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CRITERIA FOR VANISHING OF TOR OVER COMPLETE INTERSECTIONS

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We exploit properties of Dao's η-pairing (see *Trans. Amer. Math. Soc.* 365:6 (2013), 2803–2821), as well as techniques of Huneke, Jorgensen, and Wiegand (*J. Algebra* 238:2 (2001), 684–702), to study the vanishing of $\operatorname{Tor}_i(M, N)$ for finitely generated modules M, N over complete intersections. We prove vanishing of $\operatorname{Tor}_i(M, N)$ for all $i \geq 1$ under depth conditions on M, N, and $M \otimes N$. Our arguments improve a result of Dao and establish a new connection between the vanishing of Tor and the depth of tensor products.

1. Introduction

In a seminal paper, Auslander [1961] proved that if R is a local ring and M and N are nonzero finitely generated R-modules such that $\operatorname{pd}(M) < \infty$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then

$$\text{(1.0.1)} \qquad \operatorname{depth}(M) + \operatorname{depth}(N) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes_R N),$$

that is, the *depth formula* holds. Huneke and Wiegand [1994, Theorem 2.5] established the depth formula for Tor-independent modules (not necessarily of finite projective dimension) over complete intersection rings. Christensen and Jorgensen [2015] extended that result to AB rings [Huneke and Jorgensen 2003], a class of Gorenstein rings strictly containing the class of complete intersections. The depth formula is important for the study of depths of tensor products of modules [Auslander 1961; Huneke and Wiegand 1994], as well as of complexes [Foxby 1980; Iyengar 1999]. We seek conditions on the modules M, N and $M \otimes_R N$ forcing such a formula to hold, in particular, conditions implying $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. The following conjecture — implicit in the work of Huneke, Jorgensen, and Wiegand — guides our search.

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Conjecture 1.1 [Huneke et al. 2001]. Let M, N be finitely generated modules over a complete intersection R of codimension c. If $M \otimes_R N$ is a $(c+1)$-st syzygy and M has rank, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The conjecture is true if $c = 0$ or $c = 1$, by [Lichtenbaum 1966, Corollary 1] and [Huneke and Wiegand 1994, Theorem 2.7], respectively. Without the assumption of rank, there are easy counterexamples, e.g., $R = k[[x, y]]/(xy)$ and $M = N = R/(x)$; M is an n-th syzygy for all n, but the odd index Tor modules are nonzero.

A finitely generated module over a complete intersection is an n-th syzygy of some finitely generated module if and only if it satisfies *Serre's condition* (S_n); see §2.6. Our methods yield a sharpening of the following theorem due to Dao:

Theorem 1.2 [Dao 2007]. *Let R be a complete intersection in an unramified regular local ring, of relative codimension c, and let M, N be finitely generated R-modules. Assume*

(i) *M and N satisfy* (S_c),

(ii) *$M \otimes_R N$ satisfies* (S_{c+1}), *and*

(iii) *$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p}$ of height at most c.*

Then $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$* (*and hence the depth formula holds*).

By analyzing Serre's conditions, we remove Dao's assumption that the ambient regular local ring be unramified; see Corollary 3.14. Even though complete intersections in unramified regular local rings suffice for many applications, our conclusion is of interest: Dao's proof uses the nonnegativity of partial Euler characteristics, but nonnegativity remains unknown for the ramified case; see [Dao 2007, Theorem 6.3 and the proof of Lemma 7.7].

If the ambient regular local ring *is* unramified, we can replace c with $c - 1$ in both hypotheses (i) and (ii), remove hypothesis (iii), and still conclude that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$ *provided* that $\eta_c^R(M, N) = 0$; see §3.1 for the definition of $\eta_c^R(-, -)$ and Theorem 3.10 for our result.

Moore, Piepmeyer, and Spiroff [Moore et al. 2013] and Walker [2014] have proved vanishing of the η-pairing in several important cases. These, in turn, yield results on vanishing of Tor. See Proposition 4.1, Theorem 4.2, and Corollary 4.3.

Our proofs rely on a reduction technique using *quasiliftings*; see §2.8. Quasiliftings were initially defined and studied in [Huneke et al. 2001]. The key ingredient for our argument is Lemma 3.9. It shows that if $R = S/(f)$ and S is a complete intersection of codimension $c - 1$, and if $\eta_c^R(M, N) = 0$, then $\eta_{c-1}^S(E, F) = 0$, where E and F are quasiliftings of M and N to S, respectively. By induction, we get that $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$. This allows us to prove the vanishing of $\operatorname{Tor}_i^R(M, N)$ from the depth and syzygy relations between the pairs E, F and M, N.

In the Appendices we revisit [Huneke and Wiegand 1994] and use our work to obtain one of the main results there. Moreover, we point out an oversight in [Miller 1998] and state the author's result in its corrected form as Corollary B.3.

2. Preliminaries

We review a few concepts and results, especially universal pushforwards and quasi-liftings [Huneke et al. 2001; Huneke and Wiegand 1994]. Throughout R will be a commutative noetherian ring.

Let $\nu_R(M)$ denote the minimal number of generators of the R-module M. If $(R, \mathfrak{m})$ is local, then the *codimension* of R is $\operatorname{codim}(R) := \nu_R(\mathfrak{m}) - \dim(R)$, a nonnegative integer. We have $\operatorname{codim}(\hat{R}) = \operatorname{codim}(R)$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

2.1. Complete intersections. R is a *complete intersection in a local ring* $(Q, \mathfrak{n})$ if there a surjection $\pi : Q \twoheadrightarrow R$ with $\ker(\pi)$ generated by a Q-regular sequence in $\mathfrak{n}$; the length of this regular sequence is the *relative codimension of R in Q*. A *hypersurface in Q* is a complete intersection of relative codimension one in Q.

Assume $\hat{R}$ is a complete intersection in a regular local ring $(Q, \mathfrak{n})$, of relative codimension c. Then $\hat{R} = Q/(\underline{f})$ for a regular sequence $\underline{f} = f_1, \dots, f_c$, where $\operatorname{codim}(R) \leq c$. Moreover, the codimension of R is c if and only if $(\underline{f}) \subseteq \mathfrak{n}^2$.

A ring is a *complete intersection* (resp., *hypersurface*) if it is local and its completion is a complete intersection (resp., hypersurface) in a regular local ring.

2.2. Ramified regular local rings. A regular local ring $(Q, \mathfrak{n}, k)$ is said to be *unramified* if either (i) Q is equicharacteristic, i.e., contains a field, or else (ii) $Q \supset \mathbb{Z}$, $\operatorname{char}(k) = p$, and $p \notin \mathfrak{n}^2$. In contrast, the regular local ring $R = V[x]/(x^2 - p)$, where V is the ring of p-adic integers, is *ramified*. Every localization, at a prime ideal, of an unramified regular local ring is again unramified; see [Auslander 1961, Lemma 3.4].

Let $(Q, \mathfrak{n}, k)$ be a d-dimensional complete regular local ring. If Q is ramified, then k has characteristic p. Further, there is a complete unramified discrete valuation ring (V, pV) such that $Q \cong T/(p - f)$, where $T = V[\![x_1, \dots, x_d]\!]$ and f is contained in the square of the maximal ideal of T; see for example [Bourbaki 2006, Chaper IX, §3]. Hence every complete regular local ring is a hypersurface in an unramified one. Consequently, when R is a complete intersection, $\hat{R}$ is a complete intersection in an unramified regular local ring Q such that

$$\operatorname{codim} R \leq c \leq \operatorname{codim} R + 1,$$

where c is the relative codimension of $\hat{R}$ in Q.

2.3. The depth formula [Huneke and Wiegand 1994, Theorem 2.5]. Let R be a complete intersection and let M, N be finitely generated R-modules. If

$\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then the *depth formula* (1.0.1) holds, that is,

$$\operatorname{depth}(M) + \operatorname{depth}(N) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes_R N).$$

Recall that $\operatorname{depth}(0) = \infty$, so the formula holds trivially if a zero module appears.

