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We prove some estimates for convex ancient solutions (the existence time for the solution starts at $-\infty$) to the power-of-mean curvature flow, when the power is strictly greater than $\frac{1}{2}$. As an application, we prove that in dimension two, the blow-down of an entire convex translating solution, namely $u_h = \frac{1}{h} u(h^{\frac{1}{1+\alpha}} x)$, locally uniformly converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$ as $h \to \infty$. Another application is that for the generalized curve shortening flow (convex curve evolving in its normal direction with speed equal to a power of its curvature), if the convex compact ancient solution sweeps the whole space $\mathbb{R}^2$, it must be a shrinking circle. Otherwise the solution must be defined in a strip region.

1. Introduction

Classifying ancient convex solutions to mean curvature flow is very important in studying the singularities of mean curvature flow. Translating solutions arise as a special case of ancient solutions when one uses a proper procedure to blow up the mean convex flow near type II singular points, and general ancient solutions arise at general singularities. Some important progress was made by Wang [2011], and Daskalopoulos, Hamilton and Sesum [Daskalopoulos et al. 2010]. Wang proved that in dimension $n = 2$, an entire convex translating solution to mean curvature flow must be rotationally symmetric in an appropriate coordinate system, which was a conjecture formulated explicitly by White [2000], but for $n \geq 3$ such solutions are not necessarily rotationally symmetric.

Wang also constructed some entire convex translating solutions with level sets neither spherical nor cylindrical in dimension greater or equal to 3. In the same paper, Wang also proved that if a convex ancient solution to the curve shortening flow sweeps the whole space $\mathbb{R}^2$, then it must be a shrinking circle — otherwise the convex ancient solution must be defined in a strip region, and he indeed constructed such solutions by a compactness argument. Daskalopoulos et al. [2010] showed that apart from the shrinking circle, the so called *Angenent oval* (a convex ancient

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solution of the curve shortening flow discovered by Angenent that decomposes into two translating solutions of the flow) is the only other embedded convex compact ancient solution of the curve shortening flow. That means that the corresponding curve shortening solution defined in a strip region constructed by Wang is exactly the "Angenent oval".

The power-of-mean curvature flow, in which a hypersurface evolves in its normal direction with speed equal to a power α of its mean curvature H, is well-studied [Andrews 1998; 2003; 2002; Schulze 2005; Chou and Zhu 2001; Sheng and Wu 2009]. Schulze [2005] called it H^α-flow. In the following, we will also call the one dimensional power-of-curvature flow the *generalized curve shortening flow*. It would be very interesting if one could classify the ancient convex solutions. In this paper, we use the method developed in [Wang 2011] to study the geometric asymptotic behavior of ancient convex solutions to H^α-flow. The general equation for H^α-flow is

$$\frac{\partial F}{\partial t} = -H^\alpha \nu,$$

where $F : M \times [0, T) \to \mathbb{R}^{n+1}$ is a time-dependent embedding of the evolving hypersurface, ν is the unit normal vector to the hypersurface $F(M, t)$ in $\mathbb{R}^{n+1}$, and H is its mean curvature. If the evolving hypersurface can be represented as a graph of a function $u(x, t)$ over some domain in $\mathbb{R}^n$, then we can project the evolution equation to the $(n + 1)$-st coordinate direction of $\mathbb{R}^{n+1}$ and the equation becomes

$$u_t = \sqrt{1 + |Du|^2}\left(\operatorname{div} \frac{Du}{\sqrt{1 + |Du|^2}}\right)^\alpha.$$

Then a translating solution to the H^α-flow will satisfy the equation

$$\sqrt{1 + |Du|^2}\left(\operatorname{div} \frac{Du}{\sqrt{1 + |Du|^2}}\right)^\alpha = 1,$$

which is equivalent to the special case $\sigma = 1$ of the following:

$$L_\sigma(u) = \left(\sqrt{\sigma + |Du|^2}\,\right)^{\frac{1}{\alpha}} \operatorname{div} \frac{Du}{\sqrt{\sigma + |Du|^2}} \tag{1}$$

$$= \left(\sigma + |Du|^2\right)^{\frac{1}{2\alpha} - \frac{1}{2}} \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{\sigma + |Du|^2}\right) u_{ij} \tag{2}$$

$$= 1, \tag{3}$$

where $\sigma \in [0, 1]$, $\alpha \in \left(\frac{1}{2}, \infty\right]$ is a constant, $n = 2$ is the dimension of $\mathbb{R}^2$. If u is a convex solution of (3), then $u + t$, as a function of $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$, is a translating solution to the flow

$$u_t = \sqrt{\sigma + |Du|^2}\left(\operatorname{div} \frac{Du}{\sqrt{\sigma + |Du|^2}}\right)^{\alpha}. \tag{4}$$

When $\sigma = 1$, Equation (4) is the nonparametric power-of-mean curvature flow. When $\sigma = 0$, Equation (3) is the level set flow. That is, if u is a solution of (3) with $\sigma = 0$, then the level set $\{u = -t\}$, where $-\infty < t < -\inf u$, evolves by the power-of-mean curvature.

In the following we will assume $\sigma \in [0, 1]$, $\alpha \in \left(\frac{1}{2}, \infty\right]$, and the dimension $n = 2$, although some of the estimates do hold in higher dimension. The main results of this paper are the following theorems.

Theorem 1. *Let u be an entire convex solution of* (3). *Let*

$$u_h(x) = h^{-1}u(h^{\frac{1}{1+\alpha}} x).$$

Then, u_h locally uniformly converges to

$$\tfrac{1}{1+\alpha}|x|^{1+\alpha} \quad \text{as } h \to \infty.$$

Theorem 2. *Let u_σ be an entire convex solution of* (3). *Then,*

$$u_0(x) = \tfrac{1}{1+\alpha}|x|^{1+\alpha}$$

up to a translation of the coordinate system. When $\sigma \in (0, 1]$, if

$$|D^2u(x)| = O(|x|^{\beta}) \quad \text{as } |x| \to \infty$$

for some fixed constant β satisfying $\beta < 3\alpha - 2$, then u_σ is rotationally symmetric after a proper translation of the coordinate system.

Corollary 3. *A convex compact ancient solution to the generalized curve shortening flow which sweeps the whole space $\mathbb{R}^2$ must be a shrinking circle.*

Remark 4. The condition $\alpha > \frac{1}{2}$ is necessary for our results. One can consider the translating solution $v(x)$ to (3) with $\sigma = 1$ in one dimension. In fact, when $\alpha \le \frac{1}{2}$, the translating solution $v(x)$ is a convex function defined on the entire real line [Chou and Zhu 2001, p. 28]. Then one can construct a function $u(x, y) = v(x) - y$, defined on the entire plane, and u will satisfy (3) with $\sigma = 0$; it is obviously not rotationally symmetric. We can also let $u(x, y) = v(x)$, which is an entire solution to (3) with $\sigma = 1$, and it is not rotationally symmetric.

When the dimension is at least two, similar examples can be given: we can take an entire rotationally symmetric solution $v(x)$ to (3) with $n \ge 2$ and $\sigma = 1$, and again let $u(x, y) = v(x) - y$ (here, y is the $(n + 1)$-st coordinate for $\mathbb{R}^{n+1}$). It is easy to see that u will satisfy (3) with n replaced by $n + 1$ and $\sigma = 0$, and the level set of u is neither a sphere nor a cylinder.

We would also like to point out that this elementary construction can be used to give a slight simplification of the proof of [Wang 2011, Theorem 2.1] (corresponding to our Corollary 10 for $\alpha = 1$). Let v_σ be an entire convex solution to (3) in dimension n with $\sigma \in (0, 1]$. Then $u(x, y) = v_\sigma(x) - \sqrt{\sigma}\, y$ will be an entire convex solution to (3) in dimension $n+1$ with $\sigma = 0$. Hence if one has proved the estimate in Corollary 10 for $\sigma = 0$ in all dimensions, the estimate for $\sigma \in (0, 1]$ follows immediately from the above construction. The remainder of the paper is divided into four sections. Sections 2 and 3 contain the proof of Theorem 1 and the first part of Theorem 2. Section 4 is devoted to the proof of Corollary 10, and the last section completes the proof of Theorem 2.

2. Power growth estimate

In this section, we prove a key estimate, which says that any entire convex solution u to (3) must satisfy

$$u(x) \le C(1 + |x|^{1+\alpha}),$$

where the constant C depends only on the upper bound of $u(0)$ and $|Du(0)|$. When $\alpha = 1$, the estimate was proved by Wang [2011, Theorem 2.1]. To apply Wang's method, the main difficulty is that now the speed function is nonlinear in the curvature. We overcome this difficulty by further exploiting some elementary convexity properties.

