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THE JOHNSON–MORITA THEORY FOR THE RING OF FRICKE CHARACTERS OF FREE GROUPS

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For a free group F_n of rank n, we consider the ring $\mathfrak{X}_{\mathbb{Q}}(F_n)$ of $\mathrm{SL}(2, \mathbb{C})$-characters of F_n generated by all Fricke characters $\mathrm{tr}\, x$ for $x \in F_n$. Its ideal J generated by $\mathrm{tr}\, x - 2$ for all $x \in F_n$ is $\mathrm{Aut}\, F_n$-invariant. We denote by $\mathcal{E}_n(1)$ the subgroup of the automorphism group $\mathrm{Aut}\, F_n$ of F_n consisting of all automorphisms which act on J/J^2 trivially. The group $\mathcal{E}_n(1)$ is regarded as a Fricke character analogue of the IA-automorphism group of F_n and the Torelli subgroup of the mapping class group of a surface. In our previous work, we constructed a homomorphism η_1 from $\mathcal{E}_n(1)$ into $\mathrm{Hom}_{\mathbb{Q}}(J/J^2, J^2/J^3)$ as a Fricke character analogue of the first Johnson homomorphisms of the mapping class group and $\mathrm{Aut}\, F_n$.

In this paper, according to Morita's work for the extension of the first Johnson homomorphism of the mapping class group, we extend η_1 to $\mathrm{Aut}\, F_n$ as a crossed homomorphism. We see that the obtained crossed homomorphism η is not null cohomologous. We also compute the images of Nielsen's generators of $\mathrm{Aut}\, F_n$ by η.

1. Introduction

In a series of works, Dennis Johnson [1980; 1983; 1985a; 1985b] established a remarkable method to investigate the group structure of the Torelli subgroup of the mapping class group of a surface. In particular, he constructed in [Johnson 1985b] a homomorphism τ to determine the abelianization of the Torelli subgroup.

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Today, his homomorphism τ is called the first Johnson homomorphism and has been generalized to those of higher degrees. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the work of many authors including Morita [1993a], Hain [1997] and others.

Let F_n be a free group generated by $x_1, x_2, \dots, x_n$. As is well known, for any $g \geq 1$, the mapping class group $\mathcal{M}_{g,1}$ of a compact oriented surface $\Sigma_{g,1}$ with one boundary component can be embedded into $\mathrm{Aut}\, F_{2g}$ by a classical work of Dehn and Nielsen. This embedding is induced from the action of $\mathcal{M}_{g,1}$ on the fundamental group of $\Sigma_{g,1}$. The definition of the Johnson homomorphisms of $\mathcal{M}_{g,1}$ can be naturally generalized to those of $\mathrm{Aut}\, F_n$. Let H be the abelianization of F_n. The kernel of the homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, H \cong \mathrm{GL}(n, \mathbb{Z})$ induced from the action of $\mathrm{Aut}\, F_n$ on H, is called the IA-automorphism group of F_n and is denoted by IA_n. The group IA_n is a free group analogue of the Torelli subgroup $\mathcal{I}_{g,1}$ of $\mathcal{M}_{g,1}$. Andreadakis [1965] introduced a central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of IA_n, and showed that each graded quotient $\mathrm{gr}^k \mathcal{A}_n := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. We call the above filtration the Andreadakis–Johnson filtration of $\mathrm{Aut}\, F_n$. Johnson studied this type of filtration for the mapping class groups in 1980s. The general linear group $\mathrm{GL}(n, \mathbb{Z})$ naturally acts on each $\mathrm{gr}^k \mathcal{A}_n$. In order to investigate the $\mathrm{GL}(n, \mathbb{Z})$-module structure of $\mathrm{gr}^k \mathcal{A}_n$, the k-th Johnson homomorphism

$$\tau_k : \mathrm{gr}^k \mathcal{A}_n \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

is a powerful and useful tool. However, even the $\mathrm{GL}(n, \mathbb{Q})$-structure of $(\mathrm{gr}^k \mathcal{A}_n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is not determined in general (see Section 2C for notation, and [Satoh 2009; 2013b] for basic materials concerning the Andreadakis–Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\, F_n$).

Now, we study a Fricke character analogue of the Andreadakis–Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. Let $R(F_n)$ be the set of all $\mathrm{SL}(2, \mathbb{C})$-representations of F_n, and $\mathcal{F}(R(F_n), \mathbb{C})$ the set of all complex-valued functions on $R(F_n)$. Then $\mathcal{F}(R(F_n), \mathbb{C})$ naturally has the $\mathbb{C}$-algebra structure coming from the pointwise product, and $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{F}(R(F_n), \mathbb{C})$ from the right. For any $x \in F_n$, define $\mathrm{tr}\, x \in \mathcal{F}(R(F_n), \mathbb{C})$ by

$$(\mathrm{tr}\, x)(\rho) := \mathrm{tr}\, \rho(x), \qquad \rho \in R(F_n).$$

Here tr on the right-hand side means the usual trace of 2×2 matrix $\rho(x)$. The element $\mathrm{tr}\, x$ is called the Fricke character of $x \in F_n$. Classically, Fricke characters were introduced by Fricke and Klein [1897] to study the moduli space of compact

Riemann surfaces. In this paper, however, we focus on purely algebraic properties of the Fricke characters. Let $\mathfrak{X}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n), \mathbb{C})$ generated by all $\operatorname{tr} x$ for $x \in F_n$. We call $\mathfrak{X}_{\mathbb{Q}}(F_n)$ the ring of Fricke characters of F_n over $\mathbb{Q}$. Horowitz [1972] showed that $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is finitely generated by

$$\{\operatorname{tr} x_{i_1} \cdots x_{i_l} \mid 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n\}.$$

In order to establish the Johnson–Morita theory for $\mathfrak{X}_{\mathbb{Q}}(F_n)$, to begin with, we need a descending filtration of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ consisting of $\operatorname{Aut} F_n$-invariant ideals. Consider an ideal

$$J := (\operatorname{tr}' x_{i_1} \cdots x_{i_l} \mid 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n) \subset \mathfrak{X}_{\mathbb{Q}}(F_n),$$

where $\operatorname{tr}' x := \operatorname{tr} x - 2$ for any $x \in F_n$. Magnus [1980] showed that J is $\operatorname{Aut} F_n$-invariant for the case where $n = 3$, and studied a representation of the quotient group of IA_3 by the inner automorphism group $\operatorname{Inn} F_3$ of F_3. On the other hand, it is easily seen that J is $\operatorname{Aut} F_n$-invariant for general $n \ge 3$. In this paper we use this ideal and the descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

in order to define a descending filtration of $\operatorname{Aut} F_n$ as an analogy of the Andreadakis–Johnson filtration. Although each graded quotient $\operatorname{gr}^k J := J^k / J^{k+1}$ is an $\operatorname{Aut} F_n$-invariant finite-dimensional $\mathbb{Q}$-vector space, by combinatorial complexities, it is quite difficult to give a basis of $\operatorname{gr}^k J$ for $n \ge 3$ in general. In [Hatakenaka and Satoh 2014] — henceforth abbreviated [HS 2014] — we explicitly give bases of $\operatorname{gr}^k J$ for $k = 1$ and 2 (see Section 4 for details).

