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A POINTWISE A-PRIORI ESTIMATE FOR THE $\bar\partial$-NEUMANN PROBLEM ON WEAKLY PSEUDOCONVEX DOMAINS

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The main result is a pointwise a-priori estimate for the $\bar\partial$-Neumann problem that holds on an *arbitrary* weakly pseudoconvex domain D. It is shown that for $(0,q)$-forms f in the domain of the adjoint $\bar\partial^*$ of $\bar\partial$, the pointwise growth of the derivatives of each coefficient of f with respect to $\overline{z_j}$ and in complex tangential directions is carefully controlled by the sum of the suprema of f, $\bar\partial f$, and $\bar\partial^* f$ over D. These estimates provide a pointwise analog of the classical basic estimate in the L^2 theory that has been the starting point for all major work in this area involving L^2 and Sobolev norm estimates for the complex Neumann and related operators.

1. Introduction

The L^2 theory of the $\bar\partial$-Neumann problem on pseudoconvex domains has been highly developed for many years. In particular, J. J. Kohn [1979] introduced the technique of subelliptic multipliers which led to the proof of subelliptic estimates in the case where the boundary is of finite type [D'Angelo 1982; Catlin 1987; Siu 2010]. The starting point for these and other investigations has been the following basic estimate, valid on any smoothly bounded pseudoconvex domain D (see [Folland and Kohn 1972; Kohn 1979] for more details). Let us fix a point $P \in bD$ and a smooth orthonormal frame for $(1,0)$-forms $\omega_1, \omega_2, \ldots, \omega_n$ on a small neighborhood U of P with $\omega_n = \gamma(\zeta)\partial r$, where r is a defining function for D. Let $L_1, \ldots, L_n$ be the corresponding dual frame for $(1,0)$ vector fields. One defines

$$\mathfrak{D}_q(D) = C^\infty_{(0,q)}(\bar D) \cap \operatorname{dom} \bar\partial^*,$$

and one denotes by $\mathfrak{D}_{qU}$ those forms in $\mathfrak{D}_q(D)$ which have compact support in $\bar D \cap U$. Then $f \in \mathfrak{D}_{qU}$ can be written as $\sum'_J f_J \bar\omega^J$, where the summation is over strictly increasing q-tuples J. Since $f \in \operatorname{dom} \bar\partial^*$, one has $f_J = 0$ on $bD \cap U$

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whenever $n \in J$. In case $q = 1$, the "L^2 basic estimate" states that there exists a constant C such that

$$\sum_{j,k} \|\overline{L_j} f_k\|^2 + \int_{bD\cap U} \mathcal{L}(r,\zeta; f^\sharp)\, dS(\zeta) \le C\big[\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2 + \|f\|^2\big]$$

for all $f = \sum f_k \overline{\omega_k} \in \mathfrak{D}_{1U}$, where $f^\sharp = (f_1, \dots, f_n)$. The norms here are the standard L^2 norms over $D \cap U$, and $\mathcal{L}$ is the Levi form of the defining function r with respect to the frame $\{L_1, \dots, L_n\}$. Since $f \in \operatorname{dom} \bar\partial^*$, one has $f_n = 0$ on bD, so that pseudoconvexity implies that $\mathcal{L}(r,\zeta; f^\sharp) \ge 0$ on bD. Furthermore, it readily follows from $f_n|_{bD\cap U} = 0$ that one then also has the estimate

$$\|f_n\|_1^2 \le C_1 \|\bar\partial f_n\|^2 \le C_2\big[\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2 + \|f\|^2\big].$$

Here $\|f_n\|_1$ is the full 1-Sobolev norm, i.e., $\|f_n\|_1^2$ is the sum of the squares of the L^2 norms of all first order derivatives of f_n.

Over the years there has been much interest in obtaining corresponding results involving pointwise and Hölder estimates. Techniques of integral representations have been most successful on *strictly* pseudoconvex domains, where the Levi polynomial provides a simple explicit local holomorphic support function (see [Range 1986] for a systematic exposition). Holomorphic support functions also exist on convex domains, and some results have been obtained in that setting in the case of finite type [Cumenge 1997; Diederich et al. 1999]. However, it has long been known that in general there are no analogous holomorphic support functions, even in very simple pseudoconvex domains of finite type [Kohn and Nirenberg 1973]. This obstruction has blocked any progress on these questions in the case of more general pseudoconvex domains.

Recently the author has introduced a *nonholomorphic* modification of the Levi polynomial to obtain new Cauchy–Fantappié kernels on arbitrary weakly pseudoconvex domains which reflect the complex geometry of the boundary and satisfy some significant partial estimates [Range 2013]. In this paper we use the new kernels in the integral representation formula developed by I. Lieb and the author in the strictly pseudoconvex case (see [Lieb and Range 1983; 1986]) to prove a pointwise analog of the classical basic L^2-estimate, as follows. This result was already announced in [Range 2011]. We define

$$\mathfrak{D}_q^k(D) = C_{(0,q)}^k(\overline{D}) \cap \operatorname{dom} \bar\partial^*$$

for $k = 1, 2, \dots$, and we denote by $\mathfrak{D}_{qU}^k$ those forms in $\mathfrak{D}_q^k(D)$ that have compact support in $\overline{D} \cap U$. We shall use the frames $\omega_1, \omega_2, \dots, \omega_n$ and $L_1, \dots, L_n$ as above. Vector fields V act on forms coefficientwise, i.e., if $f = \sum_J' f_J \bar\omega^J$, then

$V(f) = \sum_J{}' V(f_J)\bar\omega^J$. For a C^1-form f of type $(0,q)$ on $\bar D$ we define the norm

$$Q_0(f) = |f|_0 + |\bar\partial f|_0 + |\vartheta f|_0,$$

where ϑ is the formal adjoint of $\bar\partial$, and $|\varphi|_0$ denotes the sum of the supremum norms over D of the coefficients of φ. For $0 < \delta \le 1$, $|\varphi|_\delta$ denotes the corresponding Hölder norm of order δ.

Main Theorem. *There exists an integral operator $S^{bD} : C_{(0,q)}(bD) \to C^\infty_{(0,q)}(D)$ which has the following properties. If bD is (Levi) pseudoconvex in a neighborhood U of the point $P \in bD$ and if U is sufficiently small, there exist constants C_δ depending on $\delta > 0$, so that one has the following uniform estimates for all $f \in \mathcal{D}^1_{qU}$, $1 \le q \le n$, and $z \in D \cap U$:*

(i) $|f - S^{bD}(f)|_\delta \le C_\delta Q_0(f)$ *for any* $\delta < 1$.

(ii) $|\overline{L_j} S^{bD}(f)(z)| \le C_\delta \operatorname{dist}(z, bD)^{\delta-1} Q_0(f)$ *for* $j = 1, \dots, n$ *and any* $\delta < \frac{1}{2}$.

(iii) $|L_j S^{bD}(f)(z)| \le C_\delta \operatorname{dist}(z, bD)^{\delta-1} Q_0(f)$ *for* $j = 1, \dots, n-1$ *and any* $\delta < \frac{1}{3}$.

Furthermore, if $f_J\bar\omega^J$ is a normal component of f with respect to the frame $\bar\omega_1, \dots, \bar\omega_n$, one has

$$|f_J|_\delta \le C_\delta Q_0(f) \quad \textit{for any } \delta < \tfrac{1}{2} \textit{ if } n \in J.$$

Note that if one also had an estimate analogous to (iii) for the normal derivative $L_n S^{bD}(f)(z)$ for *some* $\delta > 0$ (with $\delta < \frac{1}{3}$), standard results would imply the Hölder estimate $|S^{bD}(f)|_\delta \le C_\delta Q_0(f)$; by using (i) one therefore would obtain an estimate

$$|f|_\delta \le C_\delta Q_0(f),$$

i.e., the Hölder analog of a subelliptic estimate. It is known that such an estimate does not hold on arbitrary pseudoconvex domains. On the other hand, the Main Theorem provides a starting point in a general setting which, combined with additional suitable properties of the boundary such as finite type, might be useful to obtain appropriate estimates for $L_n S^{bD}(f)$. In particular, the author is investigating analogs of Kohn's subelliptic multipliers in the integral representation setting underlying the Main Theorem (see [Range 2011] for an outline of such potential applications).

2. Integral representations

We briefly recall some fundamentals of the integral representation machinery. We follow the terminology and notation from [Range 1986], where full details may be found. A (kernel) generating form $W(\zeta, z)$ for the smoothly bounded domain

$D \subset \mathbb{C}^n$ is a $(1,0)$-form $W = \sum_{j=1}^n w_j \, d\zeta_j$ defined on $bD \times D$ with coefficients of class C^1 which satisfies $\sum w_j(\zeta_j - z_j) = 1$. For $0 \le q \le n-1$, the associated Cauchy–Fantappié (= CF) form of order q is defined by

$$\Omega_q(W) = c_{nq} W \wedge (\overline{\partial_\zeta} W)^{n-q-1} \wedge (\overline{\partial_z} W)^q.$$ [1]

$\Omega_q(W)$ is a double form on $bD \times D$ of type $(n, n-q-1)$ in ζ and type $(0,q)$ in z. One also sets $\Omega_{-1}(W) = \Omega_n(W) = 0$.

With $\beta = |\zeta - z|^2$, the form $B = \partial\beta/\beta = \sum_{j=1}^n \overline{(\zeta_j - z_j)}/|\zeta - z|^2 \, d\zeta_j$ is the generating form for the Bochner–Martinelli–Koppelman (= BMK) kernels. One has the following BMK formula (here and in the following, the integration variable is always ζ): if $f \in C^1_{(0,q)}(\bar{D})$ then for $z \in D$,

$$(1)\quad f(z) = \int_{bD} f(\zeta) \wedge \Omega_q(B) - \overline{\partial_z} \int_D f(\zeta) \wedge \Omega_{q-1}(B) - \int_D \overline{\partial_\zeta} f(\zeta) \wedge \Omega_q(B).$$

The next formula, due to W. Koppelman, describes how to replace $\Omega_q(B)$ by some other CF kernel $\Omega_q(W)$ on the boundary bD. Since $\Omega_n(B) \equiv 0$, we shall assume $q < n$ from now on. Given any generating form W on $bD \times D$, one has

$$f(z) = \int_{bD} f(\zeta) \wedge \Omega_q(W) + \overline{\partial_z} T_q^W(f) + T_{q+1}^W(\bar\partial f) \quad \text{for } f \in C^1_{(0,q)}(\bar{D}), z \in D.$$

Here the integral operator $T_q^W : C^1_{(0.q)}(\bar{D}) \to C_{(0,q-1)}(D)$ is defined by

$$T_q^W(f) = \int_{bD} f \wedge \Omega_{q-1}(W, B) - \int_D f(\zeta) \wedge \Omega_{q-1}(B)$$

for any $0 \le q < n$, where the "transition" kernels $\Omega_{q-1}(W, B)$ involve explicit expressions in terms of W and B which will be recalled later on.

