automorphism of K which takes $e_i \mapsto \epsilon_i$ and $\epsilon_i \mapsto -e_i$ while preserving all other generators e_j and ϵ_j with $j \neq i$, lies in $\mathrm{Sp}(K)$. In terms of coordinates, it has the effect of interchanging a_i and α_i up to sign. Using this automorphism, we may arrange that $v(a_i) \le v(\alpha_i)$ for each i. Therefore, there exists an integer b_i such that

$$b_i a_i \equiv \alpha_i \mod p^{\lambda_i}.$$

Let $B_i : A \to \hat{A}$ be the homomorphism which takes e_i to $b_i \epsilon_i$ and all other generators e_j with $j \neq i$ to 0. Then the automorphism of K which takes (a, α) to $(a, \alpha - B_i(a))$ also lies in $\mathrm{Sp}(K)$. This has the effect of changing α_i to 0. Repeating this process for each i allows us to reduce (a, α) to $(a, 0)$ as claimed.

Now, for every automorphism g of A, the automorphism $(a,\alpha)\mapsto(g(a),\hat{g}^{-1}(\alpha))$ lies in $\mathrm{Sp}(K)$ (here $\hat{g}$ is the automorphism of $\hat{A}$ defined by $\hat{g}(\chi)(a)=\chi(g(a))$ for $a\in A$ and $\chi\in\hat{A}$). Such automorphisms can be used to reduce $(a,0)$ further to an element of the form $(a(I),0)$ for some order ideal $I\subset P_\lambda$. Since, for distinct I, these elements are in distinct $\mathrm{Aut}(K)$-orbits, they must also be in distinct $\mathrm{Sp}(K)$-orbits. $\square$

5. Multiplicity one

5A. ***Relation to commutativity.*** Suppose that the decomposition of the Weil representation into irreducible representations is given by

$$L^2(A)=\bigoplus_{\pi\in\mathrm{Sp}(K)^\wedge} m_\pi\mathscr{H}_\pi. \tag{5.1}$$

Then by Schur's lemma, the endomorphism algebra of $L^2(A)$ is a direct sum of matrix algebras:

$$\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)=\bigoplus_{\pi\in\mathrm{Sp}(K)^\wedge} M_{m_\pi\times m_\pi}(\mathbb{C}).$$

It follows that $m_\pi\le 1$ for every $\pi\in\mathrm{Sp}(K)^\wedge$ if and only if the ring $\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)$ of endomorphisms of the Weil representations is commutative. For each order ideal $I\subset P_\lambda$, let O_I denote the $\mathrm{Sp}(K)$-orbit of $(a(I),0)$ in K and let

$$T_I=\sum_{k\in O_I}W_k.$$

By Theorems 3.3 and 4.5, the set of all T_I as I varies over the order ideals $I\subset P_\lambda$ is a basis of $\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)$.

Let $K_I=A_I\times\hat{A}_I$ ($\hat{A}_I$ as in Section 4B) and define

$$\Delta_I=\sum_{k\in K_I}W_k. \tag{5.2}$$

Since $K_I=\bigsqcup_{J\subset I}O_J$,

$$\Delta_I=\sum_{J\subset I}T_J.$$

Thus, the elements Δ_I are obtained from the basis elements T_I of $\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)$ by a unipotent upper-triangular transformation. Hence,

$$\{\Delta_I\mid I\in J(P_\lambda)\}$$

is also a basis of $\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)$. Thus if Δ_I commutes with Δ_J for all $I,J\in J(P_\lambda)$, then $\mathrm{End}_{\mathrm{Sp}(K)}L^2(A)$ is a commutative algebra.

Therefore, in order to show that $m_\pi \le 1$ for each $\pi \in \mathrm{Sp}(K)^\wedge$, it suffices to show that for any two order ideals $I, J \subset P_\lambda$, Δ_I and Δ_J commute. This will follow from the calculation in Section 5B.

5B. *Calculation of the product.*

Lemma 5.3. *For any two order ideals $I, J \subset P_\lambda$,*

$$\Delta_I \Delta_J = |K_{I\cap J}| \Delta_{(I\cap J)^\perp \cap (I\cup J)}.$$

Proof. The coefficient of W_k in $\Delta_I \Delta_J$ is

$$\sum_{\substack{x\in K_I, y\in K_J\\ x+y=k}} c(x, y). \tag{5.4}$$

From the definition of $I(a)$, it is easy to see that $I(a+b) \subset I(a) \cup I(b)$. Therefore, $A_I + A_J \subset A_{I\cup J}$ and hence $K_I + K_J \subset K_{I\cup J}$. It follows that the sum (5.4) is 0 unless $k \in K_{I\cup J}$. Suppose $x_0 \in K_I$ and $y_0 \in K_J$ are such that $x_0 + y_0 = k$. Then the sum (5.4) becomes

$$\begin{aligned}\sum_{l\in K_I\cap K_J} c(x_0+l, y_0-l) &= c(x_0, y_0) \sum_{l\in K_I\cap K_J} c(l, y_0)c(l, x_0)\\ &= c(x_0, y_0) \sum_{l\in K_I\cap K_J} c(l, k)\\ &= \begin{cases} c(x_0, y_0)|K_I\cap K_J| & \text{if } k \in (K_I\cap K_J)^\perp,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Observe that $K_I \cap K_J = K_{I\cap J}$. It remains to show that, for every $k \in K_{I\cup J}$, there exist $x_0 \in K_I$ and $y_0 \in K_J$ such that $k = x_0 + y_0$ and $c(x_0, y_0) = 1$. Since $\Delta_I \Delta_J$ is constant on $\mathrm{Sp}(K)$-orbits in K, we may use Theorem 4.5 to assume that $k = (a(I'), 0)$ for some order ideal $I' \subset I \cup J$. We have $\max I' \subset I \cup J$. Let I'_1 be the order ideal generated by $(\max I') \cap I$, and I'_2 be the order ideal generated by $(\max I') - I$. Then $a(I') = a(I'_1) + a(I'_2)$. Clearly $(a(I'_1), 0)$ and $(a(I'_2), 0)$ have the properties required of x_0 and y_0. ∎

5C. *Multiplicity one.* We have proved:

Theorem 5.5. *In the decomposition (5.1) of the Weil representation of* $\mathrm{Sp}(K)$, $m_\pi \le 1$ *for every isomorphism class* π *of irreducible representations of* $\mathrm{Sp}(K)$.