For any constant $h > 0$, we denote

$$\Gamma_h = \{x \in \mathbb{R}^n : u(x) = h\},$$
$$\Omega_h = \{x \in \mathbb{R}^n : u(x) < h\},$$

so that Γ_h is the boundary of Ω_h. Let κ be the curvature of the level curve Γ_h. We have

$$L_\sigma(u) = \left(\sigma + u_\gamma^2\right)^{\frac{1}{2\alpha} - \frac{1}{2}} \left(\kappa u_\gamma + \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}\right) \tag{5}$$

$$\ge \kappa u_\gamma^{\frac{1}{\alpha}} = L_0(u), \tag{6}$$

where γ is the unit outward normal to Ω_h, and $u_{\gamma\gamma} = \gamma_i \gamma_j u_{ij}$.

Before starting the proof of our main results, we recall a well known convergence result for the generalized curve shortening flow.

Lemma 5 [Andrews 2003, Theorems 1.3, 1.4, 1.5]. *Let ℓ_t be a time-dependent family of closed curves in $\mathbb{R}^2$ evolving under the generalized curve shortening flow with $\alpha > \frac{1}{3}$. Suppose the initial curve ℓ_0 is convex. Then the curve converges in finite time T to a round point P in the sense that $((1+\alpha)(T-t))^{-\frac{1}{1+\alpha}}(\ell_t - P)$ is asymptotic to the unit circle.*

Next, we prove a lemma which will be used to control the shape of the level set of a complete convex solution to (3).

Lemma 6. *Let u be a complete convex solution of* (3). *Suppose that $u(0) = 0$ and that the infimum $\inf\{|x| : x \in \Gamma_1\}$ is attained at $x_0 = (0, -\delta) \in \Gamma_1$ for some $\delta > 0$ sufficiently small. Let D_1 be the projection of Γ_1 onto the axis $\{x_2 = 0\}$. Then, D_1 contains the interval $(-R, R)$, and when $\alpha \le 1$, R satisfies*

$$R \ge C_1(-\log\delta - C_2)^{\frac{\alpha}{\alpha+1}}, \tag{7}$$

where $C_1, C_2 > 0$ are independent of δ; when $\alpha > 1$, $R \ge C$ for some positive constant C.

The proof of this lemma follows that of [Wang 2011, Lemma 2.4] with minor modifications; the for reader's convenience, we give some details here.

Proof. First, we prove the lemma when $\frac{1}{2} < \alpha \le 1$. Suppose that near x_0, Γ_1 is given by $x_2 = g(x_1)$. Then, g is a convex function satisfying $g(0) = -\delta$ and $g'(0) = 0$. Let $b > 0$ be a constant such that $g'(b) = 1$. To prove (7), it suffices to prove

$$b \ge C_1(-\log\delta - C_2)^{\frac{\alpha}{\alpha+1}}. \tag{8}$$

For any $y = (y_1, y_2) \in \Gamma_1$, where $y_1 \in [0, b]$, as in the proof of [Wang 2011, Lemma 2.4] we have

$$u_\gamma(y) \ge \frac{\sqrt{1+g'^2}}{y_1 g' - y_2}, \tag{9}$$

where γ is the unit normal of the sublevel set Ω_1. Since $L_0 u \le 1$, we have

$$\frac{g''}{(1+g'^2)^{\frac{3}{2}}} \frac{(1+g'^2)^{\frac{1}{2\alpha}}}{(y_1 g' - y^2)^{\frac{1}{\alpha}}} \le \kappa u_\gamma^{\frac{1}{\alpha}} \le 1, \tag{10}$$

where κ is the curvature of the level curve Γ_1. Hence,

$$g''(y_1) \le (1+g'^2)^{\frac{3}{2}-\frac{1}{2\alpha}}(y_1 g' - y_2)^{\frac{1}{\alpha}} \tag{11}$$

$$\le 10 y_1^{\frac{1}{\alpha}} g' + 10\delta \tag{12}$$

where $y_2 = g(y_1)$ and $g'(y_1) \le 1$ for $y_1 \in (0, b)$. The inequality from (11) to (12) is trivial when $y_2 \ge 0$. When $y_2 \le 0$, since $|y_2| \le \delta$, we have either $y_1 g' \le \delta$ or $y_1 g' > \delta$. For the former we have

$$(y_1 g' - y_2)^{\frac{1}{\alpha}} \le (2\delta)^{\frac{1}{\alpha}} \le 4\delta;$$

for the latter, since $g'(y_1) \le 1$, we have

$$(y_1 g' - y_2)^{\frac{1}{\alpha}} \le (2 y_1 g')^{\frac{1}{\alpha}} \le 4 y_1^{\frac{1}{\alpha}} g'.$$

We consider the equation

$$\rho''(t) = 10t^{\frac{1}{\alpha}}\rho' + 10\delta \tag{13}$$

with initial conditions $\rho(0) = -\delta$ and $\rho'(0) = 0$. Then for $t \in (0, b)$, we have

$$\rho'(t) = 10\delta e^{\frac{10\alpha}{\alpha+1}t^{\frac{\alpha+1}{\alpha}}} \int_0^t e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}}\,ds. \tag{14}$$

Since $\int_0^\infty e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}}\,ds$ is bounded above by some constant C, we have

$$1 \le \rho'(b) = 10\delta e^{\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}} \int_0^b e^{-\frac{10\alpha}{\alpha+1}s^{\frac{\alpha+1}{\alpha}}}\,ds \tag{15}$$

$$\le C_1\delta e^{\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}}, \tag{16}$$

from which (8) follows.

When $\alpha > 1$, the situation is different. First, we introduce a number a such that $g'(a) = \frac{1}{2}$. Then, we can follow the proof above until (11). For (12) the inequality becomes

$$g''(y_1) \le 10y_1^{\frac{1}{\alpha}}g' + 10\delta^{\frac{1}{\alpha}},$$

for $y_1 \in [a, b]$. Now (16) becomes

$$e^{-\frac{10\alpha}{\alpha+1}b^{\frac{\alpha+1}{\alpha}}}\rho'(b) - e^{-\frac{10\alpha}{\alpha+1}a^{\frac{\alpha+1}{\alpha}}}\rho'(a) \le C_1\delta^{\frac{1}{\alpha}}.$$

Then, it is easy to see that when δ is small, $b \ge C$, for some fixed constant C. □

Remark 7. When $\alpha \le 1$, it follows from Lemma 6 that when δ is sufficiently small, by convexity and in view of Figure 1, we have that Ω_1 contains the shadowed region. Then it is easy to check that Ω_1 contains an ellipse

$$E = \left\{(x_1, x_2) \;\middle|\; \frac{x_1^2}{\left(\frac{R}{6}\right)^2} + \frac{\left(x_2 - \frac{7\delta^* - 5\delta}{12}\right)^2}{\left(\frac{\delta^* + \delta}{4}\right)^2} = 1\right\}, \tag{17}$$

where δ^* is a positive constant such that $u(0, \delta^*) = 1$ and R is defined in Lemma 5.

When $\alpha > 1$, if δ^* is very large, in the part $\{x : u(x) \le 1,\ x_1 \ge 0\}$, by convexity we can find an ellipse which has the length of short axis bounded from below and the length of long axis very large, and if we let the ellipse evolve under the generalized curve shortening flow, it will take time more than 1 for it to converge to a round point. When δ^* is less than some fixed constant, we need to consider two cases.

Case 1: The set $\{u \le 1\}$ is not compact. In this case when we project $\{u(x) = 1\}$ to the axis $\{x_2 = 0\}$, and denote the leftmost and rightmost points as $(-l, 0)$ and $(r, 0)$, respectively. Then either l or r is very large, which guarantees that one can

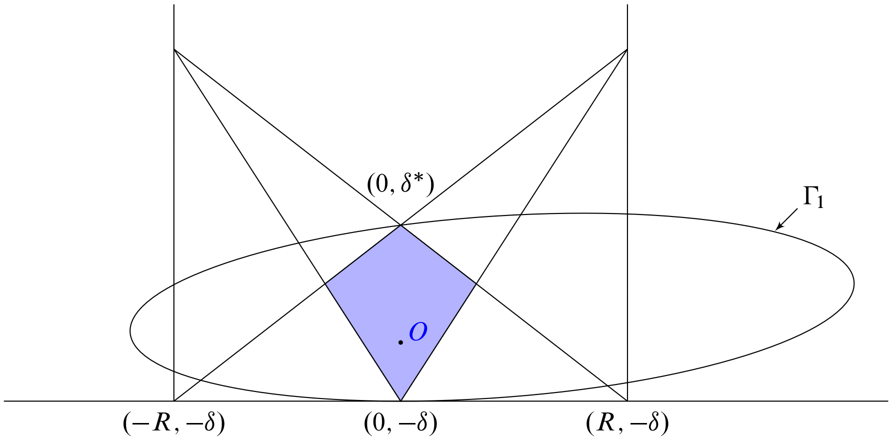

Figure 1. Γ_1 contains the shadow part.

still find an ellipse inside $\{x : u(x) \le 1,\ x_1 \le 0\}$ (or $\{x : u(x) \le 1,\ x_1 \ge 0\}$) with the similar property as before.