Remark. For D strictly pseudoconvex, Henkin and Ramirez have constructed a generating form $W^{HR}(\zeta, z)$ that is holomorphic in z, so $\Omega_q(W^{HR}) = 0$ on bD for $q \ge 1$. Consequently, if f is a $\bar\partial$-closed $(0,q)$-form on $\bar{D}$, one has $f = \overline{\partial_z} T_q^{W^{HR}}(f)$, with an explicit solution operator $T_q^{W^{HR}}$. Based on the critical information that the Levi form of the boundary is positive definite in this case, it is well known that this solution operator is bounded from L^∞ into $\Lambda_{1/2}$. Furthermore, one also has the a-priori Hölder estimate $|f|_{1/2} \le CQ_0(f)$ for all $f \in \mathcal{D}^1_{qU}$ (see [Lieb and Range 1986]). Attempts to prove corresponding estimates on more general domains ultimately run into the obstruction of the example by Kohn and Nirenberg [1973] mentioned above, i.e., in general it is not possible to find a corresponding reasonably explicit *holomorphic* generating form on weakly pseudoconvex domains—even if of finite type—except under very restrictive geometric conditions.

[1] $c_{nq} = ((-1)^{q(q-1)/2}/(2\pi i)^n)\binom{n-1}{q}$

In case $f \in \mathfrak{D}^1_q(D)$, one may transform formula (1) into

$$f = \int_{bD} f \wedge \Omega_q(B) + (\bar\partial f, \bar\partial \omega_q) + (\vartheta f, \vartheta \omega_q),$$

where ω_q denotes the fundamental solution of $\Box$ on $(0,q)$-forms, ϑ denotes the formal adjoint of $\bar\partial$, so that $\vartheta f = \bar\partial^* f$, and $(\cdot\,,\cdot)$ denotes the standard L^2 inner product of forms over D (see [LR 1983] and [Range 1986]). The fundamental solution ω_q is an isotropic kernel whose regularity properties are well understood. In particular, the operator

$$S^{\text{iso}} : f \to S^{\text{iso}}(f) = (\bar\partial f, \bar\partial \omega_q) + (\vartheta f, \vartheta \omega_q)$$

satisfies a Hölder estimate

$$|S^{\text{iso}}(f)|_\delta \le C_\delta Q_0(f) \quad \text{for all } f \in C^1_{(0,q)}(D) \text{ and any } \delta < 1. \tag{2}$$

Consequently, the essential information regarding all pointwise estimations is contained in the boundary integral $S^{bD}(f) = \int_{bD} f \wedge \Omega_q(B)$. The kernel of $\Omega_q(B)$ is isotropic; it treats derivatives in all directions equally, and direct differentiation under the integral in $\int_{bD} f \wedge \Omega_q(B)$ leads to an expression that will in general blow up like $\text{dist}(z, bD)^{-1}$. So this general representation of the operator S^{bD} does not provide any useful information.

Note that since $\Omega_n(B) \equiv 0$, the Main Theorem holds trivially with $S^{bD} \equiv 0$ when $q = n$.

By the Koppelman formulas, given any generating form W on $bD \times D$, one can transform $S^{bD}(f)$ into

$$S^{bD}(f) = \int_{bD} f \wedge \Omega_q(W) + \int_{bD} \bar\partial f \wedge \Omega_q(W, B) + \int_{bD} f \wedge \overline{\partial_z} \Omega_{q-1}(W, B). \tag{3}$$

The proof of the Main Theorem relies on formula (3) on a weakly pseudoconvex domain $D \Subset \mathbb{C}^n$, applied to the *nonholomorphic* generating form $W^{\mathcal{L}}(\zeta, z)$ introduced in [Range 2013]. Let us briefly recall the key properties of $W^{\mathcal{L}}(\zeta, z)$. Given a sufficiently small neighborhood $U = U(P)$, on $(bD \cap U) \times (D \cap U)$ the form $W^{\mathcal{L}}(\zeta, z)$ is represented explicitly by

$$W^{\mathcal{L}}(\zeta, z) = \frac{\sum_{j=1}^n g_j(\zeta, z)\, d\zeta_j}{\Phi_K(\zeta, z)},$$

where $\Phi_K(\zeta, z) = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j)$ for $\zeta \in bD$. The (nonholomorphic) support function Φ_K is defined by

$$\Phi_K(\zeta, z) = F^{(r)}(\zeta, z) - r(\zeta) + K|\zeta - z|^3,$$

where $F^{(r)}(\zeta,z)$ is the Levi polynomial of a suitable defining function r, and $K>0$ is a suitably chosen large constant. We note that $W^{\mathcal{L}}(\zeta,z)$ is C^∞ in z for $z\neq\zeta$. Recall from [Range 2013] that the neighborhood U, the constant K, and $\varepsilon>0$ can be chosen so that for all $\zeta,z\in\overline{D}\cap U$ with $|\zeta-z|<\varepsilon$, one has

$$(4)\quad |\Phi_K(\zeta,z)|\gtrsim\big[|\operatorname{Im}F^{(r)}(\zeta,z)|+|r(\zeta)|+|r(z)|+\mathcal{L}(r,\zeta;\pi^t_\zeta(\zeta-z))+K|\zeta-z|^3\big].$$

Here $\pi^t_\zeta(\zeta-z)$ denotes the projection of $(\zeta-z)$ onto the complex tangent space of the level surface $M_{r(\zeta)}$ through the point ζ, and $\mathcal{L}(r,\zeta;\pi^t_\zeta(\zeta-z))$ denotes the Levi form of r at the point ζ. As shown in [Range 2013], the defining function r can be chosen so that $\mathcal{L}(r,\zeta;\pi^t_\zeta(\zeta-z))\geq 0$ for all $\zeta\in\overline{D}\cap U$. We also recall that—as in the classical strictly pseudoconvex case—$r(\zeta)$ and $\operatorname{Im}F^{(r)}(\zeta,z)$ can be used as (real) coordinates in a neighborhood of a fixed point z.

Note that since $|\zeta-z|^3$ is real and symmetric in ζ and z, it follows from the known case $K=0$ (see [Range 1986], for example) that if one defines $\Phi^*_K(\zeta,z)=\overline{\Phi_K(z,\zeta)}$, one has the approximate symmetry

$$(5)\qquad \Phi^*_K-\Phi_K=\mathcal{E}_3.^{2}$$

In the following, we simplify the notation by dropping the subscript K, i.e., we will write Φ instead of Φ_K.

For $0\leq q<n$ we thus consider the integral representation formula

$$(6)\qquad f=S^{bD}(f)+S^{\text{iso}}(f)\quad\text{for }\ f\in\mathfrak{D}^1_{0,q}(\overline{D}),$$

where the boundary operator S^{bD} is given by

$$(7)\quad \int_{bD}f\wedge\Omega_q(W^{\mathcal{L}})+\int_{bD}\bar\partial f\wedge\Omega_q(W^{\mathcal{L}},B)+\int_{bD}f\wedge\bar\partial_z\Omega_{q-1}(W^{\mathcal{L}},B),$$

for $f\in\mathfrak{D}^1_{0,q}(\overline{D})$. Corresponding formulas hold locally on $U\cap bD$ whenever the boundary is Levi pseudoconvex in U. It is then clear that property (i) in the Main Theorem is satisfied. The main difficulty involves establishing the estimates (ii) and (iii).

The proof of the Main Theorem involves a careful analysis of the boundary integrals in formula (7). In contrast to [Lieb and Range 1983], which we henceforth abbreviate [LR 1983], for the most part we deal directly with the integrals over bD, thereby simplifying the analysis. However, for the most critical terms we will need to apply Stokes' theorem and introduce the Hodge $*$ operator as in [LR 1983] to transform the integrals into standard L^2 inner products of forms over $D\cap U$, and exploit certain approximate symmetries in the kernels.

[2] $\mathcal{E}_j$ denotes a smooth expression which satisfies $|\mathcal{E}_j|\leq C|\zeta-z|^j$.

3. The integral $\int_{bD} f \wedge \Omega_q(W^{\mathcal{L}})$

When D is strictly pseudoconvex, $W^{\mathcal{L}}$ can be chosen to be holomorphic in z for ζ close to z, so that the estimations become trivial if $q \geq 1$, since then $\overline{\partial_z} W = 0$ near the singularity. In the general case considered here, this integral needs to be carefully estimated as well. The analysis of this integral involves straightforward modifications of the case $q = 0$ discussed in [Range 2013], as follows.

We only consider ζ, z with $|\zeta - z| < \frac{1}{2}\varepsilon$, so that we can use the explicit form of $W^{\mathcal{L}} = g/\Phi$ recalled above, and the local frames $\{\omega_1, \ldots, \omega_n\}$ and $\{L_1, \ldots, L_n\}$. Recall that for $j = 0, 1, 2$, an expression $\mathcal{E}_j^\sharp$ denotes a form which is smooth for $\zeta \neq z$ and that satisfies a uniform estimate $|\mathcal{E}_j^\sharp| \lesssim |\zeta - z|^j$, and whose precise formula may change from place to place. While Φ is not holomorphic in z, one has $\overline{\partial_z}\Phi = \mathcal{E}_2^\sharp$; furthermore, one has $L_j^z \Phi = \mathcal{E}_1^\sharp$ for $j < n$, while $L_n^z \Phi \neq 0$ at $\zeta = z$.

By the properties of CF forms, on bD one has

$$\Omega_q(W^{\mathcal{L}}) = c_{nq} \frac{g \wedge (\overline{\partial_\zeta} g)^{n-q-1} \wedge (\overline{\partial_z} g)^q}{\Phi^n}.$$

The coefficients g_j of $g = \sum g_j \, d\zeta_j$ are given by

$$g_j = \frac{\partial r}{\partial \zeta_j} - \frac{1}{2} \sum_k \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k} (\zeta_k - z_k) + \mathcal{E}_2^\sharp.$$

The form of g implies that

$$g = \partial r(\zeta) + \mathcal{E}_1 + \mathcal{E}_2^\sharp, \quad \bar\partial_\zeta g = \bar\partial \partial r(\zeta) + \mathcal{E}_1^\sharp, \quad \text{and} \quad \bar\partial_z g = \mathcal{E}_1^\sharp.$$

It follows readily that for $0 \leq t \leq n-1$ one has

$$g \wedge (\overline{\partial_\zeta} g)^t = \partial r(\zeta) \wedge \sum_{k=0}^{t} [\bar\partial \partial r(\zeta)]^k (\mathcal{E}_1^\sharp)^{t-k} + \sum_{k=0}^{t} [\bar\partial \partial r(\zeta)]^k (\mathcal{E}_1^\sharp)^{t-k+1},$$

where $(\mathcal{E}_1^\sharp)^s$ denotes a generic form of appropriate degree whose coefficients are products of s terms of type $\mathcal{E}_1^\sharp$ in the case $s \geq 1$, or a term of type $\mathcal{E}_0^\sharp$ for $s = 0, -1$.