Every $\mathrm{Sp}(K)$-invariant subspace is completely determined by the subset of $\mathrm{Sp}(K)^\wedge$ consisting of representations that occur in it. Therefore:

Corollary 5.6. *The set of* $\mathrm{Sp}(K)$*-invariant subspaces of* $L^2(A)$, *partially ordered by inclusion, forms a finite Boolean lattice.*

6. Elementary invariant subspaces

In this section, we construct some elementary invariant subspaces for the Weil representation of $\mathrm{Sp}(K)$ on $L^2(A)$. In Section 8, we will use these subspaces and the results of Section 7 to construct enough invariant subspaces to carve out all the irreducible subspaces.

6A. *Small order ideals.*

Definition 6.1 (small order ideal). An order ideal $I \subset P_\lambda$ is said to be small if $I \subset I^\perp$, with $I^\perp$ as in Section 4B.

For example, the order ideal I in Figure 2 is small.

6B. *Interpreting some Δ_I.*

Lemma 6.2. *For each order ideal $I \subset P_\lambda$, let Δ_I be as in (5.2).*

(6.2.1) $\Delta_{P_\lambda} f(u) = |A| f(-u)$ *for all* $f \in L^2(A)$ *and* $u \in A$.

(6.2.2) *For every small order ideal $I \subset P_\lambda$, $|A_I|^{-2}\Delta_I$ is the orthogonal projection onto the subspace of $L^2(A)$ consisting of functions supported on $A_{I^\perp}$ and invariant under translations in A_I.*

Proof. For any order ideal $I \subset P_\lambda$, we have

$$\Delta_I f(u) = \sum_{x\in A_I}\sum_{\chi\in\hat{A}_I} \chi(u - x/2) f(u-x).$$

The inner sum is $f(u-x)$ times the sum of values of a character of $\hat{A}_I$, which vanishes if this character is nontrivial, namely if $u - x/2 \notin A_{I^\perp}$, and is $|A_I|$ otherwise. Therefore,

$$\Delta_I f(u) = |A_I| \sum_{x\in A_I\cap(2u+A_{I^\perp})} f(u-x) = |A_I| \sum_{x\in(u+A_I)\cap(-u+A_{I^\perp})} f(x). \tag{6.3}$$

Taking $I = P_\lambda$ in (6.3) gives (6.2.1).

Now suppose that $I \subset I^\perp$. If $u \notin A_{I^\perp}$ then $(u + A_I)\cap(-u + A_{I^\perp}) = \varnothing$, so that $\Delta_I f(u) = 0$. If $u \in A_{I^\perp}$, then the sum (6.3) is over $u + A_I$, so $|A_I|^{-2}\Delta_I$ is the averaging over A_I-cosets, from which (6.2.2) follows. □

6C. *Even and odd functions.*

Theorem 6.4. *The subspaces of $L^2(A)$ consisting of even and odd functions are invariant under* $\mathrm{Sp}(K)$.

Proof. By (6.2.1),

$$\left[\tfrac{1}{2}(\mathrm{Id}_{L^2(A)} \pm |A|^{-1}\Delta_{P_\lambda})\right] f(u) = \tfrac{1}{2}(f(u) \pm f(-u)). \tag{6.5}$$

These operators are the orthogonal projections onto the subspaces of even and odd functions in $L^2(A)$. Since these operators commute with $\mathrm{Sp}(K)$ (by Theorem 3.3), their images are $\mathrm{Sp}(K)$-invariant subspaces of $L^2(A)$. $\square$

Remark 6.6 (the Weil representation is an ordinary representation). The subspaces of even and odd functions on A have dimensions $(|A|+1)/2$ and $(|A|-1)/2$, respectively. For each $g \in \mathrm{Sp}(K)$, let W_g^+ and W_g^- denote the restrictions of W_g to these spaces. Taking the determinants of the identities

$$W_{g_1}^{\pm} W_{g_2}^{\pm} = c(g_1, g_2) W_{g_1 g_2}^{\pm}$$

gives the identities

$$\det W_{g_1}^{+} \det W_{g_2}^{+} = c(g_1, g_2)^{(|A|+1)/2} \det W_{g_1 g_2}^{+},$$
$$\det W_{g_1}^{-} \det W_{g_2}^{-} = c(g_1, g_2)^{(|A|-1)/2} \det W_{g_1 g_2}^{-}.$$

Dividing the first equation by the second and rearranging gives

$$c(g_1, g_2) = \frac{\alpha(g_1)\alpha(g_2)}{\alpha(g_1 g_2)}$$

for all $g_1, g_2 \in \mathrm{Sp}(K)$, where $\alpha : G \to U(1)$ is defined by

$$\alpha(g) = \frac{\det(W_g^+)}{\det(W_g^-)} \quad \text{for all } g \in \mathrm{Sp}(K).$$

Therefore, if each W_g is replaced by $\alpha(g)^{-1} W_g$, then $g \mapsto W_g$ is a representation of $\mathrm{Sp}(K)$ on $L^2(A)$. This argument seems to be well known. It has appeared before in [Adler and Ramanan 1996, Appendix I] and again in [Cliff et al. 2000].

6D. *Invariant spaces corresponding to small order ideals.* Since Δ_I commutes with $\mathrm{Sp}(K)$, its image is an $\mathrm{Sp}(K)$-invariant subspace of $L^2(A)$. An immediate consequence of (6.2.2) is the following theorem:

Theorem 6.7. *For each small order ideal $I \subset P_\lambda$, the subspace of $L^2(A)$ consisting of functions supported on $A_{I^\perp}$ which are invariant under translations in A_I is an $\mathrm{Sp}(K)$-invariant subspace of $L^2(A)$.*

Remark 6.8 (alternative description). For $f \in L^2(A)$, recall that its Fourier transform is the function on $\hat{A}$ defined by

$$\hat{f}(\chi) = \sum_{a \in A} f(a)\overline{\chi(a)} \quad \text{for each } \chi \in \hat{A}.$$

For any subgroup B of A, the Fourier transforms of functions invariant under translations in B are the functions supported on the annihilator subgroup $B^\perp$ of A (consisting of characters which vanish on B), and Fourier transforms of functions supported on B are the functions which are invariant under $B^\perp$. Therefore, the functions supported on $A_{I^\perp}$ which are invariant under A_I are precisely the functions supported on $A_{I^\perp}$ whose Fourier transforms are supported on $\hat{A}_{I^\perp}$. They are also the functions invariant under translations in A_I whose Fourier transforms are invariant under translations in $\hat{A}_I$.