2.4. Torsion submodule. The *torsion submodule* $\top_R M$ of M is the kernel of the natural homomorphism $M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R) = \{\text{non-zerodivisors}\}^{-1} R$ is the total quotient ring of R. The module M is *torsion* if $\top_R M = M$, and *torsion-free* if $\top_R M = 0$. To restate, M is torsion-free if and only if every non-zerodivisor of R is a non-zerodivisor on M, that is, if and only if $\bigcup \operatorname{Ass} M \subseteq \bigcup \operatorname{Ass} R$. Similarly, M is torsion if and only if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$. For notation, the inclusion $\top_R M \subseteq M$ has cokernel $\perp_R M$:

$$0 \longrightarrow \top_R M \longrightarrow M \longrightarrow \perp_R M \longrightarrow 0. \tag{2.4.1}$$

2.5. Torsionless and reflexive modules. Let M be a finitely generated R-module; M^* denotes its dual $\operatorname{Hom}_R(M, R)$. The module M is *torsionless* if it embeds in a free module, equivalently, the canonical map $M \to M^{**}$ is injective. Torsionless modules are torsion-free, and the converse holds if $R_{\mathfrak{p}}$ is Gorenstein for every associated prime $\mathfrak{p}$ of R; see [Vasconcelos 1968, Theorem A.1]. The module M is *reflexive* provided the map $M \to M^{**}$ is an isomorphism.

2.6. Serre's conditions (see [Leuschke and Wiegand 2012, Appendix A, §1] and [Evans and Griffith 1985, Theorem 3.8]). Let M be a finitely generated R-module and let n be a nonnegative integer. Then M is said to satisfy *Serre's condition* (S_n) provided that

$$\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{n, \operatorname{height}(\mathfrak{p})\} \quad \text{for all } \mathfrak{p} \in \operatorname{Supp}(M).$$

A finitely generated module M over a local ring R is *maximal Cohen–Macaulay* if $\operatorname{depth}(M) = \dim(R)$; necessary for this equality is that $M \neq 0$.

If M satisfies (S_1), then M is torsion-free, and the converse holds if R has no embedded primes, e.g., is reduced or Cohen–Macaulay; see §2.4. If R is Gorenstein, then M satisfies (S_2) if and only if M is reflexive; see §2.5 and [Evans and Griffith 1985, Theorem 3.6]. Moreover, if R is Gorenstein, then M satisfies (S_n) if and only if M is an n-th syzygy module; see [Leuschke and Wiegand 2012, Corollary A.12].

A localization of a torsion-free module need not be torsion-free; see, for example, [Epstein and Yao 2012, Example 3.9]. However, over Cohen–Macaulay rings, we have the following.

Remark 2.7. Assume that R is Cohen–Macaulay and M is a finitely generated R-module. Let $\mathfrak{p}$ be a prime ideal of R. Note that, since $\top_R M$ is killed by a non-zerodivisor of R, $(\top_R M)_{\mathfrak{p}}$ is a torsion $R_{\mathfrak{p}}$-module. Next, $\perp_R M$ satisfies (S_1) as R is Cohen–Macaulay, and so $(\perp_R M)_{\mathfrak{p}}$ is a torsion-free $R_{\mathfrak{p}}$-module; see §2.6.

Localizing the exact sequence (2.4.1) at $\mathfrak{p}$, we see that $(\top_R M)_{\mathfrak{p}} \cong \top_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. In particular, if M is a torsion-free R-module, then $M_{\mathfrak{p}}$ is a torsion-free $R_{\mathfrak{p}}$-module.

We recall a technique from [Huneke et al. 2001, §1] for lowering the codimension.

2.8. Pushforward and quasilifting [Huneke et al. 2001, §1]. Let R be a Gorenstein local ring and let M be a finitely generated torsion-free R-module. Choose a surjection $\varepsilon\colon R^{(\nu)} \twoheadrightarrow M^*$ with $\nu = \nu_R(M^*)$. Applying $\operatorname{Hom}(-, R)$ to this surjection, we obtain an injection $\varepsilon^*\colon M^{**} \hookrightarrow R^{(\nu)}$. Let M_1 be the cokernel of the composition $M \hookrightarrow M^{**} \hookrightarrow R^{(\nu)}$. The exact sequence

$$0 \to M \to R^{(\nu)} \to M_1 \to 0 \tag{2.8.1}$$

is called a *pushforward* of M. The extension (2.8.1) and the module M_1 are unique up to noncanonical isomorphism; see [Celikbas 2011, pp. 174–175]. We refer to such a module M_1 as the pushforward of M. Note $M_1 = 0$ if and only if M is free.

Assume $R = S/(f)$ where $(S, \mathfrak{n})$ is a local ring and f is a non-zerodivisor in $\mathfrak{n}$. Let $S^{(\nu)} \twoheadrightarrow M_1$ be the composition of the canonical map $S^{(\nu)} \twoheadrightarrow R^{(\nu)}$ and the map $R^{(\nu)} \twoheadrightarrow M_1$ in (2.8.1). The *quasilifting* of M to S is the module E in the exact sequence of S-modules:

$$0 \to E \to S^{(\nu)} \to M_1 \to 0\,. \tag{2.8.2}$$

The quasilifting of M is unique up to isomorphism of S-modules.

Proposition 2.9 is from [Huneke et al. 2001, Propositions 1.6 and 1.7]; while Proposition 2.10 is embedded in the proofs of [Huneke et al. 2001, Propositions 1.8 and 2.4] and is recorded explicitly in [Celikbas 2011, Proposition 3.2(3)(b)]. We will use Proposition 2.10 in the proofs of Theorem 3.10 and Theorem B.2 below.

Proposition 2.9 [Huneke et al. 2001]. *Let R be a Gorenstein local ring and let M be a finitely generated torsion-free R-module. Let M_1 denote the pushforward of M.*

(i) *Let $n \geq 0$. Then M satisfies (S_{n+1}) if and only if M_1 satisfies (S_n).*

(ii) *Let $\mathfrak{p}$ be a prime ideal. If $M_{\mathfrak{p}}$ is a maximal Cohen–Macaulay $R_{\mathfrak{p}}$-module, then $(M_1)_{\mathfrak{p}}$ is either zero or a maximal Cohen–Macaulay $R_{\mathfrak{p}}$-module.*

Proposition 2.10 [Huneke et al. 2001]. *Let $R = S/(f)$ where S is a complete intersection and f is a non-zerodivisor in S. Let N be a finitely generated torsion-free R-module such that $M \otimes_R N$ is reflexive. Assume $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$ and for all primes $\mathfrak{p}$ of R with $\operatorname{height}(\mathfrak{p}) \leq 1$.*

(i) *Then $M_1 \otimes_R N$ is torsion-free.*

(ii) *Let E and F denote the quasiliftings of M and N to S, respectively; see §2.8. Assume $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$. Then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Serre's conditions (S_n) need not ascend along flat local homomorphisms. This can be problematic:

Example 2.11. The ring $\mathbb{C}[\![x, y, u, v]\!]/(x^2, xy)$ has depth two and therefore, by Heitmann's theorem [1993, Theorem 8], it is the completion $\hat{R}$ of a unique factorization domain $(R, \mathfrak{m})$. Then R, being normal, satisfies (S_2), but $\hat{R}$ does not even satisfy (S_1), since the localization at the height-one prime ideal (x, y) has depth zero.

For flat local homomorphisms between Cohen–Macaulay rings, and more generally when the fibers are Cohen–Macaulay, however, (S_n) *does* ascend and descend:

Lemma 2.12. *Let R be a local ring, $\mathfrak{p}$ a prime ideal of R, and let M be a finitely generated R-module.*

(1) *If M is reflexive, then so is the $R_\mathfrak{p}$-module $M_\mathfrak{p}$.*

(2) *Suppose R is Cohen–Macaulay. Then $(\top_R M)_\mathfrak{p} = \top_{R_\mathfrak{p}} M_\mathfrak{p}$; in particular, if M is torsion-free, then so is $M_\mathfrak{p}$.*

(3) *Suppose $R \to S$ is a flat local homomorphism. If $S \otimes_R M$ satisfies (S_n) as an S-module, then M satisfies (S_n) as an R-module; the converse holds when the fibers of the map $R \to S$ are Cohen–Macaulay.*

Proof. For part (1), localize the isomorphism $M \to M^{**}$. Part (2) is Remark 2.7. Part (3) can be proved along the same lines as [Matsumura 1989, Theorem 23.9]: For any $\mathfrak{q}$ in Spec S with $\mathfrak{p} = \mathfrak{q} \cap R$, it follows from [Matsumura 1989, Theorems 15.1 and 23.3] that

$$\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}),$$
$$\operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

When $S \otimes_R M$ satisfies (S_n), for $\mathfrak{q}$ minimal in $S/\mathfrak{p}S$, these equalities give

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} \geq \min\{n, \operatorname{height}(\mathfrak{q})\} = \min\{n, \operatorname{height}(\mathfrak{p})\}.$$

Thus M satisfies (S_n). Conversely, if $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is Cohen–Macaulay and the R-module M satisfies (S_n), one gets

$$\operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} \geq \min\{n, \operatorname{height}(\mathfrak{p})\} + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \min\{n, \operatorname{height}(\mathfrak{q})\}.$$

This completes the proof of part (3). □

3. Main theorem

Our main result, Theorem 3.10, is here. We use the θ- and η-pairings introduced by Hochster [1981] and Dao [2007]. After preliminaries on these, we focus on complete intersections; see §2.1, the setting of our applications.

