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CONSTRUCTIONS OF PERIODIC MINIMAL SURFACES AND MINIMAL ANNULI IN SOL_3

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We construct two one-parameter families of minimal properly embedded surfaces in the Lie group Sol_3 using a Weierstrass-type representation. These surfaces are not invariant by a one-parameter group of ambient isometries. The first one can be viewed as a family of helicoids, and the second one as a family of minimal annuli called catenoids. Finally we study limits of these catenoids, and in particular we show that one of these limits is a new minimal entire graph.

1. Introduction

The aim of this paper is to construct two one-parameter families of examples of properly embedded minimal surfaces in the Lie group Sol_3, endowed with its standard metric. This Lie group is a homogeneous Riemannian manifold with a 3-dimensional isometry group and is one of the eight Thurston geometries. There is no rotation in Sol_3, and so no surface of revolution.

The Hopf differential, which exists on surfaces in every 3-dimensional space form, has been generalized by Abresch and Rosenberg [2004; 2005] to every 3-dimensional homogeneous Riemannian manifold with 4-dimensional isometry group. This tool leads to a lot of works in the field of constant mean curvature (CMC) surfaces in Nil_3, $\widetilde{PSL}_2(\mathbb{R})$ and in the Berger spheres. More precisely, Abresch and Rosenberg [2005] proved that the generalized Hopf differential exists in a simply connected Riemannian 3-manifold if and only if its isometry group has at least dimension 4.

Berdinskii and Taimanov [2005] gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors, but they pointed out some difficulties for using this theory in the case of Sol_3. Nevertheless, some explicit simple examples of minimal surfaces in Sol_3 have been constructed in the past decade. Masaltsev [2006] and Daniel and Mira [2013] gave some basic examples of minimal graphs in Sol_3: $x_1 = ax_2 + b$, $x_1 = ae^{-x_3}$, $x_1 = ax_2e^{-x_3}$ and $x_1 = x_2e^{-2x_3}$ (and their images by ambient isometries). López and Munteanu

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[2011; 2012; 2014], López [2014] and Masaltsev [2006] studied minimal surfaces in Sol_3 invariant by a one-parameter group of ambient isometries. Finally, Ana Menezes [2014] constructed singly and doubly periodic Scherk minimal surfaces in Nil_3 and Sol_3, and Minh Hoang Nguyen [2014] gave conditions for the Dirichlet problem for the minimal surface equation in Sol_3 to have solutions.

The method that we use in this paper is the one used by Daniel and Hauswirth [2009] in Nil_3 to construct minimal embedded annuli: We first construct a one-parameter family of embedded minimal surfaces called *helicoids* and we calculate its Gauss map g. A result of Inoguchi and Lee [2008] shows that this map is harmonic for a certain metric on $\overline{\mathbb{C}}$. Then we seek another family of maps g with separated variables that still satisfies the harmonic map equation, and we use a Weierstrass-type representation given by Inoguchi and Lee to construct a minimal immersion whose Gauss map is g. We prove that these immersions are periodic, so we get minimal annuli. As far as the authors know, these annuli are the first examples of nonsimply connected minimal surfaces with finite topology (that is, diffeomorphic to a compact surface without a finite number of points) in Sol_3.

The model we use for Sol_3 is described in Section 2. In the third section, we give some properties of the Gauss map of a conformal minimal immersion in Sol_3 (see [Daniel and Mira 2013]). In the fourth section, we construct the family $(\mathcal{H}_K)_{K\in]-1;1[}$ of helicoids, and finally we construct the family $(\mathcal{C}_\alpha)_{\alpha\in]-1;1[\setminus\{0\}}$ of embedded minimal annuli. The study of the limit case of the parameter of this family gives another example of a minimal surface in Sol_3, which is an entire graph. None of these surfaces is invariant by a one-parameter family of isometries.

Theorem. *There exists a one-parameter family* $(\mathcal{C}_\alpha)_{\alpha\in]-1;1[\setminus\{0\}}$ *of properly embedded minimal annuli in* Sol_3, *called* catenoids, *having the following properties*:

(1) *The intersection of* $\mathcal{C}_\alpha$ *with any plane* $\{x_3=\lambda\}$ *is a nonempty closed embedded convex curve.*

(2) *The annulus* $\mathcal{C}_\alpha$ *is conformally equivalent to* $\mathbb{C}\setminus\{0\}$.

(3) *The annulus* $\mathcal{C}_\alpha$ *has three symmetries fixing the origin*: *rotation by* π *around the* x_3*-axis*, *reflection in* $\{x_1=0\}$ *and reflection in* $\{x_2=0\}$.

2. The Lie group Sol_3

Definition. The Lie group Sol_3 is $\mathbb{R}^3$ with the multiplication $*$ defined by

$$(x_1,x_2,x_3)*(y_1,y_2,y_3)=(y_1e^{-x_3}+x_1,\,y_2e^{x_3}+x_2,\,x_3+y_3)$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3)\in\mathbb{R}^3$. The identity element is 0 and the inverse element of (x_1,x_2,x_3) is $(x_1,x_2,x_3)^{-1}=(-x_1e^{x_3},-x_2e^{-x_3},-x_3)$. The Lie group is noncommutative.

The left multiplication l_a by an element $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is given for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\begin{aligned} l_a(x) = a * x &= (x_1 e^{-a_3} + a_1, x_2 e^{a_3} + a_2, a_3 + x_3) \\ &= a + M_a x, \end{aligned}$$

where

$$M_a = \begin{pmatrix} e^{-a_3} & 0 & 0 \\ 0 & e^{a_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the metric $(\cdot, \cdot)$ on Sol$_3$ to be left-invariant, it has to satisfy

$$(M_a X, M_a Y)_{a*x} = (X, Y)_x$$

for all $a, x, X, Y \in \mathbb{R}^3$. We define a left-invariant Riemannian metric for $x, X, Y \in \mathbb{R}^3$ by the formula

$$(X, Y)_x = \langle M_{x^{-1}} X, M_{x^{-1}} Y \rangle, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^3$ and x^{-1} is the inverse element of x in Sol$_3$. The formula (1) leads to the expression of the previous metric

$$ds_x^2 = e^{2x_3}\, dx_1^2 + e^{-2x_3}\, dx_2^2 + dx_3^2, \tag{2}$$

where (x_1, x_2, x_3) are canonical coordinates of $\mathbb{R}^3$. Since the translations are isometries now, Sol$_3$ is a homogeneous manifold with this metric.

Remark. This metric is not the only possible left-invariant one on Sol$_3$. In fact, there exists a two-parameter family of nonisometric left-invariant metrics on Sol$_3$. One of these parameters is a homothetic one. The metrics that are nonhomothetic to (2) have no reflections; see [Meeks and Pérez 2012].