For any $k \ge 1$, set

$$\mathcal{E}_n(k) := \ker\bigl(\operatorname{Aut} F_n \to \operatorname{Aut}(J/J^{k+1})\bigr),$$

where the homomorphism is induced by the action of $\operatorname{Aut} F_n$ on J/J^{k+1}. The groups $\mathcal{E}_n(k)$ define a descending filtration

$$\mathcal{E}_n(1) \supset \mathcal{E}_n(2) \supset \cdots$$

of $\operatorname{Aut} F_n$. In [HS 2014], we showed this filtration is central, and $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$ for any $k \ge 1$. Furthermore, we determined the first term $\mathcal{E}_n(1)$ to be $\operatorname{Inn} F_n \cdot \mathcal{A}_n(2)$ where $\operatorname{Inn} F_n$ is the inner automorphism group of F_n. Thus, each graded quotient $\operatorname{gr}^k \mathcal{E}_n := \mathcal{E}_n(k)/\mathcal{E}_n(k+1)$ is an abelian group for any $k \ge 1$. In order to study the structures of $\operatorname{gr}^k \mathcal{E}_n$, we have introduced homomorphisms

$$\eta_k : \operatorname{gr}^k \mathcal{E}_n \to \operatorname{Hom}_{\mathbb{Q}}(\operatorname{gr}^1 J, \operatorname{gr}^{k+1} J)$$

defined by

$$\sigma \pmod{\mathcal{E}_n(k+1)} \mapsto \bigl(f \pmod{J^2} \mapsto f^\sigma - f \pmod{J^{k+1}}\bigr).$$

The homomorphism η_k is a Fricke character analogue of the k-th Johnson homomorphism τ_k. In [HS 2014], we showed that each η_k is an $\mathrm{Aut}\, F_n/\mathcal{E}_n(1)$-equivariant injective homomorphism. This implies that each of $\mathrm{gr}^k\mathcal{E}_n$ is torsion-free, and that $\dim_{\mathbb{Q}}(\mathrm{gr}^k\mathcal{E}_n \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$.

In this paper, we concentrate on the homomorphism

$$\tilde{\eta}_1 : \mathcal{E}_n(1) \to \mathrm{gr}^1\mathcal{E}_n \xrightarrow{\eta_1} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J).$$

Morita [1993b] showed that the composition

$$\mathcal{I}_{g,1} \xrightarrow{\tau_1} H^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}(k+1) \to (H^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}(k+1)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right]$$

of the first Johnson homomorphism of the mapping class group $\mathcal{M}_{g,1}$ the natural projection naturally extends to $\mathcal{M}_{g,1}$ as a crossed homomorphism. He also showed that this extension is unique up to 1-coboundary. Here $\mathbb{Z}\left[\frac{1}{2}\right]$ means a subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by attaching $\frac{1}{2}$. The analogous result for $\mathrm{Aut}\, F_n$ was obtained by Kawazumi [2006], who showed the composition

$$\mathrm{IA}_n \to \mathrm{gr}^1\mathcal{A}_n \xrightarrow{\tau_1} H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to (H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{2}\right]$$

of the first Johnson homomorphism of $\mathrm{Aut}\, F_n$ with the natural projection naturally and uniquely extends to $\mathrm{Aut}\, F_n$ as a crossed homomorphism up to 1-coboundary. Furthermore, very recently, Day [2013] showed that each Johnson homomorphism τ_k of $\mathrm{Aut}\, F_n$ can be extended to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.

The main purpose of the paper is to give a Fricke character analogue of these results. Namely, according to Morita's work, we extend $\tilde{\eta}_1$ to $\mathrm{Aut}\, F_n$ as a crossed homomorphism:

Theorem 3.6. *There is a crossed homomorphism $\eta : \mathrm{Aut}\, F_n \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ such that the restriction of η to $\mathcal{E}_n(1)$ is η_1.*

At the present stage, we do not know whether the extension η is unique up to 1-coboundary or not since we cannot determine the first cohomology group of $\mathrm{Aut}\, F_n$ with coefficients in $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ due to the combinatorial complexity. By using his extended crossed homomorphisms, Kawazumi [2006] constructed twisted higher cocycles of $\mathrm{Aut}\, F_n$, and investigated its restriction to the mapping class group. In particular, he expressed the Morita–Mumford classes as a cup product of the twisted cocycles. In order to study the twisted cohomology groups of $\mathrm{Aut}\, F_n$ with coefficients in modules of the Fricke characters, we are convinced that our work establishes a foothold as a first step.

In Section 2, we review the definitions of Fricke characters, the Johnson homomorphisms τ_k and the homomorphisms η_k. In Section 3, we extend the homomorphism η_1 to $\mathrm{Aut}\, F_n$ as a crossed homomorphism. In Section 4, we show that the crossed homomorphism η is not null cohomologous in the first cohomology group of $\mathrm{Aut}\, F_n$ with coefficients in $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$. Furthermore, in Section 5, we calculate the image of Nielsen's generators of $\mathrm{Aut}\, F_n$ by η.

2. Preliminaries

2A. ***Notation and conventions.*** Throughout the paper, we use the following notation and conventions: Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab} unless otherwise noted.
- The automorphism group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^{σ}.
- For an element $g \in G$, we also denote the coset class of g in G/N by g if there is no confusion in context.
- For any elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2B. ***The rings of Fricke characters.*** In this subsection, we review the rings of Fricke characters of the free group F_n. Let $R(F_n)$ be the set $\mathrm{Hom}(F_n, \mathrm{SL}(2, \mathbb{C}))$ of all $\mathrm{SL}(2, \mathbb{C})$-representations of F_n, and $\mathcal{F}(R(F_n), \mathbb{C})$ the set of all complex-valued functions on $R(F_n)$. Then $\mathcal{F}(R(F_n), \mathbb{C})$ naturally has a $\mathbb{C}$-algebra structure by the operations defined by

$$(\chi + \chi')(\rho) := \chi(\rho) + \chi'(\rho),$$
$$(\chi\chi')(\rho) := \chi(\rho)\chi'(\rho),$$
$$(\lambda\chi)(\rho) := \lambda(\chi(\rho)),$$

for any $\chi, \chi' \in \mathcal{F}(R(F_n), \mathbb{C})$, $\lambda \in \mathbb{C}$ and $\rho \in R(F_n)$. The group $\mathrm{Aut}\, F_n$ naturally acts on $R(F_n)$ and $\mathcal{F}(R(F_n), \mathbb{C})$ from the right by

$$\rho^{\sigma}(x) := \rho(x^{\sigma^{-1}}) \quad \text{for } \rho \in R(F_n) \text{ and } x \in F_n,$$
$$\chi^{\sigma}(\rho) := \chi(\rho^{\sigma^{-1}}) \quad \text{for } \chi \in \mathcal{F}(R(F_n), \mathbb{C}) \text{ and } \rho \in R(F_n),$$

for any $\sigma \in \mathrm{Aut}\, F_n$. For any $x \in F_n$, we define an element $\mathrm{tr}\, x$ of $\mathcal{F}(R(F_n), \mathbb{C})$ by

$$(\mathrm{tr}\, x)(\rho) := \mathrm{tr}\, \rho(x)$$

for any $\rho \in R(F_n)$. Here tr on the right-hand side means the trace of 2×2 matrix $\rho(x)$. The element $\mathrm{tr}\, x$ is called the Fricke character of $x \in F_n$. The action of an element $\sigma \in \mathrm{Aut}\, F_n$ on $\mathrm{tr}\, x$ is given by $\mathrm{tr}\, x^{\sigma}$. We have the following well-known formulae:

(1) $\mathrm{tr}\, xy + \mathrm{tr}\, xy^{-1} = (\mathrm{tr}\, x)(\mathrm{tr}\, y)$,

(2) $\mathrm{tr}\, xyz + \mathrm{tr}\, yxz = (\mathrm{tr}\, x)(\mathrm{tr}\, yz) + (\mathrm{tr}\, y)(\mathrm{tr}\, xz) + (\mathrm{tr}\, z)(\mathrm{tr}\, xy) - (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, z)$,

(3) $\mathrm{tr}[x, y] = (\mathrm{tr}\, x)^2 + (\mathrm{tr}\, y)^2 + (\mathrm{tr}\, xy)^2 - (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, xy) - 2$,

$$(4)\quad 2\,\mathrm{tr}\,xyzw = (\mathrm{tr}\,x)(\mathrm{tr}\,yzw) + (\mathrm{tr}\,y)(\mathrm{tr}\,zwx) + (\mathrm{tr}\,z)(\mathrm{tr}\,wxy) + (\mathrm{tr}\,w)(\mathrm{tr}\,xyz)$$
$$+ (\mathrm{tr}\,xy)(\mathrm{tr}\,zw) - (\mathrm{tr}\,xz)(\mathrm{tr}\,yw) + (\mathrm{tr}\,xw)(\mathrm{tr}\,yz)$$
$$- (\mathrm{tr}\,x)(\mathrm{tr}\,y)(\mathrm{tr}\,zw) - (\mathrm{tr}\,y)(\mathrm{tr}\,z)(\mathrm{tr}\,xw) - (\mathrm{tr}\,x)(\mathrm{tr}\,w)(\mathrm{tr}\,yz)$$
$$- (\mathrm{tr}\,z)(\mathrm{tr}\,w)(\mathrm{tr}\,xy) + (\mathrm{tr}\,x)(\mathrm{tr}\,y)(\mathrm{tr}\,z)(\mathrm{tr}\,w)$$

for any $x, y, z, w \in F_n$. Equations (2) and (4) are due to Vogt [1889]. (For details, see [Maclachlan and Reid 2003, Section 3.4] for example.) The point of (2) is that $\mathrm{tr}\,yxz$ can be written as a sum of $-\mathrm{tr}\,xyz$ and a polynomial in $\mathrm{tr}\,v$, with v a word in x, y, z of length at most two. Similarly, the point of (4) is that $\mathrm{tr}\,xyzw$ can be written as a polynomial in $\mathrm{tr}\,v$ with v a word in x, y, z, w of length at most three.

