

*Pacific
Journal of
Mathematics*

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WITH A SMALL TENSOR GENERATOR**

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We consider semisimple super Tannakian categories generated by an object whose symmetric or alternating tensor square is simple up to trivial summands. Using representation theory, we provide a criterion to identify the corresponding Tannaka super groups that applies in many situations. As an example we discuss the tensor category generated by the convolution powers of an algebraic curve inside its Jacobian variety.

1. Introduction

The goal of this paper is to classify reductive super groups with a representation which is *small* in the sense that its symmetric or alternating square is irreducible or splits into an irreducible plus a trivial representation. This discussion fits into the general framework of small objects in tensor categories over an algebraically closed field k of characteristic zero, where by definition a *tensor category over k* is a rigid symmetric monoidal k -linear abelian category \mathbf{C} whose unit object $\mathbf{1} \in \mathbf{C}$ satisfies $\text{End}(\mathbf{1}) = k$. Recall that the structure of a monoidal category is given by a k -linear exact bifunctor $- \otimes - : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ together with a unit object and associativity constraints $a_{U,V,W} : U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W$ for $U, V, W \in \mathbf{C}$ such that the usual compatibilities hold. A monoidal category is called symmetric if it is equipped with symmetry constraints $s_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$ which are compatible with the previous structure and satisfy $s_{V,U} \circ s_{U,V} = \text{id}$. It is called rigid if to every $V \in \mathbf{C}$ one may functorially attach an object $V^\vee \in \mathbf{C}$ with natural isomorphisms

$$\text{Hom}(U \otimes V, W) \xrightarrow{\sim} \text{Hom}(U, V^\vee \otimes W).$$

Tensor categories are ubiquitous in many areas of mathematics like representation theory, topology and algebraic geometry [Deligne et al. 1982; Gabber and Loeser 1996; Krämer and Weissauer 2015; Krämer 2014]. The typical example is the category $\mathbf{C} = \text{Rep}_k(G)$ of finite-dimensional algebraic super representations of an affine super group scheme G over k . Here the representation spaces are super vector

MSC2010: primary 20G05, 20C15; secondary 18D10.

Keywords: Tannakian category, tensor generator, symmetric square, alternating square, irreducible representation, reductive super group.

spaces, i.e., $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces $V = V_0 \oplus V_1$ of finite dimension over k , and the symmetry constraints are defined by the sign rule $s_{U,V}(u \otimes v) = (-1)^{\alpha\beta} v \otimes u$ for $u \in U_\alpha$, $v \in V_\beta$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$.

We say that a tensor category \mathbf{C} over k is *algebraic* if any object $V \in \mathbf{C}$ is of finite length $\ell(V)$ and if the length $\ell(V^{\otimes n})$ of tensor powers grows at most polynomially in n . Over an algebraically closed field k of characteristic zero, Deligne has shown [2002] that every algebraic tensor category is equivalent as a tensor category over k to the category $\text{Rep}_k(G, \varepsilon)$ of finite-dimensional algebraic super representations V of an affine super group scheme G over k with the property that a certain element $\varepsilon \in G(k)$ acts via the parity automorphism on V . Here the parity automorphism of a super vector space $V = V_0 \oplus V_1$ is given by $(-1)^\alpha$ on V_α for $\alpha \in \mathbb{Z}/2\mathbb{Z}$. Note that the above framework includes the usual representation categories of algebraic groups G by taking $\varepsilon = 1$. Since by definition $\text{Rep}_k(G, \varepsilon)$ is a full subcategory of the algebraic tensor category $\text{Rep}_k(G)$ of all algebraic super representations, for the study of small objects it suffices to consider the latter.