Note that since $\iota^*(\omega_n \wedge \overline{\omega_n}) = 0$ on bD, the pullback of $\partial r(\zeta) \wedge [\bar\partial \partial r(\zeta)]^k$ to bD involves only *tangential* components $\tan[\bar\partial \partial r(\zeta)]$, while the pullback of $[\bar\partial \partial r(\zeta)]^k$ alone will involve exterior products of at least $k-1$ different tangential components.

When estimating integrals involving these expressions, we make use of the fact that — in suitable z-diagonalizing coordinates (see [Range 2013]) — each *tangential* component $\tan[\bar\partial \partial r(\zeta)]$ in the numerator of the kernel reduces the order of the vanishing of the corresponding factor Φ in the denominator from three to an estimate $\gtrsim |\zeta_l - z_l|^2$, i.e.,

$$|\tan[\bar\partial \partial r(\zeta)]/\Phi| \lesssim 1/(|r(z)| + |\zeta_l - z_l|^2), \tag{8}$$

where ζ_l is an appropriate complex tangential coordinate. Similarly,

$$|\mathcal{E}_1^\sharp/\Phi| \lesssim 1/(|r(z)|+|\zeta-z|^2).$$

In order to keep track of these estimates, we introduce forms $\mathcal{L}[\mu]$ of *Levi weight* μ as follows. If $\mu \geq 1$, we say $\mathcal{L}[\mu]$ has Levi weight μ if each summand of $\mathcal{L}[\mu]$ contains at least μ factors which either are (different) purely tangential components of $\bar\partial\partial r(\zeta)$, or of type $\mathcal{E}_1^\sharp$. We also set $\mathcal{L}[\mu] = \mathcal{E}_0$ if $\mu \leq 0$. It then follows that

$$g \wedge (\overline{\partial_\zeta} g)^t = \mathcal{L}[t]$$

on the boundary, and consequently the numerator of $\Omega_q(W^{\mathcal{L}})$ satisfies

$$g \wedge (\overline{\partial_\zeta} g)^{n-q-1} \wedge (\overline{\partial_z} g)^q = \mathcal{L}[n-1]. \tag{9}$$

Proposition 1. *For any q with $0 \leq q \leq n-1$, the operator*

$$T_q^{\mathcal{L}} : C_{(0,q)}(bD) \to C^\infty_{(0,q)}(D),$$

defined by

$$T_q^{\mathcal{L}} f(z) = \int_{bD} f(\zeta) \wedge \Omega_q(W^{\mathcal{L}})(\zeta, z),$$

satisfies the estimates

$$\left|\overline{L_j^z}(T_q^{\mathcal{L}} f(z))\right| \leq C_\delta |f|_0 \operatorname{dist}(z, bD)^{\delta-1} \qquad \text{for } \delta < \tfrac{2}{3} \text{ and } 1 \leq j \leq n, \tag{10}$$

$$\left|L_j^z(T_q^{\mathcal{L}} f(z))\right| \leq C_\delta |f|_0 \operatorname{dist}(z, bD)^{\delta-1} \qquad \text{for } \delta < \tfrac{1}{3} \text{ and } j \leq n-1, \tag{11}$$

for suitable constants C_δ.

Given the estimation (9) of the numerator of $\Omega_q(W^{\mathcal{L}})$, the proof given in [Range 2013] for the case $q = 0$ and for the derivatives $\overline{L_j^z}$ carries over to the general case. To prove the estimate (11), one uses $L_j^z\Phi = \mathcal{E}_1^\sharp$ for $j \leq n-1$, which implies that $|L_j^z\Phi/\Phi| \preceq \operatorname{dist}(z, bD)^{-2/3}$. The estimations then proceed as in [Range 2013].

Remark. There is no corresponding estimate for the differentiation with respect to L_n^z, i.e., in the normal direction, since $L_n^z\Phi \neq 0$ for $\zeta = z$; therefore the operator $T_q^{\mathcal{L}}$ is not smoothing, i.e., there is no Hölder estimate

$$|T_q^{\mathcal{L}} f|_\delta \leq C_\delta |f|_0 \quad \text{for any } \delta > 0.$$

Proposition 1 provides a *partial* smoothing property:

Definition 2. A kernel $\Gamma(\zeta, z)$, or the integral operator $T_\Gamma : C_*(\bar D) \to C^1_*(D)$ defined by it, is $\bar z$-smoothing of order $\delta > 0$ if T_Γ satisfies the estimates (10). Similarly, we say that Γ (or T_Γ) is tangentially smoothing of order δ if T_Γ satisfies the estimates (11) for L_j^z and $\overline{L_j^z}$ for $j = 1, \ldots, n-1$.

Here C_* denotes spaces of forms of appropriate type.

4. Boundary admissible kernels

Before proceeding with the analysis of the integrals involving the transition kernels, we introduce admissible kernels and their weighted order by suitably modifying corresponding notions from [LR 1983]. We say that a kernel $\Gamma(\zeta, z)$ defined on

$$bD \times \bar{D} - \{(\zeta, \zeta) : \zeta \in bD\}$$

is simple *admissible* if for each $P \in bD$, there exists a neighborhood U of P, such that on $(bD \cap U) \times (\bar{D} \cap U)$ there is a representation of the form

$$\Gamma = \frac{\mathcal{L}[\mu](\mathcal{E}_1^{\sharp})^j}{\Phi^{t_1} \beta^{t_0}},$$

where all exponents are ≥ 0. Note that j may be a noninteger, in which case $(\mathcal{E}_1^{\sharp})^j$ denotes a form which is estimated by $C|\zeta - z|^j$. Such a representation is said to have (weighted) *boundary* order $\geq \lambda$ $(\lambda \in \mathbb{R})$ provided:

i) if $t_1 \geq 1$ and $\mu \geq 1$, then

$$2n - 1 + j - 1 - 2\max(0, \min(t_1 - 1, \mu)) - 3\max(t_1 - 1 - \mu, 0) - 2t_0 \geq \lambda;$$

while if $\mu \leq 0$ then

$$2n - 1 + j - 1 - 3\max(t_1 - 1, 0) - 2t_0 \geq \lambda;$$

or

ii) if $t_1 = 0$, then

$$2n - 1 + j - 2t_0 \geq \lambda.$$

This definition of order takes into account that the dimension of bD is $2n - 1$, and that *one* factor Φ may be counted with weight 1, since by estimate (4) one has $|\Phi| \gtrsim |\operatorname{Im} F^r|$, and $\operatorname{Im} F^r(\cdot, z)$ serves as a local coordinate on the boundary in a neighborhood of z.

A kernel Γ is admissible of boundary order $\geq \lambda$ if it is a finite sum of simple admissible kernels with representations of boundary order $\geq \lambda$.

The results in the previous section show that $\Omega_q(W^{\mathcal{L}})$ is admissible of boundary order ≥ 0.

As in the strictly pseudoconvex case considered in [LR 1983], an admissible kernel Γ of boundary order $\lambda \geq 1$ is smoothing of some positive order δ. This follows from an estimate

$$|V^z T_\Gamma(f)(z)| \leq C_\delta |f|_0 \operatorname{dist}(z, bD)^{-1+\delta},$$

for any vector field V^z of unit length acting in z. On the other hand, admissible kernels of boundary order $\lambda = 0$ are not smoothing in general.

More precisely, we have:

Theorem 3. *Let Γ_λ be an admissible kernel of boundary order $\geq \lambda$, and let*

$$J_\lambda(z) = \int_{bD} |\Gamma_\lambda(\zeta, z)| \, dS(\zeta).$$

$J_\lambda(z)$ has the following properties:

(a) *If $\lambda > 0$, then* $\sup_D J_\lambda(z) < \infty$.

(b) *If $\lambda = 0$, then $J_0(z) \lesssim \operatorname{dist}(z, bD)^{-\alpha}$ for any $\alpha > 0$.*

(c) *If $\lambda \geq 1$, then Γ_λ is smoothing of order δ for any $\delta < \frac{1}{3}$, tangentially smoothing of order $\delta < \frac{2}{3}$, and $\bar{z}$-smoothing of order $\delta < 1$.*

(d) *If $\lambda \geq 2$, then Γ_λ is smoothing of order δ for any $\delta < \frac{2}{3}$.*

Proof. Part (a) was essentially proved in [Range 2013] for the kernel $\Omega_0(W^{\mathcal{L}})$. The general case follows by the same arguments. Part (b) follows from (a) by noting that $\operatorname{dist}(z, bD)^\alpha \leq |\zeta - z|^\alpha$ for $\zeta \in bD$. For (c), note that given a vector field V^z, all terms in $V^z\Gamma_\lambda$ are of boundary order $\geq \lambda - 1 \geq 0$, except those where differentiation is applied to Φ; in that case use $V^z(\Phi^{-s}) = (\Phi^{-s})[\mathcal{E}_0^\sharp/\Phi]$ and $|1/\Phi| \lesssim |r(z)|^{-2/3}/|\zeta - z)|$ to see that $V^z\Gamma$ is estimated by $|r(z)|^{-2/3}$ multiplied with a kernel of order ≥ 0. Similarly, in the case where V^z is tangential, one can replace $|r(z)|^{-2/3}$ by $|r(z)|^{-1/3}$, and in the case $V^z = \overline{L_j^z}$, one uses $\overline{L_j^z}(\Phi^{-s}) = (\Phi^{-s})[\mathcal{E}_2^\sharp/\Phi]$ to see that $\overline{L_j^z}\Gamma_\lambda$ is of order ≥ 0. The required estimates then follow from (b). Finally, (d) follows by appropriately modifying the proof of (c). □

The most significant part of this paper is the analysis of the kernels of order zero. Such kernels are not smoothing in general. However, as we saw for $\Omega_q(W^{\mathcal{L}})$, it turns out that in many cases they are at least $\bar{z}$-smoothing and tangentially smoothing of some positive order. On the other hand, one readily checks that kernels of type such as $\mathcal{E}_1^\sharp/\beta^n$ (e.g., those appearing in the BMK kernels) or $1/(\Phi\beta^{n-1})$, which are of boundary order zero, do not give preference to tangential or $\bar{z}$-derivatives, and consequently such kernels are not $\bar{z}$-smoothing of any positive order. Therefore one needs to analyze the kernels of boundary order ≥ 0 that arise in the current setting more carefully in order to obtain the estimates stated in the Main Theorem. It will be convenient to introduce the following notation:

Definition 4. The symbol Γ_λ denotes an admissible kernel of (boundary) order $\geq \lambda$. We denote by $\Gamma_{0,1/2}^{\bar{z}}$ (resp. $\Gamma_{0,2/3}^{\bar{z}}$) an admissible kernel of order ≥ 0 which is $\bar{z}$-smoothing of any order $\delta < \frac{1}{2}$ (resp. $\delta < \frac{2}{3}$) and tangentially smoothing of any order $\delta < \frac{1}{3}$.