Let $\mathfrak{X}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-vector subspace of $\mathcal{F}(R(F_n), \mathbb{C})$ generated by all $\mathrm{tr}\,x$ for $x \in F_n$. From (1), it turns out that $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is a $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n), \mathbb{C})$. We call $\mathfrak{X}_{\mathbb{Q}}(F_n)$ the ring of Fricke characters of F_n over $\mathbb{Q}$. Let $\mathbb{Q}[t]$ be the rational polynomial ring

$$\mathbb{Q}[t_{i_1\cdots i_l} \mid 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n]$$

of $n + \binom{n}{2} + \binom{n}{3}$ indeterminates. Consider the ring homomorphism

$$\pi : \mathbb{Q}[t] \to \mathcal{F}(R(F_n), \mathbb{C})$$

defined by

$$\pi(1) := \tfrac{1}{2}(\mathrm{tr}\,1_{F_n}), \qquad \pi(t_{i_1\cdots i_l}) := \mathrm{tr}\,x_{i_1}\cdots x_{i_l}.$$

Clearly, $\mathrm{Im}\,\pi \subset \mathfrak{X}_{\mathbb{Q}}(F_n)$. By a classical result due to Horowitz [1972], we have:

Theorem 2.1. *For any $n \ge 2$, the homomorphism $\pi : \mathbb{Q}[t] \to \mathfrak{X}_{\mathbb{Q}}(F_n)$ is surjective.*

More precisely, Horowitz obtained a generating set of the ring of Fricke characters of G over $\mathbb{Z}$. Using this and (4), we can obtain the above theorem. Set

$$I := \mathrm{Ker}\,\pi = \{ f \in \mathbb{Q}[t] \mid f(\mathrm{tr}\,\rho(x_{i_1}\cdots x_{i_l})) = 0 \text{ for any } \rho \in R(F_n)\}.$$

Then π induces an isomorphism $\mathbb{Q}[t]/I \to \mathfrak{X}_{\mathbb{Q}}(F_n)$. In this paper, we always identify $\mathbb{Q}[t]/I$ with $\mathfrak{X}_{\mathbb{Q}}(F_n)$ through this isomorphism, and also call $\mathbb{Q}[t]/I$ the ring of Fricke characters of F_n over $\mathbb{Q}$. We define an action of $\mathrm{Aut}\,F_n$ on $\mathbb{Q}[t]/I$ such that the isomorphism $\mathbb{Q}[t]/I \xrightarrow{\cong} \mathfrak{X}_{\mathbb{Q}}(F_n)$ is $\mathrm{Aut}\,F_n$-equivariant.

We remark that the structure of the ideal I is quite complicated in general. It is an open problem to find a generating set for I when $n \ge 4$. Horowitz [1972] showed that $I = (0)$ for $n = 1$ and 2, and that I is the principal ideal generated by a quadratic polynomial

$$t_{123}^2 - P_{123}(t)t_{123} + Q_{123}(t),$$

where

$$P_{abc}(t) := t_{ab}t_c + t_{ac}t_b + t_{bc}t_a - t_a t_b t_c,$$
$$Q_{abc}(t) := t_a^2 + t_b^2 + t_c^2 + t_{ab}^2 + t_{ac}^2 + t_{bc}^2 - t_a t_b t_{ab} - t_a t_c t_{ac} - t_b t_c t_{bc} + t_{ab} t_{bc} t_{ac} - 4.$$

For $n \geq 4$, Whittemore [1973] showed that the ideal I is not principal. However, very little is known further in this case.

Although we can obtain a representation of $\mathrm{Aut}\, F_n$ by the action of $\mathrm{Aut}\, F_n$ on $\mathfrak{X}_{\mathbb{Q}}(F_n) \cong \mathbb{Q}[t]/I$, it is an infinite-degree representation, and hence it is not so easy to handle. In order to construct finite-dimensional representations of $\mathrm{Aut}\, F_n$, we consider a descending filtration of $\mathrm{Aut}\, F_n$-invariant ideals of $\mathbb{Q}[t]/I$, and take its graded quotients. Set $t'_{i_1 \cdots i_l} := t_{i_1 \cdots i_l} - 2 \in \mathbb{Q}[t]$. For simplicity, we also denote by $t'_{i_1 \cdots i_l}$ its coset class in $\mathbb{Q}[t]/I$ by abuse of notation. Consider the ideal

$$J := (t'_{i_1 \cdots i_l} \mid 1 \leq l \leq 3,\ 1 \leq i_1 < i_2 < \cdots < i_l \leq n) \subset \mathbb{Q}[t]/I$$

of $\mathbb{Q}[t]/I$ generated by all $t'_{i_1 \cdots i_l}$s. Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of $\mathrm{Aut}\, F_n$-invariant ideals of $\mathbb{Q}[t]/I$. Set $\mathrm{gr}^k J := J^k / J^{k+1}$ for each $k \geq 1$. Then each $\mathrm{gr}^k J$ is an $\mathrm{Aut}\, F_n$-invariant finite-dimensional $\mathbb{Q}$-vector space. In order to describe the action of $\mathrm{Aut}\, F_n$ on $\mathrm{gr}^k J$ precisely, we have to find a basis of it. In general, however, by combinatorial complexities, it is quite a difficult problem. In [HS 2014], we explicitly gave bases of $\mathrm{gr}^k J$ for $k = 1$ and 2 (see Section 4). For $k \geq 3$, even the dimension of $\mathrm{gr}^k J$ is not determined.

In [Hatakenaka and Satoh 2015], we studied the rings of Fricke characters of free abelian groups by a parallel argument as above. We review that work briefly. Let H be the abelianization of F_n. The (coset classes of) $x_1, x_2, \ldots, x_n$ form a basis of H. Let $\mathfrak{X}_{\mathbb{Q}}(H)$ be the ring of Fricke characters of H over $\mathbb{Q}$. Since

$$2\,\mathrm{tr}\, xyz = (\mathrm{tr}\, x)(\mathrm{tr}\, yz) + (\mathrm{tr}\, y)(\mathrm{tr}\, xz) + (\mathrm{tr}\, z)(\mathrm{tr}\, xy) - (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, z)$$

in $\mathfrak{X}_{\mathbb{Q}}(H)$ for any $x, y, z \in H$, by using Horowitz's result as mentioned above, we can see that $\mathfrak{X}_{\mathbb{Q}}(H)$ is generated by $\mathrm{tr}\, x_i$ for $1 \leq i \leq n$ and $\mathrm{tr}\, x_i x_j$ for $1 \leq i < j \leq n$. Let J_H be the ideal generated by all $\mathrm{tr}'\, x_i := \mathrm{tr}\, x_i - 2$ for $1 \leq i \leq n$ and $\mathrm{tr}'\, x_i x_j := \mathrm{tr}\, x_i x_j - 2$ for $1 \leq i < j \leq n$. Then we have a descending filtration $J_H \supset J_H^2 \supset \cdots$. We obtained a basis of the graded quotient $\mathrm{gr}^k J_H$ as a $\mathbb{Q}$-vector space for any $k \geq 1$:

Theorem 2.2 [Hatakenaka and Satoh 2015]. *For any $n \geq 2$ and $k \geq 1$,*

$$\bigcup_{l=0}^{k} \Big\{ (\mathrm{tr}'\, x_{p_1} x_{q_1}) \cdots (\mathrm{tr}'\, x_{p_l} x_{q_l})(\mathrm{tr}'\, x_{i_{l+1}}) \cdots (\mathrm{tr}'\, x_{i_k}) \\ \Big| \ 1 \leq p_1 < q_1 < \cdots < p_l < q_l \leq n,\ 1 \leq i_{l+1} \leq \cdots \leq i_k \leq n \Big\}$$

is a basis of $\mathrm{gr}^k J_H$.