If such an algebraic tensor category $\mathbf{C} = \text{Rep}_k(G)$ has a *tensor generator* X in the sense that any object is a subquotient of a tensor power $(X \oplus X^\vee)^{\otimes r}$ for some $r \in \mathbb{N}$, then the super group scheme G is of finite type over k and will be called the *Tannaka super group* of the category. We then have a faithful algebraic super representation $G \hookrightarrow \text{GL}(V)$ on the finite-dimensional super vector space V associated to X . In what follows, by an *algebraic super group* over k we mean an affine super group scheme of finite type over k . Coming back to the general case, any algebraic tensor category \mathbf{C} is the direct limit of tensor subcategories with a tensor generator, so the corresponding affine super group scheme G is an inverse limit of algebraic super groups over k , and for the study of small objects it suffices to consider algebraic super groups. Unfortunately, in contrast to the situation for ordinary algebraic groups, the representation theory of algebraic super groups is hardly understood. Even for the general linear super groups $G = \text{GL}_{m|n}(k)$ over k the categories $\text{Rep}_k(G)$ are not semisimple, and their tensor structure seems to be rather complicated. For example, in general the tensor product of irreducible objects is not a direct sum of irreducible objects. This often makes it desirable to replace \mathbf{C} by some quotient category with simpler properties.

For any algebraic tensor category \mathbf{C} over k , a general construction due to André and Kahn [2002, Section 8] together with the above result by Deligne implies that there is a universal k -linear (though in general not exact) quotient functor $\pi : \mathbf{C} \rightarrow \mathbf{C}^{\text{red}}$ of algebraic tensor categories such that \mathbf{C}^{red} is semisimple. An indecomposable object $V \in \mathbf{C}$ becomes isomorphic to zero in the quotient category \mathbf{C}^{red} if and only if its super dimension $\dim(V_0) - \dim(V_1)$ is zero. Furthermore the functor π maps indecomposable objects to irreducible or zero objects, so it maps small objects to small objects. Bearing this in mind, we say an affine super group

scheme G over k is *reductive* if the category $\text{Rep}_k(G)$ is semisimple. The reductive algebraic super groups over an algebraically closed field k of characteristic zero have been classified in [Weissauer 2009]. In particular, they are all isogenous to products of ordinary reductive groups and orthosymplectic super groups $\text{OSp}_{1|2m}(k)$, and their representation theory may be understood in terms of the representation theory of ordinary connected reductive groups and finite groups. In general, an affine super group scheme over k is reductive if and only if it is an inverse limit of reductive algebraic super groups. The above construction then associates to any affine super group scheme G over k a reductive super group scheme G^{red} over k , where $\mathbf{C}^{\text{red}} = \text{Rep}_k(G^{\text{red}})$ for the category $\mathbf{C} = \text{Rep}_k(G)$.

While from a theoretical point of view this seems to give a rather satisfying picture, in concrete applications the algebraic tensor categories arising from the construction of André and Kahn are often hard to approach. The case of a classical algebraic group G over k , where $G^{\text{red}} = G/U$ for the unipotent radical $U \trianglelefteq G^0$ of the connected component, is not typical. In general there may be no simple relation between G^{red} and G . For the general linear super groups $G = \text{GL}_{m|n}(k)$ with $m, n > 1$ the associated reductive super group schemes G^{red} are not even of finite type over k . One of the motivations for studying small objects in algebraic tensor categories is to get a better understanding of the construction of André and Kahn in such situations.

Apart from examples in representation theory, this is useful also in algebraic geometry, especially in the context of Brill–Noether sheaves [Krämer and Weissauer 2013; Weissauer 2007; Weissauer 2008]. For a smooth complex projective variety X , the image of X in its Albanese variety defines a distinguished object V of a semisimple algebraic tensor category $\mathbf{C} = \mathbf{C}(X)$ which is constructed via convolutions of perverse sheaves, see [Krämer and Weissauer 2015]. The corresponding Tannaka super group $G = G_X$ is a classical reductive complex algebraic group which is an intrinsic invariant of the variety X . If the object $V \in \mathbf{C}$ is small, our main result (Theorem 1.1) gives a criterion to determine this group. In Section 6 we illustrate this for a smooth curve X of genus $g \geq 1$. It has been shown in [Weissauer 2007] that in this case

$$G_X = \begin{cases} \text{Sp}_{2g-2}(\mathbb{C}) & \text{if } X \text{ is hyperelliptic,} \\ \text{SL}_{2g-2}(\mathbb{C}) & \text{otherwise;} \end{cases}$$

our criterion leads to a very short and much simpler proof of this result.