According to this definition, Proposition 1 states that $\Omega_q(W^{\mathcal{L}})$ is a kernel of type $\Gamma_{0,2/3}^{\bar{z}}$. Similarly, we note that by Theorem 3 a kernel of type Γ_1 is (better than) of type $\Gamma_{0,2/3}^{\bar{z}}$, and, in fact, is smoothing of order $\delta < \frac{1}{3}$ in *all* directions.

5. The integrals $\int_{bD} \bar\partial f \wedge \Omega_q(W^{\mathcal{L}}, B)$ and $\int_{bD} f \wedge \overline{\partial_z}\Omega_{q-1}(W^{\mathcal{L}}, B)$

We recall (see [LR 1983] and [Range 1986], for example) that for $0 \le q \le n-2$ the transition kernels $\Omega_q(W^{\mathcal{L}}, B)$ are defined by

$$(12)\quad \Omega_q(W^{\mathcal{L}}, B) = (2\pi i)^{-n} \sum_{\mu=0}^{n-q-2} \sum_{k=0}^{q} a_{\mu,k,q} W^{\mathcal{L}} \wedge B \wedge (\overline{\partial_\zeta} W^{\mathcal{L}})^\mu \wedge (\overline{\partial_\zeta} B)^{n-q-2-\mu} \wedge (\overline{\partial_z} W^{\mathcal{L}})^k \wedge (\overline{\partial_z} B)^{q-k},$$

where the coefficients $a_{\mu,k,q}$ are certain rational numbers, while $\Omega_{n-1}(W^{\mathcal{L}}, B) \equiv 0$. Again, it is enough to consider $|\zeta - z| \le \frac{1}{2}\varepsilon$, so that $W^{\mathcal{L}} = g/\Phi$. It then follows from (12) and standard results about CF form that on bD the form $\Omega_q(W^{\mathcal{L}}, B)$ is given by a sum of terms

$$(13)\quad A_{q,\mu k} = \frac{a_{\mu,k,q}}{(2\pi i)^n} \frac{g \wedge \partial\beta \wedge (\overline{\partial_\zeta} g)^\mu \wedge (\overline{\partial_z} g)^k \wedge (\overline{\partial_\zeta}\partial\beta)^{n-q-2-\mu} \wedge (\overline{\partial_z}\partial\beta)^{q-k}}{\Phi^{1+\mu+k}\ \beta^{n-\mu-k-1}},$$

where $a_{\mu,k,q} \in \mathbb{Q}$, $0 \le \mu \le n-q-2$ and $0 \le k \le q$. As in the case of the kernel $\Omega_q(W^{\mathcal{L}})$, it follows that

$$A_{q,\mu k}(W^{\mathcal{L}}, B) = \frac{\mathcal{L}[\mu](\mathcal{E}_1^\sharp)^k}{\Phi^{1+\mu+k}} \frac{\mathcal{E}_1}{\beta^{n-\mu-k-1}}.$$

Consequently the kernels $A_{q,\mu k}(W^{\mathcal{L}}, B)$ are admissible of boundary order ≥ 1. The integral $\int_{bD} \bar\partial f \wedge \Omega_q(W^{\mathcal{L}}, B)$ is therefore covered by Theorem 3(c); in particular, its kernel is smoothing of order $\delta < \frac{1}{3}$.

Next, one checks that

$$\overline{\partial_z} A_{q,\mu k}(W^{\mathcal{L}}, B) = \frac{\mathcal{L}[\mu](\mathcal{E}_1^\sharp)^k}{\Phi^{1+\mu+k}} \frac{\mathcal{E}_0}{\beta^{n-\mu-k-1}} + + \frac{\mathcal{L}[\mu](\mathcal{E}_1^\sharp)^k \mathcal{E}_2^\sharp}{\Phi^{1+\mu+k+1}} \frac{\mathcal{E}_1}{\beta^{n-\mu-k-1}} + \frac{\mathcal{L}[\mu](\mathcal{E}_1^\sharp)^k}{\Phi^{1+\mu+k}} \frac{\mathcal{E}_2}{\beta^{n-\mu-k}}.$$

This shows that the kernels $\overline{\partial_z} A_{q,\mu k}(W^{\mathcal{L}}, B)$ are admissible, and one easily verifies that $\overline{\partial_z} A_{q,\mu k}$ is of boundary order ≥ 0.

Recall that $0 \le \mu \le n-q-2$ and $0 \le k \le q$, so that $0 \le \mu + k \le n-2$. We first consider the case $\mu + k \ge 1$, which occurs only when $n \ge 3$.

Lemma 5. *Suppose $\mu + k \ge 1$. Then $\overline{\partial_z} A_{q,\mu k}(W^{\mathcal{L}}, B)$ is a kernel of type $\Gamma^{\bar z}_{0,1/2}$.*

Proof. We saw that $\overline{\partial_z} A_{q,\mu k}(W^{\mathcal{L}}, B)$ is of order ≥ 0. Applying a derivative with respect to $\overline{z_j}$ to a factor $1/\Phi$ in any of the summands of $\overline{\partial_z} A_{q,\mu k}(W^{\mathcal{L}}, B)$ results in a term estimated by a kernel Γ_0 of order ≥ 0 multiplied by $|\mathcal{E}_2^\sharp/\Phi|$, and since $|\Phi| \gtrsim |r(z)|^{1-\delta}|\zeta - z|^{3\delta}$, the factor $|\mathcal{E}_2^\sharp/\Phi|$ will be bounded by $|\zeta - z|^\alpha |r(z)|^{-1+\delta}$ for some $\alpha > 0$ if $\delta < \frac{2}{3}$. By Theorem 3(a), the kernel $|\zeta - z|^\alpha |\Gamma_0|$ is integrable uniformly in z if $\alpha > 0$. For the other differentiations, note that since $\mu + k \ge 1$, each summand in $\overline{\partial_z} A_{q,\mu k}$ has at least one factor $\mathcal{L}[1]/\Phi$ or $\mathcal{E}_1^\sharp/\Phi$ with weight ≥ -2 in addition to the one factor $1/\Phi$ which is counted with weight ≥ -1 in the calculation of the order of $\overline{\partial_z} A_{q,\mu k}$. Differentiating the numerator of any such factor of weight ≥ -2 results in a term $\mathcal{E}_0/\Phi$, which can be estimated by $(1/|\zeta - z|^{3\delta})(1/|r(z)|^{1-\delta})$, where the factor $1/|\zeta - z|^{3\delta}$ is of weight > -2 for any $\delta < \frac{2}{3}$. If the differentiation is applied to any of the remaining factors in each of the summands, the order of the kernel decreases at most by one without affecting any of the factors $\mathcal{L}[1]/\Phi$ or $\mathcal{E}_1^\sharp/\Phi$. In order to compensate for this decrease, one must extract a factor $\mathcal{E}_1^\sharp$ from such a factor of weight -2, leaving a factor of weight -2 multiplied with a suitable power $|r(z)|^{-1+\delta}$. This follows as before for a factor $\mathcal{E}_1^\sharp/\Phi$, since $1/\Phi$ is estimated by $1/(|\zeta - z|^2|r(z)|^{1/3})$. For factors $\mathcal{L}[1]/\Phi$, note that according to (8) — after introducing z-diagonalizing coordinates — one may estimate $|\tan \bar\partial\partial r/\Phi|$ by terms of the type

$$\frac{1}{|r(z)| + |\zeta_l - z_l|^2} \lesssim \frac{1}{|\zeta_l - z_l|^{2\delta}|r(z)|^{1-\delta}} \lesssim \frac{1}{|\zeta_l - z_l|^{2\delta+1}|r(z)|^{1-\delta}} \mathcal{E}_1^\sharp$$

for a suitable $l \leq n-1$ (see [Range 2013] for details). Here the factor $1/|\zeta_l - z_l|^{2\delta+1}$ is of weight ≥ -2 for any $\delta < \frac{1}{2}$. Altogether we thus proved that $\overline{\partial_z} A_{q,\mu k}$ is $\bar z$-smoothing of any order $\delta < \frac{1}{2}$. Finally, if one considers a tangential derivative L_j^z, $j \leq n-1$, the same arguments apply as long as $\delta < \frac{1}{3}$. □

Remark. Note that this last argument restricts the order of $\bar z$-smoothing to $\delta < \frac{1}{2}$, while in all other previous instances one has the stronger estimates of order $\delta < \frac{2}{3}$.

6. The kernel $\overline{\partial_z} A_{q-1,00}$

We are thus left with $\overline{\partial_z} A_{q-1,00}$. This is the critical and most delicate case. Note that this kernel contains a term $\mathcal{E}_0/(\Phi\beta^{n-1})$ (of order ≥ 0); however, differentiation with respect to $\overline{z_j}$ will result, among others, in a term $\mathcal{E}_1/(\Phi\beta^n)$ which is estimated, at best, by $|\mathcal{E}_1/\beta^n||r(z)|^{-1}$. We see that $\overline{\partial_z} A_{q-1,00}$ contains terms which are not $\bar z$-smoothing of any positive order $\delta > 0$. In order to proceed we need to identify these critical terms and exploit certain approximate symmetries in analogy to the method introduced in [LR 1983].

We begin by applying Stokes' theorem to replace the integral

$$\int_{bD} f \wedge \overline{\partial_z} A_{q-1,00}$$

by an integral over D. For this purpose we first extend $A_{q-1,00}$ — that is, $W^{\mathcal{L}}$ and B — from the boundary into D without introducing any new singularities, as follows. By the estimate (4) for $|\Phi|$, as long as $-\varepsilon < r(\zeta) < 0$, one has $|\Phi| \gtrsim |r(\zeta)|$. Choose a C^∞ function φ on $\bar D$ so that $\varphi(\zeta) \equiv 1$ for $-\frac{1}{2}\varepsilon \leq r(\zeta)$ and $\varphi(\zeta) \equiv 0$ for

$r(\zeta) \le -\frac{3}{4}\varepsilon$. We then define the (0, 1)-form

$$\widehat{W^{\mathcal{L}}}(\zeta, z) = \varphi(\zeta) W^{\mathcal{L}}(\zeta, z)$$

on $\bar D \times \bar D - \{(\zeta, \zeta) : \zeta \in bD\}$, so that $\widehat{W^{\mathcal{L}}}(\zeta, z) = W^{\mathcal{L}}(\zeta, z)$ for $\zeta \in bD$.