As a corollary, we obtain a lower bound on the dimension of $\mathrm{gr}^k J$ by

$$\dim_{\mathbb{Q}}(\mathrm{gr}^k J) \geq \dim_{\mathbb{Q}}(\mathrm{gr}^k J_H) = \sum_{l=0}^{k} \binom{n}{2l} \binom{n+k-l-1}{k-l}$$

through the surjective homomorphism $\mathrm{gr}^k J \to \mathrm{gr}^k J_H$ induced from the abelianization $F_n \to H$.

2C. ***Johnson homomorphisms of* Aut F_n.** In this subsection, we review the Johnson homomorphisms of Aut F_n (for details, see [Satoh 2013b], for example). Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. We identify Aut H with the general linear group $\mathrm{GL}(n, \mathbb{Z})$ by fixing the basis $x_1, \ldots, x_n$ of H. The kernel IA_n of ρ is called the IA-automorphism group of F_n. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \qquad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n. For each $k \geq 1$, the natural action of Aut F_n on $\mathcal{L}_n(k)$ induces that of $\mathrm{GL}(n, \mathbb{Z})$ since IA_n acts on $\mathcal{L}_n(k)$ trivially.

For each $k \geq 1$, the action of Aut F_n on the nilpotent quotient group $F_n/\Gamma_n(k+1)$ induces a homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote its kernel by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \mathcal{A}_n(3) \supset \cdots$$

of IA_n. We call this the Andreadakis–Johnson filtration of Aut F_n; historically, it was introduced by Andreadakis [1965], who showed:

Theorem 2.3. (1) *For any $k, l \geq 1$, $\sigma \in \mathcal{A}_n(k)$ and $x \in \Gamma_n(l)$, $x^{-1}x^{\sigma} \in \Gamma_n(k+l)$.*

(2) *For any k and $l \geq 1$, $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$.*

(3) $\bigcap_{k \geq 1} \mathcal{A}_n(k) = 1$.

In the 1980s, Dennis Johnson studied this type of filtration for the mapping class group of a surface, and this became known as the Johnson filtration of the mapping class group.

For each $k \geq 1$, the group Aut F_n acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbb{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$ on the graded quotients $\mathrm{gr}^k \mathcal{A}_n := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ by Theorem 2.3(2). In order to study the $\mathrm{GL}(n, \mathbb{Z})$-module structure of $\mathrm{gr}^k\mathcal{A}_n$, we consider the Johnson homomorphisms of $\mathrm{Aut}\,F_n$. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of $\tilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k\mathcal{A}_n \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1).$$

The homomorphisms $\tilde{\tau}_k$ and τ_k are called the k-th Johnson homomorphisms of $\mathrm{Aut}\,F_n$. We can see each τ_k is $\mathrm{GL}(n, \mathbb{Z})$-equivariant.

Now, we have a question to ask whether the first Johnson homomorphism can be extended to $\mathrm{Aut}\,F_n$ or not. Such study and results were given by Morita [1993b] for the mapping class group of a surface, and by Kawazumi [2006] for the automorphism group of a free group as mentioned in the introduction. In particular, Kawazumi showed the first rational Johnson homomorphism

$$\tilde{\tau}_1 : \mathrm{IA}_n \to H^* \otimes_{\mathbb{Z}} \Lambda^2 H \to (H^* \otimes_{\mathbb{Z}} \Lambda^2 H) \otimes_{\mathbb{Z}} \mathbb{Q}$$

can be extended to $\mathrm{Aut}\,F_n$ as a crossed homomorphism. In [Satoh 2013a], we showed that

$$H^1(\mathrm{Aut}\,F_n, (H^* \otimes_{\mathbb{Z}} \Lambda^2 H) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong \mathbb{Q}^{\oplus 2},$$

and that the extension of the first Johnson homomorphism and a crossed homomorphism obtained from the Magnus representation form a basis of the above first cohomology group. These results are free group analogues of Morita's work [1993b] for the mapping class group of a surface.

2D. *Homomorphisms η_k.* In [HS 2014], we introduced homomorphisms η_k which are Fricke character analogue of the Johnson homomorphisms τ_k. Here we review the definition of η_k.

For any $k \geq 1$, let

$$\mathcal{E}_n(k) := \mathrm{Ker}(\mathrm{Aut}\,F_n \to \mathrm{Aut}(J/J^{k+1}))$$

be the kernel of a homomorphism $\mathrm{Aut}\,F_n \to \mathrm{Aut}(J/J^{k+1})$ induced from the action of $\mathrm{Aut}\,F_n$ on J/J^{k+1}. Then the groups $\mathcal{E}_n(k)$ define a descending filtration

$$\mathcal{E}_n(1) \supset \mathcal{E}_n(2) \supset \cdots \supset \mathcal{E}_n(k) \supset \cdots$$

of $\mathrm{Aut}\,F_n$.

Theorem 2.4 [HS 2014]. *For any $n \geq 3$, we have*:

(1) *For any $k, l \geq 1$, $[\mathcal{E}_n(k), \mathcal{E}_n(l)] \subset \mathcal{E}_n(k+l)$.*

(2) *$\mathcal{E}_n(1) = \mathrm{Inn}\,F_n \cdot \mathcal{A}_n(2)$, where $\mathrm{Inn}\,F_n$ is the inner automorphism group of F_n.*

(3) *For any $k \geq 1$, $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$.*

From Theorem 2.4(1), the graded quotients $\mathrm{gr}^k\mathcal{E}_n := \mathcal{E}_n(k)/\mathcal{E}_n(k+1)$ are abelian groups for any $k \geq 1$. Since each $\mathcal{E}_n(k)$ is a normal subgroup of $\mathrm{Aut}\,F_n$, the group $\mathrm{Aut}\,F_n$ naturally acts on $\mathrm{gr}^k\mathcal{E}_n$ by the conjugation from the right. Namely, for any $\sigma \in \mathrm{Aut}\,F_n$ and $\tau \in \mathcal{E}_n(k)$, the action of σ on $\tau \pmod{\mathcal{E}_n(k+1)}$ is given by

$$(\tau \pmod{\mathcal{E}_n(k+1)}) \cdot \sigma := \sigma^{-1}\tau\sigma \pmod{\mathcal{E}_n(k+1)}.$$

Furthermore, since $\{\mathcal{E}_n(k)\}$ is a central filtration, the action of $\mathcal{E}_n(1)$ on $\mathrm{gr}^k\mathcal{E}_n$ is trivial. Hence we can consider each $\mathrm{gr}^k\mathcal{E}_n$ as an $\mathrm{Aut}\,F_n/\mathcal{E}_n(1)$-module. In order to investigate the $\mathrm{Aut}\,F_n/\mathcal{E}_n(1)$-module structure of $\mathrm{gr}^k\mathcal{E}_n$, we have introduced

$$\eta_k : \mathrm{gr}^k\mathcal{E}_n \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^{k+1} J)$$

defined by

$$\sigma \pmod{\mathcal{E}_n(k+1)} \mapsto \big(f \pmod{J^2} \mapsto f^\sigma - f \pmod{J^{k+1}}\big).$$

In [HS 2014], we showed that each η_k is an $\mathrm{Aut}\,F_n/\mathcal{E}_n(1)$-equivariant injective homomorphism. However, the structure of the image of η_k is not well-understood even in the case where $k = 1$.