3.1. The θ- and η-pairings [Hochster 1981; Dao 2013a; Dao 2007]. Let R be a local ring and let M and N be finitely generated R-modules. Assume that there exists an integer f (depending on M and N), such that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \geq f$.

If R is a hypersurface, then $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ for all $i \gg 0$; see [Eisenbud 1980]. Hochster [1981] introduced the θ pairing for $n \gg 0$ by

$$\theta^R(M, N) = \operatorname{length}(\operatorname{Tor}_{2n}^R(M, N)) - \operatorname{length}(\operatorname{Tor}_{2n-1}^R(M, N))$$

When R is any complete intersection, Dao [2007, Definition 4.2.] made the definition

$$\eta_e^R(M, N) = \lim_{n\to\infty} \frac{1}{n^e} \sum_{i=f}^{n} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M, N)).$$

The η-pairing is a natural extension to complete intersections of the θ-pairing. Moreover the following statements hold; see [Dao 2007, Theorem 4.3].

(i) $\eta_e^R(M, -)$ and $\eta_e^R(-, N)$ are additive on short exact sequences, provided η_e^R is defined on the pairs of modules involved.

(ii) If R is a hypersurface, then $\eta_1^R(M, N) = \frac{1}{2}\theta^R(M, N)$. Hence $\eta_1^R(M, N) = 0$ if and only if $\theta^R(M, N) = 0$.

Assume R is a complete intersection.

(iii) $\eta_e^R(M, N) = 0$ if $e \geq \operatorname{codim} R$ and either M or N has finite length.

(iv) η_e^R is finite when $e = \operatorname{codim}(R)$, and η_e^R is zero when $e > \operatorname{codim} R$.

The next result [Dao 2007, Theorem 6.3], on *Tor-rigidity*, shows the utility of the η-pairing.

Theorem 3.2 [Dao 2007]. *Let R be a local ring whose completion is a complete intersection, of relative codimension $c \geq 1$, in an* unramified *regular local ring. Let M, N be finitely generated R-modules. Assume* $\operatorname{Tor}_i^R(M, N)$ *has finite length for all $i \gg 0$, and that $\eta_c^R(M, N) = 0$. Then the pair M, N is c-Tor-rigid, that is, if $s \geq 0$ and* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i = s, \ldots, s + c - 1$, then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq s$.*

The following conjectures have received quite a bit of attention:

Conjectures 3.3. Assume R is a local ring which is an isolated singularity, i.e., $R_{\mathfrak{p}}$ is a regular local ring for all nonmaximal prime ideals $\mathfrak{p}$ of R.

(i) [Dao 2013a, Conjecture 3.15] If R is an equicharacteristic hypersurface of even dimension, then $\eta_1^R(M, N) = 0$ for all finitely generated R-modules M and N.

(ii) [Moore et al. 2013, Conjecture 2.4] If R is a complete intersection of codimension $c \geq 2$, then $\eta_c^R(M, N) = 0$ for all finitely generated R-modules M and N.

Moore, Piepmeyer, Spiroff and Walker [2011] have settled Conjecture 3.3(i) in the affirmative for certain types of affine algebras. Polishchuk and Vaintrob [2012, Remark 4.1.5], as well as Buchweitz and Van Straten [2012, Main Theorem], have since given other proofs, in somewhat different contexts, of this result; see Theorem 4.2 for a recent result of Walker [2014] concerning Conjecture 3.3(ii), and Corollary 4.3 for an application of his result.

Our proofs of Lemma 3.6 and Theorem B.2 use the following (see [Auslander 1961, Lemma 3.1] or [Huneke and Wiegand 1994, Lemma 1.1]).

Remark 3.4. Let R be a local ring, and let M and N be nonzero finitely generated R-modules. Assume $M \otimes_R N$ is torsion-free. Then $M \otimes_R N \cong M \otimes \perp_R N$. Moreover, if $\operatorname{Tor}_1^R(M, \perp_R N) = 0$, then $\top_R N = 0$, and hence N is torsion-free.

We encounter the same hypotheses often enough to warrant a piece of notation.

Notation 3.5. Let c be a positive integer. A pair M, N of finitely generated modules over a ring R satisfies (SP_c) provided the following conditions hold:

(i) M and N satisfy Serre's condition (S_{c-1}).

(ii) $M \otimes_R N$ satisfies (S_c).

(iii) $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \gg 0$.

Hypersurfaces. We begin with a lemma analogous to [Dao 2008, Proposition 3.1]; however, we do not assume any depth properties on either M or N; see §2.1 and Notation 3.5.

Lemma 3.6. *Let R be a local ring whose completion is a hypersurface in an* unramified *regular local ring, and let M, N be finitely generated R-modules. Assume that the following hold*:

(i) $\dim(R) \geq 1$.

(ii) *The pair M, N satisfies* (SP_1).

(iii) $\operatorname{Supp}_R(\top_R N) \subseteq \operatorname{Supp}_R(M)$.

(iv) $\theta^R(M, N) = 0$.

Then $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$, and N is torsion-free.*

Proof. Consider the following conditions for a prime ideal $\mathfrak{p}$ of R:

(3.6.1) $$(\top_R N)_{\mathfrak{p}} \text{ has finite length over } R_{\mathfrak{p}} \quad \text{and} \quad \dim(R_{\mathfrak{p}}) \geq 1.$$

Claim: If $\mathfrak{p}$ is as in (3.6.1), then $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = 0$ for all $i \geq 1$.

We may assume that $M_{\mathfrak{p}} \neq 0$. We know from (ii) that $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ has finite length over $R_{\mathfrak{p}}$ for all $i \gg 0$. Since $(\top_R N)_{\mathfrak{p}}$ has finite length, the exact sequence (2.4.1) for N, localized at $\mathfrak{p}$, shows that $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}})$ has finite length over $R_{\mathfrak{p}}$ for all $i \gg 0$.

Using the additivity of $\theta^{R_\mathfrak{p}}$ along the same exact sequence, we see that

$$\theta^{R_\mathfrak{p}}(M_\mathfrak{p}, (\perp_R N)_\mathfrak{p}) = -\theta^{R_\mathfrak{p}}(M_\mathfrak{p}, (\top_R N)_\mathfrak{p}) = 0, \tag{3.6.2}$$

the last by §3.1.

Since $\perp_R N$ is a torsionless R-module (see §2.5), there exists an exact sequence

$$0 \to \perp_R N \to R^{(n)} \to Z \to 0. \tag{3.6.3}$$

Localizing this sequence at $\mathfrak{p}$, we see that, for $i \gg 0$, $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$ has finite length and hence (since $\dim(R_\mathfrak{p}) \geq 1$) is torsion. Now Corollary A.2 forces $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$ to be torsion for all $i \geq 1$.

From (3.6.3), we see that $\operatorname{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$ embeds into $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} (\perp_R N)_\mathfrak{p}$. But $\operatorname{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$ is torsion, and (by Remarks 2.7 and Remark 3.4) $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} (\perp_R N)_\mathfrak{p}$ is torsion-free; therefore $\operatorname{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p}) = 0$.

Next we note that $\theta^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p}) = -\theta^{R_\mathfrak{p}}(M_\mathfrak{p}, (\perp_R N)_\mathfrak{p}) = 0$; see (3.6.3) and (3.6.2). This implies, by Theorem 3.2, that $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p}) = 0$ for all $i \geq 1$; see §3.1. The claim now follows from (3.6.3).