We also define

$$\widehat{B}(\zeta, z) = \frac{\partial_\zeta \beta}{\mathcal{P}(\zeta, z)}, \quad \text{where } \mathcal{P}(\zeta, z) = \beta(\zeta, z) + \frac{2r(\zeta)r(z)}{\|\partial r(\zeta)\| \|\partial r(z)\|}$$

on $\bar D \times \bar D - \{(\zeta, \zeta) : \zeta \in bD\}$. (Note that $r(\zeta)r(z) \ge 0$ for $\zeta, z \in D$.) Clearly $\widehat{B}(\zeta, z) = B(\zeta, z)$ for $\zeta \in bD$. By replacing $W^{\mathcal{L}}$ with $\widehat{W^{\mathcal{L}}}$ and B with $\widehat{B}$ in $A_{q-1,00}$, one can therefore assume that $A_{q-1,00}$ extends to $\bar D \times \bar D - \{(\zeta, \zeta) : \zeta \in bD\}$ without any singularities.

It then follows that

$$\int_{bD} f \wedge \overline{\partial_z} A_{q-1,00} = \int_D \overline{\partial_\zeta} f \wedge \overline{\partial_z} A_{q-1,00} + (-1)^q \int_D f \wedge \overline{\partial_\zeta}\, \overline{\partial_z} A_{q-1,00}.$$

Remark. When one considers kernels that are integrated over D, the definition of admissible kernels and of their weighted order needs to be modified appropriately. First of all, the dimension of the domain of integration is now $2n$, which leads to an increase in order by one. Also, each factor $r(z)$ or $r(\zeta)$ in the numerator increases the order by at least one. Furthermore, since both $r(\zeta)$ and $\operatorname{Im} F^{(r)}(\zeta, z)$ are used as coordinates in a neighborhood of z for $\operatorname{dist}(z, bD) < \varepsilon$, the weighted order is adjusted to account for the fact that by estimate (4), now up to *two* factors Φ are counted with weight ≥ -1 (see [LR 1983, Definition 4.2] for more details). In particular, it follows that the (extended) kernel $\overline{\partial_z} A_{q-1,00}$, which is admissible of boundary order ≥ 0, is admissible of order ≥ 1 over D.

It is straightforward to prove the analogous version of Theorem 3 for kernels integrated over D. In particular, one then obtains:

Lemma 6. *The operator $T_{q-1,00} : C_{(0,q+1)}(\bar D) \to C^1_{(0,q)}(D)$ defined by*

$$T_{q-1,00}(\psi) = \int_D \psi \wedge \overline{\partial_z} A_{q-1,00}$$

is smoothing of any order $\delta < \frac{1}{3}$ and $\bar z$-smoothing of any order $\delta < 1$.

Note that because of the term $K|\zeta - z|^3$ contained in Φ, the kernel $\overline{\partial_z} A_{q-1,00}$ is only of class C^1 jointly in (ζ, z) near points where $\zeta = z$.

We are left with estimating integrals of the kernel $\overline{\partial_\zeta}\, \overline{\partial_z} A_{q-1,00} = \overline{\partial_z}\, \overline{\partial_\zeta} A_{q-1,00}$ over D. This kernel is readily seen to be of order ≥ 0, but it contains terms that are not $\bar z$-smoothing of any order $\delta > 0$.

Proceeding as in [LR 1983], we introduce the kernel $\overline{L_{q-1}} = (-1)^q * A_{q-1,00}$ for $1 \le q \le n-1$, where $*$ is the Hodge operator acting on the variable ζ with respect to the standard inner product of forms in $\mathbb{C}^n$. Note that in [LR 1983] the definition of L_{q-1} involved $\Omega_{q-1}(\widehat{W^{\mathcal{L}}}, \widehat{B})$, while here we only take those summands $A_{q,\mu k}$ with $\mu + k = 0$. Since $A_{q-1,00} = -*\overline{L_{q-1}}$, one then obtains

$$(-1)^q \overline{\partial_\zeta} A_{q-1,00} = **\overline{\partial_\zeta} A_{q-1,00} = *(*\overline{\partial_\zeta})(-*\overline{L_{q-1}}) = *\overline{\vartheta_\zeta}\,\overline{L_{q-1}},$$

where $\vartheta_\zeta = -*\partial_\zeta *$ is the (formal) adjoint of $\bar\partial$. It follows that

$$(-1)^q \int_D f \wedge \bar\partial_\zeta A_{q-1,00} = \int_D f \wedge *\overline{\vartheta_\zeta}\,\overline{L_{q-1}} = (f, \vartheta_\zeta L_{q-1})_D,$$

where the inner product is taken by integrating the pointwise inner product of forms over D. Since $\overline{\partial_z}$ commutes with $*_\zeta$, one has

$$(-1)^q \int_D f \wedge \overline{\partial_z}\,\overline{\partial_\zeta} A_{q-1,00} = (f, \partial_z \vartheta_\zeta L_{q-1})_D.$$

Let us introduce the Hermitian transpose K^* of a double form $K = K(\zeta, z)$ by

$$K^*(\zeta, z) = \overline{K(z, \zeta)}.$$

Note that $K^*(\zeta, z)$ is the kernel of the adjoint T^* of $T : f \to (f, K(\cdot\,, z))_D$, i.e., $T^*(f) \to (f, K^*(\cdot\,, z))_D$.

One now writes

$$(f, \partial_z \vartheta_\zeta L_{q-1})_D = (f, \partial_z \vartheta_\zeta L_{q-1} - [\partial_z \vartheta_\zeta L_{q-1}]^*)_D + (f, [\partial_z \vartheta_\zeta L_{q-1}]^*)_D.$$

Since $[\partial_z \vartheta_\zeta L_{q-1}]^* = \overline{\partial_\zeta}\,\overline{\vartheta_z} L^*_{q-1}$, if $f \in \operatorname{dom} \bar\partial^*$, one may integrate by parts in the second inner product, resulting in $(f, \overline{\partial_\zeta}\,\overline{\vartheta_z} L^*_{q-1})_D = (\bar\partial^* f, \overline{\vartheta_z} L^*_{q-1})_D$.

Expanding the definition of admissible kernels to allow for factors Φ^*, with corresponding definition of order, one verifies that the kernel $\vartheta_z \overline{L^*_{q-1}}$ is admissible of order ≥ 1, and consequently it is smoothing of order $\delta < \frac{1}{3}$, and $\bar z$-smoothing of order $\delta < 1$.

7. The critical singularities

We now carefully examine $\partial_z \vartheta_\zeta L_{q-1} - [\partial_z \vartheta_\zeta L_{q-1}]^*$ and verify that there is a cancellation of critical terms, so that the conjugate of this kernel is partially smoothing as required for the Main Theorem.

We use the standard orthonormal frame $\omega_1, \ldots, \omega_n$ for $(1,0)$-forms on a neighborhood U of $P \in bD$, with $\omega_n = \partial r(\zeta)/\|\partial r(\zeta)\|$. After shrinking ε, we may assume that $B(P, \varepsilon) \subset U$. As usual, we shall focus on estimating integrals for fixed $z \in D$ with $|z - P| < \frac{1}{2}\varepsilon$, and integration over $\zeta \in D \cap U$ with $r(\zeta) \geq -\frac{1}{2}\varepsilon$ and $|\zeta - z| < \frac{1}{2}\varepsilon$. Let $L_1, \ldots, L_n$ be the corresponding dual frame of $(1,0)$-vectors, acting on the ζ variables. If $L = \sum_{k=1}^n a_k(\zeta)\partial/\partial\zeta_k$, we denote by $L^z = L^z_\zeta = \sum_{k=1}^n a_k(\zeta)\partial/\partial z_k$

the corresponding vector field acting on the z variables. A similar convention applies to ω_j^z, which we denote by θ_j.

One then readily verifies the following equations:

(a) $\bar\partial r(\zeta) = \overline{\omega_n}\|\partial r(\zeta)\|$;

(b) $\partial\beta = \sum_{j=1}^n (L_j\beta)\omega_j$ and $\bar\partial\beta = \sum_{j=1}^n (\overline{L_j}\beta)\overline{\omega_j}$;

(c) $\bar\partial\partial\beta = 2\sum_{j=1}^n \overline{\omega_j}\wedge\omega_j + \mathcal{E}_1$;

(d) $\overline{\partial_z}\partial\beta = -2\sum_{j=1}^n \overline{\theta_j}\wedge\omega_j + \mathcal{E}_1$;

(e) $L_j^z\beta = -L_j\beta + \mathcal{E}_2$ and $L_jL_k\beta = \mathcal{E}_1$;

(f) $L_j\mathcal{P} = \mathcal{E}_1$ and $L_j^z\mathcal{P} = \mathcal{E}_1$ for $j < n$.

Somewhat more delicate are the following two formulas. They are the analogs of [LR 1983, Lemmas 5.9 and 5.35], with the differences due to the fact that in the present paper the defining function is not normalized, as it is restricted to a special form so that its level surfaces remain pseudoconvex. The definition of the extension $\mathcal{P}$ has been modified accordingly. Since both formulas require *exact* identification of the leading terms, we include the details of the proofs.