Returning to representation theory, let k again be an algebraically closed field of characteristic zero. The main goal of this paper is to classify all reductive super groups G over k that arise as the Tannaka super group of a semisimple tensor category with a small tensor generator; see Theorem 1.1. For simplicity, in what follows the term *representation* refers to a representation on a super vector space in

the case of true super groups, but to an ordinary representation otherwise. For V in $\text{Rep}_k(G)$ we denote by

$$T_\epsilon(V) = \begin{cases} S^2(V) & \text{for } \epsilon = +1, \\ \Lambda^2(V) & \text{for } \epsilon = -1, \end{cases}$$

the symmetric and the alternating squares with respect to the symmetry constraint for super vector spaces. If $T_\epsilon(V)$ is irreducible or a direct sum of an irreducible and a one-dimensional trivial representation $\mathbf{1}$, we say V is ϵ -small (or just *small*). Small representations are irreducible. If the trivial direct summand $\mathbf{1}$ occurs in $T_\epsilon(V)$, then V is isomorphic to its dual V^\vee and hence carries a nondegenerate symmetric or alternating bilinear form. We say that V is *very small* if both $S^2(V)$ and $\Lambda^2(V)$ are irreducible. Since $\dim_k(\text{End}_G(V \otimes V)) = \dim_k(\text{End}_G(V \otimes V^\vee))$, this is the case if and only if $V \otimes V^\vee \cong W \oplus \mathbf{1}$ for some irreducible representation $W \in \text{Rep}_k(G)$.

By definition a super group is *quasisimple* if it is a perfect central extension of a (nonabelian) simple super group. For the finite quasisimple groups G very small and self-dual small faithful representations have been classified by Magaard, Malle and Tiep [2002, Theorem 7.14], using earlier results of Magaard and Malle [1998]. In a more general setup the list of very small representations has been extended by Guralnick and Tiep [2005, Theorem 1.5] to arbitrary reductive groups. In particular, except for the standard representation of the special linear group, very small representations of G only exist if the quotient $G/Z(G)$ by the center $Z(G)$ is finite. The class of small representations is much richer and contains several cases with $\dim(G/Z(G)) > 0$.

To state our main result we use the following notation. For super groups G_i and representations $V_i \in \text{Rep}_k(G_i)$, define $G_1 \otimes G_2 \subset \text{GL}(V_1 \boxtimes V_2)$ to be the image of the exterior tensor product representation. If a group of automorphisms of $G_1 \otimes G_2$ contains elements that interchange the two subgroups $G_1 \otimes \{1\}$ and $\{1\} \otimes G_2$, we say that it *flips the two factors*. If a group acts transitively on a set X and if the action on the set of 2-element subsets of X is still transitive, we say that the group acts *2-homogeneously* on X . If for $V \in \text{Rep}_k(G)$ the restriction $V|_K$ to some normal abelian subgroup $K \trianglelefteq G$ splits into a direct sum of pairwise distinct characters that are permuted 2-homogeneously and faithfully by the adjoint action of G/K , we say that the representation V is *2-homogeneous monomial*. Finally, a finite p -group E is called *extraspecial* if $E/Z(E)$ is elementary abelian and $Z(E) = [E, E]$ is cyclic of order p . Then $|E| = p^{1+2n}$ for some $n \in \mathbb{N}$, and for any nontrivial character ω of $Z(E) \cong \mathbb{Z}/p\mathbb{Z}$ there is a unique irreducible representation $V_\omega \in \text{Rep}_k(E)$ with dimension p^n on which $Z(E)$ acts via ω [Dornhoff 1971, Theorem 31.5].