Case 2: The set $\{u \le 1\}$ is compact. For this case, we will always assume 0 is the minimum point of u, and $u(0) = 0$. We claim that when δ is very small, for the purpose of the proof of Corollary 10, we can assume one of l or r is very large. Indeed, if the claim is not true, then we have a sequence of functions u_i such that $\{u_i \le 1\}$ has width bounded by some constant independent of i, and $\operatorname{dist}(0, \{u_i \le 1\}) \to 0$ as $i \to \infty$. In view of the following proof of Corollary 10, we can assume u_i satisfies (3) with $\sigma_i \to 0$. Then by passing to a subsequence, we can assume that $\{u_i = 1\}$ converges to a convex curve C_0 in hausdorff distance. Let C_0 evolve under the generalized curve shortening flow; by Lemma 5, it will converge to a point P, but by the above discussion we see that P is on C_0, which is clearly impossible. Once l or r is very large, we can find an ellipse with the similar property as in the case 1.

Remark 8. One can also establish a similar lemma in higher dimensions, which says that D_1 (a convex set with dimension greater than 1) contains a ball centered at the origin with radius

$$R \ge C_n(-\log\delta - C)^{\frac{\alpha}{\alpha+1}},$$

where C_n is a constant depending only on n and C is a positive constant independent of δ. The proof can be reduced to the two dimensional case; for the details, refer to the proof of [Wang 2011, Lemma 2.6].

Lemma 9. *Let u be a complete convex solution of* (3). *Suppose $u(0) = 0$, δ and δ^* are defined as in Lemma 6 and Remark 7. If δ and δ^* are sufficiently small, then u is defined in a strip region.*

When $\alpha = 1$, this lemma is proved by Wang [2011, Corollary 2.2]. The proof of Lemma 9 is based on a careful study of the shape of the level set of u. Before giving the proof, we will give an important corollary first.

Corollary 10. *Let u be an entire convex solution of* (3) *in* $\mathbb{R}^2$, *then*

$$u(x) \le C(1 + |x|^{1+\alpha}), \tag{18}$$

where the constant C depends only on the upper bound for $u(0)$ and $|Du(0)|$.

Proof. The proof of this Corollary follows the proof of [Wang 2011, Theorem 2.1]. We only record some necessary changes here. First, the rescaling $u_h(x) = \frac{1}{h}u(h^{\frac{1}{2}}x)$ used that proof should be replaced by $u_h(x) = \frac{1}{h}u(h^{\frac{1}{1+\alpha}}x)$. Note that u_h solves (3) with $\sigma = \sigma_h \to 0$ as $h \to \infty$. Second, the ellipse used in that proof when applying the comparison argument should be replaced by the one discussed in Remark 7. □

Proof of Lemma 9. By a rotation of coordinates we may assume that the axial directions of E in Remark 7 coincide with those of the coordinate system. Let $\mathcal{M}_u$ be the graph of u, which consists of two parts, $\mathcal{M}_u = \mathcal{M}^+ \cup \mathcal{M}^-$, where

$$\mathcal{M}^+ = \{(x, u(x)) \in \mathbb{R}^3 : \partial_{x_2} u \ge 0\} \quad \text{and} \quad \mathcal{M}^- = \{(x, u(x)) \in \mathbb{R}^3 : \partial_{x_2} u \le 0\}.$$

Then $\mathcal{M}^\pm$ can be represented as the graphs of functions $g^\pm$ of the form $x_2 = g^\pm(x_1, x_3)$, for $(x_1, x_3) \in D$ where D is the projection of $\mathcal{M}_u$ onto the plane $\{x_2 = 0\}$. The functions g^+ and g^- are concave and convex, respectively, and we have $x_3 = u(x_1, g^\pm(x_1, x_3))$. Set

$$g = g^+ - g^-. \tag{19}$$

Then g is a positive, concave function on D, vanishing on ∂D. For any $h > 0$ let $g_h(x_1) = g(x_1, h)$, $g_h^\pm(x_1) = g^\pm(x_1, h)$, and $D_h = \{x_1 \in \mathbb{R}^1 : (x_1, h) \in D\}$. Then g_h is a positive, concave function in D_h, vanishing on ∂D_h, and $D_h = (-\underline{a}_h, \bar{a}_h)$ is an interval containing the origin. Let $b_h = g_h(0)$. We consider the case $\sigma = 0$ first.

Claim 1: Suppose h is large, $g_1(0) = \delta^* + \delta$ is small, $b_h \le 4$, and $\underline{a}_h, \bar{a}_h \ge b_h$. Then,

$$\bar{a}_h \ge \frac{1}{1000}\frac{h}{b_h^\alpha} \text{ for } \alpha \le 1 \quad \text{and} \quad \bar{a}_h \ge \frac{1}{1000}\frac{h^{1/(2\alpha-1)}}{b_h^{1/(2\alpha-1)}} \text{ for } \alpha > 1.$$

Proof. Without loss of generality, we assume $\bar{a}_h \le \underline{a}_h$. Let $U_h = \Omega_h \cap \{x_1 > 0\}$. By the convexity of U_h and the assumption $\underline{a}_h, \bar{a}_h \ge b_h$, we have $\underline{a}_s, \bar{a}_s \ge \frac{1}{2}b_h$ for all $s \in \left(\frac{1}{2}h, h\right)$. Hence by the concavity of g,

$$\left|\frac{d}{dx_1} g_s(0)\right| \le 2 \quad \text{for } s \in \left(\tfrac{1}{2}h, h\right),$$

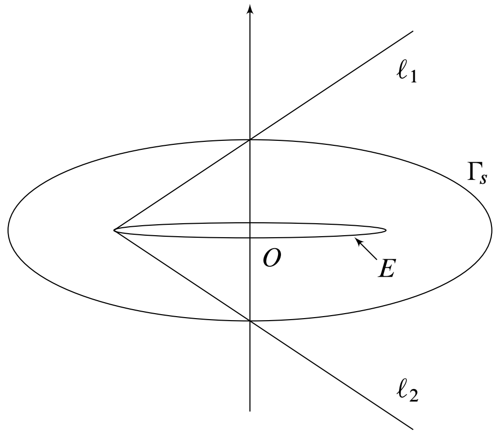

Figure 2. $\Gamma_s \cap \{x_1 > 0\}$ is trapped between two lines.

which means the arclength of the image of $\Gamma_s \cap \{x_1 > 0\}$ under the Gauss map is bigger then $\frac{\pi}{6}$. Notice that Ω_1 contains E, which was defined in Remark 7. When δ and δ^* are very small, E is very thin and long. The center of E is very close to the origin; in fact, for our purpose we can just pretend E is centered at the origin. By convexity of Ω_h and in view of Figure 2, we see that $\Gamma_s \cap \{x_1 > 0\}$ is trapped between two lines ℓ_1 and ℓ_2, and the slopes of ℓ_1 and ℓ_2 are very close to 0 when E is very long and thin. Then it is clear that the largest distance from the points on $\Gamma_s \cap \{x_1 > 0\}$ to the origin can not be bigger than $10\bar{a}_h$. By convexity of u,

$$u_\gamma(x) \geq \frac{h}{20\bar{a}_h} \quad \text{for } x \in \Gamma_s \cap \{x_1 > 0\}.$$

Since $\Gamma_s \cap \{x_1 > 0\}$ evolves under the generalized curve shortening flow, when $\alpha \leq 1$ we have the estimate

$$\frac{d}{ds}|U_s| = \int_{\Gamma_s \cap \{x_1 > 0\}} \kappa^\alpha \, d\xi \tag{20}$$

$$= \int_{\Gamma_s \cap \{x_1 > 0\}} u_\gamma^{\frac{1}{\alpha}-1} \kappa \, d\xi \tag{21}$$

$$\geq \frac{1}{50}\Big(\frac{h}{\bar{a}_h}\Big)^{\frac{1}{\alpha}-1} \frac{\pi}{6}, \tag{22}$$

where from (20) to (21) we used the equation $\kappa u_\gamma^{\frac{1}{\alpha}} = 1$. The claim follows by the simple fact that

$$\frac{3}{2} b_h \bar{a}_h \geq |U_h| \geq \frac{1}{50}\Big(\frac{h}{\bar{a}_h}\Big)^{\frac{1}{\alpha}-1} \frac{\pi}{6}\frac{h}{2}.$$

When $\alpha > 1$, let l_s denote the arclength of $\Gamma_s \cap \{x_1 > 0\}$. Then, by the above discussion, it is not hard to see that $l_s \approx C\bar{a}_h$. Then by a simple application of

Jensen's inequality,

$$\begin{aligned}\frac{d}{ds}(|U_s|) &= \int_{\Gamma_s\cap\{x_1>0\}} \kappa^\alpha \, d\xi \\ &= l_s \int_{\Gamma_s\cap\{x_1>0\}} \kappa^\alpha \frac{1}{l_s}\, d\xi \\ &\geq l_s \left(\int_{\Gamma_s\cap\{x_1>0\}} \frac{\kappa}{l_s}\, d\xi\right)^\alpha \geq C l_s^{1-\alpha} \geq C\bar{a}_h^{1-\alpha}.\end{aligned}$$

Again by the simple fact that $\frac{3}{2}b_h\bar{a}_h \geq |U_h|$, we can complete the proof in the same way as in the previous case. □

From here until (55) we will prove the lemma for the case $\frac{1}{2} < \alpha \leq 1$, and then we will give the details for the case $\alpha > 1$.