3. An extension of η_1 as a crossed homomorphism

In this section, we extend the homomorphism

$$\tilde{\eta}_1 : \mathcal{E}_n(1) \to \mathrm{gr}^1\mathcal{E}_n \xrightarrow{\eta_1} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$$

to $\mathrm{Aut}\,F_n$ as a crossed homomorphism, following [Morita 1993b]. Furthermore, according to the usual convention in homological algebra, for any group G and G-module M, we consider that G acts on M from the left unless otherwise noted. Hence, the right actions mentioned above are read as left actions in the natural way. For example, for any $\sigma \in \mathrm{Aut}\,F_n$ and $x \in F_n$, the left action of σ on the Fricke character $\mathrm{tr}\,x$ is given by

$$\sigma \cdot (\mathrm{tr}\,x) = \mathrm{tr}\,x^{\sigma^{-1}}.$$

For basic materials for cohomology of associative algebras, see [Cartan and Eilenberg 1999, Chapter IX], for example.

First, consider an extension

$$0 \to J^2/J^3 \to \mathfrak{X}_{\mathbb{Q}}(F_n)/J^3 \to \mathfrak{X}_{\mathbb{Q}}(F_n)/J^2 \to 0 \tag{5}$$

of associative $\mathbb{Q}$-algebras. For an associative ring R, we denote by $\mathrm{Aut}_{(\mathrm{Ring})}(R)$ the ring automorphism group of R. For a $\mathbb{Q}$-vector space M, we denote by $\mathrm{Aut}(M)$ the $\mathbb{Q}$-linear automorphism group of M.

Proposition 3.1. *The natural homomorphism*

$$\Phi : \mathrm{Aut}_{(\mathrm{Ring})}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^3) \to \mathrm{Aut}_{(\mathrm{Ring})}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2)$$

is surjective.

Proof. First, observe the characteristic class $\theta(\mathrm{id}_{J/J^2})$ of the extension (5), where

$$\theta : \mathrm{Hom}_{E(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2)}(J^2/J^3, J^2/J^3) \to H^2(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2, J^2/J^3)$$

is the connecting homomorphism, and $E(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2)$ is the enveloping algebra of $\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2$. Choose a 2-cocycle c of the algebra $\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2$ with coefficients in J^2/J^3, which represents the cohomology class $\theta(\mathrm{id}_{J/J^2})$. Then $\mathfrak{X}_{\mathbb{Q}}(F_n)/J^3$ can be explicitly described as the product $J^2/J^3 \times \mathfrak{X}_{\mathbb{Q}}(F_n)/J^2$ equipped with the multiplication given by

$$(\xi, \tau)(\xi', \tau') = (\xi\tau' + \tau\xi' + c(\tau, \tau'), \tau\tau')$$

for any $\xi, \xi' \in J^2/J^3$ and $\tau, \tau' \in \mathfrak{X}_{\mathbb{Q}}(F_n)/J^2$. We denote by Λ_c this associative algebra.

For any $\alpha \in \mathrm{Aut}_{(\mathrm{Ring})}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^2)$, since the cohomology class of c is the characteristic class induced from id_{J/J^2}, the 2-cocycle $\alpha^{\sharp}(c)$ should be cohomologous to c where $\alpha^{\sharp}$ is the induced homomorphism from α. Hence there exists a 1-chain $d : \mathfrak{X}_{\mathbb{Q}}(F_n)/J^2 \to J^2/J^3$ such that $\alpha^{\sharp}(c) - c = \delta d$, where δ is the coboundary homomorphism. Namely, we have

$$\alpha^{-1} \cdot c(\alpha(\tau), \alpha(\tau')) - c(\tau, \tau') = \tau d(\tau') - d(\tau\tau') + d(\tau)\tau'$$

for any $\tau, \tau' \in \mathfrak{X}_{\mathbb{Q}}(F_n)/J^2$. Define the map $\tilde{\alpha} : \Lambda_c \to \Lambda_c$ to be

$$(\xi, \tau) \mapsto (\alpha(\xi) - \alpha(d(\tau)), \alpha(\tau)).$$

Then $\tilde{\alpha} \in \mathrm{Aut}_{(\mathrm{Ring})}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^3)$ under the identification $\Lambda_c = \mathfrak{X}_{\mathbb{Q}}(F_n)/J^3$, and $\Phi(\tilde{\alpha}) = \alpha$. □

For any $f \in J$, we denote the coset class of f in J/J^k by $[f]_k$. For any $k \geq 2$, set

$$\overline{\mathrm{Aut}}(J/J^k) := \{\sigma \in \mathrm{Aut}(J/J^k) \mid \sigma([\gamma\gamma']_k) = \sigma([\gamma]_k)\,\sigma([\gamma']_k),\ \gamma, \gamma' \in J\}.$$

Note that $\overline{\mathrm{Aut}}(J/J^2) = \mathrm{Aut}(J/J^2) = \mathrm{GL}_{\mathbb{Q}}(J/J^2)$.

Lemma 3.2. *The group homomorphism*

$$\Psi : \mathrm{Aut}_{(\mathrm{Ring})}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^k) \to \overline{\mathrm{Aut}}(J/J^k),$$

defined by $\sigma \mapsto \sigma|_{J/J^k}$, *is an isomorphism.*

Proof. Consider the polynomial ring

$$\mathbb{Q}[t'] := \mathbb{Q}[t'_{i_1\cdots i_l} \mid 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n]$$

with indeterminates $t'_{i_1\cdots i_l}$, and the natural surjection $\pi' : \mathbb{Q}[t'] \to \mathfrak{X}_{\mathbb{Q}}(F_n)$ given by $1 \mapsto \frac{1}{2}(\mathrm{tr}\, 1_{F_n})$ and $t'_{i_1\cdots i_l} \mapsto \mathrm{tr}'\, x_{i_1} \cdots x_{i_l}$. The kernel I' of π' is contained in the ideal J' generated by all $t'_{i_1\cdots i_l}$.

We construct the inverse map of Ψ. For any $\beta \in \overline{\mathrm{Aut}}(J/J^k)$, define the $\mathbb{Q}$-algebra homomorphism $\tilde{\beta} : \mathbb{Q}[t'] \to \mathfrak{X}_{\mathbb{Q}}(F_n)/J^k$ to be

$$\tilde{\beta}(1) := \left[\tfrac{1}{2}(\mathrm{tr}\, 1_{F_n})\right]_k, \quad \tilde{\beta}(t'_{i_1\cdots i_l}) := \beta([\pi'(t'_{i_1\cdots i_l})]_k).$$

Since $I' \subset \mathrm{Ker}\, \tilde{\beta}$ and $\tilde{\beta}(J') = J$, the above $\tilde{\beta}$ induces a ring homomorphism $\mathfrak{X}_{\mathbb{Q}}(F_n)/J^k \to \mathfrak{X}_{\mathbb{Q}}(F_n)/J^k$, say $\tilde{\beta}$, by abuse of notation. Since β is an automorphism, so is $\tilde{\beta}$. Then we have the homomorphism $\Psi' : \overline{\mathrm{Aut}}(J/J^k) \to \mathrm{Aut}(\mathfrak{X}_{\mathbb{Q}}(F_n)/J^k)$ defined by $\beta \mapsto \tilde{\beta}$, and see that Ψ' is the inverse of Ψ. $\square$

From Proposition 3.1 and Lemma 3.2, we obtain the induced surjective homomorphism

$$\varphi : \overline{\mathrm{Aut}}(J/J^3) \to \overline{\mathrm{Aut}}(J/J^2).$$

Next, we consider an embedding of $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ into $\overline{\mathrm{Aut}}(J/J^3)$. For any $f \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$, define the map $\tilde{f} : J/J^3 \to J/J^3$ by

$$\tilde{f}([\gamma]_3) := [\gamma]_3 + f([\gamma]_2)$$

for any $\gamma \in J$.