Identify $L^2(A_{I^\perp}/A_I)$ with the space of functions in $L^2(A)$ which are supported on $A_{I^\perp}$ and invariant under translations in A_I. Let $K(I) = A_{I^\perp}/A_I \times (A_{I^\perp}/A_I)^\wedge$. $K(I)$ can be identified with $({A_I}^\perp \times \hat{A}_{I^\perp})/(A_I \times \hat{A}_I)$. Thus $K(I)$ is a quotient of one characteristic subgroup of K by another. Therefore the action $\mathrm{Sp}(K)$ on K descends to an action on $K(I)$, giving rise to a homomorphism $\mathrm{Sp}(K) \to \mathrm{Sp}(K(I))$. The defining condition (1.3) for the Weil representation ensures:

Theorem 6.9. *For every small order ideal $I \subset P_\lambda$, the Weil representation of* $\mathrm{Sp}(K)$ *on $L^2(A_{I^\perp}/A_I)$ is projectively equivalent to the representation obtained by composing the Weil representation of* $\mathrm{Sp}(K(I))$ *on $L^2(A_{I^\perp}/A_I)$ with the homomorphism* $\mathrm{Sp}(K) \to \mathrm{Sp}(K(I))$.

7. Component decomposition

7A. ***Connected components of a partially ordered set.*** A partially ordered set is said to be connected if its Hasse diagram is a connected graph. A connected component of a partially ordered set is a maximal connected induced subposet. Every partially ordered set can be written as the disjoint union of its connected components in the sense of [Stanley 1997, Section 3.2]. Denote the set of connected components of a poset P by $\pi_0(P)$.

7B. ***Connected components of $J - I$.*** Suppose that $I \subset J$ are two order ideals in P_λ. Each connected component $C \in \pi_0(J - I)$ determines a segment (namely, a contiguous set of integers) S_C in $\{1, \dots, l\}$:

$$S_C = \{1 \le k \le l \mid (v, k) \in C \text{ for some } v\}.$$

The S_C are pairwise disjoint, but their union may be strictly smaller than $\{1, \dots, l\}$. Write S_0 for the complement of $\bigsqcup_{C \in \pi_0(I^\perp - I)} S_C$ in $\{1, \dots, l\}$. It will be convenient to write

$$\tilde{\pi}_0(I^\perp - I) = \pi_0(I^\perp - I) \sqcup \{0\}.$$

Define partitions $\lambda(C) = (\lambda_k \mid k \in S_C)$ for each $C \in \tilde{\pi}_0(I^\perp - I)$. Then $P_{\lambda(C)}$ is the induced subposet of P_λ consisting of those pairs $(v, k) \in P_\lambda$ for which $k \in S_C$. Let $I(C)$ and $J(C)$ be the ideals in $P_{\lambda(C)}$ obtained by intersecting I and J, respectively, with $P_{\lambda(C)}$.

For example, if $\lambda = (5, 4, 4, 1)$ and I is the order ideal in the diagram on the left in Figure 2 and $J = I^\perp$, then $I^\perp - I$ is depicted in the diagram on the left in

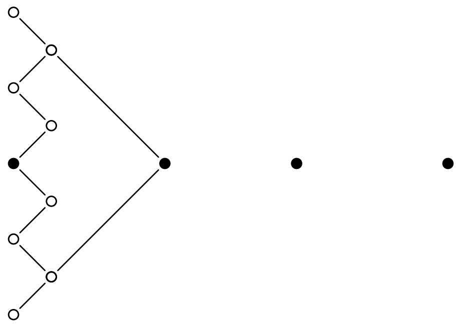


Figure 3. Left: $I^\perp - I$ inside $P_{5,4,4,1}$. Right: $I^\perp - I$ by itself.

Figure 3. As the diagram on the right shows, the induced subposet $I^\perp - I$ has two connected components, C_1 and C_2, with $\lambda(C_1) = (5)$ and $\lambda(C_2) = (1)$. Moreover, $\lambda(0) = (4, 4)$.

Lemma 7.1. *Let $I \subset J$ be two order ideals in P_λ. For each $C \in \pi_0(J - I)$ let $L(C)$ be an order ideal in C. Let*

$$L = I \sqcup \bigsqcup_{C \in \pi_0(J-I)} L(C).$$

Then L is an order ideal in P_λ.

Proof. Since $\bigsqcup_{C \in \pi_0(J-I)} L(C)$ is an order ideal in $J - I$, its union with I is an order ideal in P_λ. □

Corollary 7.2. *If $I \subset J$ are two order ideals in P_λ and C and D are distinct components of $J - I$, then the intersection with $P_{\lambda(C)}$ of the order ideal in P_λ generated by $J(D)$ is contained in $I(C)$.*

Proof. By Lemma 7.1, $J(D) \cup I$ is an order ideal in P_λ. Therefore, it contains the order ideal in P_λ generated by $J(D)$. If C and D are distinct connected components of $J - I$, then $(J(D) \cup I) \cap P_\lambda(C) = I(C)$. Therefore the intersection with $P_{\lambda(C)}$ of the order ideal in P_λ generated by $J(D)$ is contained in $I(C)$. □

7C. *Decomposition of endomorphisms.* Suppose that A has the form (4.1). Then define A_C to be the subgroup

$$A_C = \{(a_1, \dots, a_l) \mid a_k = 0 \text{ if } k \notin S_C\}.$$

Thus A_C is a finite abelian p-group of type $\lambda(C)$. We have a decomposition

$$A = \prod_{C \in \tilde{\pi}_0(J-I)} A_C. \tag{7.3}$$

Denote the characteristic subgroups of A_C corresponding to $I(C)$ and $J(C)$

(which are order ideals in $P_{\lambda(C)}$) by $A_{I,C}$ and $A_{J,C}$ respectively. The decomposition (7.3) induces a decomposition

$$A_J/A_I = \prod_{C\in\pi_0(J-I)} A_{J,C}/A_{I,C}. \tag{7.4}$$

There is no contribution from A_0 since $A_{I,0} = A_{J,0}$.

With respect to the decomposition (7.3), every endomorphism of A can be written as a square matrix $\{\phi_{CD}\}$, where $\phi_{CD} : A_D \to A_C$ is a homomorphism.

Lemma 7.5. *Let $I \subset J$ be order ideals in P_λ. Then every endomorphism ϕ of A induces an endomorphism*

$$\bar{\phi} : A_J/A_I \to A_J/A_I$$

such that $\bar{\phi}(A_{J,C}/A_{I,C}) \subset A_{J,C}/A_{I,C}$ for each $C \in \pi_0(J-I)$, and

$$\bar{\phi} = \bigoplus_{C\in\pi_0(J-I)} \overline{\phi_{CC}},$$

where $\overline{\phi_{CC}}$ is the endomorphism of $A_{J,C}/A_{I,C}$ induced by ϕ_{CC}.