Claim 2: Let $h_k = 2^k$, $\bar{a}_k = \bar{a}_{h_k}$, $b_k = b_{h_k}$, $g_k = g_{h_k}$, and $D_k = D_{h_k}$. Then,

$$g_k(0) \leq g_{k-1}(0) + C_0 2^{-k/C} \quad \text{for all } k \text{ large}, \tag{23}$$

where C_0 is a fixed constant and C depends only on α.

Lemma 9 follows from Claims 1 and 2 in the following way. Let the convex set P be the projection of the graph of g onto the plane $\{x_3 = 0\}$. By Claim 2 and the fact that P contains x_1-axis (it follows from Claim 1), P must equal $I \times \mathbb{R}$ for some interval

$$I \subset \left[0, \lim_{k\to\infty} g_k(0)\right].$$

Then, by (19), $\mathcal{M}_u$ is also contained in a strip region as stated in Lemma 9.

Proof of Claim 2. To prove (23), observe that since g is positive and concave,

$$g_k(0) \leq h_k g_0(0) \leq 2^k(\delta + \delta^*).$$

Hence, we can start from sufficiently large k_0, satisfying $g_{k_0}(0) \leq 1$ and

$$g_{k_0} + C_0 \sum_{j=k_0}^{\infty} 2^{-j/C} \leq 2. \tag{24}$$

Suppose (23) holds up to k. Then by (24), we have $g_k(0) \leq 2$. By the concavity of g and and the fact that $g \geq 0$, we have $g_{k+1}(0) \leq 2g_k(0) \leq 4$. By Claim 1, we have $\bar{a}_{k+1} \geq \frac{1}{10000}h_k$. To prove (23) at $k+1$,

$$L_k = \left\{x_1 \in \mathbb{R}^1 : -\frac{C_1}{4}h_k < x_1 < \frac{C_1}{4}h_k\right\}, \quad C_1 = \frac{1}{10000}, \tag{25}$$

$$Q_k = L_k \times [h_k, h_{k+1}] \subset D. \tag{26}$$

Since $g > 0$ and g is concave, we have the estimates

$$g(x_1, h) \le 8, \tag{27}$$

$$|\partial_h g(x_1, h)| \le \frac{16}{h_k}, \tag{28}$$

$$|\partial_{x_1} g(x_1, h)| \le \frac{16}{h_k} \quad \text{for all } (x_1, h) \in Q_k. \tag{29}$$

Let $\mathscr{X}^{\pm} = \{(x_1, h) \in Q_k : |\partial_{x_1 x_1} g^{\pm}(x_1, h)| \ge h_k^{-\beta}\}$, where β is chosen such that $\frac{1}{\alpha} < \beta < 2$. For any $h \in (h_k, h_{k+1})$, by (29),

$$|\{x_1 \in L_k : (x_1, h) \in \mathscr{X}^+\}|\, h_k^{-\beta} \le \int_{L_k} \partial_{x_1 x_1} g^+ \tag{30}$$

$$\le \int_{L_k} \partial_{x_1 x_1} g \tag{31}$$

$$\le 2 \sup_{L_k} |\partial_{x_1} g| \tag{32}$$

$$\le \frac{C}{h_k}. \tag{33}$$

So, $|\mathscr{X}^+| \le C h_k^{\beta}$. Similarly, we have $|\mathscr{X}^-| \le C h_k^{\beta}$.

For any given $y_1 \in L_k$, let $\mathscr{X}^{\pm}_{y_1} = \mathscr{X}^+ \cap \{x_1 = y_1\}$. Then, by the estimate above, there is a set $\widetilde{L}^{\pm} \subset L_k$ with measure

$$|\widetilde{L}^{\pm}| \le C h_k^{\beta/2},$$

such that for any $y_1 \in L_k - \widetilde{L}^{\pm}$, we have $|\mathscr{X}^{\pm}_{y_1}| \le h_k^{\beta}/2$. When k is large, we can always find $y_1 = C h^{\beta/2} \in L_k - \widetilde{L}^{\pm}$, where the constant C is under control. For such y_1, we have

$$\begin{aligned} & g(y_1, h_{k+1}) - g(y_1, h_k) \\ &\qquad = g^+(y_1, h_{k+1}) - g^+(y_1, h_k) + |g^-(y_1, h_{k+1}) - g^-(y_1, h_k)|. \end{aligned} \tag{34}$$

In the following, we will estimate $g^+(y_1, h_{k+1}) - g^+(y_1, h_k)$. The estimate for $|g^-(y_1, h_{k+1}) - g^-(y_1, h_k)|$ is analogous. By the same reason as that for [Wang 2011, §2.21], we have

$$\begin{cases} (\partial_h g^+)^{-1} = (1 + \varepsilon_1) u_\gamma, \\ \partial_{x_1 x_1} g^+ = (1 + \varepsilon_2) \kappa. \end{cases} \tag{35}$$

Then, by the equation $u_\gamma^{\frac{1}{\alpha}} \kappa = 1$, we have

$$\partial_h g_h^+(y_1, h) \le C (\partial_{x_1 x_1} g^+)^{\alpha} \le C h_k^{-\beta\alpha}. \tag{36}$$

Now,

$$g^+(y_1,h_{k+1})-g^+(y_1,h_k)=\int_{h_k}^{h_{k+1}}\partial_h g^+(y_1,h)\,dh \tag{37}$$

$$=\int_{\mathscr{X}_{y_1}^+}\partial_h g^+(y_1,h)\,dh+\int_{[h_k,h_{k+1}]-\mathscr{X}_{y_1}^+}dh \tag{38}$$

$$\le C_1 h_k^{\frac{\beta}{2}}\frac{1}{h_k}+C_2 h_k^{-\beta\alpha}h_k. \tag{39}$$

Recall that β satisfies $\frac{1}{\alpha}<\beta<2$, and we have $\eta:=\min\{1-\frac{\beta}{2},\beta\alpha-1\}>0$. From (34) and (39), we have the estimate

$$g(y_1,h_{k+1})-g(y_1,h_k)\le\frac{C}{h_k^{\eta}},$$

for some fixed constant C. Then, we will assume $\partial_{x_1}g(0,h_k)<0$ (otherwise we can replace x_1 by $-x_1$); therefore, by the above estimate,

$$g(y_1,h_{k+1})\le g(y_1,h_k)+\frac{C}{h_k^{\eta}}\le g(0,h_k)+\frac{C}{h_k^{\eta}}.$$

Since g is positive, concave, and defined on $[0,\bar a_{k+1}]$, with $\bar a_{k+1}\ge Ch_{k+1}$,

$$\frac{g_{k+1}(0)}{g_{k+1}(y_1)}\le\frac{\bar a_{k+1}}{\bar a_{k+1}-y_1}\le 1+Ch_{k+1}^{\frac{\beta}{2}-1}.$$

Therefore, by the two estimates above,

$$g_{k+1}(0)\le g_k(0)+Ch_k^{-\eta},$$

which implies (23) immediately. □

For the proof of Lemma 6 when $\sigma\in(0,1]$, we need to use (5) and (6). In fact, by (6) we see that Γ_h is moving at a velocity greater than or equal to its curvature to the power α. Hence, we still have the lower bound of $\frac{d}{ds}(|U_s|)$ as in the proof of Claim 1. Then we can follow the above proof for the case $\sigma=0$ until (37), replacing the equalities "=" in (20) and (21) with inequalities "≥". As in [Wang 2011], when $\sigma=0$, in order to control the second integral in (38) we used the equation $\kappa u_\gamma^{1/\alpha}=1$. But when $\sigma\neq 0$, by (28) and (35) we have

$$u_\gamma\ge C(\partial_h g^+)^{-1}\ge Ch_k. \tag{40}$$

Hence, we may assume that u_γ is as large as we want, which means that in formula (5), the only important extra term is

$$(\sigma+u_\gamma^2)^{\frac{1}{2\alpha}-\frac12}\frac{\sigma u_{\gamma\gamma}}{\sigma+u_\gamma^2}.$$

To handle this term, we divide the integral (39) into three parts:

$$g^+(y_1,h_{k+1})-g+(y_1,h_k)=\int_{h_k}^{h_{k+1}}\partial_h g^+(y_1,h)\,dh \tag{41}$$

$$=\left(\int_{I_1}+\int_{I_2}+\int_{I_3}\right)\partial_h g^+(y_1,h)\,dh, \tag{42}$$

where

$$I_1=\mathscr{X}^+_{y_1}, \tag{43}$$

$$I_2=\left\{h\in[h_k,h_{k+1}]-I_1 : (\sigma+u_\gamma^2)^{\frac{1}{2\alpha}-\frac{1}{2}}\frac{\sigma u_{\gamma\gamma}}{\sigma+u_\gamma^2}\le\frac{1}{2}\right\}, \tag{44}$$

$$I_3=[h_k,h_{k+1}]-I_1\cup I_2. \tag{45}$$

For the first integral, we can do exactly the same thing as we have done from (38) to (39), namely,

$$\int_{I_1}\partial_h g^+(y_1,h)\,dh\le\frac{C}{h_k}h_k^{\frac{\beta}{2}}=Ch_k^{\frac{\beta}{2}-1}.$$

Note that the power $\frac{\beta}{2}-1$ is a negative number.