Proposition 3.3. *With the above notation, for any* $f \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$, *we see* $\tilde{f} \in \overline{\mathrm{Aut}}(J/J^3)$, *and the map*

$$\iota : \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J) \to \overline{\mathrm{Aut}}(J/J^3),$$

defined by $f \mapsto \tilde{f}$, *is injective.*

Proof. First, we show $\tilde{f}$ is a homomorphism. For any $\gamma, \gamma' \in J$,

$$\begin{aligned}\tilde{f}([\gamma]_3 + [\gamma']_3) &= \tilde{f}([\gamma + \gamma']_3) = [\gamma + \gamma']_3 + f([\gamma + \gamma']_2)\\ &= ([\gamma]_3 + f([\gamma]_2)) + ([\gamma']_3 + f([\gamma']_2))\\ &= \tilde{f}([\gamma]_3) + \tilde{f}([\gamma']_3).\end{aligned}$$

Thus $\tilde{f}$ is a homomorphism. Furthermore, $\tilde{f}$ satisfies

$$\begin{aligned}\tilde{f}([\gamma]_3[\gamma']_3) &= \tilde{f}([\gamma\gamma']_3) = [\gamma\gamma']_3 + f([\gamma\gamma']_2) = [\gamma]_3[\gamma']_3\\ &= ([\gamma]_3 + f([\gamma]_2))([\gamma']_3 + f([\gamma']_2)) = \tilde{f}([\gamma]_3)\tilde{f}([\gamma']_3)\end{aligned}$$

for any $\gamma, \gamma' \in J$. On the other hand, we have $\widetilde{f+g} = \tilde{f} \circ \tilde{g}$ for any $f, g \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$. In fact, for any $\gamma \in J$,

$$\begin{aligned}(\widetilde{f+g})([\gamma]_3) &= [\gamma]_3 + f([\gamma]_2) + g([\gamma]_2),\\ (\tilde{f} \circ \tilde{g})([\gamma]_3) &= \tilde{f}([\gamma]_3 + g([\gamma]_2)) = [\gamma]_3 + g([\gamma]_2) + f([\gamma]_2).\end{aligned}$$

This shows that ι is a homomorphism. On the other hand, for the zero map $0 \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$, it is obvious that $\tilde{0} = \mathrm{id}_{J/J^3}$. Hence each $\tilde{f}$ has its inverse map $\tilde{f}^{-1} = \widetilde{-f}$. This means $\tilde{f}$ an automorphism on J/J^3.

Finally, we show that ι is injective. Assume that $\tilde{f} = \mathrm{id}_{J/J^3}$ for some $f \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$. Then for any $[\gamma]_3 \in J/J^3$, we have

$$\tilde{f}([\gamma]_3) = [\gamma]_3 + f([\gamma]_2) = [\gamma]_3.$$

Hence $f([\gamma]_2) = 0$ for any $[\gamma]_2 \in J/J^2$, and $f = 0$. This shows ι is injective. □

Proposition 3.4. *The sequence*

$$0 \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J) \xrightarrow{\iota} \overline{\mathrm{Aut}}(J/J^3) \xrightarrow{\varphi} \overline{\mathrm{Aut}}(J/J^2) \to 1 \tag{6}$$

is a split group extension.

Proof. First, we show the above sequence is exact. Namely, it suffices to show $\mathrm{Im}\,\iota = \mathrm{Ker}\,\varphi$. The fact that $\mathrm{Im}\,\iota \subset \mathrm{Ker}\,\varphi$ follows from

$$\begin{aligned}(\varphi \circ \iota)(f)([\gamma]_2) &= \varphi(\tilde{f})([\gamma]_2) = [\gamma]_3 + f([\gamma]_2) \pmod{J^2}\\ &= [\gamma]_2\end{aligned}$$

for any $f \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ and $\gamma \in J$. To show $\mathrm{Im}\,\iota \supset \mathrm{Ker}\,\varphi$, take any $f \in \mathrm{Ker}\,\varphi$. Define $g \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ to be

$$g([\gamma]_2) := f([\gamma]_3) - [\gamma]_3$$

for any $\gamma \in J$. The map g is well-defined. In fact, for any $\gamma, \gamma' \in J$ such that $[\gamma]_2 = [\gamma']_2$, if we set $\gamma' - \gamma = \varepsilon \in J^2$, then we have

$$\begin{aligned}g([\gamma']_2) &= f([\gamma']_3) - [\gamma']_3 = f([\gamma + \varepsilon]_3) - [\gamma + \varepsilon]_3\\ &= f([\gamma]_3) - [\gamma]_3 + f([\varepsilon]_3) - [\varepsilon]_3 = f([\gamma]_3) - [\gamma]_3\\ &= g([\gamma]_2).\end{aligned}$$

Here we remark that $f([\varepsilon]_3) - [\varepsilon]_3 = 0 \in J^2/J^3$ since $\varepsilon \in J^2$ and $f \in \mathrm{Ker}\,\varphi$. It is easy to show that g is a homomorphism. Furthermore, for any $\gamma \in J$,

$$\tilde{g}([\gamma]_3) = [\gamma]_3 + g([\gamma]_2) = f([\gamma]_3).$$

This shows $f = \tilde{g} = \iota(g) \in \mathrm{Im}\,\iota$.

Finally, we construct the section $s : \overline{\mathrm{Aut}}(J/J^2) \to \overline{\mathrm{Aut}}(J/J^3)$ of (6). Take elements $\gamma_1, \gamma_2, \dots, \gamma_p \in J$, $\gamma_{p+1}, \dots, \gamma_{p+q} \in J^2$ such that $([\gamma_1]_2, [\gamma_2]_2, \dots, [\gamma_p]_2)$ and $([\gamma_{p+1}]_3, \dots, [\gamma_{p+q}]_3)$ form bases of $\mathrm{gr}^1 J$ and $\mathrm{gr}^2 J$, respectively. Then $([\gamma_1]_3, [\gamma_2]_3, \dots, [\gamma_{p+q}]_3)$ is a basis of J/J^3.

For any $\beta \in \overline{\mathrm{Aut}}(J/J^2)$, there exists an element $\tilde{\beta} \in \overline{\mathrm{Aut}}(J/J^3)$ such that $\varphi(\tilde{\beta}) = \beta$. In general, for any $1 \le j \le p$, the image $\tilde{\beta}([\gamma_j]_3)$ can be written as

$$\tilde{\beta}([\gamma_j]_3) = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3$$

for some $a_{ij} \in \mathbb{Q}$. Since $\beta \in \mathrm{Aut}(J/J^2)$, if for any $1 \le j \le p$ we set

$$v_j := a_{1j}[\gamma_1]_2 + \cdots + a_{pj}[\gamma_p]_2,$$

then $(v_1, v_2, \dots, v_p)$ is a basis of $\mathrm{gr}^1 J$. Let $\delta = \delta_{\tilde{\beta}} : \mathrm{gr}^1 J \to \mathrm{gr}^2 J$ be the $\mathbb{Q}$-linear map given by

$$\delta(v_j) = -(a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3)$$

for any $1 \le j \le p$. Then we obtain

$$\begin{aligned}
&(\tilde{\delta} \circ \tilde{\beta})([\gamma_j]_3) \\
&\quad = \tilde{\delta}(a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3) \\
&\quad = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3 + \delta(v_j) \\
&\quad = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3
\end{aligned}$$

for any $1 \le j \le p$. Consider the map $s : \overline{\mathrm{Aut}}(J/J^2) \to \overline{\mathrm{Aut}}(J/J^3)$ defined by $\beta \mapsto \tilde{\delta} \circ \tilde{\beta}$. We can see that s is a homomorphism and is the required section. Hence the exact sequence (6) splits. □

Now, we construct a crossed homomorphism of $\mathrm{Aut}\, F_n$ which is an extension of $\tilde{\eta}_1$. We provide an easy exercise:

Lemma 3.5. *Let*

$$0 \to K \to G \to N \to 1$$

be a split extension of groups over N with an additive abelian group K. For any g, there exist unique elements $k_g \in K$ and $n_g \in N$ such that $g = k_g n_g$. Then the map $k : G \to K$ defined by $g \mapsto k_g$ is a crossed homomorphism.

By using the above lemma, we obtain:

Theorem 3.6. *There is a crossed homomorphism $\eta : \mathrm{Aut}\, F_n \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$ such that the restriction of η to $\mathcal{E}_n(1)$ is $\tilde{\eta}_1$.*

Proof. By applying Lemma 3.5 to the split extension (6), we obtain a crossed homomorphism $k : \overline{\mathrm{Aut}}(J/J^3) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$. Hence by composing k and the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}(J/J^3)$, we obtain a crossed homomorphism

$$\eta : \mathrm{Aut}\, F_n \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J).$$

This is the required homomorphism. □

4. Nontriviality of η as a 1-cocycle

In this section, we give a few remarks about the image of η_1 and the nontriviality of η as a 1-cocycle.