Theorem 1.1. *Let G be a reductive super group and $V \in \text{Rep}_k(G)$ an ϵ -small faithful representation of super dimension $d > 0$. Then one of the following holds:*

- (a) *The connected component $G^0 \subseteq G$ is quasisimple and the restriction $V|_{G^0}$ remains ϵ -small. In this case the possible Dynkin types of G^0 and the highest weights of $V|_{G^0}$ are given in [Theorem 4.1](#).*
- (b) *$(G^0, V|_{G^0}) \cong (G_1 \otimes G_1, W \boxtimes W)$ where $G_1 \in \{\mathrm{SL}_m(k), \mathrm{GL}_m(k)\}$ and where W is the m -dimensional standard representation or its dual. Here G flips the two factors so that $G \cong G^0 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\epsilon = -1$.*
- (c) *There exists an embedding $G \hookrightarrow \mathrm{GO}_4(k)$ such that V is the restriction of the four-dimensional orthogonal standard representation, and $\epsilon = +1$.*
- (d) *The representation V is 2-homogeneous monomial; then $\epsilon = -1$ unless V has (nonsuper) dimension $\dim_k(V) \leq 2$.*
- (e) *The group $G = Z(G) \cdot S$ is a (not necessarily direct, but commuting) product of its center and some finite subgroup $S \subseteq G$. Furthermore we have an exact sequence $0 \rightarrow H \rightarrow S \rightarrow \mathrm{Out}(H)$ where*
- (e₁) *either H is quasisimple,*
 - (e₂) *or $(H, V|_H) \cong (G_1 \otimes G_1, W \boxtimes W)$ for some very small $W \in \mathrm{Rep}_k(G_1)$, in which case S flips the two factors and $\epsilon = -1$,*
 - (e₃) *or H is a finite p -group for some prime p and contains a G -stable extraspecial subgroup E of order p^{2n+1} for some $n \in \mathbb{N}$. In this case $V|_E$ is irreducible with dimension p^n .*

By definition of the symmetry constraint, the parity flip $W = \Pi V$ with $W_0 = V_1$ and $W_1 = V_0$ satisfies $S^2(W) = \Lambda^2(V)$ and $\Lambda^2(W) = S^2(V)$. This parity flip changes the sign of the super dimension; since the super dimension of an irreducible representation of a reductive super group is always nonzero [[Weissauer 2009](#), Lemma 15], this explains why we assumed $d > 0$ in [Theorem 1.1](#).

Note that for any faithful irreducible $V \in \mathrm{Rep}_k(G)$, Schur's lemma implies that the center $Z(G)$ acts on V via scalar matrices. So either $Z(G) = \mathbb{G}_m$ or $Z(G)$ is a finite cyclic group. If the restriction $V|_{G^0}$ to the connected component remains irreducible, then the conclusion of Schur's lemma also holds with the center of G replaced by the centralizer $Z_G(G^0) \subseteq G$. Thus in the situation of case (a) the group of connected components is easily controlled since $G/(G^0 \cdot Z_G(G^0)) \hookrightarrow \mathrm{Out}(G^0)$ must be a subgroup of outer automorphisms fixing the isomorphism type of the representation $V|_{G^0}$ in the table of [Theorem 4.1](#).

For the converse of [Theorem 1.1](#) one readily checks that all representations V in case (a), (b), (e₂) are small. Concerning (c), recall that the group of orthogonal similitudes $\mathrm{GO}_4(k)$ is the product of its center with $\mathrm{GSO}_4(k) \cong \mathrm{GL}_2(k) \otimes \mathrm{GL}_2(k)$, and that for the latter any small representation must be a product of two very small ones. As a typical example of (d), for any 2-homogeneous subgroup F of the symmetric group \mathfrak{S}_d we have the 2-homogeneous monomial small representation