Then we estimate the second integral, note that when $(y_1,h)\in I_2$, we have

$$(\sigma+u_\gamma^2)^{\frac{1}{2\alpha}}\kappa u_\gamma\ge\tfrac{1}{2}.$$

By (40) u_γ is large, so we have $\kappa u_\gamma^{\frac{1}{\alpha}}\ge\frac{1}{4}$, hence by (35) we have

$$\partial_h g^+\le C(\partial_{x_1x_1}g^+)^\alpha\le Ch_k^{-\alpha\beta}. \tag{46}$$

Therefore,

$$\int_{I_2}\partial_h g^+(y_1,h)\,dh\le Ch_k^{-\beta\alpha}h_k=Ch_k^{1-\beta\alpha}.$$

Note that $1-\beta\alpha$ is a negative number. Observe that we can assume I_2 is on the right hand side of I_3, since by the concavity of g^+ we know that when $h\ge\inf I_2$, $\partial_h g^+(y_1,h)$ will satisfy the estimate (46).

For the third integral, notice that by the same argument as that for [Wang 2011, §2.24],

$$\begin{cases}u_\gamma(y_1,h)=u_{x_2}(1+\varepsilon_1),\\ u_{\gamma\gamma}(y_1,h)=u_{x_2x_2}(1+\varepsilon_2)+\varepsilon_3u_{x_2}.\end{cases} \tag{47}$$

Hence, by (47),

$$(\sigma+u_{x_2}^2)^{\frac{1}{2\alpha}-\frac{1}{2}}\frac{\sigma u_{x_2x_2}}{\sigma+u_\gamma^2}\ge\tfrac{1}{3}. \tag{48}$$

Since $\sigma \in [0,1]$ and u_γ is large, we have

$$u'' = u_{x_2x_2} \ge \tfrac{1}{4}(u')^{3-\frac{1}{\alpha}}. \tag{49}$$

By differentiating the equation $u(x_1, g^+(x_1,h)) = h$ twice with respect to h,

$$(g^+)'' = -u''(g^+)'^3 \le -\tfrac{1}{4}(g^+)'^{\frac{1}{\alpha}-3}(g^+)'^3 = -\tfrac{1}{4}(g^+)'^{\frac{1}{\alpha}}. \tag{50}$$

Note that (50) is for points with corresponding $h \in I_3$. By the discussion after (46) we only need to estimate

$$\int_{[h_k + h_k^{\frac{\beta+2}{4}}, \inf I_2]} (g^+)'\, dh.$$

Therefore, by (50) and noticing that $(g^+)' \ge 0$,

$$\tfrac{\alpha}{\alpha-1}(g^+)'^{\frac{\alpha-1}{\alpha}}(h) \le \tfrac{\alpha}{\alpha-1}(g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) - \tfrac{1}{4}\big|I_3 \cap [h_k, h]\big|. \tag{51}$$

Hence, when $h \in [h_k^{(\beta+2)/4}, \inf I_2]$,

$$(g^+)'(h) \le \big((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) + C(h-h_k)\big)^{\frac{\alpha}{\alpha-1}}. \tag{52}$$

Finally,

$$\int_{[h_k + h_k^{\frac{\beta+2}{4}}, \inf I_2]} (g^+)'\, dh \le \int_{h_k}^{h_{k+1}} \big((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) + C(h-h_k)\big)^{\frac{\alpha}{\alpha-1}}\, dh \tag{53}$$

$$\le \tfrac{\alpha-1}{2\alpha-1}\big((g^+)'^{\frac{\alpha-1}{\alpha}}(h_k) + C(h-h_k)\big)^{\frac{\alpha}{\alpha-1}+1}\Big|_{h_k}^{2h_k} \tag{54}$$

$$\le C(g^+)'^{\frac{2\alpha-1}{\alpha}} \le C h_k^{\frac{1-2\alpha}{\alpha}}. \tag{55}$$

Note that $\frac{1-2\alpha}{\alpha} < 0$ when $\alpha > \frac{1}{2}$, so we can complete the proof as in the $\sigma = 0$ case.

When $\alpha > 1$, we need to choose the constants and exponents more carefully. First of all, in view of the Lemma 9 for $\alpha > 1$, in order to have properties (35) and (47), we need only to replace the number 2 in (24) with some number much smaller than the constant C in Lemma 9. The definition of L_k in (25) should be modified to

$$L_k = \Big\{x_1 \in \mathbb{R}^1 : -\frac{C_1}{4} h_k^{\frac{1}{2\alpha-1}} < x_1 < \frac{C_1}{4} h_k^{\frac{1}{2\alpha-1}}\Big\}, \quad C_1 = \frac{1}{10000},$$

and the definition of Q_k in (26) remains the same. It is easy to see that we still have the estimates (27)–(28), but (29) becomes

$$|\partial_{x_1} g(x_1,h)| \le 16 h_k^{-\frac{1}{2\alpha-1}} \quad \text{for all } (x_1,h) \in Q_k.$$

Then for the definition of

$$\mathscr{X}^{\pm} = \{(x_1,h) \in Q_k : |\partial_{x_1x_1} g^{\pm}(x_1,h)| \ge h_k^{-\beta}\},$$

we need to choose the exponent β so that $\frac{1}{\alpha} < \beta < \frac{2}{2\alpha-1}$. By doing the same computation as (30)–(32),

$$\big|\{x_1 \in L_k : (x_1,h) \in \mathscr{X}^+\}\big| h_k^{-\beta} \le \int_{L_k} \partial_{x_1x_1} g^+ \le C h_k^{-\frac{1}{2\alpha-1}}.$$

Hence,

$$|\mathscr{X}^+| \le C h_k^{1+\beta-\frac{1}{2\alpha-1}} \quad \text{and} \quad |\mathscr{X}^-| \le C h_k^{1+\beta-\frac{1}{2\alpha-1}}.$$

Then, by the above estimate there is a set $\widetilde{L}^{\pm} \subset L_k$ with measure

$$|\widetilde{L}^{\pm}| \le C h_k^{\beta+\varepsilon-\frac{1}{2\alpha-1}}$$

such that for any $y_1 \in L_k - \widetilde{L}^{\pm}$, we have $|\mathscr{X}^{\pm}_{y_1}| \le h_k^{1-\varepsilon}$, where ε is chosen such that $\beta + \varepsilon < \frac{2}{2\alpha-1}$. Now, (35)–(38) remain the same, and (39) becomes

$$g^+(y_1, h_{k+1}) - g^+(y_1, h_k) \le C_1 h_k^{1-\varepsilon} \frac{1}{h_k} + C_2 h_k^{-\beta\alpha} h_k.$$

By the choice of β, all the exponents of h_k are negative. We do not need to change anything from (40) to (49). Finally from (50) we need to replace the computation in the case $\alpha \le 1$ with the following computation.

First, we have $(g^+)'' \le -\frac{1}{4}(g^+)'^{\frac{1}{\alpha}} \le -\frac{1}{4}(g^+)'$, and we only need to bound

$$\int_{[h_k + h_k^{1-\varepsilon/2}, \inf I_2]} (g^+)'\, dh.$$

Note that $(g^+)' \ge 0$. By integrating the above differential inequality, we have

$$(g^+)'(h) \le (g^+)'(h_k) e^{-\frac{1}{4}|I_3|} \le (g^+)'(h_k) e^{\frac{1}{8}(h-h_k)}$$

when $h \in [h_k + h_k^{1-\varepsilon/2}, \inf I_2]$. Therefore, we have

$$\begin{aligned}\int_{[h_k + h_k^{1-\varepsilon/2}, \inf I_2]} (g^+)'\, dh &\le \int_{h_k}^{h_{k+1}} (g^+)'(h_k) e^{\frac{1}{8}(h-h_k)}\, dh \\ &\le C (g^+)'(h_k) \le \frac{C}{h_k}.\end{aligned}$$

□

3. Blow-down of an entire convex ancient solutions converges to a power function

In this section we prove that the blow-down of an entire convex solution to (3) converges to a power function.