Proof. By Theorem 4.3 and Corollary 7.2, if $C \neq D$ then $\phi_{CD}(A_{J,D}) \subset A_{I,C}$. Therefore, $\bar{\phi}$ remains unchanged if ϕ_{CD} is replaced by 0 for all $C \neq D$. This amounts to replacing ϕ by $\oplus_C \phi_{CC}$, and the lemma follows. □

7D. ***Tensor product decomposition of invariant subspaces.*** Let I be a small order ideal. We shall use the notation of Section 7C with $J = I^\perp$. For each $C \in \pi_0(I^\perp - I)$ let $K_C = A_C \times \hat{A}_C$, and let $\mathrm{Sp}(K_C)$ be the corresponding symplectic group. Just as (by Theorem 6.7) $L^2(A_{I^\perp}/A_I)$ is an invariant subspace for the Weil representation of $\mathrm{Sp}(K)$ on $L^2(A)$, $L^2(A_{I^\perp,C}/A_{I,C})$ is an invariant subspace for the Weil representation of $\mathrm{Sp}(K_C)$ on $L^2(A_C)$.

Now, if $g \in \mathrm{Sp}(K)$, we may write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

with respect to the decomposition $K = A \times \hat{A}$. For convenience, we identify $\hat{A}$ with A using $e_i \mapsto \epsilon_i$ for $i = 1, \dots, l$, where e_i and ϵ_i are as in Section 4C. Hence, we may think of each g_{ij} as an endomorphism of A. By Lemma 7.5, the resulting endomorphism $\bar{g}_{ij}$ of $A_{I^\perp}/A_I$ preserves $A_{I^\perp,C}/A_{I,C}$ for each C. It follows that the image of $\mathrm{Sp}(K)$ in $\mathrm{Sp}(K(I))$ (see Section 6D) is the product of the images of the $\mathrm{Sp}(K_C)$ in the $\mathrm{Sp}(K_C(I \cap C))$ as C ranges over $\pi_0(I^\perp - I)$.

Thus, by Theorems 2.4 and 6.9, we have:

Corollary 7.6. *The Weil representation of* $\mathrm{Sp}(K)$ *on* $L^2(A_{I^\perp}/A_I)$ *is projectively equivalent to the tensor product of the Weil representations of the* $\mathrm{Sp}(K_C)$ *on the* $L^2(A_{I^\perp,C}/A_{I,C})$ *as* C *ranges over* $\pi_0(I^\perp - I)$.

8. Poset of invariant subspaces

8A. ***The invariant subspaces.*** Let

$Q_\lambda = \{(I, \phi) \mid I \subset P_\lambda$ a small order ideal, $\phi : \pi_0(I^\perp - I) \to \mathbb{Z}/2\mathbb{Z}$ any function$\}$.

For each $(I, \phi) \in Q_\lambda$, use the decomposition of Corollary 7.6 to define $L^2(A)_{I,\phi}$ as the subspace of $L^2(A_{I^\perp}/A_I)$ given by

$$L^2(A)_{I,\phi} = \bigotimes_{C \in \pi_0(I^\perp - I)} L^2(A_{I^\perp,C}/A_{I,C})_{\phi(C)},$$

where $L^2(A_{I^\perp,C}/A_{I,C})_{\phi(C)}$ denotes the space of even or odd functions on the quotient $A_{I^\perp,C}/A_{I,C}$ when $\phi(C)$ is 0 or 1, respectively. In other words, $L^2(A)_{I,\phi}$ consists of functions on $A_{I^\perp}/A_I$ which, under the decomposition

$$A_{I^\perp}/A_I = \prod_{C \in \pi_0(I^\perp - I)} A_{I^\perp,C}/A_{I,C}, \tag{8.1}$$

are even in the components where $\phi(C) = 0$ and odd in the components where $\phi(C) = 1$. By Theorems 6.4 and 6.7, and by Corollary 7.6, $L^2(A)_{I,\phi}$ is an $\mathrm{Sp}(K)$-invariant subspace of $L^2(A)$ for each $(I, \phi) \in Q_\lambda$.

8B. ***The partial order.*** Clearly:

Lemma 8.2. *For* (I, ϕ) *and* (I', ϕ') *in* Q_λ, $L^2(A)_{I',\phi'} \subset L^2(A)_{I,\phi}$ *if and only if the following conditions are satisfied*:

(8.2.1) $I \subset I'$.

(8.2.2) *For each* $P \in \pi_0(I^\perp - I)$,

$$\phi(P) = \sum_{\substack{P' \in \pi_0(I'^\perp - I') \\ P' \subset P}} \phi'(P').$$

Thus, the conditions (8.2.1) and (8.2.2) define a partial order on Q_λ (which is obviously independent of p). Recall that the multiplicity $m(x)$ of an element $x = (v, k) \in P_\lambda$ is the number of times k occurs in the partition λ. For any subset $S \subset P_\lambda$, let $[S]$ denote the number of elements of S, counted with multiplicity:

$$[S] = \sum_{x \in S} m(x).$$

Lemma 8.3. *For each* $(I, \phi) \in Q_\lambda$,

$$\dim L^2(A)_{I,\phi} = \prod_{C \in \pi_0(I^\perp - I)} \frac{p^{[C]} + (-1)^{\phi(C)}}{2}.$$

9. Irreducible subspaces

9A. ***A bijection between $J(P_\lambda)$ and Q_λ.*** Let $J(P_\lambda)$ denote the lattice of order ideals in P_λ.

Lemma 9.1. *For each partition* λ, $|J(P_\lambda)| = |Q_\lambda|$.

Proof. We construct an explicit bijection $Q_\lambda \to J(P_\lambda)$. To $(I, \phi) \in Q_\lambda$, associate the ideal (see Lemma 7.1)

$$\Theta(I, \phi) = I \cup \bigsqcup_{C \in \pi_0(I^\perp - I)} I_{\phi(C)},$$

where

$$I_{\phi(C)} = \begin{cases} I \cap C & \text{if } \phi(C) = 0, \\ I^\perp \cap C & \text{if } \phi(C) = 1. \end{cases}$$

In the other direction, given an ideal $J \subset P_\lambda$, $I = J \cap J^\perp$ is a small order ideal. We have $I^\perp = J \cup J^\perp$. For each $C \in \pi_0(I^\perp - I)$, define

$$\phi_J(C) = \begin{cases} 0 & \text{if } I \cap C = J \cap C, \\ 1 & \text{if } I \cap C = J^\perp \cap C. \end{cases}$$

Define $\Psi : Q_\lambda \to J(P_\lambda)$ by $\Psi(J) = (J \cap J^\perp, \phi_J)$. It is easy to verify that Φ and Ψ are mutual inverses. $\square$

9B. ***Existence lemma.***

Lemma 9.2. *For every* $(I, \phi) \in Q_\lambda$, *there exists* $f \in L^2(A)_{I,\phi}$ *such that* $f \notin L^2(A)_{I',\phi'}$ *for any* $(I', \phi') < (I, \phi)$.