By setting

$$E_1(x) = e^{-x_3}\, \partial_1, \quad E_2(x) = e^{x_3}\, \partial_2, \quad \text{and} \quad E_3(x) = \partial_3,$$

we obtain a left-invariant orthonormal frame (E_1, E_2, E_3). Thus, we now have two frames to express the coordinates of a vector field on Sol$_3$; we will use brackets to denote the coordinates in the frame (E_1, E_2, E_3); then at a point $x \in$ Sol$_3$, we have

$$a_1\, \partial_1 + a_2\, \partial_2 + a_3\, \partial_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} e^{x_3} a_1 \\ e^{-x_3} a_2 \\ a_3 \end{bmatrix}. \tag{3}$$

The following property holds (see [Daniel and Mira 2013]):

Proposition 1. *The isotropy group of the origin of* Sol_3 *is isomorphic to the dihedral group* $\mathcal{D}_4$ *and generated by orientation-reversing isometries*

$$\sigma : (x_1, x_2, x_3) \longmapsto (x_2, -x_1, -x_3) \quad \textit{and} \quad \tau : (x_1, x_2, x_3) \longmapsto (-x_1, x_2, x_3),$$

whose orders are 4 *and* 2, *respectively.*

While τ is simply a reflection in the plane $\{x_1 = 0\}$, the generator σ can be described as a rotation by $-\pi/3$ around E_3 composed with reflection in $\{x_3 = 0\}$. The cyclic group $\langle\sigma\rangle$ also contains $\sigma^3 = \sigma^{-1}$ and σ^2, the reflection in E_3 (rotation by π around E_3). The remaining nonidentity elements of the isotropy group of the origin are $\sigma\tau$ and $\sigma^3\tau$, which are respectively the reflections in the lines $\{(x_1, x_1, 0)\}$ and $\{(x_1, -x_1, 0)\}$, and $\sigma^2\tau$, which is reflection in the plane $\{x_2 = 0\}$.

We deduce the following theorem:

Theorem 2. *The isometry group of* Sol_3 *has dimension* 3.

Finally, we express the Levi-Civita connection ∇ of Sol_3 associated to the metric given by (2) in the frame (E_1, E_2, E_3). First, we calculate the Lie brackets of the vectors of the frame. The Lie bracket in the Lie algebra $\mathfrak{sol}_3$ of Sol_3 is given by

$$[X, Y] = (Y_3 X_1 - X_3 Y_1, X_3 Y_2 - Y_3 X_2, 0)$$

for all $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$. Then we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = -E_2.$$

Hence,

$$\begin{aligned} &\nabla_{E_1} E_1 = -E_3, && \nabla_{E_2} E_1 = 0, && \nabla_{E_3} E_1 = 0, \\ &\nabla_{E_1} E_2 = 0, && \nabla_{E_2} E_2 = E_3, && \nabla_{E_3} E_2 = 0, \\ &\nabla_{E_1} E_3 = E_1, && \nabla_{E_2} E_3 = -E_2, && \nabla_{E_3} E_3 = 0. \end{aligned}$$

3. The Gauss map

Let Σ be a Riemann surface and $z = u + iv$ local complex coordinates in Σ. Let $x : \Sigma \to \mathrm{Sol}_3$ be a conformal immersion. We set

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and we define $\lambda \in \mathbb{R}_+^*$ by

$$2(x_z, x_{\bar{z}}) = \|x_u\|^2 = \|x_v\|^2 = \lambda.$$

Thus, a unit normal vector field is $N : \Sigma \to T\mathrm{Sol}_3$ defined by

$$N = -\frac{2i}{\lambda} x_z \wedge x_{\bar{z}} := \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Hence we define $\widehat{N} : \Sigma \to \mathbb{S}^2 \subset \mathbb{R}^3$ by the formula $M_{x^{-1}} N = \widehat{N}$, that is,

$$\widehat{N} = \begin{pmatrix} e^{x_3} & 0 & 0 \\ 0 & e^{-x_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} N_1 e^{-x_3} \\ N_2 e^{x_3} \\ N_3 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}.$$

Definition. The Gauss map of the immersion x is the application

$$g = \sigma \circ \widehat{N} : \Sigma \longrightarrow \mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}},$$

where σ is the stereographic projection with respect to the southern pole, i.e.,

$$N = \frac{1}{1+|g|^2} \begin{bmatrix} 2\Re(g) \\ 2\Im(g) \\ 1-|g|^2 \end{bmatrix}, \tag{4}$$

$$g = \frac{N_1 + i N_2}{1 + N_3}. \tag{5}$$

The following result is due to [Inoguchi and Lee 2008]. It can be viewed as a Weierstrass representation in Sol$_3$.

Theorem 3. *Let $x : \Sigma \to \mathrm{Sol}_3$ be a conformal minimal immersion and $g : \Sigma \to \overline{\mathbb{C}}$ its Gauss map. Then, whenever g is neither real nor purely imaginary, it is nowhere antiholomorphic ($g_z \neq 0$ for every point for any local conformal parameter z on Σ), and it satisfies the second order elliptic equation*

$$g_{z\bar{z}} = \frac{2g g_z g_{\bar{z}}}{g^2 - \bar{g}^2}. \tag{6}$$

Moreover, the immersion $x = (x_1, x_2, x_3)$ can be expressed in terms of g by the representation formulas

$$x_{1z} = e^{-x_3} \frac{(\bar{g}^2 - 1) g_z}{g^2 - \bar{g}^2}, \quad x_{2z} = i e^{x_3} \frac{(\bar{g}^2 + 1) g_z}{g^2 - \bar{g}^2}, \quad x_{3z} = \frac{2\bar{g} g_z}{g^2 - \bar{g}^2} \tag{7}$$

whenever it is well-defined.

Conversely, given a map $g : \Sigma \to \overline{\mathbb{C}}$ defined on a simply connected Riemann surface Σ satisfying (6), *then the map $x : \Sigma \to \mathrm{Sol}_3$ given by the representation formulas* (7) *is a conformal minimal immersion with possibly branched points whenever it is well-defined, and its Gauss map is g.*

Remark. (1) There exists a similar result for the case of CMC H-surfaces ; see [Daniel and Mira 2013].

(2) Equation (6) is the harmonic map equation for maps $g:\Sigma\to(\overline{\mathbb{C}},ds^2)$ equipped with the metric

$$ds^2=\frac{|d\omega|^2}{|\omega^2-\bar{\omega}^2|}.$$

This is a singular metric, not defined on the real and pure imaginary axes. See [Inoguchi and Lee 2008] for more details.

(3) Equation (6) can be only considered at points where $g\neq\infty$. But if g is a solution of (6), i/g is also a solution at points where $g\neq 0$. The conjugate map $\bar{g}$ and every $g\circ\phi$, with ϕ a locally injective holomorphic function, are solutions too. Moreover, if g is a nowhere antiholomorphic solution of (6), and x is the induced conformal minimal immersion, then ig and $1/g$ induce conformal minimal immersions given by σx and τx. Finally, $\bar{g}$ is the Gauss map of $\sigma^2\tau x$ after a change of orientation.

Definition. The Hopf differential of the map g is the quadratic form

$$Q=q\,dz^2=\frac{g_z\bar{g}_z}{g^2-\bar{g}^2}\,dz^2.$$

Remark. (1) The function q depends on the choice of coordinates, whereas Q does not.

(2) As stated in the introduction, the Hopf differential (or its Abresch–Rosenberg generalization) is not defined on Sol_3. If we apply the definition of the Hopf differential of the harmonic maps on $(\overline{\mathbb{C}},ds^2)$, we get

$$Q=\frac{g_z\bar{g}_z}{|g^2-\bar{g}^2|}\,dz^2,$$

but this leads to a nonsmooth differential. Because $g^2-\bar{g}^2$ is purely imaginary on each quarter of the complex plane, the definitions are related by multiplication by i or $-i$, depending on the quarter. Thus, this “Hopf differential” is defined and holomorphic only on each of the four quarters delimited by the real and purely imaginary axes.

4. Construction of the helicoids in Sol_3

In this section we construct a one-parameter family of helicoids in Sol_3: we define a helicoid to be a minimal surface containing the x_3-axis whose intersection with every plane $\{x_3=\text{constant}\}$ is a straight line and which is invariant by left multiplication by an element of Sol_3 of the form $(0,0,T)$ for some $T\neq 0$.