Proof of Theorem 1 and the first part of Theorem 2. First, we prove that there is a subsequence of u_h converging to $\frac{1}{1+\alpha}|x|^{1+\alpha}$, where $u_h(x) = h^{-1} u(h^{\frac{1}{1+\alpha}} x)$.

By adding a constant we may suppose $u(0)=0$. Let $x_{n+1}=a\cdot x$ be the equation of the tangent plane of u at 0. By Corollary 10 and the convexity of u we have

$$a\cdot x \le u(x) \le C(1+|x|^{1+\alpha}).$$

Hence,

$$h^{-\frac{\alpha}{1+\alpha}} a\cdot x \le u_h(x) \le C\left(\tfrac{1}{h}+|x|^{1+\alpha}\right).$$

By convexity, Du_h is locally uniformly bounded. Hence, u_h subconverges to a convex function u_0 which satisfies $u_0(0)=0$, and

$$0 \le u_0(x) \le C|x|^{1+\alpha}.$$

It is easy to check that u_0 is an entire convex viscosity solution to (3) with $\sigma=0$, and the comparison principle holds on any bounded domain.

Now we will prove that $\{u_0(x)=0\}=\{0\}$. In fact, if $\{x : u_0(x)=0\}$ is a bounded set, then $\{u_0(x)=h\}$ is a closed, bounded convex curve which evolves under the generalized curve shortening flow; from [Andrews 1998] it follows that $\{u_0(x)=0\}=\{0\}$. If $\{u_0(x)=0\}$ contains a straight line, say the line $\{(t,0) : t\in\mathbb{R}\}$, then by convexity, u is independent of x_1, which is impossible. So we only need to rule out the possibility that $\{u_0(x)=0\}$ contains a ray but no straight lines. In this case, for fixed $h>0$, we can find an ellipse E inside $\{u_0(x)<h\}$, with the short axis bounded from below by a constant depending only on h and with the long axis as long as we want (one only needs to look at the asymptotic cone of $\{u_0(x)=h\}$), but since $\{u_0(x)=h\}$ evolves under the generalized curve shortening flow and $E\subset\{u_0(x)\le h\}$, this is impossible by comparison principle.

Then since $\{u_0(x)=0\}=\{0\}$, $\Gamma_{1,u_0}=\{u_0(x)=1\}$ is a bounded convex curve, and the level set $\{u_0(x)=-t\}$ evolves under the generalized curve shortening flow, with time $t\in(-\infty,0)$. From [Andrews 1998; 2003] we have the following asymptotic behavior of the convex solution u_0 of $L_0 u=1$:

$$u_0(x)=\tfrac{1}{1+\alpha}|x|^{1+\alpha}+\varphi(x), \tag{56}$$

where $\varphi(x)=o(|x|^{\alpha+1})$ for $x\neq 0$ near the origin. In fact, if the initial level curve is in a sufficiently small neighborhood of circle, by Lemma 13, $|\varphi(x)|\le C|x|^{1+\alpha+\eta}$ for some small positive η, where C is a constant depending only on the initial closeness to the circle. Hence, given any $\epsilon>0$, for sufficiently small $h'>0$,

$$B_{(1-\epsilon)r}(0)\subset\Omega_{h',u_0}\subset B_{(1+\epsilon)r}(0),$$

where $r=((1+\alpha)h')^{\frac{1}{1+\alpha}}$. Hence, there is a sequence $h_m\to\infty$ such that

$$B_{(1-\frac{1}{m})r_{m,i}}(0)\subset\Omega_{h_m,u}\subset B_{(1+\frac{1}{m})r_{m,i}}(0),$$

where

$$r_{m,i}=((1+\alpha)ih_m)^{\frac{1}{1+\alpha}},\quad i=1,\dots,m.$$

Then u_{h_m} subconverges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$.

Since u_0 is an entire convex solution to $L_0 u = 1$ (we still use the notation u_0, but it means an arbitrary entire convex solution), from the above argument, we can find a sequence h_m such that

$$u_{0h_m}(x) = \frac{1}{h_m} u_0\big(h_m^{\frac{1}{1+\alpha}} x\big)$$

locally uniformly converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$. Hence, the sublevel set $\Omega_{\frac{1}{1+\alpha}, u_{0h_m}}$ satisfies

$$B_{1-\epsilon_m}(0) \subset \Omega_{\frac{1}{1+\alpha}, u_{0h_m}} \subset B_{1+\epsilon_m}(0),$$

where $\epsilon_m \to 0$ as $m \to \infty$. By the discussion below (56),

$$u_{0h_m}(x) = \tfrac{1}{1+\alpha}|x|^{1+\alpha} + \varphi(x),$$

where $|\varphi(x)| \le C|x|^{1+\alpha+\eta}$ for some fixed small positive η, and the constant C is independent of m. Replacing x by $h_m^{-1/(1+\alpha)} x$ in the asymptotic formula above,

$$u_0(x) = \tfrac{1}{1+\alpha}|x|^{1+\alpha} + h_m \varphi\big(h_m^{-\frac{1}{1+\alpha}} x\big),$$

where for any fixed x, $h_m \varphi(h_m^{-\frac{1}{1+\alpha}} x) \to 0$. Hence $u_0(x) = \frac{1}{1+\alpha}|x|^{1+\alpha}$. So we have proved Theorem 1 and the first part of Theorem 2. □

4. One-dimensional entire convex ancient solution must be a shrinking circle

This section is devoted to the proof of Corollary 3, which is completed by combining the following lemma (corresponding to [Wang 2011, Lemma 4.1]) and Theorem 2.

Lemma 11. *Let Ω be a smooth, bounded, convex domain in $\mathbb{R}^2$. Let u be the solution of* (3) *with $\sigma = 0$, vanishing on $\partial\Omega$. Then for any constant h satisfying* $\inf_\Omega u < h < 0$, *the level set $\Gamma_{h,u} = \{u = h\}$ is convex. Moreover*, $\log(-u)$ *is a concave function.*

Proof. Observe that $\varphi := -\log(-u)$ satisfies

$$|D\varphi|^{\frac{1}{\alpha}-1} \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{\varphi_i \varphi_j}{|D\varphi|^2}\right) \varphi_{ij} = e^{\frac{1}{\alpha}\varphi}.$$

Since $\varphi(x) \to +\infty$ as $x \to \partial\Omega$, [Kawohl 1985, Theorem 3.13] implies that φ is convex. □

With the previous lemma and [Wang 2011, Lemma 4.4], we know that any convex compact ancient solution to the generalized curve shortening flow can be represented as a convex solution u to (3) with $\sigma = 0$, and if the solution to the flow sweeps the whole space, then the corresponding u will be an entire solution. Thus, Theorem 2 implies Corollary 3 immediately.

Remark 12. We can also use the method in [Wang 2011, Section 4] to construct a rotationally nonsymmetric convex compact ancient solution for generalized curve shortening flow with power $\alpha \in \left(\frac{1}{2}, 1\right)$. Indeed, for mean curvature flow, a rotationally nonsymmetric convex compact ancient solution is constructed in Lemmas 4.1–4.4 of that reference. By examining the proofs of these lemmas, we can see that they work well for the generalized curve shortening flow considered here.

5. Two-dimensional entire convex translating solution

In this section, by using the previous results and a delicate iteration argument, we prove that under some extra condition on the asymptotic behavior of the solution at infinity the translating solution must be rotationally symmetric.

First of all, we would like to point out that instead of using Gage and Hamilton's exponential convergence of the curve shortening flow [1986], we need to use the corresponding exponential convergence for the generalized curve shortening flow and we will state it as a lemma, corresponding to [Wang 2011, Lemma 3.2].

Lemma 13. *Let $\{\ell_t\}$ be a convex solution to the generalized curve shortening flow with initial curve $\{\ell_0\}$ uniformly convex. Suppose $\{\ell_0\}$ is in the δ_0-neighborhood of a unit circle, $\{\ell_t\}$ shrinks to the origin at $t = \frac{1}{1+\alpha}$. Let*

$$\tilde{\ell}_t = (1-(1+\alpha)t)^{-\frac{1}{1+\alpha}}\ell_t$$

be the normalization of ℓ_t. Then $\tilde{\ell}_t$ is in the δ_t-neighborhood of the unit circle centered at the origin,

$$\tilde{\ell}_t \subset N_{\delta_t} S^1,$$

with

$$\delta_t \le C\delta_0\left(\tfrac{1}{1+\alpha} - t\right)^{\iota}$$

for some small positive constant ι.

Remark 14. Exponential convergence of the standard curve shortening flow (when $\alpha = 1$) was proved by Gage and Hamilton [1986]. For the general case (when $\alpha > \frac{1}{3}$), as discussed in the following proof, Gage and Hamilton's method combined with Andrews' estimates [1998, Propositions II1.1 and II1.2] can still be used to prove the corresponding exponential convergence result.