If $\top_R N \neq 0$, then there is a prime $\mathfrak{p}$, minimal in $\operatorname{Supp}_R(\top_R N)$, and so $(\top_R N)_\mathfrak{p}$ is a nonzero module of finite length. Moreover $\dim(R_\mathfrak{p}) \geq 1$: otherwise $\mathfrak{p} \in \operatorname{Ass}(R)$ and hence $(\top_R N)_\mathfrak{p} = 0$; see § 2.4. Thus $\mathfrak{p}$ satisfies (3.6.1) and, by our claim, $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, (\perp_R N)_\mathfrak{p}) = 0$ for $i \geq 1$. The hypothesis (iii) on supports implies that $M_\mathfrak{p} \neq 0$, and now Remark 3.4 yields a contradiction. We conclude that $\top_R N = 0$.

Applying the claim to the maximal ideal $\mathfrak{p}$ of R yields the required vanishing. □

Remark 3.7. (i) The hypothesis (iii) of Lemma 3.6 holds when, for example, the support of N is contained in that of M. Moreover, if R is a domain and M and N are nonzero, then, since $M \otimes_R N$ is torsion-free, we see that $\operatorname{Supp}(M \otimes_R N) = \operatorname{Spec}(R)$, whence $\operatorname{Supp}(M) = \operatorname{Spec}(R)$.

(ii) Most of the hypotheses in Lemma 3.6 are essential; see the discussion after [Huneke and Wiegand 1997, Remark 1.5]. Notice, without the assumption that $\dim(R) \geq 1$, the lemma would fail. Take, for example, $R = \mathbb{C}[x]/(x^2)$ and $M = R/(x) = N$. The vanishing of θ is also essential: let $R = \mathbb{C}[\![x, y]\!]/(xy)$, $M = R/(x)$ and $N = R/(x^2)$. Then the pair M, N satisfies conditions (ii) and (iii) of Lemma 3.6. On the other hand $\operatorname{Tor}_{2i+1}^R(M, N) \cong k$ for all $i \geq 0$, and $\operatorname{Tor}_{2i}^R(M, N) = 0$ for all $i \geq 1$. (Thus $\theta^R(M, N) = -1$.)

The completion of any regular ring is a hypersurface in an unramified regular local ring; see §2.2. Hence the following consequence of Lemma 3.6 extends [Lichtenbaum 1966, Corollary 3], which in turn builds on [Auslander 1961, Theorem 3.2]; see C. Miller's result recorded as Corollary B.3 here.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a d-dimensional local ring whose completion is a hypersurface in an unramified regular local ring, with $d \geq 1$, and let M be a* *finitely generated R-module. Assume* $\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ *for all prime ideals* $\mathfrak{p} \neq \mathfrak{m}$ *and that* $\theta^R(M, -) = 0$*. If* $\bigotimes_R^n M$ *is torsion-free for some integer* $n \geq 2$*, then* $\mathrm{pd}(M) \leq (d-1)/n$*. Consequently, if M is not free, then* $\bigotimes_R^n M$ *has torsion for each* $n \geq \max\{2, d\}$.

Proof. We may assume $M \neq 0$. Iterating Lemma 3.6 shows that $\bigotimes_R^p M$ is torsion-free for $p = 1, \ldots, n$, and that $\mathrm{Tor}_i^R(M, \bigotimes_R^{p-1} M) = 0$ for all $i \geq 1$. Taking $p = 2$, we see from [Huneke and Wiegand 1997, Theorem 1.9] that $\mathrm{pd}(M) < \infty$. Since $\mathrm{depth}(\bigotimes_R^n M) \geq 1$, one obtains, using [Auslander 1961, Corollary 1.3] and the Auslander–Buchsbaum formula [1957, Theorem 3.7],

$$n \cdot \mathrm{pd}(M) = \mathrm{pd}\Big(\bigotimes_R^n M\Big) = d - \mathrm{depth}\Big(\bigotimes_R^n M\Big) \leq d - 1. \qquad \square$$

Complete intersections. Hypersurfaces in complete intersections give the inductive step for our proof of Theorem 3.10; see §2.8 on pushforwards.

Lemma 3.9. *Let* $(S, \mathfrak{n})$ *be a complete intersection, and let R be a hypersurface in S. Let M and N be finitely generated torsion-free R-modules, and let E and F be the quasiliftings of M and N, respectively, to S. Assume* $\mathrm{Tor}_i^R(M, N)$ *has finite length for all* $i \gg 0$*. Let e be an integer with* $e \geq \max\{2, \mathrm{codim}(S) + 1\}$*. Then*

(i) $\mathrm{Tor}_i^S(E, F)$ *has finite length for all* $i \gg 0$*, and*

(ii) $\eta_{e-1}^S(E, F) = 2e \cdot \eta_e^R(M, N)$.

Proof. By hypothesis, $R \cong S/(f)$, where f is a non-zerodivisor in S. The spectral sequence associated to the change of rings $S \to R$ yields the following exact sequence — see [Lichtenbaum 1966, pp. 223–224] or [Murthy 1963, p. 561] — for all $n \geq 1$:

$$\cdots \to \mathrm{Tor}_{n-1}^R(M, N) \to \mathrm{Tor}_n^S(M, N) \to \mathrm{Tor}_n^R(M, N) \to \cdots.$$

Consequently $\mathrm{Tor}_i^S(M, N)$ has finite length for $i \gg 0$. Let M_1 and N_1 be the pushforwards of M and N, respectively. Since $\mathrm{Tor}_i^S(R, -) = 0$ for all $i \geq 2$, the sequences (2.8.2) and (2.8.1) yield isomorphisms

$$\mathrm{Tor}_i^S(E, N) \cong \mathrm{Tor}_{i+1}^S(M_1, N) \cong \mathrm{Tor}_i^S(M, N) \text{ for all } i \geq 2.$$

Arguing in the same vein, one gets isomorphisms

$$\mathrm{Tor}_i^S(E, F) \cong \mathrm{Tor}_i^S(E, N) \text{ for all } i \geq 2.$$

Hence the length of $\mathrm{Tor}_i^S(E, F)$ is finite for all $i \gg 0$, and so (i) holds.

Similar arguments show the η-pairing, over both R and S, as appropriate, is defined for all pairs (X, Y) with $X \in \{M, M_1, E\}$ and $Y \in \{N, N_1, F\}$.

By hypothesis, $\operatorname{codim}(S) \leq e-1$, and hence $\operatorname{codim}(R) \leq e$; see §2.1. Additivity of η along the exact sequences (2.8.1) and (2.8.2) thus gives

$$\eta_e^R(M,N) = -\eta_e^R(M_1,N) = \eta_e^R(M_1,N_1),$$
$$\eta_{e-1}^S(E,F) = -\eta_{e-1}^S(M_1,F) = \eta_{e-1}^S(M_1,N_1).$$

Our assumption that $e \geq \max\{2, \operatorname{codim} S+1\}$, together with Theorem 4.1(3) from [Dao 2007], allow us to invoke Theorem 4.3(3) from the same reference, which says that

$$2e \cdot \eta_e^R(M_1,N_1) = \eta_{e-1}^S(M_1,N_1).$$

This gives (ii), completing the proof. □

The next theorem is our main result. As its hypotheses are technical, several of its consequences are discussed in Section 4; see Section 2 for background.

Theorem 3.10. *Let R be a local ring whose completion is a complete intersection in an* unramified *regular local ring, of relative codimension $c \geq 1$. Let M, N be finitely generated R-modules. Assume the following hold*:

(i) $\dim(R) \geq c$.

(ii) *The pair (M,N) satisfies* (SP_c).

(iii) $\operatorname{Supp}_R(\top_R N) \subseteq \operatorname{Supp}_R(M)$.

(iv) $\eta_c^R(M,N) = 0$.

Then, $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$.

Proof. The case $c=1$ is Lemma 3.6. For $c \geq 2$, proceed by induction on c. We can assume R is complete, so that $R = Q/(\underline{f})$, where Q is an unramified regular local ring and $\underline{f} = f_1, \dots, f_c$ is a Q-regular sequence; see §2.2 and Lemma 2.12. Let $R = S/(f)$, where $S = Q/(f_1, \dots, f_{c-1})$ and $f = f_c$.