Proof. Take as f the unique element in $L^2(A)_{I,\phi}$ whose value at $a(I^\perp) + A_I$ (using the notation of Section 4A) is 1, and which vanishes on all elements of $A_{I^\perp}/A_I$ not obtained from $a(I^\perp) + A_I$ by changing the signs of some of its components under the decomposition (8.1). $\square$

9C. ***The irreducible invariant subspaces.*** The two lemmas above are enough to give us the main theorem:

Theorem 9.3. *For each* $(I, \phi) \in Q_\lambda$, *there is a unique irreducible subspace for the Weil representation of* $\mathrm{Sp}(K)$ *on* $L^2(A)$ *which is contained in* $L^2(A)_{I,\phi}$ *but not* $L^2(A)_{I',\phi'}$ *for any* $(I', \phi') < (I, \phi)$. *As p varies, the dimension of this representation is a polynomial in p of degree* $[I^\perp - I]$ *with leading coefficient* $2^{-|\pi_0(I^\perp - I)|}$ *and all coefficients in* $\mathbb{Z}_{(2)}$.

Proof. By Corollary 5.6, the $\mathrm{Sp}(K)$-invariant subspaces of $L^2(A)$ form a Boolean lattice Λ. Let R denote the set of minimal nontrivial $\mathrm{Sp}(K)$-invariant subspaces of $L^2(A)$. These are the atoms of Λ. By Corollary 3.4 and Theorem 4.5, the cardinality of R is the same as that of $J(P_\lambda)$. Each invariant subspace is determined by the atoms which are contained in it. The map $(I, \phi) \mapsto L^2(A)_{I,\phi}$ is an order-preserving map $Q_\lambda \to \Lambda$. Let $R_{I,\phi}$ be the set of atoms which occur in $L^2(A)_{I,\phi}$ but not in $L^2(A)_{I',\phi'}$ for any $(I', \phi') < (I, \phi)$. The subsets $R_{I,\phi}$ are $|Q_\lambda|$ pairwise disjoint subsets of R, and by Lemma 9.2, each of them is nonempty. Therefore, by Lemma 9.1, each of them must be a singleton, and these subspaces exhaust R. It follows that there is a unique irreducible representation of $\mathrm{Sp}(K)$ that occurs in $L^2(A)_{I,\phi}$ but not in $L^2(A)_{I',\phi'}$ for any $(I', \phi') < (I, \phi)$. Let $V_{I,\phi}$ denote this irreducible subspace.

By Lemma 8.3,

$$\sum_{(I',\phi')\leq(I,\phi)} \dim V_{I',\phi'} = \prod_{P\in\pi_0(I^\perp - I)} \frac{p^{[C]} + (-1)^{\phi(C)}}{2}.$$

By the Möbius inversion formula [Stanley 1997, Section 3.7],

$$\dim V_{I,\phi} = \sum_{(I',\phi')\leq(I,\phi)} \mu((I,\phi),(I',\phi')) \prod_{C\in\pi_0(I'^\perp - I')} \frac{p^{[C]} + (-1)^{\phi(C)}}{2}, \tag{9.4}$$

where μ is the Möbius function of Q_λ. Since $\mu((I,\phi),(I,\phi)) = 1$ and the Möbius function is integer-valued, the right-hand side of (9.4) is indeed a polynomial in p with leading coefficient $2^{-|\pi_0(I^\perp - I)|}$. Clearly, the other coefficients do not have denominators other than powers of 2. $\square$

9D. *A combinatorial lemma.*

Lemma 9.5. *Let P be a poset and $J(P)$ be its lattice of order ideals. Let $m : P \to \mathbb{N}$ be any function (called the multiplicity function). For each subset $S \subset P$, let $[S] = \sum_{x\in S} m(x)$, the elements of S counted with multiplicity, and let* $\max S$ *denote the set of maximal elements of S. If $\alpha : J(P) \to \mathbb{C}[t]$ is a function such that*

$$\sum_{J\subset I} \alpha(J) = t^{[I]} \text{ for every order ideal } I \subset P,$$

then

$$\alpha(I) = t^{[I]} \prod_{x\in\max I} (1 - t^{-m(x)}). \tag{9.6}$$

Proof. By the Möbius inversion formula for a finite distributive lattice [Stanley 1997, Example 3.9.6],

$$\alpha(I) = \sum_{I-\max I\subset J\subset I} (-1)^{|I-J|} t^{[J]} = t^{[I]} \sum_{S\subset\max I} (-1)^{|\max I - S|} t^{-[\max I - S]}. \tag{9.7}$$

Each term in the expansion of the product

$$\prod_{x\in\max I}(1-t^{-m(x)})$$

is obtained by choosing a subset $S\subset\max I$ and taking

$$\prod_{x\notin S}(-t^{-m(x)})=(-1)^{|\max I-S|}t^{-[\max I-S]}.$$

Therefore, the expression (9.7) for $\alpha(I)$ reduces to (9.6) as claimed. □

9E. ***Explicit formula for the dimension.*** Recall (from Section 9C) that for each $(I,\phi)\in Q_\lambda$, $V_{I,\phi}$ denotes the unique irreducible $\mathrm{Sp}(K)$-invariant subspace of $L^2(A)$ which lies in $L^2(A)_{I,\phi}$ but not in any proper subspace of the form $L^2(A)_{I',\phi'}$. We shall obtain a nice expression for $\dim V_{I,\phi}$ by applying Lemma 9.5 to the induced subposet of P_λ given by

$$P_\lambda^+=\{(v,k)\in P_\lambda\mid v<(k-1)/2\}.$$

For each small order ideal $I\subset P_\lambda$, let $I^+=I^\perp\cap P_\lambda^+$. Then $I\mapsto I^+$ is an order-reversing isomorphism from the partially ordered set of small order ideals in P_λ to the partially ordered set $J(P_\lambda^+)$ of all order ideals in P_λ^+.

Let

$$V_I=\bigoplus_{\phi:\pi_0(I^\perp-I)\to\mathbb{Z}/2\mathbb{Z}}V_{I,\phi}. \tag{9.8}$$

Denote by V_I^0 and V_I^1 the subspaces of even or odd functions in V_I respectively.