The first Johnson homomorphism $\tau_1 : \mathrm{gr}^1\mathcal{A}_n \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(l+1)$ is surjective, and hence an isomorphism. We can easily see this fact by calculating the images of Magnus's generators of IA_n, given by

$$K_{ij} : \begin{cases} x_i \mapsto {x_j}^{-1} x_i x_j, \\ x_t \mapsto x_t \quad (t \neq i), \end{cases}$$

for distinct $i, j \in \{1, 2, \dots, n\}$, and

$$K_{ijk} : \begin{cases} x_i \mapsto x_i x_j x_k {x_j}^{-1} {x_k}^{-1}, \\ x_t \mapsto x_t \quad (t \neq i), \end{cases}$$

for distinct $i, j, k \in \{1, 2, \dots, n\}$ such that $j < k$. From a viewpoint of a comparative study, it is a natural problem for us to determine the image of η_1. However, this is complicated for two reasons. One is that we do not have any generating set of $\mathcal{E}_n(1)$. From our result $\mathcal{E}_n(1) = \mathrm{Inn}\, F_n \cdot \mathcal{A}_n(2)$, it suffices to obtain a generating set of $\mathcal{A}_n(2)$. This seems, however, quite difficult in general. The other is that the basis of $\mathrm{gr}^2 J$ obtained in [HS 2014] is too lengthy to handle. In fact, consider

$$T_1 := \{t'_i \mid 1 \le i \le n\} \cup \{t'_{ij} \mid 1 \le i < j \le n\} \cup \{t'_{ijk} \mid 1 \le i < j < k \le n\} \subset J$$

and

$$\begin{aligned} T_2 := &\{t'_i t'_j \mid 1 \le i \le j \le n\} \cup \{t'_i t'_{ab} \mid 1 \le i \le n,\ 1 \le a < b \le n\} \\ &\cup \{t'_i t'_{abc} \mid 1 \le i \le n,\ 1 \le a < b < c \le n\} \\ &\cup \{t'_{ij} t'_{ab} \mid 1 \le i < j \le n,\ 1 \le a < b \le n,\ (i, j) \le (a, b)\} \\ &\cup \{t'_{ab} t'_{abc}, t'_{ac} t'_{abc}, t'_{bc} t'_{abc} \mid 1 \le a < b < c \le n\} \\ &\cup \{t'_{ia} t'_{abc}, t'_{ib} t'_{abc}, t'_{ic} t'_{abc}, t'_{ia} t'_{ibc}, t'_{ab} t'_{iac}, t'_{ab} t'_{ibc}, t'_{ac} t'_{ibc}, t'_{ib} t'_{iac} \mid \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i < a < b < c \le n\} \\ &\cup \{t'_{ja} t'_{ibc}, t'_{jb} t'_{iac}, t'_{jc} t'_{iab}, t'_{ab} t'_{ijc}, t'_{ac} t'_{ijb}, t'_{bc} t'_{ija} \mid 1 \le i < j < a < b < c \le n\} \\ \subset &\ J^2. \end{aligned}$$

We showed in [HS 2014] that the T_k form a basis of $\mathrm{gr}^k J$ for $k = 1, 2$. We cannot write down the total image of η_1 explicitly by hand, because T_2 consists of too many monomials.

Finally, we remark on the nontriviality of the coset class of η in the first cohomology group of $\mathrm{Aut}\, F_n$ with coefficients in $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)$. Consider the automorphism $\sigma := [K_{21}, K_{23}] \in \mathcal{A}_n(2) \subset \mathcal{E}_n(1)$. It satisfies

$$x_i^\sigma = \begin{cases} [x_1^{-1}, x_3^{-1}]x_2[x_3^{-1}, x_1^{-1}] & \text{if } i = 2, \\ x_i & \text{if } i \neq 2. \end{cases}$$

Hence we see that σ maps $t'_{123} \in \mathbb{Q}[t]/I$ to

$$t'_{123} - 2(t'_1)^2 - 4t'_1t'_2 + 4t'_2t'_3 + 2(t'_3)^2 + 2t'_{12}t'_1 - 2t'_{12}t'_3 + 6t'_{13}t'_1 + 6t'_{13}t'_2 \\ + 2t'_{13}t'_3 + 2t'_{23}t'_1 - 2t'_{23}t'_3 - 4(t'_{13})^2 - 6t'_{12}t'_{13} - 2t'_{13}t'_{23} + 4t'_{13}t'_{123}$$

modulo J^3 by a hand calculation. By using the basis T_2 of $\mathrm{gr}^2 J$, we can see that

$$\tilde{\eta}_1(\sigma)(t'_{123}) = (t'_{123})^\sigma - t'_{123} \neq 0 \in \mathrm{gr}^2 J.$$

Thus the restriction $\tilde{\eta}_1$ of η to $\mathcal{E}_n(1)$ is a nontrivial homomorphism, and so is the cohomology class of η in

$$H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J)).$$

It seems to be a natural and interesting problem to determine the above first cohomology group, and show whether η generates it or not.

5. The image of the crossed homomorphism η

In order to calculate the image of η, it is important to know $\eta(\sigma)$ with σ generators of $\sigma \in \mathrm{Aut}\, F_n$. In this section, we calculate the images of Nielsen's generators of $\mathrm{Aut}\, F_n$ by the crossed homomorphism η.

For any $k \geq 2$, let $\rho_k : \mathrm{Aut}\, F_n \to \overline{\mathrm{Aut}}(J/J^k)$ be the natural homomorphism induced from the action of $\mathrm{Aut}\, F_n$ on J/J^k. For any $\sigma \in \mathrm{Aut}\, F_n$, by identifying $\overline{\mathrm{Aut}}(J/J^3)$ with the semidirect product $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathrm{gr}^2 J) \rtimes \overline{\mathrm{Aut}}(J/J^2)$, we have

$$\rho_3(\sigma) = (\eta(\sigma), \rho_2(\sigma)) = (\eta(\sigma), 1)(0, \rho_2(\sigma)) = (\eta(\sigma), 1)s(\rho_2(\sigma)) \in \overline{\mathrm{Aut}}(J/J^3),$$

and hence

$$(\eta(\sigma), 1) = \rho_3(\sigma)s(\rho_2(\sigma))^{-1}.$$

From this, in order to compute $\eta(\sigma)$ for any $\sigma \in \mathrm{Aut}\, F_n$, it suffices to compute $\rho_3(\sigma)s(\rho_2(\sigma))^{-1}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_p \in J$ and $\gamma_{p+1}, \ldots, \gamma_{p+q} \in J^2$ be elements such that $([\gamma_1]_2, [\gamma_2]_2, \dots, [\gamma_p]_2)$ and $([\gamma_{p+1}]_3, \dots, [\gamma_{p+q}]_3)$ form bases of $\mathrm{gr}^1 J$ and $\mathrm{gr}^2 J$. By recalling the arguments in the previous section, we see that for any $1 \le j \le p$, if $\rho_3(\sigma)$ satisfies

$$\rho_3(\sigma)([\gamma_j]_3) = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3$$

for some $a_{ij} \in \mathbb{Q}$, then

$$s(\rho_2(\sigma))([\gamma_j]_3) = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3$$

for any $1 \le j \le n$. By using these facts, we can compute the images $\eta(\sigma)$ for any $\sigma \in \mathrm{Aut}\, F_n$.

Here let us recall Nielsen's generators of $\mathrm{Aut}\, F_n$. Let P, Q, S and U be the automorphisms of F_n defined as follows:

	x_1	x_2	x_3	$\cdots$	x_{n-1}	x_n
P	x_2	x_1	x_3	$\cdots$	x_{n-1}	x_n
Q	x_2	x_3	x_4	$\cdots$	x_n	x_1
S	x_1^{-1}	x_2	x_3	$\cdots$	x_{n-1}	x_n
U	x_1x_2	x_2	x_3	$\cdots$	x_{n-1}	x_n

Namely, P is the automorphism of F_n induced from the permutation of x_1 and x_2, Q is induced from the cyclic permutation of the basis, and so on. Nielsen [1924] showed that $\mathrm{Aut}\, F_n$ is generated by P, Q, S and U for any $n \ge 2$, and also gave finitely many relations among the generators P, Q, S and U. This is the first finite presentation for $\mathrm{Aut}\, F_n$.