of $G = (\mathbb{G}_m)^d \rtimes F$ on $V = k^d$ with the natural action. Apart from a single extra case, the 2-homogeneous permutation groups on $d \geq 4$ letters are precisely the doubly transitive ones [Kantor 1969, Proposition 3.1; 1972], and the finite doubly transitive groups have been classified by Huppert, Hering and others [Dixon and Mortimer 1996, Section 7.7]. In the extraspecial case (e_3) the analysis of the smallness condition is more subtle and we postpone it to the remarks after the proof of Proposition 3.1. Thus altogether Theorem 1.1 gives an essentially complete picture except for the case (e_1) of finite quasisimple groups, which would require a close analysis of the representations of finite groups of Lie type generalizing the methods of Guralnick, Magaard, Malle and Tiep.

For the sake of brevity, in what follows the term *group* will always be taken to include super groups. However, until Section 4 the term *dimension* will still refer to the ordinary dimension (as opposed to the super dimension).

2. Clifford–Mackey theory

Let us say that $V \in \text{Rep}_k(G)$ is *strongly irreducible* if for any noncentral normal subgroup $H \trianglelefteq G$ of finite index the restriction $V|_H$ is irreducible.

Proposition 2.1. *For any faithful ϵ -small representation $V \in \text{Rep}_k(G)$ one of the following cases occurs:*

- (a) *The representation V is strongly irreducible.*
- (b) *V is a 2-homogeneous monomial representation. In this case $\epsilon = -1$ or V has dimension $\dim_k(V) \leq 2$.*
- (c) *There exists an embedding $G \hookrightarrow \text{GO}_4(k)$ such that V is the restriction of the four-dimensional orthogonal standard representation.*

Proof. Let $H \trianglelefteq G$ be a normal subgroup. If the restriction $V|_H$ is not isotypic, let $V|_H = W_1 \oplus \cdots \oplus W_n$ be its isotypic decomposition. Then $V \cong \text{Ind}_{H_1}^G(W_1)$ is induced from a representation of the stabilizer $H_1 \leq G$ of W_1 , and we get a splitting into two G -stable summands

$$T_\epsilon(V) \cong \text{Ind}_{H_1}^G(T_\epsilon(W_1)) \oplus \left[\bigoplus_{i \neq j} W_i \otimes W_j \right]_\epsilon,$$

where the subscript ϵ in the second summand indicates the ϵ -eigenspace of the symmetry constraint which flips the two factors of the tensor product. Since in the nonisotypic case we have $n > 1$, ϵ -smallness implies that $\dim_k(W_1) = 1$, and $\epsilon = -1$ or $\dim_k(V) = n = 2$. All W_i have dimension one, so $V|_H$ splits as a sum of pairwise distinct characters. Now G acts by conjugation on the set X of these characters, and the kernel K of this permutation representation of G is a normal subgroup which is abelian since V is faithful. So (b) holds.

Now suppose that $V|_H$ is isotypic. Then, as in [Dornhoff 1971, Theorem 25.9], there are projective representations U_1, U_2 of G such that $V \cong U_1 \otimes U_2$, where the restriction $U_1|_H$ is irreducible and where every $h \in H$ acts as the identity on U_2 . Then

$$T_{\pm}(V) \cong (T_+(U_1) \otimes T_{\pm}(U_2)) \oplus (T_-(U_1) \otimes T_{\mp}(U_2)),$$

and since V is small, one of the summands $T_{\epsilon_1}(U_1) \otimes T_{\epsilon_2}(U_2)$ must have dimension at most one. By direct inspection this can happen only if either $d_i = \dim_k(U_i) = 1$ for some $i \in \{1, 2\}$, or $d_1 = d_2 = 2$. Now $V|_H \cong U_1 \oplus \cdots \oplus U_1 = d_2 \cdot U_1$ so that for $d_1 = 1$ the group H is contained in the center $Z(G)$, which acts on V via scalar matrices. For $d_2 = 1$ the restriction $V|_H$ remains irreducible. For $d_1 = d_2 = 2$ case (c) occurs since $U_1, U_2 \in \text{Rep}_k(H)$ extend to projective representations of the whole group G whose image then is contained in the product of its center with the special orthogonal similitude group $\text{GL}_2(k) \otimes \text{GL}_2(k) \cong \text{GSO}_4(k)$. \square