Lemma 9.9. *If $I\subset P_\lambda$ is a small order ideal, then for $\epsilon\in\{0,1\}$,*

$$\dim V_I^\epsilon=\begin{cases}(p^{[I^\perp-I]}+(-1)^\epsilon)/2 & \text{if } I^+=\varnothing,\\ p^{[I^\perp-I]}\prod_{x\in\max I^+}(1-p^{-2m(x)})/2 & \text{otherwise.}\end{cases}$$

Proof. Suppose $I\subset P_\lambda$ is a small order ideal. Then

$$L^2(A_{I^\perp}/A_I)=\bigoplus_{\substack{J\supset I\\ J\text{ small}}}V_J=\bigoplus_{J^+\subset I^+}V_J. \tag{9.10}$$

Define $\alpha: J(P_\lambda^+)\to\mathbb{C}$ by $\alpha(J^+)=\dim V_J$. Comparing dimensions,

$$\sum_{J^+\subset I^+}\alpha(J^+)=p^{[I^\perp-I]}. \tag{9.11}$$

Let $E=\{(v,k)\in I^\perp-I\mid v=(k-1)/2\}$, the set of points in $I^\perp-I$ which lie on its axis of symmetry. Then $[I^\perp-I]=[E]+2[I^+]$. Therefore (9.11) becomes

$$\sum_{J^+\subset I^+}\alpha(J^+)=p^{[E]}p^{2[I^+]}.$$

Taking $P = P_\lambda^+$ and setting $t = p^2$ in Lemma 9.5 gives

$$\dim V_I = p^{[E]+2[I^+]} \prod_{x \in \max I^+} (1 - p^{-2m(x)}) = p^{[I^\perp - I]} \prod_{x \in \max I^\perp} (1 - p^{-2m(x)}).$$

In order to obtain Lemma 9.9, it remains to find the dimensions of the spaces of even and odd functions in V_I. If $I^+ = \varnothing$ then $E = I^\perp - I$. In this case, $V_{I,\phi}$ is just the set of even or odd functions in $L^2(A_{I^\perp}/A_I)$ and has dimension as claimed.

Otherwise, we proceed by induction on I^+. Thus, assume that Lemma 9.9 holds for small order ideals $I' \supsetneq I$. The space of even functions in $L^2(A_{I^\perp}/A_I)$ has dimension one more than the space of odd functions. Breaking up the spaces in (9.10) into even and odd functions, we see this difference is accounted for by the summand corresponding to $J^+ = \varnothing$, as discussed above. By the induction hypothesis, the dimensions of even and odd parts of the summands corresponding to $\varnothing \subsetneq J^+ \subsetneq I^+$ are equal. Therefore, the even and odd parts of V_I must have the same dimension. $\square$

Theorem 9.12. *If $I \subset P_\lambda$ is a small order ideal, then*

$$\dim V_{I,\phi} = \prod_{C \in \pi_0(I^\perp - I)} \dim V_{I(C),\phi(C)},$$

where, since $I(C)^\perp - I(C)$ is connected, $\dim V_{I(C),\phi(C)}$ *is given by Lemma 9.9.*

9F. *Examples.* We begin with the case $A = (\mathbb{Z}/p^k\mathbb{Z})^l$, corresponding to the partition $\lambda = (k, \ldots, k)$ (repeated l times). P_λ is then a linear order, with k points. Q_λ has two linear components, consisting of the even and odd parts. An informative way to display the decomposition of $L^2(A)$ is as the Hasse diagram of Q_λ, but with the dimension of the corresponding irreducible invariant subspace in place of each vertex. In this case we get

$$\begin{array}{ccc}
\dfrac{p^{lk}(1-p^{-2})}{2} & \qquad & \dfrac{p^{lk}(1-p^{-2})}{2} \\
\Big| & & \Big| \\
\dfrac{p^{l(k-2)}(1-p^{-2})}{2} & & \dfrac{p^{l(k-2)}(1-p^{-2})}{2} \\
\vdots & & \vdots \\
\dfrac{p^{(k-2(\lfloor k/2\rfloor-1))}(1-p^{-2})}{2} & & \dfrac{p^{(k-2(\lfloor k/2\rfloor-1))}(1-p^{-2})}{2} \\
\Big| & & \Big| \\
\dfrac{p^{(k-2\lfloor k/2\rfloor)}+1}{2} & & \dfrac{p^{(k-2\lfloor k/2\rfloor)}-1}{2}
\end{array}$$

The entry at the bottom right is zero when k is even and should be omitted. This is consistent with the previously known results in [Prasad 1998; Cliff et al. 2000]. The picture for $\lambda = (2, 1)$ is the same as that for $\lambda = (3)$. Perhaps the simplest nontrivial example is $\lambda = (3, 1)$ (it is the smallest example where $J(P_\lambda)$ is not a chain). We get

$$\begin{tikzcd}
 & \frac{p^4-p^2}{2} \arrow[dl, dash] \arrow[d, dash] & \frac{p^4-p^2}{2} \arrow[d, dash] \arrow[dr, dash] & \\
\frac{(p+1)^2}{4} & \frac{(p-1)^2}{4} & \frac{p^2-1}{4} & \frac{p^2-1}{4}
\end{tikzcd}$$