Lemma 7. $L_n^z\mathcal{P} = -\dfrac{2}{\|\partial r(\zeta)\|}\overline{\Phi} + \mathcal{E}_0 r(\zeta)r(z) + \mathcal{E}_1 r(\zeta) + \mathcal{E}_2.$

Proof. We fix $\zeta\in U$. After a unitary change of coordinates in the ζ variables, one may assume that $\partial r/\partial\zeta_j(\zeta) = 0$ for $j<n$ and $\partial r/\partial\zeta_n(\zeta) > 0$, so that $\|\partial r(\zeta)\| = \sqrt{2}\partial r/\partial\zeta_n(\zeta)$ and $(L_n^z)_z = \sqrt{2}\partial/\partial z_n + \mathcal{E}_1$. Then $L_n^z r(z) = \sqrt{2}\partial r/\partial z_n(\zeta) + \mathcal{E}_1 = \|\partial r(\zeta)\| + \mathcal{E}_1$. In this coordinate system one has

$$
\begin{aligned}
\sqrt{2}\overline{\Phi}(\zeta,z) &= \sqrt{2}\frac{\partial r}{\partial\overline{\zeta_n}}(\zeta)(\overline{\zeta_n - z_n}) + \mathcal{E}_2 - \sqrt{2}r(\zeta)\\
&= \|\partial r(\zeta)\|(\overline{\zeta_n - z_n}) - \sqrt{2}r(\zeta) + \mathcal{E}_2,
\end{aligned}
$$

and

$$
\begin{aligned}
L_n^z P(\zeta,z) &= -\sqrt{2}(\overline{\zeta_n - z_n}) + \mathcal{E}_2 + \frac{2r(\zeta)}{\|\partial r(\zeta)\|}\frac{L_n^z r(z)}{\|\partial r(z)\|} + \mathcal{E}_0 r(\zeta)r(z)\\
&= -\sqrt{2}(\overline{\zeta_n - z_n}) + \mathcal{E}_2 + \frac{2r(\zeta)}{\|\partial r(\zeta)\|}[1+\mathcal{E}_1] + \mathcal{E}_0 r(\zeta)r(z)\\
&= -\sqrt{2}(\overline{\zeta_n - z_n}) + \frac{2r(\zeta)}{\|\partial r(\zeta)\|} + \mathcal{E}_2 + \mathcal{E}_1 r(\zeta) + \mathcal{E}_0 r(\zeta)r(z)\\
&= -\frac{\sqrt{2}}{\|\partial r(\zeta)\|}[\|\partial r(\zeta)\|(\overline{\zeta_n - z_n}) - \sqrt{2}r(\zeta)]\\
&\qquad\qquad + \mathcal{E}_2 + \mathcal{E}_1 r(\zeta) + \mathcal{E}_0 r(\zeta)r(z).
\end{aligned}
$$

The proof is completed by combining these two equations. □

Lemma 8. $2\mathcal{P}-\sum_{j=1}^{n-1}|L_j\beta|^2=\frac{4}{\|\partial r(\zeta)\|\|\partial r(z)\|}|\Phi|^2+\mathcal{E}_3+\mathcal{E}_2 r(\zeta).$

Proof. Here we fix z, and after a unitary coordinate change in ζ we may assume that $L_j=\sqrt{2}\partial/\partial\zeta_j+\mathcal{E}_1$, so that $L_j\beta=\sqrt{2}(\overline{\zeta_j-z_j})+\mathcal{E}_2$. Hence $\sum_{j=1}^{n-1}|L_j\beta|^2=2\sum_{j=1}^{n-1}|\zeta_j-z_j|^2+\mathcal{E}_3$, and therefore

$$2\mathcal{P}-\sum_{j=1}^{n-1}|L_j\beta|^2=2|\zeta_n-z_n|^2+\frac{4r(\zeta)r(z)}{\|\partial r(\zeta)\|\|\partial r(z)\|}+\mathcal{E}_3. \tag{14}$$

Furthermore,

$$\begin{aligned}\sqrt{2}\overline{\Phi}(\zeta,z)&=\sqrt{2}\frac{\partial r}{\partial\overline{\zeta_n}}(z)(\overline{\zeta_n-z_n})+\mathcal{E}_2-\sqrt{2}r(\zeta)\\&=\|\partial r(z)\|(\overline{\zeta_n-z_n})-\sqrt{2}r(\zeta)+\mathcal{E}_2,\end{aligned}$$

It follows that

$$\begin{aligned}&2|\Phi(\zeta,z)|^2\\&\quad=\|\partial r(z)\|(\overline{\zeta_n-z_n})\sqrt{2}\Phi-2r(\zeta)\Phi+\mathcal{E}_2\Phi\\&\quad=\|\partial r(z)\|^2|\zeta_n-z_n|^2+\|\partial r(z)\|(\overline{\zeta_n-z_n})[-\sqrt{2}r(\zeta)+\mathcal{E}_2]-2r(\zeta)\Phi+\mathcal{E}_2\Phi\\&\quad=\|\partial r(z)\|^2|\zeta_n-z_n|^2-\sqrt{2}r(\zeta)[\|\partial r(z)\|(\overline{\zeta_n-z_n})+\sqrt{2}\Phi]+\mathcal{E}_3+\mathcal{E}_2\Phi.\end{aligned}$$

By (5) one has

$$\begin{aligned}\sqrt{2}\Phi=\sqrt{2}\Phi^*+\mathcal{E}_3&=\|\partial r(\zeta)\|(\overline{z_n-\zeta_n})-\sqrt{2}r(z)+\mathcal{E}_2\\&=-\|\partial r(z)\|(\overline{\zeta_n-z_n})-\sqrt{2}r(z)+\mathcal{E}_2,\end{aligned}$$

where we used that $\|\partial r(\zeta)\|=\|\partial r(z)\|+\mathcal{E}_1$. Inserting this equation into the previous one and using $\mathcal{E}_2\Phi=\mathcal{E}_3+\mathcal{E}_2 r(\zeta)$, results in

$$\begin{aligned}2|\Phi(\zeta,z)|^2&=\|\partial r(z)\|^2|\zeta_n-z_n|^2-\sqrt{2}r(\zeta)(-\sqrt{2}r(z)+\mathcal{E}_2)+\mathcal{E}_3+\mathcal{E}_2 r(\zeta)\\&=\|\partial r(\zeta)\|\|\partial r(z)\||\zeta_n-z_n|^2+2r(\zeta)r(z)+\mathcal{E}_3+\mathcal{E}_2 r(\zeta)\\&=\tfrac{1}{2}\|\partial r(\zeta)\|\|\partial r(z)\|\left[2|\zeta_n-z_n|^2+\frac{4r(\zeta)r(z)}{\|\partial r(\zeta)\|\|\partial r(z)\|}\right]+\mathcal{E}_3+\mathcal{E}_2 r(\zeta).\end{aligned}$$

The lemma follows after inserting (14) and rearranging. □

We now identify precisely the kernel L_{q-1} and the critical highest order singularity of $\partial_z\vartheta_\zeta L_{q-1}$ with respect to the standard frames introduced above. To simplify notation we replace $q-1$ with q and consider $A_{q,00}$ and L_q for $0\le q\le n-2$. The computations follow closely those for the case $\mu=0$ in [LR 1983]; therefore we just state the relevant formulas, and provide more details only where critical differences arise.

From (13) one sees that

$$A_{q,00} = \frac{a_q}{(2\pi i)^n}\frac{g\wedge\partial\beta\wedge(\overline{\partial_\zeta}\partial\beta)^{n-q-2}\wedge(\overline{\partial_z}\partial\beta)^q}{\Phi\mathcal{P}^{n-1}}$$
$$= \frac{a_q}{(2\pi i)^n}\frac{\partial r\wedge\partial\beta\wedge(\overline{\partial_\zeta}\partial\beta)^{n-q-2}\wedge(\overline{\partial_z}\partial\beta)^q+\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}}.$$

Then

$$\overline{L_q} = (-1)^{q+1} * A_{q,00} = \overline{C_q} + \frac{\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}},$$

where

$$\overline{C_q} = (-1)^{q+1}\frac{a_q}{(2\pi i)^n} *_\zeta \frac{\partial r\wedge\partial\beta\wedge(\overline{\partial_\zeta}\partial\beta)^{n-q-2}\wedge(\overline{\partial_z}\partial\beta)^q}{\Phi\mathcal{P}^{n-1}}.$$

It follows that

$$\overline{\partial_z}\overline{\vartheta_\zeta}\overline{L_q} = \overline{\partial_z}\overline{\vartheta_\zeta}\overline{C_q} + \overline{\partial_z}\overline{\vartheta_\zeta}\frac{\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}}.$$

Remark. Note that in contrast to [LR 1983], the kernels $L_q = C_q + \mathcal{E}_2/(\bar\Phi\mathcal{P}^{n-1})$ and $\partial_z\vartheta_\zeta L_q$ analyzed here only involve the term corresponding to $\mu = 0$ in the same reference. Since in this paper we are concerned with $\bar z$-smoothing, we need to consider the conjugates $\overline{L_q}$, $\overline{C_q}$, and $\overline{\partial_z}\overline{\vartheta_\zeta}\overline{L_q}$, which are the kernels that appear in the integral $\int f\wedge *\overline{\partial_z}\overline{\vartheta_\zeta}\overline{L_q} = (f,\partial_z\vartheta_\zeta L_q)_D$.

Lemma 9. $\overline{\partial_z}\overline{\vartheta_\zeta}\dfrac{\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}}$ *is of type* Γ_1.

Proof. The proof of this lemma involves a straightforward verification. Note that $\mathcal{E}_2/(\Phi\mathcal{P}^{n-1})$ is of order ≥ 3. Differentiation with respect to ζ reduces the order by one only, since after differentiating $1/\Phi$ the resulting factor $1/\Phi^2$ has weight -2. Similarly, since $\overline{\partial_z}\Phi = \mathcal{E}_2^\sharp$, subsequent application of $\overline{\partial_z}$ also reduces the order by one only. □

Next we represent $\overline{C_q}$ in terms of the local orthonormal frames. By utilizing the various formulas recalled above, it follows that

$$\overline{C_q} = \frac{\gamma_q\|\partial r(\zeta)\|}{i^n\Phi\mathcal{P}^{n-1}} *_\zeta \sum_{\substack{|Q|=q\\ j<n}} \omega_n\wedge(L_j\beta)\omega_j\wedge(\bar\omega\wedge\omega)^J\wedge\omega^Q\wedge\bar\theta^Q + \frac{\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}},$$

where γ_q is a *real* constant. The summation is over all strictly increasing q-tuples Q with $n\notin Q$, over $j<n$ with $j\notin Q$, and J is the ordered $(n-q-2)$-tuple complementary to njQ in $\{1,\dots,n\}$. Since $*[\omega^{njQ}\wedge(\bar\omega\wedge\omega)^J] = b_q i^n\omega^{njQ}$, where b_q is real, it follows that

$$\overline{C_q} = \widetilde{\gamma_q}\|\partial r(\zeta)\| \sum_{\substack{|Q|=q\\ j<n}} \frac{L_j\beta}{\Phi\mathcal{P}^{n-1}}\omega^{njQ}\wedge\bar\theta^Q + \frac{\mathcal{E}_2}{\Phi\mathcal{P}^{n-1}},$$

for another *real* constant $\widetilde{\gamma_q}$. By using Lemma 9, it then follows that

$$\overline{\partial_z}\,\overline{\vartheta_\zeta}\,\overline{C_q} = \widetilde{\gamma_q}\|\partial r(\zeta)\|\overline{\partial_z}\,\overline{\vartheta_\zeta}\Bigg[\sum_{\substack{|Q|=q\\ j<n}} \frac{L_j\beta}{\Phi\mathcal{P}^{n-1}}\omega^{njQ}\wedge\bar{\theta}^Q\Bigg] + \Gamma_1.$$

Let us introduce

$$\overline{C_q^0} = \sum_{\substack{|Q|=q\\ j<n}} \frac{L_j\beta}{\Phi\mathcal{P}^{n-1}}\omega^{njQ}\wedge\bar{\theta}^Q.$$

The heart of the analysis of $\partial_z\vartheta_\zeta L_q - [\partial_z\vartheta_\zeta L_q]^*$ is contained in:

Theorem 10. *For $0\le q\le n-2$ the kernel*

$$\overline{\partial_z}\,\overline{\vartheta_\zeta}\,\overline{C_q^0} - [\overline{\partial_z}\,\overline{\vartheta_\zeta}\,\overline{C_q^0}]^*$$

is of type $\Gamma^{\bar{z}}_{0,2/3}$.