Hypothesis (ii) implies $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$; see 3.5. Hence Corollary A.3 implies that, for all primes $\mathfrak{p}$ with $\operatorname{height}(\mathfrak{p}) \leq c-1$,

(3.10.1) $$\operatorname{Tor}_i^R(M,N)_{\mathfrak{p}} = 0 \text{ for all } i \geq 1.$$

Condition (ii) also implies M and N are torsion-free since $c \geq 2$; see 3.5. Hence quasiliftings E and F of M and N to S, respectively, exist; see §2.8. Using the vanishing of Tor modules in (3.10.1) and [Huneke et al. 2001, Theorem 4.8] — compare [Celikbas 2011, Proposition 3.1(7)] — one gets that

(3.10.2) $$E \otimes_S F \text{ satisfies } (\mathrm{S}_{c-1}) \text{ as an } S\text{-module}.$$

It follows from [Huneke et al. 2001, Propositions 1.6 and 1.7] (see also [Celikbas 2011, Propositions 3.1(2) and 3.1(6)]) that the assumptions in (i) of (SP_c) pass to

E and F; see Notation 3.5. So,

(3.10.3) $$E \text{ and } F \text{ satisfy } (\mathrm{S}_{c-1}) \text{ as } S\text{-modules.}$$

Lemma 3.9 guarantees that $\operatorname{Tor}_i^S(E, F)$ has finite length for all $i \gg 0$ and that $\eta_{c-1}(E, F) = 0$. In particular the pair E, F satisfies (SP_{c-1}) over the ring S. Moreover, E and F, being syzygies, are torsion-free, so we indeed have that $\operatorname{Supp}_S(\top_S F) \subseteq \operatorname{Supp}_S(E)$. Now the inductive hypothesis implies that

(3.10.4) $$\operatorname{Tor}_i^S(E, F) = 0 \text{ for all } i \geq 1.$$

Condition (ii) also implies that $M \otimes_R N$ is reflexive since $c \geq 2$; see §2.6. Furthermore, $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{height}(\mathfrak{p}) \leq 1$; see (3.10.1). Thus Proposition 2.10 and (3.10.4) yield $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. □

Remark 3.11. In Theorem 3.10, if $c \geq 2$, hypothesis (ii) implies that N is torsion-free, i.e., $\top_R N = 0$; see §2.6 and Notation 3.5. Thus, when $c \geq 2$, hypothesis (iii) of Theorem 3.10 is redundant.

When $\dim(R) > c$, the equivalence of (i) and (ii) in the following corollary seems interesting; see also §2.3. Actually, in that case the equivalence of (ii) and (iii) holds without the assumption that $\eta_c^R(M, N) = 0$. See [Celikbas 2011, Corollary 2.4].

Corollary 3.12. *Let R be an isolated singularity whose completion is a complete intersection in an unramified regular local ring, of relative codimension c. Let M and N be maximal Cohen–Macaulay R-modules. Assume* $\dim(R) \geq c$. *Assume further that* $\eta_c^R(M, N) = 0$. *The following conditions are equivalent*:

(i) *$M \otimes_R N$ satisfies* (S_c).

(ii) *$M \otimes_R N$ is maximal Cohen–Macaulay.*

(iii) $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$, and hence the depth formula holds.*

Over a complete intersection, vanishing of Ext is closely related to vanishing of Tor: $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$; see [Avramov and Buchweitz 2000, Remark 6.3]. Our next example shows the hypotheses of Theorem 3.10 do *not* force the vanishing of $\operatorname{Ext}_R^i(M, N)$ for all $i \geq 1$.

Example 3.13. Let $(R, \mathfrak{m}, k)$ be a complete intersection with $\operatorname{codim}(R) = 2$ and $\dim(R) \geq 3$. Let N be the d-th syzygy of k, where $d = \dim(R)$, and let M be the second syzygy of $R/(\underline{x})$, where $\underline{x}$ is a maximal R-regular sequence.

Note that N is maximal Cohen–Macaulay, $\operatorname{depth}(M) = 2$, and $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for all primes $\mathfrak{p} \neq \mathfrak{m}$. It follows, since $\operatorname{pd}(M) < \infty$, that $\eta_2^R(M, N) = 0$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$; see §3.1 and Theorem A.1. Therefore the depth formula §2.3 shows that $\operatorname{depth}(M \otimes_R N) = 2$. Since M is a second syzygy, it satisfies (S_2) and hence $M \otimes_R N$ satisfies (S_2); see §2.6. In particular, the pair M, N satisfies (SP_2); see 3.5. However $\operatorname{Ext}_R^{d-2}(M, N) = \operatorname{Ext}^d(R/(\underline{x}), N) \neq 0$; see, for example, [Matsumura 1989, Chapter 19, Lemma 1(iii)].

Here is the extension of Dao's theorem [2007, Theorem 7.7] promised in the introduction (compare Theorem 1.2):

Corollary 3.14. *Let R be a local ring that is a complete intersection, and let M and N be finitely generated R-modules. Assume that the following conditions hold for some integer $e \geq \operatorname{codim}(R)$:*

(i) *M and N satisfy* (S_e).

(ii) *$M \otimes_R N$ satisfies* (S_{e+1}).

(iii) *$M_{\mathfrak{p}}$ is a free for all prime ideals $\mathfrak{p}$ of R of height at most e.*

Then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, and hence the depth formula holds.

Proof. If $e = 0$ this is a theorem in [Auslander 1961] and [Lichtenbaum 1966, Corollary 2]. Assume now that $e \geq 1$. We use induction on $\dim R$. If $\dim R \leq e$, condition (iii) implies that M is free, and there is nothing to prove. Assuming $\dim R \geq e + 1$, we note that the hypotheses localize, so $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for each $i \geq 1$ and each prime ideal $\mathfrak{p}$ in the punctured spectrum of R; that is to say, $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \geq 1$. Thus the pair M, N satisfies (SP_{e+1}). Moreover, since $\operatorname{codim} R < e + 1$, we have $\eta_{e+1}^R = 0$ by item (iv) of §3.1. The completion of R can be realized as a complete intersection, of relative codimension $e+1$, in an unramified regular local ring (see §2.2). Hence the desired result follows from Theorem 3.10. □

4. Vanishing of η

In this section we apply our results to situations where the η-pairing is known to vanish. We know, from Theorem 3.10, that, as long as the critical hypothesis $\eta_c^R(M, N) = 0$ holds, we can replace c with $c - 1$ in the hypotheses of Theorem 1.2 and still conclude the vanishing of Tor. Although it is not easy to verify vanishing of η (see Conjectures 3.3), there are several classes of rings R for which it is known that $\eta^R(M, N) = 0$ for all finitely generated R-modules M and N. For example, if R is an even-dimensional simple ("ADE") singularity in characteristic zero, then Dao observed [2013a, Corollary 3.16] that $\theta^R(M, N) = 0$; see [Dao 2013a, Corollary 3.6] and also [Dao 2013a, §3] for more examples.

Now we give a localized version of a vanishing theorem for graded rings, due to Moore, Piepmeyer, Spiroff, and Walker [2013].

Proposition 4.1. *Let k be a perfect field and $Q = k[x_1, \dots, x_n]$ the polynomial ring with the standard grading. Let $\underline{f} = f_1, \dots, f_c$ be a Q-regular sequence of* *homogeneous polynomials, with $c \geq 2$. Put $A = Q/(\underline{f})$ and $R = A_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Assume that $A_{\mathfrak{p}}$ is a regular local ring for each $\mathfrak{p}$ in* $\operatorname{Spec}(A)\setminus\{\mathfrak{m}\}$. *Then $\eta_c^R(M, N) = 0$ for all finitely generated R-modules M and N. In particular, if $n \geq 2c$ and the pair M, N satisfies* (SP_c), *then M and N are Tor-independent.*

Proof. Choose finitely generated A-modules U and V such that $U_{\mathfrak{m}} \cong M$ and $V_{\mathfrak{m}} \cong N$. For any maximal ideal $\mathfrak{n} \neq \mathfrak{m}$, the local ring $A_{\mathfrak{n}}$ is regular, and hence $\operatorname{Tor}_i^A(U, V)_{\mathfrak{n}} = 0$ for $i \gg 0$. It follows that the map $\operatorname{Tor}_i^A(U, V) \to \operatorname{Tor}_i^R(M, N)$ induced by the localization maps $U \to M$ and $V \to N$ is an isomorphism for $i \gg 0$. Also, for any A-module supported at $\mathfrak{m}$, its length as an A-module is equal to its length as an R-module. In conclusion, $\eta_c^R(M, N) = \eta_c^A(U, V)$.