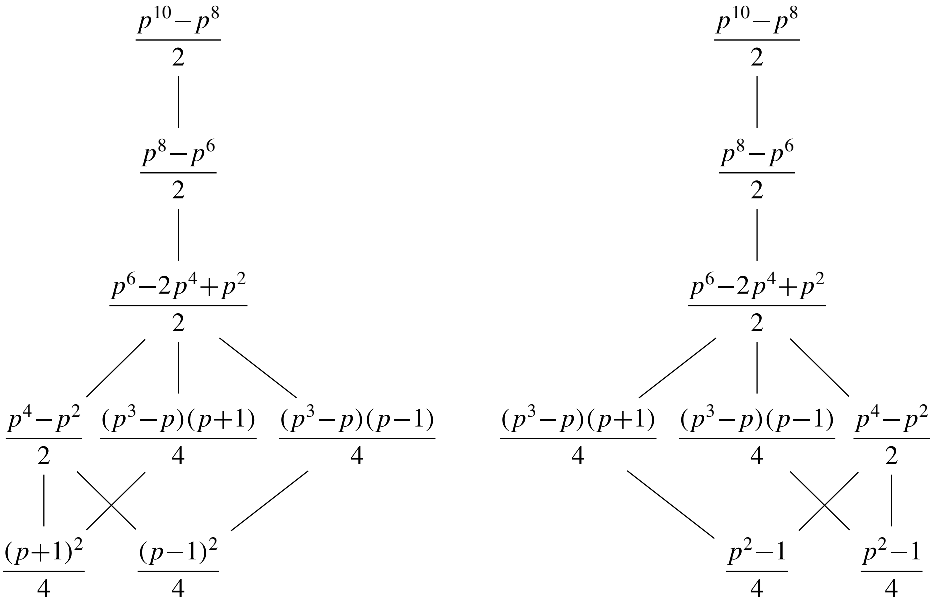
For $\lambda = (3, 2, 1)$, we get

$$\begin{tikzcd}
 & \frac{p^6-p^4}{2} \arrow[d, dash] & \frac{p^6-p^4}{2} \arrow[d, dash] & \\
 & \frac{p^4-p^2}{2} \arrow[dl, dash] \arrow[d, dash] & \frac{p^4-p^2}{2} \arrow[d, dash] \arrow[dr, dash] & \\
\frac{(p+1)^2}{4} & \frac{(p-1)^2}{4} & \frac{p^2-1}{4} & \frac{p^2-1}{4}
\end{tikzcd}$$

For $\lambda = (4, 2)$, we have

$$\begin{tikzcd}
 & \frac{p^6-p^4}{2} \arrow[d, dash] & \frac{p^6-p^4}{2} \arrow[d, dash] & \\
 & \frac{p^4-2p^2+1}{2} \arrow[dl, dash] \arrow[d, dash] & \frac{p^4-2p^2+1}{2} \arrow[d, dash] \arrow[dr, dash] & \\
\frac{p^2-1}{2} \arrow[dr, dash] & \frac{p^2-1}{2} \arrow[d, dash] & \frac{p^2-1}{2} & \frac{p^2-1}{2} \\
 & 1 & &
\end{tikzcd}$$

For $\lambda = (4, 3, 2, 1)$, we have



9G. *Projections onto the irreducible subspaces.* For each $(I, \phi) \in Q_\lambda$, let $E_{I,\phi}$ denote the projection operator onto $V_{I,\phi}$. Recall from Lemma 3.1 that the set of Weyl operators $\{W_k \mid k \in K\}$ is an orthonormal basis of $\mathrm{End}_{\mathbb{C}}\, L^2(A)$. Therefore, we may write

$$E_{I,\phi} = \sum_{k \in K} e_k(I, \phi) W_k$$

for some scalars $e_k(I, \phi)$. The goal of this section is to show that this expansion is completely combinatorial. More precisely, by Theorem 4.5, each $\mathrm{Sp}(K)$-orbit in K corresponds to an order ideal in P_λ. We shall show that if k lies in the $\mathrm{Sp}(K)$-orbit corresponding to the order ideal J, then $e_k(I, \phi)$ is a polynomial in p whose coefficients depend only on the combinatorial data I, ϕ, and J.

In Section 5A we saw that $\{\Delta_L \mid L \in J(P_\lambda)\}$ is a basis of $\mathrm{End}_{\mathrm{Sp}(K)}\, L^2(A)$. Therefore, we may write

$$E_{I,\phi} = \sum_{L \subset P_\lambda} \alpha_L(I, \phi) \Delta_L$$

for some constants $\alpha_L(I, \phi)$. If k lies in the orbit corresponding to J then

$$e_k(I, \phi) = \sum_{L \supset J} \alpha_L(I, \phi).$$

Therefore, it suffices to show that the $\alpha_L(I, \phi)$ are polynomials in p whose coefficients are determined by the combinatorial data L, I, and ϕ (Theorem 9.17). In fact, Theorems 9.18 and 9.19 compute $\alpha_L(I, \phi)$ explicitly.

To begin with, consider the case where $I^\perp - I$ is connected. If E_I is the projection operator onto V_I (defined by (9.8)), then by (6.2.2),

$$|A|^{-1} p^{[I^\perp - I]} \Delta_I = \sum_{J^+ \subset I^+} E_J.$$

Using Möbius inversion for a finite distributive lattice as in Section 9D,

$$|A| E_I = \sum_{I^+ - \max I^+ \subset J^+ \subset I^+} (-1)^{|I^+ - J^+|} p^{[J^\perp - J]} \Delta_J.$$

Since $V_{I,\phi}$ consists of even or odd functions in V_I (depending on whether $\phi(I^\perp - I)$ is 0 or 1), by (6.5), $E_{I,\phi}$ is given by

$$E_{I,\phi} = \tfrac{1}{2} E_I \big(\mathrm{Id}_{L^2(A)} + (-1)^{\phi(I^\perp - I)} |A|^{-1} \Delta_{P_\lambda}\big).$$

By Lemma 5.3,

$$(|A|^{-1} p^{[J^\perp - J]} \Delta_J)(|A|^{-1} \Delta_{P_\lambda}) = |A|^{-1} \Delta_{J^\perp}.$$

Therefore, when $I^\perp - I$ is connected,

$$2|A| E_{I,\phi} = \sum_{I^+ - \max I^+ \subset J^+ \subset I^+} (-1)^{|I^+ - J^+|} \big(p^{[J^\perp - J]} \Delta_J + (-1)^{\phi(I^\perp - I)} \Delta_{J^\perp}\big). \tag{9.13}$$

Now take $I \subset P_\lambda$ to be any small order ideal. The decomposition (7.3) gives

$$L^2(A) = \bigotimes_{C \in \tilde{\pi}_0(I^\perp - I)} L^2(A_C)$$

and

$$V_{I,\phi} = \Big(\bigotimes\nolimits_{C \in \pi_0(I^\perp - I)} V_{I(C),\phi(C)}\Big) \otimes L^2(A_{I^\perp(0)}/A_{I(0)}),$$

the last factor being one dimensional (since $I(0) = I^\perp(0)$). So we have

$$E_{I,\phi} = \Big(\bigotimes\nolimits_{C \in \pi_0(I^\perp - I)} E_{I(C),\phi(C)}\Big) \otimes \Delta_{I(0)}, \tag{9.14}$$

where, since $I(C)^\perp - I(C)$ is connected, $E_{I(C),\phi(C)}$ is determined by (9.13). A typical term in the expansion (9.14) will be of the form

$$\Big(\bigotimes\nolimits_{C \in \pi_0(I^\perp - I)} \Delta_{L(C)}\Big) \otimes \Delta_{I(0)}, \tag{9.15}$$

where for each $C \in \pi_0(I^\perp - I)$ we have $I(C) \subset L(C) \subset I^\perp(C)$, either $L(C)$ or $L(C)^\perp$ being a small order ideal in $P_{\lambda(C)}$. But this is just Δ_L, where

$$L = I \ \cup \bigsqcup_{C \in \pi_0(I^\perp - I)} L(C), \tag{9.16}$$

is an order ideal in P_λ by Lemma 7.1. We have this qualitative result:

Theorem 9.17. *For each $(I,\phi)\in Q_\lambda$, $2^{|\pi_0(I^\perp - I)|}|A|\alpha_L(I,\phi)$ is a polynomial in p whose coefficients are integers depending only on the combinatorial data I, ϕ, L.*

Let $I_L = L\cap L^\perp$. Examining (9.13) more carefully gives:

Theorem 9.18. *The coefficient $\alpha_L(I,\phi)$ is nonzero if and only if the following conditions hold*:

(9.18.1) *For each $C\in\pi_0(I^\perp - I)$, either $L(C)$ or $L(C)^\perp$ is a small order ideal in $P_{\lambda(C)}$.*

(9.18.2) $I^+ - \max I^+ \subset I_L^+ \subset I^+$.

Proof. For $\alpha_L(I,\phi)$ to be nonzero, it is necessary that L be of the form (9.16) for some order ideals $L(C)$ of $P_{\lambda(C)}$ which occur in the right hand side of (9.13). Furthermore, since each order ideal in $P_{\lambda(C)}$ appears at most once in the right hand side of (9.13), so each order ideal in P_λ appears only once in the expansion (9.14). In particular, no cancellation is possible, and for all such ideals $\alpha_L(I,\phi)\neq 0$.

Now $L(C)$ appears on the right hand side of (9.13) if and only if (9.18.1) holds, and $I(C)^+ - \max I(C)^+ \subset I_L(C)^+ \subset I(C)^+$. Since $\max I^+ = \bigsqcup_C \max I(C)^+$, this amounts to the condition (9.18.2). □

If these conditions do hold, then for each $C'\in\pi_0(I_L^\perp - I_L)$, there exists $C\in\pi_0(I^\perp - I)$ such that $C'\subset C$. Furthermore, $\phi_L(C')$ depends only on C, so we may denote its value by $\phi_L(C)$. For $I = I(C)$, the right hand side of (9.13) can be written as

$$\sum_{L(C)} (-1)^{|I(C)^+ - L(C)^+| + \phi(C)\phi_L(C)} p^{[I_L(C)^\perp - L(C)]},$$

the sum being over an appropriate set of order ideals $L(C)\subset P_{\lambda(C)}$. Let

$$\langle\phi_1,\phi_2\rangle = \sum_{C\in\pi_0(I^\perp - I)} \phi_1(C)\phi_2(C)$$

for any functions $\phi_i : \pi_0(I^\perp - I)\to\mathbb{Z}/2\mathbb{Z}$. The additive nature of the exponents in the above expression allows us to get an exact expression for $\alpha_L(I,\phi)$:

Theorem 9.19. *If an order ideal $L\subset P_\lambda$ satisfies the conditions of Theorem 9.18, then*

$$2^{|\pi_0(I^\perp - I)|}|A|\alpha_L(I,\phi) = (-1)^{|I^+ - I_L^+| + \langle\phi,\phi_L\rangle} p^{[I_L^\perp - L]}.$$

10. Finite modules over a Dedekind domain

Let F be a non-Archimedean local field with ring of integers R. Let P denote the maximal ideal of R. Assume that the residue field R/P is of odd order q. Fix a continuous character $\psi : F\to U(1)$ whose restriction to R is trivial, but whose restriction to $P^{-1}R$ is not (see, for example, Tate's thesis [1967]). Then if $\psi_x(y) = \psi(xy)$, the map $x \mapsto \psi_x$ is an isomorphism of F into $\hat{F}$. Under this isomorphism, R has image $R^\perp = (F/R)^\wedge$. More generally, P^{-n} has image $(P^n)^\perp = (F/P^n)^\wedge$ for every integer n (recall that for positive n, P^{-n} is the set of elements $x \in F$ such that $xP^n \subset R$). Thus, it gives rise to an isomorphism $P^{-n}/R \to (R/P^n)^\wedge$ for each positive integer n. Since P^{-n}/R inherits the structure of an R-module, this isomorphism also allows us to think of $(R/P^n)^\wedge$ as an R-module. Now suppose A is a finitely generated torsion module over R. Then

$$A = R/P^{\lambda_1} \times \cdots \times R/P^{\lambda_l} \tag{10.1}$$

for a unique partition λ. By the discussion above, $\hat{A}$ is also an R-module (noncanonically isomorphic to A). Let $K = A \times \hat{A}$, and $\mathrm{Sp}(K)$ be as in Theorem 1.1. Define $\mathrm{Sp}_R(K)$ to be the subgroup of $\mathrm{Sp}(K)$ consisting of R-module automorphisms.

The Weil representation of $\mathrm{Sp}_R(K)$ is simply the restriction of the Weil representation of $\mathrm{Sp}(K)$ on $L^2(A)$ to $\mathrm{Sp}_R(K)$. All the theorems and proofs in this article concerning finite abelian p-groups generalize to the Weil representation of $\mathrm{Sp}_R(K)$ on $L^2(A)$, so long as p is replaced by q in the formulas. Since every finitely generated torsion module over a Dedekind domain is a product of its primary components, and module automorphisms respect the primary decomposition, the reduction in Section 2C works for finite modules of odd order over Dedekind domains.

Singla [2010; 2011] has proved that the representation theory of $G(R/P^2)$, where G is a classical group, depends on R only through q, the order of the residue field. More precisely, if R and R' are two discrete valuation rings and an isomorphism between their residue fields is fixed (for example, take $R = \mathbb{Z}_p$, the ring of p-adic integers, and $R' = (\mathbb{Z}/p\mathbb{Z})[[t]]$, the ring of Laurent series with coefficients in $\mathbb{Z}/p\mathbb{Z}$), then there is a canonical bijection between the irreducible representations of $G(R/P^2)$ and $G(R'/P^2)$ which preserves dimensions. There is also a canonical bijection between their conjugacy classes which preserves sizes. All existing evidence points towards the existence of a similar correspondence for automorphism groups of modules of type λ (see, for example [Onn 2008, Conjecture 1.2]). The results in this paper also point in the same direction: for each partition λ, there is a canonical correspondence between the invariant subspaces of the Weil representations associated to the finitely generated torsion R-module of type λ and the finitely generated torsion R'-module of type λ which preserves dimensions.

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