Proof. The proof of Lemma 13 is similar to the proof of [Wang 2011, Lemma 3.2]. Since the initial curve ℓ_0 is uniformly convex and close to a unit circle, by [Andrews 1998, Propositions II1.1 and II1.2], the curvature of $\tilde{\ell}_t$ is bounded from below and from above by some constant depending only on δ_0, when $t \in \left(\frac{1}{4\alpha+4}, \frac{1}{2\alpha+2}\right)$.

Hence the evolution equation for $\tilde{\ell}_t$ is uniformly parabolic. Therefore, we can apply Schauder's estimates safely for $\alpha > \frac{1}{2}$, as in [Wang 2011], which says that

$$\|\tilde{\ell}_t - S^1\|_{C^k} \le C\delta_0 \quad \text{for } t \in \left(\tfrac{1}{4\alpha+4}, \tfrac{1}{2\alpha+2}\right).$$

Although the constant C will depend on the lower and upper bounds of the curvature of the initial curve, it is not a problem for our purpose, since when we blow down the solution for $\sigma = 0$, the norm of the gradient Du_h on the curve $\{u_h(x) = 1\}$ approaches 1.

By the equation $\kappa u_\gamma^{\frac{1}{\alpha}} = 1$, we see that the curvature κ is also very close to 1 on that curve. For the exponential decay rate of the derivative of curvature, one can imitate the proof in [Gage and Hamilton 1986, §§5.7.10–5.7.15], and our corresponding estimate will be $|\kappa'(\tau)| \le C\delta_0 e^{-\iota\tau}$ for some small positive number ι, where $\tau = -\frac{1}{1+\alpha}\log\left(\frac{1}{1+\alpha} - t\right)$. Indeed, in the case $\alpha > 1$, this is done by Chen and Huang [Huang 2011, Corollary 3.2], and it is easy to check that their computation also works for the case $\frac{1}{3} < \alpha < 1$ by taking ℓ small enough. This estimate immediately implies our lemma. □

In the following we will consider the case when $\sigma = 1$ and $\alpha > 1$. By translating and adding some constant we can assume $u(0) = \inf u$. Let

$$u_h(x) = \frac{1}{h} u(h^{\frac{1}{1+\alpha}} x).$$

Then u_h satisfies the equation $L_\sigma u_h = 1$ with $\sigma = h^{-\frac{2\alpha}{1+\alpha}}$. By Theorem 1, u_h converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$, and the level set $\Gamma_{\frac{1}{1+\alpha}, u_h}$ converges to the unit circle as $h \to \infty$.

Lemma 15 [Wang 2011, Lemma 3.3]. *The function u satisfies*

$$u(x) = \tfrac{1}{1+\alpha}|x|^{1+\alpha} + O(|x|^{1+\alpha-2\alpha\beta}), \tag{57}$$

where β is a constant, chosen such that $\frac{1}{2\alpha} < \beta < \min\{1, \frac{1+\alpha}{2\alpha}\}$.

Proof. For any given small $\delta_0 > 0$, take h sufficiently large such that

$$\Gamma_{\frac{1}{1+\alpha}, u_h} \subset N_{\delta_0}(S^1) \tag{58}$$

for the unit circle S^1 with center p_0. Note that when h is large, δ_0 is very close to 0. Then we will prove the following claim:

Claim 3: For small fixed τ,

$$\Gamma_{\tau, u_h} \subset ((1+\alpha)\tau)^{\frac{1}{1+\alpha}} N_{\delta_\tau}\left((1 + \tfrac{a_0}{\tau})^{\frac{1}{1+\alpha}} S^1\right) \tag{59}$$

with

$$\delta_\tau \le C_1(\tau)\sigma^\beta + C_2\delta_0\tau^\eta, \tag{60}$$

where η is a small positive constant, the constants C_1 and C_2 are independent of δ_0 and h, and C_2 is also independent of τ. Let u_0 be the solution of

$$L_0(u) = 1 \quad \text{in } \Omega_{\frac{1}{1+\alpha},u_h},$$

satisfying

$$u_0 = u_h = \tfrac{1}{1+\alpha} \quad \text{on } \partial\Omega_{\frac{1}{1+\alpha},u_h},$$

where $a_0 = |\inf u_0|$ and the center of

$$(1 + \tfrac{a_0}{\tau})^{\frac{1}{1+\alpha}} S^1$$

is the minimum point of u_0 multiplied by the factor $((1+\alpha)\tau)^{-\frac{1}{1+\alpha}}$.

Proof of Claim 3. We only need to prove that

$$\operatorname{dist}\big((1+\alpha)^{\frac{1}{1+\alpha}}(\tau + a_0)^{\frac{1}{1+\alpha}} S^1, \Gamma_{\tau,u}\big) \le C_1(\tau)\sigma^\beta + C_2\delta_0\tau^{\frac{1}{1+\alpha}+\eta}, \tag{61}$$

where η is some small positive constant and C_2 is independent of τ. By Theorem 1 we know that u_h converges to $\frac{1}{1+\alpha}|x|^{1+\alpha}$ uniformly on any compact subset of $\mathbb{R}^2$. Then by the convexity of u_h, we have that $|Du_h|$ is bounded above and below by some constants depending on τ_0 for large h when

$$x \in \big\{x \in \Omega_{\frac{1}{1+\alpha},u_h} : \tau_0 \le u_h < \tfrac{1}{1+\alpha}\big\}.$$

Hence, by the growth condition for D^2u in Theorem 2, we have $\sigma(u_h)_{\gamma\gamma} \le C\sigma^\beta$, where C is a constant depending on τ_0. Therefore, we have

$$\kappa(u_h)_\gamma^{1/\alpha} \approx 1 - C\sigma^\beta \quad \text{on } \big\{x \in \Omega_{\frac{1}{1+\alpha},u_h} : \tau \le u_h < \tfrac{1}{1+\alpha}\big\},$$

where C depends on τ_0. Let

$$\tilde{u}_0 = (1 - C\sigma^\beta)^\alpha(u_0 - \tfrac{1}{1+\alpha}) + \tfrac{1}{1+\alpha};$$

then

$$L_0(\tilde{u}_0) = 1 - C\sigma^\beta \quad \text{in } \Omega_{\frac{1}{1+\alpha},u_h}$$

with

$$\tilde{u}_0 = u_h = \tfrac{1}{1+\alpha} \quad \text{on } \partial\Omega_{\frac{1}{1+\alpha},u_h}.$$

Now by the comparison principle, $\Omega_{\tau,u_0} \supset \Omega_{\tau,u_h} \supset \Omega_{\tau,\tilde{u}_0}$, and by the asymptotic behavior of u_0,

$$\Gamma_{\tau,u_0} \subset N_\zeta\big((\tau + a_0)^{\frac{1}{1+\alpha}} S^1\big) \quad \text{and} \quad \Gamma_{\tau,\tilde{u}_0} \subset N_\zeta\big((\tau + a_0 - C\sigma^\beta)^{\frac{1}{1+\alpha}} S^1\big),$$

where $\zeta = C\delta_0(\tau + a_0)^\eta$. Let

$$\ell_1 = (\tau + a_0)^{\frac{1}{1+\alpha}} S^1 \quad \text{and} \quad \ell_2 = (\tau + a_0 - C\sigma^\beta)^{\frac{1}{1+\alpha}} S^1,$$

both centered at p_1, which is the minimum point of u_0. Hence

$$\operatorname{dist}((\tau + a_0)^{\frac{1}{1+\alpha}} S^1, \Gamma_{\tau,u_h}) \le \operatorname{dist}(\ell_1, \ell_2) + C\delta_0(\tau + a_0)^{\frac{1}{1+\alpha}+\eta}, \tag{62}$$

where $\operatorname{dist}(\ell_1, \ell_2)$ can be bounded by $C_1(\tau)\sigma^\beta$; hence (60) follows from the above discussion. □

Now, we will use an iteration argument to prove the following claim, which will enable us to simplify (59) and (60).