Theorem 4. *There exists a one-parameter family* $(\mathcal{H}_K)_{K\in]-1;1[\setminus\{0\}}$ *of properly embedded minimal helicoids in* Sol_3 *having the following properties*:

(1) *For all* $K \in]-1;1[\setminus\{0\}$, *the surface* $\mathcal{H}_K$ *contains the* x_3*-axis.*

(2) *For all* $K \in]-1;1[\setminus\{0\}$, *the intersection of* $\mathcal{H}_K$ *and any horizontal plane* $\{x_3 = \lambda\}$ *is a straight line.*

(3) *For all* $K \in]-1;1[\setminus\{0\}$, *there exists* T_K *such that* $\mathcal{H}_K$ *is invariant by left multiplication by* $(0, 0, T_K)$.

(4) *The helicoids* $\mathcal{H}_K$ *have three symmetries fixing the origin*: *rotation by* π *around the* x_3*-axis, rotation by* π *around the* $(x, x, 0)$*-axis and rotation by* π *around the* $(x, -x, 0)$*-axis.*

Let $K \in]-1, 1[$; we define a map $g : \mathbb{C} \to \overline{\mathbb{C}}$ by

$$g(z = u + iv) = e^{-u} e^{ib(v)} e^{-i\pi/4},$$

where b satisfies the ODE

$$b' = \sqrt{1 - K\cos(2b)}, \quad b(0) = 0. \tag{8}$$

Proposition 5. *The map* b *is well-defined and has the following properties*:

(1) *The function* b *is an increasing diffeomorphism from* $\mathbb{R}$ *onto* $\mathbb{R}$.

(2) *The function* b *is odd.*

(3) *There exists a real number* $W > 0$ *such that*

$$\forall v \in \mathbb{R}, \quad b(v + W) = b(v) + \pi.$$

(4) *The function* b *satisfies* $b(kW) = k\pi$, *for all* $k \in \mathbb{Z}$.

Proof. Since $K \in]-1, 1[$, there exists $r > 0$ such that $1 - K\cos(2b) \in]r, 2[$; the Cauchy–Lipschitz theorem can be applied, and b is well-defined. By (8) we have $b' > 0$ on its domain of definition, and $\sqrt{r} < b' < 2$. Since b' is bounded by two positive constants, b is defined on $\mathbb{R}$, and

$$\lim_{v\to\pm\infty} b(v) = \pm\infty.$$

The function $\hat{b} : v \mapsto -b(-v)$ satisfies (8) with $\hat{b}(0) = 0$; hence $\hat{b} = b$ and b is odd. Finally, there exists $W > 0$ such that $b(W) = \pi$; then the function $\tilde{b} : v \mapsto b(v + W) - \pi$ satisfies (8) with $\tilde{b}(0) = 0$; hence, $\tilde{b} = b$. □

Corollary 6. *We have* $b(kW/2) = k\pi/2$ *for all* $k \in 2\mathbb{Z} + 1$.

Proof. We have

$$b\Big(\frac{W}{2}\Big) = b\Big(-\frac{W}{2} + W\Big) = -b\Big(\frac{W}{2}\Big) + \pi,$$

which gives the formula for $k = 1$, and the corollary easily follows. □

Proposition 7. *The function g satisfies $(g^2-\bar{g}^2)g_{z\bar{z}} = 2gg_zg_{\bar{z}}$, and its Hopf differential is*

$$Q = \frac{iK}{8}\,dz^2. \tag{9}$$

Proof. A direct calculation shows that g satisfies the equation. Hence, the Hopf differential is given by

$$Q = \frac{g_z\bar{g}_z}{g^2-\bar{g}^2}\,dz^2 = \frac{i(1-b'^2)}{8\cos(2b)}\,dz^2 = \frac{iK}{8}\,dz^2. \qquad \square$$

Thus the map g induces a conformal minimal immersion $x = (x_1, x_2, x_3)$ such that

$$x_{1z} = e^{-x_3}\frac{(\bar{g}^2-1)g_z}{g^2-\bar{g}^2} = \frac{(1+ie^{-2u}e^{-2ib})(1-b')e^{ib}e^{i\pi/4}}{4e^{-u}\cos(2b)}e^{-x_3},$$

$$x_{2z} = ie^{x_3}\frac{(\bar{g}^2+1)g_z}{g^2-\bar{g}^2} = -\frac{(1-ie^{-2u}e^{-2ib})i(1-b')e^{ib}e^{i\pi/4}}{4e^{-u}\cos(2b)}e^{x_3},$$

$$x_{3z} = \frac{2\bar{g}g_z}{g^2-\bar{g}^2} = \frac{i(b'-1)}{2\cos(2b)}.$$

This map is an immersion since the metric induced by x is given by

$$dw^2 = \|x_u\|^2|dz|^2 = \frac{K^2}{(1+b')^2}\cosh^2 u|dz|^2.$$

We obtain immediately that x_3 is a one-variable function and satisfies

$$x_3'(v) = \frac{1-b'(v)}{\cos(2b(v))} = \frac{K}{1+b'(v)}.$$

Remark. For $K=0$, we get x_3 is constant, and the image of x is a point. In the sequel, we will always exclude this case.

By setting $x_3(0)=0$, we choose x_3 among the primitive functions.

Proposition 8. (1) *The function x_3 is defined on $\mathbb{R}$ and is bijective.*

(2) *The function x_3 is odd.*

(3) *The function x_3 satisfies*

$$x_3(v+W) = x_3(v) + x_3(W)$$

for all real numbers v.

Proof. The map x_3 is bijective on $\mathbb{R}$ since it is a primitive of a continuous function, and its derivative has the sign of K. Since the map b is odd, b' is even, so x_3' is even and x_3 is odd. Finally, we have $x_3'(v+W) = x_3'(v)$, and the result follows. $\square$

Hence, the functions

$$x_1(u+iv) = \tfrac{\sqrt{2}}{2}(\cos b(v) - \sin b(v))x_3' e^{-x_3} \sinh u,$$
$$x_2(u+iv) = \tfrac{\sqrt{2}}{2}(\cos b(v) + \sin b(v))x_3' e^{x_3} \sinh u,$$

satisfy the equations above.

Theorem 9. *Let K be a real number such that $|K| < 1$ and $K \neq 0$, and b the function defined by* (8). *We define the function x_3 by*

$$x_3' = \frac{K}{1+b'}, \quad x_3(0) = 0.$$

Then the map

$$x : u+iv \in \mathbb{C} \longmapsto \begin{pmatrix} \tfrac{\sqrt{2}}{2}(\cos b(v) - \sin b(v))x_3' e^{-x_3} \sinh u \\ \tfrac{\sqrt{2}}{2}(\cos b(v) + \sin b(v))x_3' e^{x_3} \sinh u \\ x_3(v) \end{pmatrix}$$

is a conformal minimal immersion whose Gauss map is

$$g : u+iv \in \mathbb{C} \longmapsto e^{-u} e^{ib(v)} e^{-i\pi/4}.$$

Moreover,

$$(0,0,2x_3(W)) * x(u+iv) = x(u+i(v+2W)) \tag{10}$$

for all $u, v \in \mathbb{R}$. The surface given by x is called a helicoid *of parameter K and will be denoted by $\mathcal{H}_K$.*

Proof. Equation (10) means that the helicoid is invariant by left multiplication by $(0,0,2x_3(W))$. Recall that we have the identity

$$x_3(v+2W) = x_3(v+W) + x_3(W) = x_3(v) + 2x_3(W)$$

for all real numbers v. Thus we get the result for the third coordinate and we prove in the same way that $e^{-2x_3(W)}x_1(u+iv) = x_1(u+i(v+2W))$ and $e^{2x_3(W)}x_2(u+iv) = x_2(u+i(v+2W))$. □

Remark. (1) The surface $\mathcal{H}_K$ is embedded because x_3 is bijective. It is easy to see that it is even properly embedded.