By direct computation, we can see that $\rho_3(P)s(\rho_2(P))^{-1}$ satisfies

$$t'_i \mapsto t'_i \quad \text{for any } 1 \le i \le n,$$

$$t'_{ij} \mapsto t'_{ij} \quad \text{for any } 1 \le i < j \le n,$$

$$t'_{ijk} \mapsto \begin{cases} t'_{12k} - \{t'_1t'_{2k} + t'_2t'_{1k} + t'_kt'_{12} - 2(t'_1t'_2 + t_1t'_k + t_2t'_k)\} & \text{if } (i, j) = (1, 2), \\ t'_{ijk} & \text{otherwise,} \end{cases}$$

and hence obtain

$$\eta(P) = -\sum_{k=3}^{n} (t'_{12k})^* \otimes \{t'_1t'_{2k} + t'_2t'_{1k} + t'_kt'_{12} - 2(t'_1t'_2 + t_1t'_k + t_2t'_k)\},$$

where $(t'_\bullet)^*$ means the dual basis in $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1 J, \mathbb{Q})$ of $t'_\bullet$ in $\mathrm{gr}^1 J$. Similarly, we can obtain the equalities

$$\eta(Q) = 0,$$

$$\eta(S) = -\sum_{j=2}^{n} (t'_{ij})^* \otimes t'_1 t'_j - \sum_{2\le j<k\le n} (t'_{1jk})^* \otimes t'_1 t'_{jk},$$

$$\begin{aligned}\eta(U) = &-(t'_{12})^* \otimes t'_1 t'_2 - \sum_{k=3}^{n} (t'_{12k})^* \otimes t'_2 t'_{1k} \\ &+ \sum_{3\le j<k\le n} (t'_{1jk})^* \otimes \big\{-(t'_1 t'_k + t'_2 t'_j + t'_j t'_k + 2t'_1 t'_j + 2t'_2 t'_k) \\ &\quad + (t'_1 t'_{2j} + t'_1 t'_{jk} - t'_2 t'_{1j} + t'_2 t'_{jk} + t'_j t'_{12} + t'_j t'_{1k} + t'_k t'_{12} + t'_k t'_{2j}) \\ &\quad - \tfrac{1}{2}(t'_1 t'_{2jk} - t'_2 t'_{1jk} + t'_j t'_{12k} + t'_k t'_{12j}) - \tfrac{1}{2}(t'_{12} t'_{jk} - t'_{1j} t'_{2k} + t'_{1k} t'_{2j})\big\}.\end{aligned}$$

Since P, Q, S and U generate $\mathrm{Aut}\, F_n$, by using the Leibniz rule, we can calculate $\eta(\sigma)$ for any $\sigma \in \mathrm{Aut}\, F_n$.

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References

[Andreadakis 1965] S. Andreadakis, "On the automorphisms of free groups and free nilpotent groups", *Proc. London Math. Soc.* (3) **15** (1965), 239–268. MR 32 #5746 Zbl 0135.04502

[Cartan and Eilenberg 1999] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1999. MR 2000h:18022 Zbl 0933.18001

[Day 2013] M. B. Day, "Extensions of Johnson's and Morita's homomorphisms that map to finitely generated abelian groups", *J. Topol. Anal.* **5**:1 (2013), 57–85. MR 3043442 Zbl 1270.57006

[Fricke and Klein 1897] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen, 1: Die gruppentheoretischen Grundlagen*, Teubner, Leipzig, 1897. MR 32 #1348 JFM 28.0334.01

[Hain 1997] R. Hain, "Infinitesimal presentations of the Torelli groups", *J. Amer. Math. Soc.* **10**:3 (1997), 597–651. MR 97k:14024 Zbl 0915.57001

[Hatakenaka and Satoh 2014] E. Hatakenaka and T. Satoh, "On the graded quotients of the ring of Fricke characters of a free group", 2014. arXiv 1206.1500

[Hatakenaka and Satoh 2015] E. Hatakenaka and T. Satoh, "On the rings of Fricke characters of free abelian groups", *J. Commut. Alg.* (online publication November 2015). To appear.

[Horowitz 1972] R. D. Horowitz, "Characters of free groups represented in the two-dimensional special linear group", *Comm. Pure Appl. Math.* **25** (1972), 635–649. MR 47 #3542 Zbl 1184.20009

[Johnson 1980] D. Johnson, "An abelian quotient of the mapping class group $\mathcal{I}_g$", *Math. Ann.* **249**:3 (1980), 225–242. MR 82a:57008 Zbl 0409.57009

[Johnson 1983] D. Johnson, "The structure of the Torelli group, I: A finite set of generators for $\mathcal{I}$", *Ann. of Math.* (2) **118**:3 (1983), 423–442. MR 85a:57005 Zbl 0549.57006

[Johnson 1985a] D. Johnson, "The structure of the Torelli group, II: A characterization of the group generated by twists on bounding curves", *Topology* **24**:2 (1985), 113–126. MR 86i:57011 Zbl 0571.57009

[Johnson 1985b] D. Johnson, "The structure of the Torelli group, III: The abelianization of $\mathcal{I}$", *Topology* **24**:2 (1985), 127–144. MR 87a:57016 Zbl 0571.57010

[Kawazumi 2006] N. Kawazumi, "Cohomological aspects of Magnus expansions", 2006. arXiv math. GT/0505497

[Maclachlan and Reid 2003] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic* 3*-manifolds*, Graduate Texts in Mathematics **219**, Springer, New York, 2003. MR 2004i:57021 Zbl 1025.57001

[Magnus 1980] W. Magnus, "Rings of Fricke characters and automorphism groups of free groups", *Mathematische Zeitschrift* **170**:1 (1980), 91–103.

[Morita 1993a] S. Morita, "Abelian quotients of subgroups of the mapping class group of surfaces", *Duke Math. J.* **70**:3 (1993), 699–726. MR 94d:57003 Zbl 0801.57011

[Morita 1993b] S. Morita, "The extension of Johnson's homomorphism from the Torelli group to the mapping class group", *Invent. Math.* **111**:1 (1993), 197–224. MR 93j:57001 Zbl 0787.57008

[Nielsen 1924] J. Nielsen, "Die Isomorphismengruppe der freien Gruppen", *Math. Ann.* **91**:3-4 (1924), 169–209. MR 1512188 JFM 50.0078.04

[Satoh 2009] T. Satoh, "The cokernel of the Johnson homomorphisms of the automorphism group of a free metabelian group", *Trans. Amer. Math. Soc.* **361**:4 (2009), 2085–2107. MR 2009i:20073 Zbl 1171.20024

[Satoh 2013a] T. Satoh, "First cohomologies and the Johnson homomorphisms of the automorphism group of a free group", *J. Pure Appl. Algebra* **217**:1 (2013), 137–152. MR 2965912 Zbl 1286.20040

[Satoh 2013b] T. Satoh, "A survey of the Johnson homomorphisms of the automorphism groups of free groups and related topics", preprint, 2013. Submitted to *Handbook of Teichmüller theory, volume V*. arXiv 1204.0876

[Vogt 1889] H. Vogt, "Sur les invariants fondamentaux des équations différentielles linéaires du second ordre", *Ann. Sci. École Norm. Sup.* (3) **6** (1889), 3–71. MR 1508833 JFM 21.0314.01

[Whittemore 1973] A. Whittemore, "On special linear characters of free groups of rank $n \geq 4$", *Proc. Amer. Math. Soc.* **40** (1973), 383–388. MR 48 #428 Zbl 0248.20031

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TAKAO SATOH
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE DIVISION II
TOKYO UNIVERSITY OF SCIENCE
1-3 KAGURAZAKA
SHINJUKU-KU
TOKYO 162-8601
JAPAN

takao@rs.tus.ac.jp

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University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

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Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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