Corollary 11. *The operator*

$$f \to (f, \partial_z\vartheta_\zeta L_{q-1} - [\partial_z\vartheta_\zeta L_{q-1}]^*)_D$$

is $\bar{z}$-smoothing of order $\delta < \frac{2}{3}$ and tangentially smoothing of order $\delta < \frac{1}{3}$ for $1\le q\le n-1$.

Proof. This follows from Theorem 10 by using Lemma 9 and also by observing that the differentiation of $\|\partial r(\zeta)\|$ results in an error term of type Γ_1. Similarly, when considering the difference $[\cdots]-[\cdots]^*$, the substitution $\|\partial r(\zeta)\| = \|\partial r(z)\| + \mathcal{E}_1$ leads to an error term of the same type. □

Remark. In order to be consistent with the notation and formulas in [LR 1983], in the proof of Theorem 10 we analyze $\Delta_q = \partial_z\vartheta_\zeta C_q^0 - [\partial_z\vartheta_\zeta C_q^0]^*$. In the end we must verify that its *conjugate* $\overline{\Delta_q}$ is of type $\Gamma^{\bar{z}}_{0,2/3}$.

Since C_q^0 is a double form of type $(0, q+2)$ in ζ and type $(q, 0)$ in z, the form $\partial_z\vartheta_\zeta C_q^0$ is of type $(0, q+1)$ in ζ and type $(q+1, 0)$ in z. Consequently,

$$\partial_z\vartheta_\zeta C_q^0 = \vartheta_\zeta\partial_z C_q^0 = \sum_{|L|=q+1}\Bigg(\sum_{|K|=q+1} A_{KL}\bar{\omega}^K\Bigg)\wedge\theta^L,$$

where the sums are taken over all strictly increasing $(q+1)$-tuples L and K. It follows that

$$\partial_z\vartheta_\zeta C_q^0 - [\partial_z\vartheta_\zeta C_q^0]^* = \sum_{|L|=q+1}\Bigg(\sum_{|K|=q+1}[A_{KL} - (A_{LK})^*]\bar{\omega}^K\Bigg)\wedge\theta^L.$$

In the next section we identify the coefficients A_{KL} precisely in order to verify that $[A_{KL} - (A_{LK})^*]$ is of order ≥ 0 and that its *conjugate* is at least of type $\Gamma^{\bar{z}}_{0,2/3}$.

8. The approximate symmetries

The computation of $\vartheta_\zeta \partial_z C_q^0$ uses the expressions for ∂_z and ϑ_ζ in terms of the standard adapted boundary frames plus error terms which do not involve differentiation. These error terms — in the end — result in kernels which are conjugates of admissible kernels of order ≥ 1, and hence will be ignored in the discussion that follows.

As usual ε^L_{lQ} denotes the sign of the permutation which carries the ordered $(q+1)$-tuple lQ into the ordered $(q+1)$-tuple L if $lQ = L$ as sets, and $\varepsilon^L_{lQ} = 0$ otherwise. We introduce $m_{lj} = (1/\overline{\Phi}) L^z_l(\overline{L_j}\beta/\mathcal{P}^{n-1})$ for $1 \leq j < n$ and $1 \leq l \leq n$.

Lemma 12. *For any K, L one has*

$$A_{KL} = -\sum_{\substack{Q\\ j,l,k}} \varepsilon^L_{lQ}\varepsilon^{njQ}_{kK}(L_k m_{lj}) + \overline{\Gamma^{\bar z}_{0,2/3}} + \Gamma_1.$$

Proof. This lemma is the analogue of [LR 1983, Lemma 5.6]. In the present case, the computation shows that the error term is of the form $\mathcal{E}_3/[\overline{\Phi}^3\mathcal{P}^{n-1}] + \Gamma_1$, where the first term is only of order zero. (In the case where D is strictly pseudoconvex, and hence $|\Phi| \gtrsim |\zeta - z|^2$, this term is Γ_1 as well.) However, one readily checks that its conjugate $\mathcal{E}_3/[\Phi^3\mathcal{P}^{n-1}]$ is of type $\Gamma^{\bar z}_{0,2/3}$. □

We are therefore left with

$$A^{(0)}_{KL} = -\sum_{\substack{Q\\ j,l,k}} \varepsilon^L_{lQ}\varepsilon^{njQ}_{kK}(L_k m_{lj}). \tag{15}$$

Only terms with $j < n$ appear with nonzero coefficients. In the following it will be assumed that $j < n$.

For $l < n$ one has

$$m_{lj} = \frac{-2\delta_{lj}}{\overline{\Phi}\mathcal{P}^{n-1}} - (n-1)\frac{L^z_l\beta\overline{L_j}\beta}{\overline{\Phi}\mathcal{P}^n} + \frac{\mathcal{E}_1}{\overline{\Phi}\mathcal{P}^{n-1}}, \tag{16}$$

while by using Lemma 7 one obtains

$$\begin{aligned} m_{nj} &= \frac{1}{\overline{\Phi}}\left[\frac{L^z_n\overline{L_j}\beta}{\mathcal{P}^{n-1}} - (n-1)\frac{(L^z_n\mathcal{P})\overline{L_j}\beta}{\mathcal{P}^n}\right] \\ &= \frac{2(n-1)}{\|\partial r(\zeta)\|}\frac{\overline{L_j}\beta}{\mathcal{P}^n} + \frac{\mathcal{E}_1}{\overline{\Phi}\mathcal{P}^{n-1}} + \frac{\mathcal{E}_1 r(\zeta)r(z) + \mathcal{E}_2 r(\zeta) + \mathcal{E}_3}{\overline{\Phi}\mathcal{P}^n}. \end{aligned} \tag{17}$$

Note that all the error terms are of order ≥ 2, and that they have only *one* factor Φ in the denominator. Consequently, applying L_k to the error terms results in Γ_1 terms. So only the leading terms of m_{lj} identified above need to be considered in the following analysis.

Proof of Theorem 10. Since all relevant error terms are of type $\Gamma^{\bar{z}}_{0,2/3}$ or better, it is enough to examine

$$\overline{A^{(0)}_{KL}} - \overline{A^{(0)*}_{LK}}, \quad \text{where } \overline{A^{(0)}_{KL}} = -\sum_{\substack{Q\\ j,l,k}} \varepsilon^{lQ}_{L}\varepsilon^{njQ}_{kK}(\overline{L_k m_{lj}}).$$

We need to consider separate cases, depending on whether n is in K (resp. L), or not:

Case 1. $n \in K$ and $n \in L$.

In this case the computations in [LR 1983] apply without further changes, subject to the adjustments due to the fact that the defining function r is not normalized. Combined with $\|\partial r(\zeta)\| = \|\partial r(z)\| + \mathcal{E}_1$, it follows that

$$\overline{A^{(0)}_{KL} - A^{(0)*}_{LK}} = \Gamma_1.$$

Case 2. $n \notin K$ and $n \notin L$.

In this case $\varepsilon^{njQ}_{kK} \neq 0$ only for $k = n$. Hence

$$A^{(0)}_{KL} = -\sum_{\substack{Q\\ j,l}} \varepsilon^{L}_{lQ}\varepsilon^{njQ}_{nK}(L_n m_{lj}) + \Gamma_1 = -\sum_{\substack{l\in L\\ j\in K}} \varepsilon^{L}_{lQ}\varepsilon^{jQ}_{K}(L_n m_{lj}) + \Gamma_1.$$

Lemma 5.21 in [LR 1983] needs to be replaced by:

Lemma 13. *$L_n m_{lj} - (L_n m_{jl})^* = \overline{\Gamma^{\bar{z}}_{0,2/3}}$ for $j, l < n$.*

Assuming the lemma, one obtains (after replacing j with l in the last equation)

$$\begin{aligned} A^{(0)*}_{LK} &= -\sum_{j,l<n} \varepsilon^{K}_{lQ}\varepsilon^{jQ}_{L}(L_n m_{lj})^* + \Gamma_1 = -\sum_{\substack{j\in L\\ l\in K}} \varepsilon^{K}_{lQ}\varepsilon^{jQ}_{L}(L_n m_{jl}) + \overline{\Gamma^{\bar{z}}_{0,2/3}} \\ &= A^{(0)}_{KL} + \overline{\Gamma^{\bar{z}}_{2/3}}. \end{aligned}$$

To prove the lemma, one uses (16). The calculation of $L_n m_{lj}$ proceeds as in [LR 1983] with the obvious changes. By using $\|\partial r(\zeta)\| = \|\partial r(z)\| + \mathcal{E}_1$ one obtains

$$\begin{aligned} &(L_n m_{jl})^* - L_n m_{lj} \\ &= -\frac{2\delta_{lj}\|\partial r(\zeta)\|}{\mathcal{P}^{n-1}}\left[\frac{1}{\overline{\overline{\Phi^{*2}}}} - \frac{1}{\overline{\overline{\Phi^2}}}\right] - 4\frac{\delta_{lj}(n-1)}{\|\partial r(\zeta)\|\mathcal{P}^n}\left[\frac{\Phi^*}{\overline{\overline{\Phi^*}}} - \frac{\Phi}{\overline{\overline{\Phi}}}\right] \\ &\quad -(n-1)\frac{L^z_j\beta\overline{L_l\beta}\|\partial r(\zeta)\|}{\mathcal{P}^n}\left[\frac{1}{\overline{\overline{\Phi^{*2}}}} - \frac{1}{\overline{\overline{\Phi^2}}}\right] - \frac{2n(n-1)}{\|\partial r(\zeta)\|}\frac{L^z_j\beta\overline{L_l\beta}}{\mathcal{P}^{n+1}}\left[\frac{\Phi^*}{\overline{\overline{\Phi^*}}} - \frac{\Phi}{\overline{\overline{\Phi}}}\right] + \Gamma_1. \end{aligned}$$

In the strictly pseudoconvex case the differences in $[\cdots]$ are of higher order than the terms individually, resulting in $(L_n m_{jl})^* - L_n m_{lj} = \Gamma_1$. In the present case, only a weaker result holds, as follows. Note that — after taking conjugates — one has

$$\frac{1}{\Phi^{*2}} - \frac{1}{\Phi^2} = \frac{(\Phi - \Phi^*)(\Phi + \Phi^*)}{\Phi^{*2}\Phi^2} = \frac{\mathcal{E}_3}{\Phi^{*2}\Phi} + \frac{\mathcal{E}_3}{\Phi^*\Phi^2},$$

where we have used the approximate symmetry (5) in the second equation. It now readily follows that $\mathcal{E}_3/[\mathcal{P}^{n-1}\Phi^{*2}\Phi]$ and $\mathcal{E}_3/[\mathcal{P}^{n-1}\Phi^*\Phi^2]$ (while of order zero, and hence not smoothing as in the strictly pseudoconvex case) are in fact of type $\Gamma^{\bar z}_{0,2/3}$. Since $\mathcal{E}_2/\mathcal{P}^n$ is estimated by $\mathcal{E}_0/\mathcal{P}^{n-1}$, the same argument works for the third term above. For the conjugate of the second term, note that

$$\frac{1}{\mathcal{P}^n}\left[\frac{\overline{\Phi^*}}{\Phi^*}-\frac{\bar\Phi}{\Phi}\right]=\frac{1}{\mathcal{P}^n}\left[\frac{\Phi\overline{\Phi^*}-\Phi^*\bar\Phi}{\Phi^*\Phi}\right]=\frac{1}{\mathcal{P}^n}\frac{\mathcal{E}_3}{\Phi^*\Phi}=\Gamma_1.$$

The fourth term is estimated the same way by first estimating $\mathcal{E}_2/\mathcal{P}^{n+1}$ by $\mathcal{E}_0/\mathcal{P}^n$.