(2) The surfaces $\mathcal{H}_K$ and $\mathcal{H}_{-K}$ are related; if we denote by the indices K and K' the data describing $\mathcal{H}_K$ and $\mathcal{H}_{-K}$, we get

$$\begin{cases} b_{-K}(v) = b_K(v+W/2) - \pi/2, \\ x_{3-K}(v) = -x_{3K}(v+W/2) + x_{3K}(W/2). \end{cases}$$

In particular, ${x_3}_{-K}(W) = -{x_3}_K(W)$ and both surfaces have the same period $|{x_3}_K(W)|$. Finally,

$$x_{-K}(u+iv) = (0,0,{x_3}_K(W/2)) * \sigma^3 x_K(u+i(v+W/2)).$$

Thus, there exists an isometry of Sol_3 that puts $\mathcal{H}_{-K}$ on $\mathcal{H}_K$.

Proposition 10. *For every real number T, there exists a unique helicoid $\mathcal{H}_K$ (up to isometry, i.e., up to $K \leftarrow\!\!\!\dashv -K$) whose period is T.*

Proof. We noticed that the period of the helicoid $\mathcal{H}_K$ is

$$\begin{aligned} 2x_3(W) := 2{x_3}_K(W) &= 2\int_0^W \frac{K}{1+b'(s)}\,ds \\ &= 2K\int_0^\pi \frac{du}{\sqrt{1-K\cos(2u)}\,(1+\sqrt{1-K\cos(2u)})}, \end{aligned}$$

with the change of variables $u = b(s)$ and $b(W) = \pi$. Seeing ${x_3}_K(W)$ as a function of the variable K, we get

$$\frac{\partial {x_3}_K(W)}{\partial K} = \int_0^\pi \frac{1}{(1-K\cos(2u))^{3/2}}\,du$$

(valid for K in every compact set $[0,a] \subset [0,1[$, and so in $[0,1[$). Then the function $K \mapsto {x_3}_K(W)$ is injective. Moreover, we have ${x_3}_0(W) = 0$ and

$$\begin{aligned} {x_3}_1(W) &= \int_0^\pi \frac{1}{\sqrt{1-\cos(2u)}\,(1+\sqrt{1-\cos(2u)})}\,du \\ &= \int_0^\pi \frac{1}{\sqrt{2}\sin u(1+\sqrt{2}\sin u)}\,du \\ &= \frac{1}{\sqrt{2}}\int_0^\infty \frac{1+t^2}{1+2\sqrt{2}t+t^2}\,dt = +\infty, \end{aligned}$$

so ${x_3}_K(W)$ is a bijection from $]0,1[$ onto $]0,+\infty[$. $\square$

The vector field defined by

$$\begin{aligned} N &= \frac{1}{1+|g|^2}\begin{bmatrix} 2\Re(g) \\ 2\Im(g) \\ 1-|g|^2 \end{bmatrix} \\ &= \frac{\sqrt{2}}{2\cosh u}\begin{bmatrix} \cos b + \sin b \\ \sin b - \cos b \\ \sqrt{2}\sinh u \end{bmatrix} \end{aligned}$$

is normal to the surface. We get

$$\nabla_{x_u} N = -\sin(2b)\frac{\sinh u}{\cosh u}x_u + \left(\frac{1+b'}{K\cosh^2 u} - \cos(2b)\right)x_v,$$
$$\nabla_{x_v} N = \left(\frac{1+b'}{K\cosh^2 u} - \cos(2b)\right)x_u + \sin(2b)\frac{\sinh u}{\cosh u}x_v,$$

and thus the Gauss curvature is given by

$$\mathcal{K} = -1 + \frac{1}{\cosh^2 u}\left(\frac{2(1+b')\cos(2b)}{K} - \frac{(1+b')^2}{K^2\cosh^2 u} + \sin^2(2b)\right).$$

In particular, the fundamental pieces of the helicoids have infinite total curvature since

$$\mathcal{K}dA = \left(-\frac{K^2}{(1+b')^2}\cosh^2 u + \frac{2K\cos(2b)}{1+b'} - \frac{1}{\cosh^2 u} + \frac{K^2\sin^2(2b)}{(1+b')^2}\right)du\,dv.$$

We notice that

$$x(-u+iv) = \begin{pmatrix} -x_1(u+iv) \\ -x_2(u+iv) \\ x_3(v) \end{pmatrix} = \sigma^2 x(u+iv),$$

where σ and τ are the isometries introduced in the first section: the helicoid $\mathcal{H}_K$ is symmetric by rotation by π around the x_3-axis, which is included in the helicoid as the image by x of the purely imaginary axis of $\mathbb{C}$. On this axis we have

$$g(0+iv) = -ie^{ib(v)}.$$

Hence, the straight line $\{(x,x,0) \mid x \in \mathbb{R}\}$ is included in the helicoid as the image by x of the real line. Along this line, we have

$$g(u+i0) = e^{-u}e^{-i\pi/4}.$$

Then we notice that

$$x(u-iv) = \begin{pmatrix} x_2(u+iv) \\ x_1(u+iv) \\ -x_3(v) \end{pmatrix} = \sigma\tau x(u+iv).$$

Thus, $\mathcal{H}_K$ is symmetric by rotation by π around the axis $\{(x,x,0) \mid x \in \mathbb{R}\}$.

Remark. The straight line $\{(x,x,0) \mid x \in \mathbb{R}\}$ is a geodesic of the helicoid. It's even a geodesic of Sol$_3$.

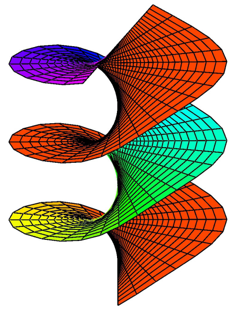

Figure 1. Helicoid for $K = 0.4$, created with Scilab.

Since the function sinh is odd, we deduce that

$$x(-u-iv) = \begin{pmatrix} -x_2(u+iv) \\ -x_1(u+iv) \\ -x_3(v) \end{pmatrix} = \sigma^3 \tau x(u+iv).$$

Thus, $\mathcal{H}_K$ is symmetric by rotation by π around the axis $\{(x,-x,0) \mid x \in \mathbb{R}\}$ (but this axis is not included in the surface).

The helicoid $\mathcal{H}_K$ has no more symmetry fixing the origin; indeed if it did, there would exist a diffeomorphism ϕ of $\mathbb{C}$ such that $x \circ \phi = \sigma^2 \circ x$ (we choose σ^2 as an example, but it is the same idea for the other elements of the isotropy group of the origin of Sol_3). By composition, the surface would have every symmetry of the isotropy group. But if $x \circ \phi = \tau x$, the decomposition $\phi = \phi_1 + i\phi_2$ leads to

$$\begin{pmatrix} x_1\big(\phi_1(u+iv) + i\phi_2(u+iv)\big) \\ x_2\big(\phi_1(u+iv) + i\phi_2(u+iv)\big) \\ x_3(\phi_2(u+iv)) \end{pmatrix} = \begin{pmatrix} -x_1(u+iv) \\ x_2(u+iv) \\ x_3(v) \end{pmatrix}.$$

Because x_3 is bijective, we get $\phi_2(u+iv) = v$ for all u, v, and then we get at the same time $\sinh(\phi_1(u+iv)) = \sinh u$ and $\sinh(\phi_1(u+iv)) = -\sinh u$, which is impossible.