Claim 4: $a_0 \le C\sigma^\beta |\log(\sigma)|$

Proof. We fix a large constant A such that $\{u_{A/\tau} = \frac{1}{1+\alpha}\}$ is very close to a unit circle. Let u_{0,τ^k} solve $L_0 u = 1$ with boundary condition $u = \tau^k$ on $\{u_h = \tau^k\}$. Denote $a_k = |\inf u_{0,\tau^k}|$. From the proof of Claim 3, we see that

$$\{u_0 < \tau\} \supset \{u_{0,\tau} < \tau\} \supset \{\tilde{u}_0 < \tau\},$$

by the comparison principle, we have $\inf u_0 < \inf u_{0,\tau} < \inf \tilde{u}_0$. So by the construction of $\tilde{u}_0$ and a simple computation, we have $a_0 - a_1 \le \inf \tilde{u}_0 - \inf u_0 \le C\sigma^\beta$. When $\tau^k \ge \frac{A}{h}$, we can iterate this argument for u_{0,τ^k} and $u_{0,\tau^{k+1}}$ by rescaling them to

$$\tfrac{1}{1+\alpha}\tau^{-k} u_{0,\tau^k}\big((1+\alpha)^{\frac{1}{1+\alpha}}\tau^{\frac{k}{1+\alpha}} x\big) \quad \text{and} \quad \tfrac{1}{1+\alpha}\tau^{-k} u_{0,\tau^{k+1}}\big((1+\alpha)^{\frac{1}{1+\alpha}}\tau^{\frac{k}{1+\alpha}} x\big),$$

respectively. After rescaling back, we have $a_k - a_{k+1} \le C\sigma^\beta$. Note that the choice of A and the condition $\tau^k \ge \frac{A}{h}$ ensure the uniform gradient bound needed in the above argument. Let k_0 be an integer satisfying $\tau^{k_0} \ge \frac{A}{h} \ge \tau^{k_0+1}$. After k_0 steps we stop the iteration, and notice that

$$\{u_h = \tfrac{A}{h}\} = h^{-\frac{1}{1+\alpha}}\{u = A\}$$

is contained in a circle with radius $Ch^{-\frac{1}{1+\alpha}}$ for some constant C. Hence it takes at most time $Ch^{-1} = C\sigma^{\frac{1+\alpha}{2\alpha}}$ for $\{u_h = \frac{A}{h}\}$ to shrink to a point. Claim 4 follows from the above discussion. □

By omitting the lower order term we can rewrite (59) and (60) as

$$\Gamma_{\tau,u_h} \subset ((1+\alpha)\tau)^{\frac{1}{1+\alpha}} N_{\delta_\tau}(S^1)$$

with

$$\delta_\tau \le C_1(\tau)\sigma^\beta + C_2\delta_0\tau^\eta. \tag{63}$$

If we take τ small such that $C_2\tau^\eta \le \frac{1}{4}$, then (63) becomes

$$\delta_\tau \le C_1(\tau)\sigma^\beta + \tfrac{1}{4}\delta_0. \tag{64}$$

Now we can carry out an iteration argument similar to that in [Wang 2011]. We start at the level $\frac{1}{1+\alpha}\tau^{-k_0}$ for some sufficient large k_0. Let

$$\Omega_k = \tau^{\frac{k}{1+\alpha}}\,\Omega_{\frac{1}{1+\alpha}\tau^{-k},u} \quad \text{and} \quad \Gamma_k = \partial\Omega_k.$$

Note that Γ_k converges to a unit circle as $k \to \infty$. Suppose that Γ_k is in the δ_k neighborhood of S^1 centered at y_k, where $\delta_k \to 0$ as $k \to \infty$ and y_k is the minimum point of the solution of $L_0 u = 1$ in Ω_k with $u = \frac{1}{1+\alpha}$ on Γ_{k+1}. By (64) we have

$$\delta_{k-1} \le C_1(\tau)\,\tau^{(k-1)\frac{2\alpha\beta}{1+\alpha}} + \tfrac{1}{4}\delta_k \tag{65}$$

for $k = k_0, k_0+1, \ldots$. Then we have

$$\Gamma_j \subset N_{\delta_j}(S^1) \tag{66}$$

with

$$\delta_j \le C\tau^{j\frac{2\alpha\beta}{1+\alpha}} \tag{67}$$

It follows that

$$\Gamma_{\frac{1}{1+\alpha}\tau^{-j},u} \subset N_{\tilde{\delta}_j}(\tau^{\frac{-j}{1+\alpha}} S^1) \tag{68}$$

with

$$\tilde{\delta}_j \le C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j}, \tag{69}$$

where $\tau^{\frac{-j}{1+\alpha}} S^1$ is centered at $z_j = \tau^{\frac{-j}{1+\alpha}} y_j$. From Lemma 13 and (64), it is not hard to see that

$$|z_j - z_{j-1}| \le C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j}. \tag{70}$$

Let $z_0 = \lim_{j\to\infty} z_j$. Then

$$|z_j - z_0| \le C\tau^{\frac{2\alpha\beta-1}{1+\alpha}j}, \tag{71}$$

which means that in (68) we can assume the circle is centered at z_0 by changing the constant C a little bit. In fact when we choose different τ, the corresponding z_0 will not change, so we can assume $z_0 = 0$. Hence, for $h = \frac{1}{1+\alpha}\tau^{-j}$,

$$\Gamma_{h,u} \subset N_\delta\big((1+\alpha)^{\frac{1}{1+\alpha}} h^{\frac{1}{1+\alpha}} S^1\big),$$

where

$$\delta \le Ch^{\frac{1-2\alpha\beta}{1+\alpha}} \tag{72}$$

and S^1 is centered at the origin. By choosing different τ, we see that the estimate holds for all large h. Lemma 15 follows from the above estimates. □

Remark 16. For the mean curvature flow (when $\alpha = 1$), I learned the proof of Claim 4 from Professor Xu-Jia Wang. Indeed, in this case one can do it as follows. Let v_k solve $L_0 w = 1$ with boundary condition $w = \tau^k$ on $\{u_h = \tau^k\}$, for $k = 1, 2, \dots$. Let $a_k = |\inf v_k|$. Then by [Wang 2011, Lemma 3.1], we have $a_0 - a_1 \le C\sigma$. By rescaling invariance, we can iterate the argument to show that $a_k - a_{k+1} \le C\sigma$, provided $\tau^k \ge \frac{1}{h}$. Hence, we stop the iteration at k_0 when $\tau^{k_0} \ge \frac{1}{h} > \tau^{k_0+1}$. Notice that

$$\{u_h \le \tau^{k_0}\} = h^{-\frac{1}{2}}\{u = h\tau^{k_0}\} \subset h^{-\frac{1}{2}}\{u \le \tfrac{1}{\tau}\}.$$

So, it is easy to see that it takes at most time $C\sigma = \frac{C}{h}$ for $\{u_h \le \tau^{k_0}\}$ to shrink to a point, namely, $a_{k_0} \le C\sigma$. Therefore,

$$a_0 = a_{k_0} + \sum_{i=0}^{k_0-1} a_i - a_{i+1} \le Ck_0\sigma \le C\sigma|\log\sigma|.$$

In order to finish the proof of Theorem 2 we need to use the following fundamental Liouville theorem by Bernstein [Simon 1997, p. 245].

Lemma 17. *Let u be an entire solution to the elliptic equation*

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = 0 \quad \textit{in } \mathbb{R}^2.$$

If u satisfies the asymptotic estimate

$$|u(x)| = o(|x|) \quad \textit{as } x \to \infty,$$

then u is a constant.

Proof of the second part of Theorem 2. Let u^* be the Legendre transform of u. Then u^* satisfies equation

$$G(x, D^2u^*) = \frac{\det D^2u^*}{\left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2}\right)F^{ij}(u^*)} = (1+|x|^2)^{\frac{1}{2\alpha}-\frac{1}{2}}, \tag{73}$$

where $F^{ij}(u^*) = \frac{\partial \det r}{\partial r_{ij}}$ at $r = D^2u^*$. We have

$$u^*(x) = C(\alpha)|x|^{1+\alpha} + O\left(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}\right), \tag{74}$$

where $C(\alpha)$ is a constant depending only on α. In fact, for big h, by Lemma 15,

$$u_h(x) = \tfrac{1}{1+\alpha}|x|^{1+\alpha} + O\left(|h|^{\frac{-2\alpha\beta}{1+\alpha}}\right)$$

in $B_1(0)$. Denote by u_h^* the Legendre transform of u_h. Then,

$$u_h^*(x) = C(\alpha)|x|^{1+\frac{1}{\alpha}} + O\left(|h|^{\frac{-2\alpha\beta}{1+\alpha}}\right),$$

where $C(\alpha)$ is a constant depending only on α and it comes from the Legendre transform of the function $\frac{1}{1+\alpha}|x|^{1+\alpha}$. Note that $u_h^*(x) = h^{-1}u^*(h^{\frac{\alpha}{1+\alpha}}x)$, we obtain (74).

Let u_0 be the unique radial solution of (3) with $\sigma = 1$, and let u_0^* be the Legendre transform of u_0. Similar to (74) we have

$$u_0^*(x) = C(\alpha)|x|^{1+\alpha} + O(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}). \tag{75}$$

Since both u^* and u_0^* satisfy (73), $v = u^* - u_0^*$ satisfies the elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)v_{ij} = 0 \quad \text{in } \mathbb{R}^2,$$

where

$$a_{ij} = \int_0^1 G^{ij}(x, D^2u_0^* + t(D^2u^* - D^2u_0^*))\,dt.$$

Here,

$$G^{ij} = \frac{\partial G(x,r)}{\partial r_{ij}}$$

for any symmetric matrix r. Note that by the choice of β, $\frac{1+\alpha-2\alpha\beta}{\alpha} < 1$; hence, by (74) and (75),

$$v = O\big(|x|^{\frac{1+\alpha-2\alpha\beta}{\alpha}}\big) = o(|x|) \quad \text{as } |x| \to \infty.$$

By Lemma 17 we conclude that v is a constant. □

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