3. Reduction to the quasisimple case

Next we study strongly irreducible $V \in \text{Rep}_k(G)$. To treat the case of finite groups simultaneously with the case of positive-dimensional reductive groups, recall from [Aschbacher 2000, Section 31] that for finite groups S the *generalized Fitting subgroup* $F^*(S)$ plays a role very similar to the one which for a reductive algebraic group is played by the derived group of the connected component. By definition $F^*(S) \leq S$ is the subgroup of S generated by the largest nilpotent normal subgroup together with the subnormal quasisimple subgroups. Here a subgroup $N \leq S$ is called *subnormal* if there is a chain $N = N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_m = S$ of subgroups where each member of the chain is a normal subgroup of the next member. To make the role of the generalized Fitting subgroup more precise, let us temporarily call a group *basic* if it is either quasisimple or a finite p -group for some prime p . For a given group G we define $H \trianglelefteq G$ as follows:

- If $G^0 \subseteq Z(G)$, then $G = Z(G) \cdot S$ for some finite normal subgroup $S \trianglelefteq G$, and fixing such a subgroup we take $H = F^*(S)$.
- Otherwise we take $H = [G^0, G^0]$ to be the derived group of the connected component. The theory of reductive groups then implies $G^0 = Z(G^0) \cdot H$.

In both cases we can find a central isogeny $\tilde{H} = H_1 \times \cdots \times H_n \twoheadrightarrow H$ such that the image of each H_i is normal in G . Choosing the labeling in a suitable way, we may furthermore assume that for each i we have a central isogeny $\tilde{H}_i = (G_i)^{s_i} \twoheadrightarrow H_i$ for s_i copies of a suitable basic group G_i and that the images of these s_i copies are permuted transitively by the adjoint action of G .

Proposition 3.1. *For any faithful ϵ -small strongly irreducible $V \in \text{Rep}_k(G)$ with dimension $\dim_k(V) > 1$ one of the following cases occurs:*

- (a) *The group H is quasisimple.*
- (b) *$(H, V|_H) \cong (G_1 \otimes G_1, W \boxtimes W)$ for some very small $W \in \text{Rep}_k(G_1)$, H flips the two factors, and we have $\epsilon = -1$.*
- (c) *H contains an extraspecial G -stable subgroup E of order p^{2n+1} for some prime p such that $V|_E$ is irreducible of dimension p^n .*
- (d) *We have an embedding $G \hookrightarrow \text{GO}_4(k)$ such that V is the restriction of the four-dimensional standard representation.*

Proof. We first claim that $H \not\subseteq Z(G)$. Indeed, for the finite group case recall that the generalized Fitting subgroup contains its own centralizer [Aschbacher 2000], so $H \subseteq Z(G)$ would imply $S = H$ and then $G = Z(G)$ would be abelian. In the infinite case where G^0 is not central, the strong irreducibility implies that $V|_{G^0}$ is irreducible so that the connected reductive group G^0 cannot be a torus. Thus indeed $H \not\subseteq Z(G)$.

Hence we can assume that the image of each H_i in G is a *noncentral* subgroup by discarding any occurring central components and saturating the other components with the center. Since $V|_{\tilde{H}} \cong U_1 \boxtimes \cdots \boxtimes U_n$ with irreducible $U_i \in \text{Rep}_k(H_i)$, we get $n = 1$ by strong irreducibility. Hence $\tilde{H} \cong (G_1)^s$ for $s = s_1$ and again we get a decomposition $V|_{\tilde{H}} \cong W_1 \boxtimes \cdots \boxtimes W_s$ with irreducible $W_i \in \text{Rep}_k(G_1)$, but now the adjoint action of G permutes the s factors G_1 transitively so that all W_i are isomorphic to a single $W \in \text{Rep}_k(G_1)$. In the decomposition