5. Catenoids in Sol_3

In this section we construct examples of minimal annuli in Sol_3. Let $\alpha \in\ ]-1;1[$. We start from a map g defined on $\mathbb{C}$ by

$$g(z = u+iv) = -ie^{-u-\gamma(v)}e^{i\rho(v)},$$

where ρ satisfies the ODE

$$\rho' = \sqrt{1-\alpha^2\sin^2(2\rho)}, \quad \rho(0) = 0, \tag{11}$$

and γ is defined by

$$\gamma' = -\alpha \sin(2\rho), \quad \gamma(0) = 0. \tag{12}$$

Proposition 11. *The map ρ is well-defined and has the following properties*:

(1) *The function ρ is an increasing diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.*

(2) *The function ρ is odd.*

(3) *There exists a real number $V > 0$ such that*

$$\forall v \in \mathbb{R}, \quad \rho(v+V) = \rho(v) + \pi.$$

(4) *The function ρ satisfies $\rho(kV) = k\pi$ for all $k \in \mathbb{Z}$.*

Proof. Since $\alpha \in \,]-1,1[$, there exists $r > 0$ such that $1-\alpha^2 \sin^2(2\rho) \in \,]r,1]$; the Cauchy–Lipschitz theorem can be applied, and ρ is well-defined. By (11) we have $\rho' > 0$ on its domain of definition, and $\sqrt{r} < \rho' < 1$. Since ρ' is bounded by two positive constants, ρ is defined on $\mathbb{R}$, and

$$\lim_{v \to \pm\infty} \rho(v) = \pm\infty.$$

The function $\hat{\rho} : v \mapsto -\rho(-v)$ satisfies (11) with $\hat{\rho}(0) = 0$; hence $\hat{\rho} = \rho$ and ρ is odd. Finally, there exists $V > 0$ such that $\rho(V) = \pi$; Then the function $\tilde{\rho} : v \mapsto \rho(v+V) - \pi$ satisfies (11) with $\tilde{\rho}(0) = 0$; hence $\tilde{\rho} = \rho$. □

Corollary 12. (1) *We have $\rho(kV/2) = k\pi/2$ for all $k \in 2\mathbb{Z}+1$.*

(2) *We have $\rho(-v+V/2) = -\rho(v) + \frac{\pi}{2}$ for all $v \in \mathbb{R}$. In particular*, $\rho(V/4) = \frac{\pi}{4}$ *and* $\rho(3V/4) = \frac{3\pi}{4}$.

Proof. (1) We have

$$\rho\Big(\frac{V}{2}\Big) = \rho\Big(-\frac{V}{2} + V\Big) = -\rho\Big(\frac{V}{2}\Big) + \pi,$$

which gives the formula for $k = 1$, and part (1) easily follows.

(2) The functions $\rho^* : v \mapsto \pi/2 - \rho(-v+V/2)$ and ρ satisfy equation (11) with $\rho^*(0) = \rho(0) = 0$, so $\rho^* = \rho$ and

$$\rho(V/4) = \rho^*(V/4) = \tfrac{\pi}{2} - \rho\big(\tfrac{\pi}{2} - \tfrac{\pi}{4}\big),$$

and the result follows. □

Proposition 13. *The function g satisfies $(g^2 - \bar{g}^2) g_{z\bar{z}} = 2g g_z g_{\bar{z}}$, and its Hopf differential is*

$$Q = -\frac{\alpha}{4}\, dz^2. \tag{13}$$

Proof. A direct calculation shows that g satisfies the equation. Hence, the Hopf differential is given by

$$\widetilde{Q} = \frac{i(1-\rho'^2-\gamma'^2-2i\gamma')}{8\sin(2\rho)}\,dz^2 = -\frac{\alpha}{4}\,dz^2. \qquad \square$$

Thus the map g induces a conformal minimal immersion $x = (x_1, x_2, x_3)$ such that

$$x_{1z} = e^{-x_3}\frac{(\bar g^2-1)g_z}{g^2-\bar g^2}, \quad x_{2z} = i e^{x_3}\frac{(\bar g^2+1)g_z}{g^2-\bar g^2}, \quad x_{3z} = \frac{2\bar g g_z}{g^2-\bar g^2}.$$

This application is an immersion since the metric induced by x is given by

$$\begin{aligned} dw^2 &= \|x_u\|^2\,|dz|^2 \\ &= (F'^2+\alpha^2)\cosh^2(u+\gamma)|dz|^2 \\ &= \left(\frac{\alpha^4\sin^2(2\rho)}{(1+\rho')^2}+\alpha^2\right)\cosh^2(u+\gamma)|dz|^2 \\ &= \frac{2\alpha^2}{1+\rho'}\cosh^2(u+\gamma)|dz|^2. \end{aligned}$$

In particular,

$$x_{3z} = \frac{i\rho'-\gamma'-i}{2\sin(2\rho)},$$

that is,

$$\begin{cases} x_{3u} = 2\Re(x_{3z}) = -\dfrac{\gamma'}{\sin(2\rho)} = \alpha, \\[2ex] x_{3v} = -2\Im(x_{3z}) = \dfrac{1-\rho'}{\sin(2\rho)} = \dfrac{\alpha^2\sin(2\rho)}{1+\rho'}. \end{cases}$$

Thus

$$x_3(u+iv) = \alpha u + \alpha^2\int^v \frac{\sin(2\rho(s))}{1+\rho'(s)}\,ds.$$

Here we have to choose an initial condition; we set

$$F(v) = \alpha^2\int_0^v \frac{\sin(2\rho(s))}{1+\rho'(s)}\,ds,$$

and define

$$x_3(u+iv) = \alpha u + F(v).$$

The function F is well-defined on $\mathbb{R}$.

Proposition 14. *The function F is even and V-periodic.*

Proof. The function F' is odd because ρ is odd and ρ' is even, so F is even. Then we get

$$F'(v+V)=\alpha^2\frac{\sin(2\rho(v)+2\pi)}{1+\rho'(v)}=F'(v),$$

so there exists a constant C such that $F(v+V)=F(v)+C$ for all $v\in\mathbb{R}$. By evaluating at zero, we get $C=F(V)$, that is,

$$\begin{aligned}C&=\alpha^2\int_0^V\frac{\sin(2\rho(s))}{1+\rho'(s)}\,ds=\alpha^2\int_0^V H(s)\,ds\\&=\alpha^2\left\{\int_0^{V/4}H(s)\,ds+\int_{V/4}^{3V/4}H(s)\,ds+\int_{3V/4}^{V}H(s)\,ds\right\}\\&:=\sum_{k=0}^{2}L_k(\alpha).\end{aligned}$$

We can now do the change of variable $u=\sin(2\rho(s))$ in each integral $L_k(\alpha)$, with

$$du=2\rho'(s)\cos(2\rho(s))\,ds=2(-1)^k\sqrt{(1-\alpha^2u^2)(1-u^2)}\,ds.$$

Thus,

$$C=\alpha^2\int_{-1}^{1}\frac{u\,du}{(1+\sqrt{1-\alpha^2u^2})\sqrt{(1-\alpha^2u^2)(1-u^2)}}=0$$

and F is V-periodic. □

Proposition 15. *The function γ is even and V-periodic.*

Proof. We prove the proposition in exactly the same way as for the function F. □

The two other equations become

$$\begin{aligned}x_{1z}&=e^{-x_3}\frac{(e^{-u-\gamma-i\rho}+e^{u+\gamma+i\rho})(1-\rho'-i\gamma')}{4\sin(2\rho)},\\x_{2z}&=-e^{x_3}\frac{(e^{u+\gamma+i\rho}-e^{-u-\gamma-i\rho})(i-i\rho'+\gamma')}{4\sin(2\rho)}.\end{aligned}$$

Those equations lead to

$$\begin{aligned}x_1&=e^{-\alpha u-F}\left(\frac{e^{u+\gamma}}{2(1-\alpha)}(F'\cos\rho-\alpha\sin\rho)-\frac{e^{-u-\gamma}}{2(1+\alpha)}(\alpha\sin\rho+F'\cos\rho)\right),\\x_2&=e^{\alpha u+F}\left(\frac{-e^{u+\gamma}}{2(1+\alpha)}(\alpha\cos\rho+F'\sin\rho)+\frac{e^{-u-\gamma}}{2(\alpha-1)}(\alpha\cos\rho-F'\sin\rho)\right).\end{aligned}$$

Remark. If $\alpha=0$, then $x(\mathbb{C})=\{0\}$. This case will be excluded in the sequel.