Case 3. The mixed case $n\in K$ and $n\notin L$.

As in [LR 1983], this is — computationally — the most complicated case. On the other hand, aside from the differences as noted, for example, in Lemmas 7 and 8, the details of the proof essentially carry over from [LR 1983] to the case considered here, with the result that one has

$$\overline{A^{(0)}_{KL}}-\overline{A^{(0)*}_{LK}}=\Gamma_1.$$

In more detail, since $n\in K$, there is exactly one ordered q-tuple J such that $K=J\cup\{n\}$, and one then has $\varepsilon^K_{nJ}A^{(0)}_{KL}=A^{(0)}_{(nJ)L}$.

Note that we need to identify the leading terms of both $\overline{A^{(0)}_{KL}}$ and $\overline{A^{(0)}_{LK}}$. Let us first consider the simpler term $\overline{A^{(0)}_{LK}}$. After interchanging L and K in (15), one has

$$A^{(0)}_{LK}=-\sum_{\substack{Q\\ j,l,k}}\varepsilon^K_{lQ}\varepsilon^{njQ}_{kL}(L_k m_{lj})+\Gamma_1.$$

Since $n\notin L$, the factor $\varepsilon^K_{lQ}\varepsilon^{njQ}_{kL}\neq 0$ only if $k=n$ and $l=n$, and furthermore $Q=J$. Therefore the leading term of $A^{(0)}_{LK}$, i.e., the sum, is different from zero only if $J\subset L$ so that $\varepsilon^{jJ}_L\neq 0$ only for that unique j for which $L=J\cup\{j\}$. It follows that for $j<n$ one has

$$A^{(0)}_{(jJ)(nJ)}=\varepsilon^{jJ}_L\,\varepsilon^K_{nJ}A^{(0)}_{LK}+\Gamma_1=-L_n m_{nj}+\Gamma_1.$$

Since $L_n\mathcal{P}=(\overline{L^z_n\mathcal{P}})^*$, Lemma 7 implies that

$$L_n\mathcal{P}=-\frac{2}{\|\partial r(z)\|}\Phi^*+\mathcal{E}_0 r(\zeta)r(z)+\mathcal{E}_1 r(z)+\mathcal{E}_2.$$

By using this equation and (17), it follows that

$$A^{(0)}_{(jJ)(nJ)}=-\frac{4n(n-1)}{\|\partial r(\zeta)\|\,\|\partial r(z)\|}\frac{\overline{L_j\beta}\Phi^*}{\mathcal{P}^{n+1}}+\Gamma_1. \tag{18}$$

Finally we calculate $\overline{A^{(0)}_{KL}}$. With J as before, one has

$$A^{(0)}_{KL} = -\sum_{\substack{Q\\ j,l,k}} \varepsilon^{L}_{lQ}\varepsilon^{njQ}_{kK}(L_k m_{lj}) + \Gamma_1 = \varepsilon^{nJ}_{K} \sum_{\substack{n\notin Q\\ j,l,k<n}} \varepsilon^{L}_{lQ}\varepsilon^{jQ}_{kJ}(L_k m_{lj}) + \Gamma_1.$$

Continuing with the intricate calculations as in [LR 1983, pp. 237–239, Case Id)] and using Lemma 8 above in place of [LR 1983, Lemma 5.35], one obtains

$$A^{(0)}_{KL} = -\varepsilon^{nJ}_{K} \frac{4n(n-1)}{\|\partial r(\zeta)\|\|\partial r(z)\|} \sum_{l<n} \varepsilon^{L}_{lJ} \frac{(L^z_l\beta)\Phi}{\mathcal{P}^{n+1}} + \Gamma_1.$$

Here the only nonzero term in the sum arises for that unique l, for which $L = J\cup\{l\}$. Consequently, the last formula implies

$$A^{(0)}_{(nJ)(lJ)} = -\frac{4n(n-1)}{\|\partial r(\zeta)\|\|\partial r(z)\|} \frac{(L^z_l\beta)\Phi}{\mathcal{P}^{n+1}} + \Gamma_1. \tag{19}$$

Let us now consider $\overline{A^{(0)}_{KL}} - \overline{A^{(0)*}_{LK}}$. As the preceding formulas show, each summand is of type Γ_1 except in the case that for the unique q-tuple $J\subset\{1,2,\dots,n-1\}$ with $K = J\cup\{n\}$, L satisfies $L = J\cup\{l\}$ as sets for some unique $l<n$. In this latter case, equations (19) and (18) imply

$$\overline{A^{(0)}_{(nJ)(lJ)}} - \overline{A^{(0)*}_{(lJ)(nJ)}} = -\frac{4n(n-1)}{\|\partial r(\zeta)\|\|\partial r(z)\|} \frac{1}{\mathcal{P}^{n+1}} \overline{[(L^z_l\beta)\Phi - (\overline{L_l\beta})^*\Phi]} + \Gamma_1 = \Gamma_1.$$

The last equation holds because $(\overline{L_l\beta})^* = L^z_l\beta$.

Case 4. $n\notin K$ and $n\in L$.

This is reduced to Case 3 by noting that

$$\overline{A^{(0)}_{KL}} - \overline{A^{(0)*}_{LK}} = -\big(\overline{A^{(0)}_{LK}} - \overline{A^{(0)*}_{KL}}\big)^*.$$

It thus follows that for all K and L one has $\overline{A^{(0)}_{KL}} - \overline{A^{(0)*}_{LK}} = \Gamma^{\bar z}_{0,2/3}$. This completes the proof of Theorem 10. □

9. Proof of the Main Theorem

In Sections 4–8, we have analyzed the integrals that appear in the representation (3) of the boundary operator S^{bD}. By combining these results, it follows that for all $f\in\mathcal{D}^1_{qU}$ and $z\in D\cap U$ one has the estimates

$$|\overline{L^z_j} S^{bD}(f)(z)| \le C_\delta \operatorname{dist}(z,bD)^{\delta-1} Q_0(f) \quad \text{for } j=1,\dots,n \text{ and any } \delta<\tfrac{1}{2},$$

$$|L^z_j S^{bD}(f)(z)| \le C_\delta \operatorname{dist}(z,bD)^{\delta-1} Q_0(f) \quad \text{for } j=1,\dots,n-1 \text{ and any } \delta<\tfrac{1}{3}.$$

We have thus completed the proof of claims (ii) and (iii) of the Main Theorem. As noted earlier, (i) follows trivially from the classical estimate (2) for S^{iso}.

Finally we prove the statement about the normal components of f. We use the following lemma, which is a routine variation of classical estimates for the BMK kernel. For $0 \le \alpha < 1$, set $C^{-\alpha}_{(0,1)}(D) = \{g \in C_{(0,1)}(D) : \sup_{z\in D} |g(z)| \operatorname{dist}(z, bD)^\alpha < \infty\}$, with the norm $|g|_{-\alpha}$ defined by the relevant supremum.

Lemma 14. *The operator $T^{BM} : C^{-\alpha}_{(0,1)}(D) \to C(D)$ defined by*

$$T^{BM}(g) = \int_D g(\zeta) \wedge \Omega_0(B)$$

satisfies the estimate

$$|T^{BM}(g)|_{1-\alpha'} \lesssim |g|_{-\alpha} \quad \text{for any } \alpha' > \alpha. \tag{20}$$

Now suppose $f \in \mathcal{D}^1_{qU}$ and let f_J be a normal component of f, so that $f_J|_{bD} = 0$. Decompose $f_J = h_J + [S^{\text{iso}}(f)]_J$, where $h = S^{bD}(f)$. We already know by estimate (2) that $|S^{\text{iso}}(f)|_\alpha \lesssim Q_0(f)$ for any $\alpha < 1$. Note that on $bD \cap U$ one has $h_J = -[S^{\text{iso}}(f)]_J$, so that $|(h_J|_{bD\cap U})|_\alpha \lesssim |S^{\text{iso}}(f)|_\alpha \lesssim Q_0(f)$ as well. By standard properties of the BM kernel $\Omega_0(B)$, it follows that $\int_{bD} h_J \Omega_0(B)$ satisfies the same estimate on $\bar D \cap U$ if $\alpha > 0$. By the case $q = 0$ of the BMK representation formula (1) applied to h_J, one has

$$h_J = \int_{bD} h_J\, \Omega_0(B) - \int_D \bar\partial h_J \wedge \Omega_0(B). \tag{21}$$

Given $\delta < \frac{1}{2}$, choose δ' with $\delta < \delta' < \frac{1}{2}$. By part (ii) of the Main Theorem, $\bar\partial h_J \in C^{-(1-\delta')}_{(0,1)}(D)$, with $|\bar\partial h_J|_{-(1-\delta')} \le C_{\delta'} Q_0(f)$. It then follows from Lemma 14 that

$$|T^{BM}(\bar\partial h_J)|_\delta \lesssim Q_0(f).$$

Each summand in the representation (21) therefore satisfies the desired Hölder estimate, so that

$$|h_J|_{\Lambda^\delta(\bar D\cap U)} \lesssim Q_0(f).$$

Since $f_J = h_J + [S^{\text{iso}}(f)]_J$, the required estimate $|f_J|_\delta \lesssim Q_0(f)$ holds as well. □

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