As k is perfect, the hypothesis on A implies that the k-algebra $A_{\mathfrak{p}}$ is smooth for each nonmaximal prime $\mathfrak{p}$ in A; see [Eisenbud 1995, Corollary 16.20]. Thus, the morphism of schemes $\operatorname{Spec}(R)\setminus\{\mathfrak{m}\} \to \operatorname{Spec}(k)$ is smooth. Now [Moore et al. 2013, Corollary 4.7] yields $\eta_c^A(U, V) = 0$, and hence $\eta_c^R(M, N) = 0$. It remains to note that if $n \geq 2c$, then $\dim R \geq c$, so Theorem 3.10 applies. □

Next, we quote a recent theorem due to Walker; it provides strong support for Conjectures 3.3, at least in equicharacteristic zero.

Theorem 4.2 [Walker 2014, Theorem 1.2]. *Let k be a field of characteristic zero, and let Q a smooth k-algebra. Let $\underline{f} = f_1, \dots, f_c$ be a Q-regular sequence, with $c \geq 2$, and put $A = Q/(f_1, \dots, f_c)$. Assume the singular locus* $\{\mathfrak{p} \in \operatorname{Spec}(A) :$ *$A_{\mathfrak{p}}$ is not regular$\}$ is a finite set of maximal ideals of A. Then $\eta_c^A(U, V) = 0$ for all finitely generated A-modules U, V.*

Corollary 4.3. *With A as in Theorem 4.2, put $R = A_{\mathfrak{m}}$, where $\mathfrak{m}$ is any maximal ideal of A. Then $\eta_c^R(M, N) = 0$ for all finitely generated R-modules M and N. In particular, if* $\dim R \geq c$ *and the pair M, N satisfies* (SP_c), *then M and N are Tor-independent.*

Proof. By inverting a suitable element of Q, we may assume that $A_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Now proceed as in the first paragraph of the proof of Proposition 4.1. □

Theorem 4.4. *Let $(R, \mathfrak{m}, k)$ be a two-dimensional, equicharacteristic, normal, excellent complete intersection of codimension c, with $c \in \{1, 2\}$, and let M and N be finitely generated R-modules. Assume k is contained in the algebraic closure of a finite field. Assume further that M and N satisfy conditions* (i) *and* (ii) *of* (SP_c). *Then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$.*

Proof. The completion $\hat{R}$ is an isolated singularity because R is excellent; see [Leuschke and Wiegand 2012, Proposition 10.9], and so $\hat{R}$ is a normal domain. Replacing R by $\hat{R}$, we may assume that $R = S/(\underline{f})$, where $(S, \mathfrak{n}, k)$ is a regular local ring and $\underline{f}$ is a regular sequence in $\mathfrak{n}^2$ of length c. Let $\bar{k}$ be an algebraic closure of k, and choose a *gonflement* $S \hookrightarrow (\bar{S}, \bar{\mathfrak{n}}, \bar{k})$ lifting the field extension $k \hookrightarrow \bar{k}$; see [2012, Chapter 10, §3]. This is a flat local homomorphism and is an inductive limit of étale extensions. Moreover, $\mathfrak{n}\bar{S} = \bar{\mathfrak{n}}$, so $\bar{S}$ is a regular local ring. By [2012, Proposition 10.15], both $\bar{S}$ and $\bar{R} := \bar{S}/(\underline{f})$ are excellent, and $\bar{R}$ is an isolated singularity. Therefore $(\bar{R}, \bar{\mathfrak{m}}, \bar{k})$ is a normal domain. Finally, we pass to the completion $\hat{S}$ of $\bar{S}$ and put $\Lambda = \hat{S}/(\underline{f})$. This is still an isolated singularity, a normal domain, and a complete intersection of codimension c. Moreover, our hypotheses on M and N ascend along the flat local homomorphism $R \to \Lambda$; see Lemma 2.12. Since Λ is an isolated singularity, $\operatorname{Tor}_i^{\Lambda}(\Lambda \otimes_R M, \Lambda \otimes_R N)$ has finite length for $i \gg 0$; thus the pair $\Lambda \otimes_R M$, $\Lambda \otimes_R N$ satisfies (SP_c).

It follows from [Celikbas and Dao 2011, Proposition 2.5 and Remark 2.6] that $G(\Lambda)/L$ is torsion, where $G(\Lambda)$ is the Grothendieck group of Λ and L is the subgroup generated by classes of modules of finite projective dimension. This implies that $\eta_c^{\Lambda}(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$; see [Dao 2013a, Corollary 3.1] and the paragraph preceding it. Now Theorem 3.10 implies that $\operatorname{Tor}_i^{\Lambda}(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$ for all $i \geq 1$: the requirement on supports is automatically satisfied, since Λ is a domain; see Remark 3.7(i). Faithfully flat descent completes the proof. □

Appendix A: An application of pushforwards

In Theorem A.4 we use pushforwards to generalize [Celikbas 2011, Theorem 3.16]. We have two preparatory results. The first one is a special case of a theorem of Jorgensen:

Theorem A.1 [Jorgensen 1999, Theorem 2.1]. *Let R be a complete intersection and let M and N be finitely generated R-modules. Assume M is maximal Cohen–Macaulay. If* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \gg 0$, then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$.*

Corollary A.2. *Let R be a complete intersection and let M, N be finitely generated R-modules. If* $\operatorname{Tor}_i^R(M, N)$ *is torsion for all $i \gg 0$, then* $\operatorname{Tor}_i^R(M, N)$ *is torsion for all $i \geq 1$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of R. By §2.4, it suffices to prove that $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $i \geq 1$. For that we may assume $M_{\mathfrak{p}} \neq 0$. Then, since $R_{\mathfrak{p}}$ is artinian, it follows that $M_{\mathfrak{p}}$ is a maximal Cohen–Macaulay $R_{\mathfrak{p}}$-module. Therefore, Theorem A.1 gives the desired vanishing. □

Corollary A.3. *Let R be a complete intersection, and let M, N be finitely generated R-modules. Assume M satisfies* (S_w), *where w is a positive integer, and that* $\operatorname{Tor}_i^R(M, N)$ *has finite length for all $i \gg 0$. Let $\mathfrak{p}$ be a nonmaximal prime ideal of R such that* $\operatorname{height}(\mathfrak{p}) \leq w$. *Then* $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ *for all $i \geq 1$.*

Proof. Serre's condition (S_w) localizes, so $M_{\mathfrak{p}}$ is either zero or a maximal Cohen–Macaulay $R_{\mathfrak{p}}$-module; see §2.6. As $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for $i \gg 0$, Theorem A.1 implies that $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $i \geq 1$. □

The next theorem generalizes [Celikbas 2011, Theorem 3.16; see also Theorems 3.4 and 3.15]; we emphasize that the ambient regular local ring in Theorem A.4 is allowed to be ramified.

Theorem A.4. *Let R be a complete intersection with* $\dim R \geq \operatorname{codim} R$, *and let M and N be finitely generated R-modules. Assume the pair M, N satisfies* (SP_c) *for some $c \geq$* $\operatorname{codim} R$. *If $c = 1$, assume further that M or N is torsion-free. If* $\operatorname{Tor}_1^R(M, N) = 0$, *then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$.*

Proof. Without loss of generality, one may assume that $c = \operatorname{codim} R$. When $c = 0$, the desired result is the rigidity theorem of Auslander [1961] and Lichtenbaum [1966], so in the remainder of the proof we assume that $c \geq 1$.