Theorem 16. *Let α be a real number such that $|\alpha| < 1$ and $\alpha \neq 0$, and ρ and γ the functions defined by* (11) *and* (12). *We define the function F by*

$$F(v) = \alpha^2 \int_0^v \frac{\sin(2\rho(s))}{1+\rho'(s)}\, ds.$$

Then the map $x : \mathbb{C} \to \mathrm{Sol}_3$ defined by

$$\begin{pmatrix} e^{-\alpha u - F}\left(\dfrac{e^{u+\gamma}}{2(1-\alpha)}(F' \cos\rho - \alpha \sin\rho) - \dfrac{e^{-u-\gamma}}{2(1+\alpha)}(\alpha \sin\rho + F' \cos\rho)\right) \\ e^{\alpha u + F}\left(\dfrac{-e^{u+\gamma}}{2(1+\alpha)}(\alpha \cos\rho + F' \sin\rho) + \dfrac{e^{-u-\gamma}}{2(\alpha-1)}(\alpha \cos\rho - F' \sin\rho)\right) \\ \alpha u + F \end{pmatrix}$$

is a conformal minimal immersion whose Gauss map is

$$g : u + iv \in \mathbb{C} \longmapsto -i e^{-u-\gamma(v)} e^{i\rho(v)}.$$

Moreover,

$$x(u + i(v + 2V)) = x(u + iv) \tag{14}$$

for all $u, v \in \mathbb{R}$. The surface given by x is called a catenoid *of parameter α and will be denoted by $\mathcal{C}_\alpha$.*

Proof. The periodicity of $\mathcal{C}_\alpha$ is an application of Propositions 11, 14 and 15. □

Remark. The surfaces $\mathcal{C}_\alpha$ and $\mathcal{C}_{-\alpha}$ are related; if we denote by the indices α and $-\alpha$ the data describing $\mathcal{C}_\alpha$ and $\mathcal{C}_{-\alpha}$, we get

$$\begin{cases} \rho_{-\alpha} = \rho_\alpha, \\ F_{-\alpha} = F_\alpha, \\ \gamma_{-\alpha} = -\gamma_\alpha. \end{cases}$$

Thus, we get

$$x_{-\alpha}(-u + iv) = \sigma^2 x_\alpha(u + iv).$$

In particular, there exists an orientation-preserving isometry of Sol_3 fixing the origin that sends $\mathcal{C}_\alpha$ on $\mathcal{C}_{-\alpha}$.

Now we show that the catenoids are embedded:

Proposition 17. *For all $\lambda \in \mathbb{R}$, the intersection of $\mathcal{C}_\alpha$ with the plane $\{x_3 = \lambda\}$ is a nonempty closed embedded convex curve.*

Proof. This intersection is nonempty: $x(\lambda/\alpha + i0) \in \mathcal{C}_\alpha \cap \{x_3 = \lambda\}$. We look at the curve in $\mathbb{C}$ defined by $x_3(u + iv) = \alpha u + F(v) = \lambda$, i.e., the curve

$$c : t \in \mathbb{R} \longmapsto \Big(\frac{\lambda - F(t)}{\alpha}, t\Big).$$

Its image by x is

$$\mathfrak{c} : t \in \mathbb{R} \longmapsto \begin{pmatrix} e^{-\lambda}\left(\dfrac{e^{\delta+\gamma}}{2(1-\alpha)}(F'\cos\rho - \alpha\sin\rho) - \dfrac{e^{-\delta-\gamma}}{2(1+\alpha)}(\alpha\sin\rho + F'\cos\rho)\right) \\ e^{\lambda}\left(\dfrac{-e^{\delta+\gamma}}{2(1+\alpha)}(\alpha\cos\rho + F'\sin\rho) + \dfrac{e^{-\delta-\gamma}}{2(\alpha-1)}(\alpha\cos\rho - F'\sin\rho)\right) \\ c \end{pmatrix}(t),$$

where $\delta = \frac{\lambda - F}{\alpha}$. The calculation leads to

$$\mathfrak{c}_1'(t) = \frac{e^{-\lambda}}{\alpha(1-\alpha^2)}\left(A(t)\cosh\left(\frac{\lambda - F}{\alpha} + \gamma\right) + B(t)\sinh\left(\frac{\lambda - F}{\alpha} + \gamma\right)\right),$$

with

$$\begin{cases} A = -F'^2\cos\rho + \alpha\gamma' F'\cos\rho - \alpha^2\rho'\cos\rho + \alpha^2 F'\sin\rho - \alpha^3\gamma'\sin\rho \\ \qquad\qquad + \alpha^2 F''\cos\rho - \alpha^2 F'\rho'\sin\rho, \\ B = \alpha F'\sin\rho - \alpha^2\gamma'\sin\rho + \alpha F''\cos\rho - \alpha F'\rho'\sin\rho - \alpha F'^2\cos\rho \\ \qquad\qquad + \alpha^2\gamma' F'\cos\rho - \alpha^3\rho'\cos\rho. \end{cases}$$

We remark that $B \equiv 0$ after simplifications, and

$$A(t) = (F'^2(t) + \alpha^2)(\alpha^2 - 1)\cos\rho(t).$$

Finally,

$$\mathfrak{c}_1'(t) = -\frac{e^{-\lambda}}{\alpha}(F'^2(t) + \alpha^2)\cos\rho(t)\cosh\left(\frac{\lambda - F(t)}{\alpha} + \gamma(t)\right).$$

In the same way, we get

$$\mathfrak{c}_2'(t) = -\frac{e^{-\lambda}}{\alpha}(F'^2(t) + \alpha^2)\sin\rho(t)\cosh\left(\frac{\lambda - F(t)}{\alpha} + \gamma(t)\right).$$

Thus

$$\mathfrak{c}_1'^2 + \mathfrak{c}_2'^2 = \frac{e^{-2\lambda}}{\alpha^2}(F'^2(t) + \alpha^2)^2\cosh^2\left(\frac{\lambda - F(t)}{\alpha} + \gamma(t)\right) > 0,$$

so the intersection $\mathcal{C}_\alpha \cap \{x_3 = \lambda\}$ is a smooth curve; moreover, it's closed since $\mathfrak{c}(t + 2V) = \mathfrak{c}(t)$ for all $t \in \mathbb{R}$.

The planes $\{x_3 = \lambda\}$ are flat: indeed, the metrics on these planes are $e^{2\lambda}dx_1^2 + e^{-2\lambda}dx_2^2$, so up to an affine transformation, we can work in euclidean coordinates, as we suppose in this proof since affinities preserve convexity.

To prove that $\mathfrak{c}$ is embedded and convex, we consider the part of $\mathfrak{c}$ corresponding to $t \in (-V/2, V/2)$. On $(-V/2, V/2)$, we have $\cos\rho(t) > 0$, thanks to Proposition 11

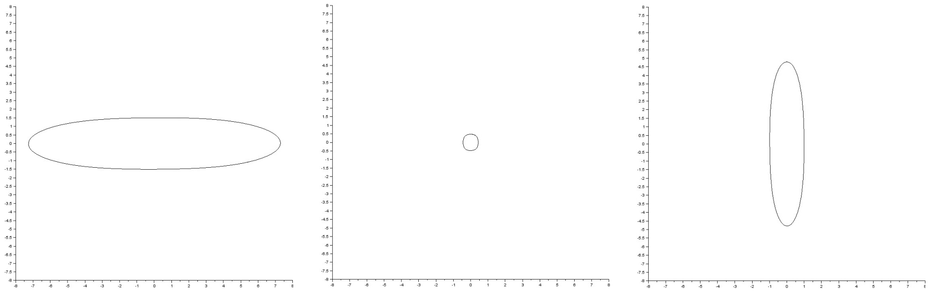

Figure 2. Sections with $\{x_3 = -1\}$, $\{x_3 = 0\}$, and $\{x_3 = 1\}$, created with Scilab.

and Corollary 12. So $\mathfrak{c}_1'(t) < 0$ if $\alpha > 0$ (and $\mathfrak{c}_1'(t) > 0$ if $\alpha < 0$) and $\mathfrak{c}_1$ is injective and decreasing if $\alpha > 0$ (and increasing if $\alpha < 0$). We get

$$\frac{d\mathfrak{c}_2}{d\mathfrak{c}_1} = \tan\rho(t),$$

so $d\mathfrak{c}_2/d\mathfrak{c}_1$ is an increasing function of t, and also of $\mathfrak{c}_1$ if $\alpha < 0$ (and a decreasing function of the decreasing function $\mathfrak{c}_1$ if $\alpha > 0$). In both cases, the curve is convex.

Then, the half of $\mathfrak{c}$ corresponding to $t \in (-V/2, V/2)$ is convex and embedded. Since $\mathfrak{c}(t+V) = -\mathfrak{c}(t)$, the entire curve is convex and embedded. $\square$

Figure 2 shows sections of the catenoid $\alpha = -0.6$ with planes $\{x_3 = \text{constant}\}$.

Corollary 18. *The surface $\mathcal{C}_\alpha$ is properly embedded for all $\alpha \in\ ]-1, 1[\setminus\{0\}$.*

Proposition 19. *For all $\alpha \in\ ]-1, 1[\setminus\{0\}$, the surface $\mathcal{C}_\alpha$ is conformally equivalent to $\mathbb{C}\setminus\{0\}$.*

Proof. The map $x : \mathbb{C}/(2iV\mathbb{Z}) \to \mathcal{C}_\alpha$ is a conformal bijective parametrization of $\mathcal{C}_\alpha$. $\square$

The vector field defined by

$$N = \frac{1}{\cosh u}\begin{bmatrix} e^{-\gamma}\sin\rho \\ -e^{-\gamma}\cos\rho \\ \sinh u \end{bmatrix}$$

is normal to the surface.

We have

$$x(u + i(v+V)) = \begin{pmatrix} -x_1(u+iv) \\ -x_2(u+iv) \\ x_3(u+iv) \end{pmatrix} = \sigma^2 x(u+iv).$$

Thus, the surface $\mathcal{C}_\alpha$ is symmetric by rotation by π around the x_3-axis.

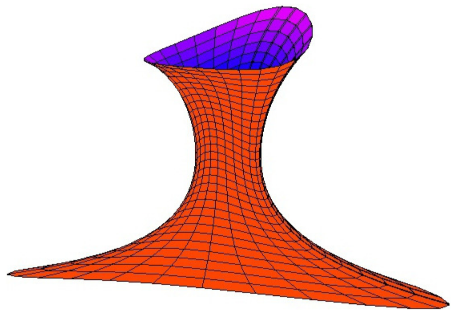

Figure 3. Catenoid for $\alpha = -0.6$, created with Scilab.

Remark. The x_3-axis is contained in the "interior" of $\mathcal{C}_\alpha$ since each curve $\mathcal{C}_\alpha \cap \{x_3 = \lambda\}$ is convex and symmetric with respect to this axis.

We also get

$$x(u-iv) = \begin{pmatrix} -x_1(u+iv) \\ x_2(u+iv) \\ x_3(u+iv) \end{pmatrix} = \tau x(u+iv),$$

and the surface $\mathcal{C}_\alpha$ is symmetric by reflection in the plane $\{x_1 = 0\}$, and finally we have

$$x(u+i(-v+V)) = \sigma^2 \tau x(u+iv),$$

and $\mathcal{C}_\alpha$ is symmetric by reflection in the plane $\{x_2 = 0\}$.

If $\mathcal{C}_\alpha$ had another symmetry fixing the origin, it would have every symmetry of the isotropy group of Sol_3, and we prove as for the helicoid that it is impossible.

6. Limits of catenoids

6.1. ***The case $\alpha = 0$.*** In this part we consider the limit surface of the catenoids $\mathcal{C}_\alpha$ when α goes to zero. For this, we do the change of parameters

$$\begin{cases} u' = u + \ln\alpha, \\ v' = v. \end{cases}$$

In these coordinates, the immersion x given in Theorem 16 takes the form

$$\begin{pmatrix} e^{\alpha\ln\alpha-\alpha u'-F}\left(\dfrac{e^{u'+\gamma}}{2\alpha(1-\alpha)}(\cos\rho F'-\alpha\sin\rho)-\dfrac{\alpha e^{-u'-\gamma}}{2(1+\alpha)}(\alpha\sin\rho+\cos\rho F')\right) \\ e^{-\alpha\ln\alpha+\alpha u'+F}\left(\dfrac{-e^{u'+\gamma}}{2\alpha(1+\alpha)}(\alpha\cos\rho+F'\sin\rho)+\dfrac{\alpha e^{-u'-\gamma}}{2(\alpha-1)}(\alpha\cos\rho-F'\sin\rho)\right) \\ -\alpha\ln\alpha+\alpha u'+F \end{pmatrix}.$$

Letting α go to zero, we get

$$\begin{cases} \rho \longrightarrow \mathrm{Id}, \\ F/\alpha \longrightarrow 0, \\ F'/\alpha \longrightarrow 0, \\ \gamma \longrightarrow 0, \end{cases}$$

and so the limit immersion is

$$\begin{pmatrix} -\dfrac{e^{u'}}{2}\sin v' \\ -\dfrac{e^{u'}}{2}\cos v' \\ 0 \end{pmatrix}.$$

Thus, we obtain a parametrization of the plane $\{x_3 = 0\}$, which is the limit of the family $(\mathcal{C}_\alpha)$ when $\alpha \to 0$.

6.2. *The case $\alpha = 1$.* We end by the study of the case $\alpha = 1$ (the case $\alpha = -1$ is exactly the same). We show that the limit surface is a minimal entire graph:

Proposition 20. *Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathrm{Sol}_3$ be defined by*

$$\boldsymbol{x}(u+iv) = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{pmatrix} = \begin{pmatrix} -\dfrac{\tanh v}{2}(1+e^{-2u}) \\ \dfrac{e^{2u}}{4} - \dfrac{u}{2} - \dfrac{\cosh(2v)}{4} \\ u + \ln(\cosh v) \end{pmatrix}.$$

Then $\boldsymbol{x}$ is a minimal immersion and there exists a C^∞-function f defined on $\mathbb{R}^2$ such that the image of x (called $\mathcal{S}$) is the $\boldsymbol{x}_2$-graph of f given by $\boldsymbol{x}_2 = f(\boldsymbol{x}_1, \boldsymbol{x}_3)$.