Assume first that $c = 1$. By hypotheses $\operatorname{Tor}_i^R(M, N)$ has finite length for $i \gg 0$ and $M \otimes_R N$ is torsion-free; see Notation 3.5. Moreover, we may assume N (say) is torsion-free. Tensoring M with the pushforward §2.8 for N gives the following:

(A.4.1) $$\operatorname{Tor}_1^R(M, N_1) \hookrightarrow M \otimes_R N,$$

(A.4.2) $$\operatorname{Tor}_i^R(M, N_1) \cong \operatorname{Tor}_{i-1}^R(M, N) \quad \text{for all } i \geq 2.$$

Equation (A.4.2) implies that $\operatorname{Tor}_i^R(M, N_1)$ has finite length for all $i \gg 0$. Therefore, since $\dim(R) \geq 1$, $\operatorname{Tor}_i^R(M, N_1)$ is torsion for all $i \gg 0$; see §2.4. Now Corollary A.2 implies that $\operatorname{Tor}_i^R(M, N_1)$ is torsion for all $i \geq 1$. As $M \otimes_R N$ is torsion-free, we deduce from (A.4.1) that $\operatorname{Tor}_1^R(M, N_1) = 0$. By (A.4.2) we have $\operatorname{Tor}_2^R(M, N_1) \cong \operatorname{Tor}_1^R(M, N) = 0$. Therefore $\operatorname{Tor}_2^R(M, N_1) = 0 = \operatorname{Tor}_1^R(M, N_1)$, and hence Murthy's rigidity theorem [1963, Theorem 1.6] implies that $\operatorname{Tor}_i^R(M, N_1) = 0$ for all $i \geq 1$. Now (A.4.2) completes the proof for the case $c = 1$.

Assume now that $c \geq 2$. We define a sequence $M_0, M_1, \dots, M_{c-1}$ of finitely generated modules by setting $M_0 = M$, and M_n to be the pushforward of M_{n-1}, for all $n = 1, \dots, c-1$. These pushforwards exist: M_0 satisfies (S_{c-1}) by Hypothesis 3.5(i), and so, by Proposition 2.9(i),

(1) each M_n satisfies (S_{c-n-1}).

For the desired result, it suffices to prove that $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for all $i \geq c$. We will, in fact, prove this for all $i \geq 1$. To this end, we establish by induction that the following hold for $n = 0, \dots, c-1$:

(2) $M_n \otimes_R N$ satisfies (S_{c-n});

(3) $\operatorname{Tor}_i^R(M_n, N)$ has finite length for all $i \gg 0$;

(4) $\operatorname{Tor}_i^R(M_n, N) = 0$ for $i = 1, \dots, n+1$.

For $n = 0$, conditions (2) and (3) are part of Hypothesis 3.5, while (4) is from our hypothesis that $\operatorname{Tor}_1^R(M, N) = 0$; recall that $M_0 = M$. Assume that (2), (3) and (4) hold for some integer n with $0 \le n \le c-2$.

Tensor the pushforward of M_n with N — see §2.8 — to obtain

$$\text{(A.4.3)} \qquad \operatorname{Tor}_i^R(M_{n+1}, N) \cong \operatorname{Tor}_{i-1}^R(M_n, N) \text{ for all } i \ge 2,$$

and the following exact sequence in which F is finitely generated and free:

$$\text{(A.4.4)} \quad 0 \to \operatorname{Tor}_1^R(M_{n+1}, N) \to M_n \otimes_R N \to F \otimes_R N \to M_{n+1} \otimes_R N \to 0.$$

Induction and (A.4.3) imply that $\operatorname{Tor}_i^R(M_{n+1}, N)$ has finite length for all $i \gg 0$, so (3) holds; furthermore, by Corollary A.2, $\operatorname{Tor}_i^R(M_{n+1}, N)$ is torsion for all $i \ge 1$. (Recall that $\dim(R) \ge \operatorname{codim}(R) = c \ge 1$ so that finite length modules are torsion.) Since $n \le c-1$, condition (2) implies that $M_n \otimes_R N$ satisfies (S_1) and hence $M_n \otimes_R N$ is torsion-free; therefore the exact sequence (A.4.4) forces $\operatorname{Tor}_1^R(M_{n+1}, N)$ to vanish. Now (A.4.3) gives (4). It remains to verify (2), namely, that $M_{n+1} \otimes_R N$ satisfies (S_{c-n-1}). To that end, let $\mathfrak{p} \in \operatorname{Supp}(M_{n+1} \otimes_R N)$. We will verify that $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} \ge \min\{c-n-1, \operatorname{height}(\mathfrak{p})\}$; see §2.6.

Suppose $\operatorname{height}(\mathfrak{p}) \ge c-n$. Recall, by Hypothesis 3.5(i), N satisfies (S_{c-1}). Hence $F \otimes_R N$, a direct sum of copies of N, satisfies (S_{c-n-1}). In particular it follows that $\operatorname{depth}_{R_\mathfrak{p}}(F \otimes_R N)_\mathfrak{p} \ge c-n-1$. Furthermore, by (2) of the induction hypothesis, we have that $\operatorname{depth}_{R_\mathfrak{p}}(M_n \otimes_R N)_\mathfrak{p} \ge c-n$. Recall that $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$. Therefore, localizing the short exact sequence in (A.4.4) at $\mathfrak{p}$, we conclude by the depth lemma that $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} \ge c-n-1$.

Next assume $\operatorname{height}(\mathfrak{p}) \le c-n-1$. We want to show that $(M_{n+1} \otimes_R N)_\mathfrak{p}$ is maximal Cohen–Macaulay. By the induction hypotheses, $\operatorname{Tor}_i^R(M_n, N)$ has finite length for all $i \gg 0$. As $n \ge 0$, we see that $\dim(R) \ge \operatorname{codim}(R) = c \ge c-n$, whence $\mathfrak{p}$ is not the maximal ideal. Thus $\operatorname{Tor}_i^R(M_n, N)_\mathfrak{p} = 0$ for all $i \gg 0$. Now, setting $w = c-n-1$ and using Corollary A.3 for the pair M_n, N, we conclude that $\operatorname{Tor}_i^R(M_n, N)_\mathfrak{p} = 0$ for all $i \ge 1$. Then (A.4.3) and the already established fact that $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$ give that $\operatorname{Tor}_i^R(M_{n+1}, N)_\mathfrak{p} = 0$ for all $i \ge 1$. Thus, the depth formula holds — see §2.3:

$$\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1})_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p}.$$

Since Serre's conditions localize, $N_\mathfrak{p}$ is maximal Cohen–Macaulay over $R_\mathfrak{p}$; see Hypothesis 3.5(i). Also, $(M_{n+1})_\mathfrak{p}$ is maximal Cohen–Macaulay whether or not $(M_n)_\mathfrak{p}$ is zero; see the pushforward sequence or Proposition 2.9(ii). By the depth formula, $(M_{n+1} \otimes_R N)_\mathfrak{p}$ is maximal Cohen–Macaulay. Thus $M_{n+1} \otimes_R N$ satisfies (2), and the induction is complete.

Now we parallel the argument for the case $c = 1$. At the end, $\operatorname{Tor}_i^R(M_{c-1}, N)$ has finite length for all $i \gg 0$, and is equal to 0 for $i = 1, \dots, c$. Tensoring M_{c-1}

with the pushforward of N, we get

$$\operatorname{Tor}_i^R(M_{c-1}, N_1) \cong \operatorname{Tor}_{i-1}^R(M_{c-1}, N) \quad \text{for all } i \geq 2, \tag{A.4.5}$$

$$\operatorname{Tor}_1^R(M_{c-1}, N_1) \hookrightarrow M_{c-1} \otimes_R N. \tag{A.4.6}$$

In view of (A.4.5), it suffices to show that $\operatorname{Tor}_1^R(M_{c-1}, N_1) = 0$: this will imply $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for all $i = 1, \dots, c+1$, and hence Murthy's rigidity theorem [1963, Theorem 1.6] will yield that $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for all $i \geq 1$, and consequently $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for all $i \geq 1$ by (A.4.5). We know that $M_{c-1} \otimes_R N$ is torsion-free. Therefore we use (A.4.6) and Corollary A.2, and obtain $\operatorname{Tor}_1^R(M_{c-1}, N_1) = 0$, as we did in the case $c = 1$. □

Appendix B: Amending the literature

We use Theorem A.4 to give a different proof of an important result of Huneke and Wiegand; see Theorem B.2 and the ensuing paragraph. We also point out a missing hypothesis in a result of C. Miller [1998, Theorem 3.1], and state the corrected form of her theorem in Corollary B.3. At the end of the paper we indicate an alternative route to the proof of [Huneke and Wiegand 1994, Theorem 3.1], the main theorem in that reference.

Theorem B.1 [Huneke and Wiegand 1994]. *Let R be a hypersurface and let M, N be finitely generated R-modules. If M or N has rank and $M \otimes_R N$ is maximal Cohen–Macaulay, then both M and N are maximal Cohen–Macaulay, and either M or N is free.*

Theorem B.1 and its variations have been analyzed, used, and studied in the literature; see [Celikbas and Wiegand 2015] and [Dao 2013b] for some history and many consequences of the theorem. The following result [Huneke and Wiegand 1994, Theorem 2.7] played an important role in its proof.