Proof. We show that this surface is (up to a translation) the limit surface of the family $(\mathcal{C}_\alpha)_{\alpha\in]-1,1[}$ when α goes to 1. For $\alpha = 1$, the Gauss map is still given by $g(z = u+iv) = -ie^{-u-\gamma(v)}e^{i\rho(v)}$, but ρ satisfies the ODE

$$\rho' = \cos(2\rho), \quad \rho(0) = 0, \tag{15}$$

and γ is still defined by

$$\gamma' = -\sin(2\rho), \quad \gamma(0) = 0. \tag{16}$$

We have explicit expressions for these functions, which are given by

$$\rho(v) = \arctan e^{2v} - \pi/4 = \arctan(\tanh v),$$
$$\gamma(v) = -\tfrac{1}{2}\ln(\cosh(2v)).$$

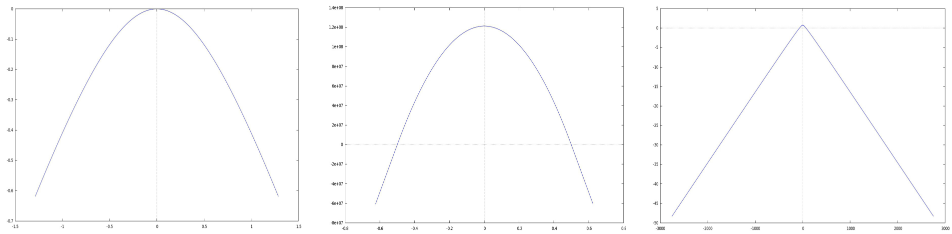

Figure 4. Sections with $\{x_3 = 0\}$, $\{x_3 = 10\}$, and $\{x_3 = -2\}$, created with Maxima.

Thus by setting

$$F(v) = \int_0^v \frac{\sin(2\rho(s))}{1+\cos(2\rho(s))}\,ds,$$

we obtain $F(v) = \ln(\cosh v)$. Then, the immersion x is given by

$$x = \begin{pmatrix} -\dfrac{e^{-2u}}{2}\tanh v + \dfrac{e^{-v}}{2\cosh v} \\ \dfrac{e^{2u}}{4} - \dfrac{u}{2} - \dfrac{\cosh(2v)}{4} \\ u + \ln(\cosh v) \end{pmatrix}.$$

A unit normal vector field is given by

$$N = \frac{1}{1+e^{-2u}\cosh(2v)}\begin{bmatrix} 2e^{-u}\sinh v \\ -2e^{-u}\cosh v \\ 1-e^{-2u}\cosh(2v) \end{bmatrix}.$$

Thus, we get

$$g(u+iv) = -ie^{-u}(\cosh v + i\sinh v),$$

which satisfies the harmonic equation (6). The metric induced by this immersion on the surface is

$$ds^2 = (e^{-4u}\tanh^2 v + e^{2u}\sinh^2 u + 1)\,|dz|^2.$$

This surface is symmetric by reflection in the plane $\{x_1 = 1/2\}$ since

$$x(u+iv) = \begin{pmatrix} \dfrac{1}{2} - \dfrac{\tanh v}{2}(1+e^{-2u}) \\ \dfrac{e^{2u}}{4} - \dfrac{u}{2} - \dfrac{\cosh(2v)}{4} \\ u + \ln(\cosh v) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \tilde{x}_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix},$$

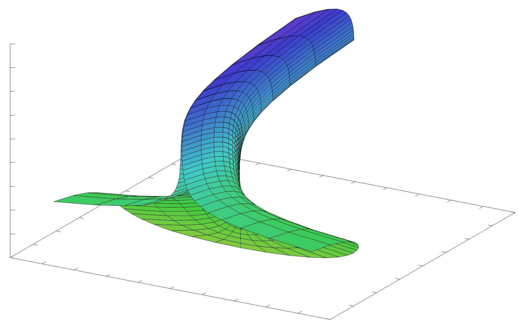

Figure 5. The surface $\mathcal{S}$, created with Maxima.

and so

$$x(u-iv)=\begin{pmatrix}\frac{1}{2}-\tilde{x}_1(u,v)\\ x_2(u,v)\\ x_3(u,v)\end{pmatrix}.$$

This property is equivalent to the property that the translated surface $(-1/2,0,0)*x(u+iv)$ is symmetric with respect to $\{x_1=0\}$. This translated surface is the image of the immersion $\boldsymbol{x}$ defined by

$$\boldsymbol{x}(u+iv)=(-1/2,0,0)*x(u+iv)=\begin{pmatrix}-\dfrac{\tanh v}{2}(1+e^{-2u})\\ \dfrac{e^{2u}}{4}-\dfrac{u}{2}-\dfrac{\cosh(2v)}{4}\\ u+\ln(\cosh v)\end{pmatrix}.$$

Then, this surface is analytic (like any minimal surface in Sol_3), so it is a local analytic $\boldsymbol{x}_2$-graph around every point where ∂_2 doesn't belong to the tangent plane, i.e., $\langle N,\partial_2\rangle\neq 0$. But

$$\langle N,\partial_2\rangle=0 \iff \cosh v e^{-u}=0,$$

which is impossible. Thus, $\mathcal{S}$ is a local analytic $\boldsymbol{x}_2$-graph around every point. Then, we consider sections of the surface $\mathcal{S}$ with planes $\{\boldsymbol{x}_3=\text{constant}\}$: on the plane $\{\boldsymbol{x}_3=\lambda\}$, we get the curve

$$c_\lambda(t)=\begin{pmatrix}-\dfrac{\tanh t}{2}(1+e^{-2\lambda}\cosh^2 t)\\ \dfrac{e^{2\lambda}}{4\cosh^2 t}-\dfrac{\lambda}{2}+\dfrac{\ln(\cosh t)}{2}-\dfrac{\cosh(2t)}{4}\end{pmatrix}:=\begin{pmatrix}\boldsymbol{x}_{1\lambda}(t)\\ \boldsymbol{x}_{2\lambda}(t)\end{pmatrix}.$$

Then,

$$\boldsymbol{x}_{1\lambda}'(t)=\frac{\tanh^2 t-1}{2}-\frac{e^{-2\lambda}}{2}(\cosh^2 t+\sinh^2 t)<0$$

for all $t \in \mathbb{R}$. Thus, the curves are injective, so the surface $\mathcal{S}$ is embedded. Moreover, by the implicit function theorem, we deduce that for every $\lambda \in \mathbb{R}$, there exists a function f_λ such that $x_{2\lambda} = f_\lambda(x_{1\lambda})$. Because the function $x_{1\lambda}$ is a decreasing diffeomorphism of $\mathbb{R}$, the function f_λ is defined on $\mathbb{R}$. This calculus is valid for all $\lambda \in \mathbb{R}$, so there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $x_2 = f(x_1, x_3)$.

Finally, this function f coincides around every point with the local C^∞-functions which give the local graphs, and so f is C^∞. □

As a conclusion, we can notice that, for a fixed x_3,

- when $x_1 \to +\infty$, $x_2 \approx -x_1 e^{2x_3}$;
- when $x_1 \to -\infty$, $x_2 \approx x_1 e^{2x_3}$.

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