Theorem B.2 [Huneke and Wiegand 1994]. *Let R be a hypersurface and let M, N be nonzero finitely generated R-modules. Assume $M \otimes_R N$ is reflexive and that N has rank. Then the following conditions hold*:

(i) $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$.

(ii) *M is reflexive, and N is torsion-free.*

Theorem B.2 was established in [Huneke and Wiegand 1994, Theorem 2.7]. However, the conclusion there was that *both* M and N are reflexive, and the proof of this stronger claim is flawed. Dao realized this, and subsequently Huneke and Wiegand corrected their oversight [2007]. A similar flaw can be found in [Miller 1998]; see Theorems 1.3 and 1.4 there and compare with our correction in Corollary B.3. The version stated above reflects our current understanding and is from [Celikbas and Piepmeyer 2014]. We do not yet know whether N is forced to be reflexive — that is, the question below remains open; cf. [Huneke and Wiegand 1994, Theorem 2.7] and [Miller 1998, Theorem 1.3].

Question. Let R be a hypersurface and M, N nonzero finitely generated R-modules. If N has rank and $M \otimes_R N$ is reflexive, must *both* M and N be reflexive?

This question has been recently studied in [Celikbas and Piepmeyer 2014], which gives partial answers using the New Intersection Theorem.

We now show how Theorem B.2 follows from Theorem A.4. In fact, one needs only the case $c = 1$ of Theorem A.4.

Proof of Theorem B.2 using Theorem A.4. Set $d = \dim R$. If $d = 0$, then N is free (since it has rank), so all is well. From now on assume $d \geq 1$. We remark at the outset that neither M nor N can be torsion, i.e., $\perp_R M \neq 0$ and $\perp_R N \neq 0$. Also, by the assumption of rank, $\operatorname{Supp}(N) = \operatorname{Spec}(R)$. Suppose first that both M and N are torsion-free; we will prove (i) by induction on $d = \dim R$. Let M_1 denote the pushforward of M; see §2.8. Then $\operatorname{Tor}_1^R(M_1, N)$ is torsion as N has rank. Since $M \otimes_R N$ is torsion-free, applying $- \otimes_R N$ to (2.8.1) shows that

$$\operatorname{Tor}_1^R(M_1, N) = 0. \tag{B.2.1}$$

Suppose for the moment that $d = 1$. Since N has rank, there is an exact sequence

$$0 \to N \to F \to C \to 0,$$

in which F is free and C is torsion; see [Huneke and Wiegand 1994, Lemma 1.3]. Note that C is of finite length since $d = 1$. Note also that $\operatorname{Tor}_2^R(M_1, C) \cong \operatorname{Tor}_1^R(M_1, N) = 0$; see (B.2.1). Therefore, Corollary 2.3 from that same reference implies that $\operatorname{Tor}_i^R(M_1, C) = 0$ for all $i \geq 2$, and hence $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. Now (2.8.1) establishes (i).

Still assuming that both M and N are torsion-free, let $d \geq 2$. The inductive hypothesis implies that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \geq 1$. In particular $\operatorname{Tor}_i^R(M, N)_{\mathfrak{q}} = 0$ for all prime ideals $\mathfrak{q}$ of R of height at most one. Therefore, Proposition 2.10 shows that $M_1 \otimes_R N$ is torsion-free, that is, $M_1 \otimes_R N$ satisfies (S_1); see §2.5 and §2.6. Furthermore, from the pushforward exact sequence (2.8.1), we see that $\operatorname{Tor}_i^R(M_1, N)$ has finite length for all $i \geq 2$. Consequently the pair M_1, N satisfies (SP_1). Now Theorem A.4, applied to M_1, N, shows that $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. By (2.8.1), we see that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. This proves (i) under the additional assumption that M and N are torsion-free.

Since $M \otimes_R N$ is torsion-free, by Remark 3.4, there are isomorphisms

$$M \otimes_R N \cong M \otimes_R \perp_R N \cong \perp_R M \otimes_R N \cong \perp_R M \otimes_R \perp_R N.$$

In particular, $\perp_R M \otimes_R \perp_R N$ is also reflexive. As noted before, neither M nor N is torsion, so $\perp_R M$ and $\perp_R N$ are nonzero. As N has rank so does $\perp_R N$, so the already established part of the result (applied to $\perp_R M$ and $\perp_R N$) yields that $\operatorname{Tor}_i^R(\perp_R M, \perp_R N) = 0$ for $i \geq 1$. Given this, since $\perp_R M \otimes_R N$ is torsion-free by the isomorphisms above, applying Remark 3.4 to the R-modules $\perp_R M$ and N gives $N = \perp_R N$; then applying Remark 3.4 to M and N yields $M = \perp_R M$. In conclusion, M and N are torsion-free, and hence $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. From the last, the depth formula holds.

The remaining step is to prove that M is reflexive. Since $\operatorname{Supp}(N) = \operatorname{Spec}(R)$, we have $\operatorname{depth}(N_{\mathfrak{p}}) \leq \operatorname{height}(\mathfrak{p})$ for all primes $\mathfrak{p}$ of R. Localizing the depth formula §2.3 shows Serre's condition (S_2) on M; see §2.6. □

The next result is due to C. Miller [1998]. In the original formulation, the essential requirement — that M have rank — is missing: for example, the module $M = R/(x)$ over the node $k[\![x, y]\!]/(xy)$ is not free, yet $M \otimes_R M$, which is just M, is maximal Cohen–Macaulay and hence reflexive. We state her result here in its corrected form and include a proof for completeness.

Corollary B.3 [Miller 1998, Theorem 3.1]. *Let R be a d-dimensional hypersurface and let M be a finitely generated R-module with rank. If $\bigotimes_R^n M$ is reflexive for some $n \geq \max\{2, d-1\}$, then M is free.*

Proof. If $d \leq 2$, then $\bigotimes_R^n M$ is maximal Cohen–Macaulay, and Theorem B.1 gives the result. Assume now that $d \geq 3$. Applying Theorem B.2 and [Huneke and Wiegand 1997, Theorem 1.9] repeatedly, we conclude the following:

(i) $\bigotimes_R^r M$ is reflexive for all $r = 1, \dots, n$.

(ii) $\operatorname{Tor}_i^R(M, \bigotimes_R^{r-1} M) = 0$ for all $i \geq 1$ and all $r = 2, \dots, n$.

(iii) $\operatorname{pd}(M) < \infty$.

It follows from (i) that $\operatorname{depth}(\bigotimes_R^r M) \geq 2$ for all $r = 1, \dots, n$; see §2.6. Also, (ii) implies the depth formula

$$\operatorname{depth}(M) + \operatorname{depth}\Big(\bigotimes_R^{r-1} M\Big) = d + \operatorname{depth}\Big(\bigotimes_R^{r} M\Big),$$

for all $r = 2, \dots, n$. One checks by induction on r that

$$r \cdot \operatorname{depth}(M) = (r-1) \cdot d + \operatorname{depth}\Big(\bigotimes_R^{r} M\Big),$$

for $r = 2, \dots, n$. By setting $r = n$, and using the inequalities $n \geq d - 1$ and $\operatorname{depth}(\bigotimes_R^n M) \geq 2$, we obtain

$$n \cdot \operatorname{depth}(M) \geq (n-1) \cdot d + 2 = n \cdot (d-1) + n - d + 2 \geq n \cdot (d-1) + 1.$$

Therefore, $\operatorname{depth}(M) \geq d$, that is, M is maximal Cohen–Macaulay. Now (iii) and the Auslander–Buchsbaum formula [1957, Theorem 3.7] imply that M is free. □

A consequence of Theorems B.1 and B.2 is the following result:

Proposition B.4 [Huneke and Wiegand 1997, Theorem 1.9]. *Suppose M and N are finitely generated modules over a hypersurface R, and assume that* $\operatorname{Tor}_i^R(M, N) = 0$ *for $i \gg 0$. Then at least one of the modules has finite projective dimension.*

At about the same time, Miller [1998] obtained the same result independently, by an elegant, direct argument. As Miller observed in that reference, one can turn things around and easily deduce Theorem B.1 from Proposition B.4 and the vanishing result Theorem B.2.

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