$$T_\epsilon(V)|_H \cong \bigoplus_{r=0}^s T_{r,\epsilon} \quad \text{with } T_{r,\epsilon} = \bigoplus_{\substack{\epsilon_1 \cdots \epsilon_s = \epsilon \\ \#\{i|\epsilon_i = +1\} = r}} T_{\epsilon_1}(W) \boxtimes \cdots \boxtimes T_{\epsilon_s}(W)$$

each summand $T_{r,\epsilon}$ is stable under the action of G . By smallness it then follows that $s \leq 2$, and for $s = 2$ the conclusions of (b) or (d) hold.

So we may assume $s = 1$ and $H = G_1$ is a basic group. If case (a) does not occur, then H is a finite p -group for some prime p . Consider then a minimal G -stable noncentral subgroup $M \trianglelefteq H$. By minimality the subgroup $[M, M]$ is contained in $A := M \cap Z(H)$ so that the quotient $U := M/A$ is abelian. Looking at the p -torsion part of this quotient one obtains, again by minimality, that U is elementary abelian. The commutator induces a bilinear map $[\cdot, \cdot] : U \times U \rightarrow A$, and if we identify A with a subgroup of \mathbb{G}_m via Schur’s lemma, $p \cdot U = 0$ implies that $[M, M]$ is contained in the subgroup $\mu_p \subseteq A$ of p -th roots of unity. So M/μ_p is abelian and in fact elementary abelian: Otherwise by minimality its p -torsion subgroup would lie in the cyclic group A/μ_p so that the abelian p -group M/μ_p would be cyclic as well. But then M would be abelian, and this is impossible since it admits the faithful irreducible representation $V|_M$ of dimension $d > 1$.

Thus M/μ_p is elementary abelian, and we claim that the extraspecial case (c) occurs. Indeed, either $A = \mu_p$ or $A = \mu_{p^2}$. For $A = \mu_p$ the subgroup $E = M$ satisfies our requirements, so suppose that $A = \mu_{p^2}$. Since M/μ_p is elementary abelian, the Frattini subgroup is $\Phi(M) = \mu_p$ by [Aschbacher 2000, (23.2)]. The Frattini subgroup is the intersection of all maximal subgroups, so it follows that there exists a maximal subgroup $E \leq M$ which contains μ_p but not μ_{p^2} . Then $M = \mu_{p^2} \cdot E$, and $E \leq M$ is an extraspecial subgroup. We will be done if we can show this subgroup is stable under the group $\text{Aut}_A(M)$ of automorphisms of M that are trivial on A . But this follows from the observation that every automorphism of E which is trivial on μ_p extends uniquely to an element of $\text{Aut}_A(M)$, which gives a natural identification $\text{Aut}_A(M) \cong \text{Aut}_{\mu_p}(E)$ compatible with the actions on M and E . \square

We remark that the only instance of case (b) in Proposition 3.1 with $\dim(H) > 0$ is $G_1 \cong \text{SL}_m(k)$, acting on $W \cong k^m$ either via the standard representation or via its dual. Indeed this will follow from Theorem 4.1 below, applied to the very small representation W of the Lie algebra of G_1 . Alternatively one could use [Guralnick and Tiep 2005].

In case (c) where H contains a G -stable extraspecial p -group E , put $|E| = p^{1+2n}$ with $n \in \mathbb{N}$. For any nontrivial character $\omega : Z(E) \cong \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{G}_m$ there exists a unique irreducible representation $V_\omega \in \text{Rep}_k(E)$ of dimension p^n on which $Z(E)$ acts via the character ω , and these are already all the irreducible representations of dimension > 1 by [Dornhoff 1971, Theorem 31.5]. Hence in case (c) we have $V|_E \cong V_\omega$ for a uniquely determined character ω . To decide which of the occurring representations are small, note that for the finite group S such that $H = F^*(S)$, we have a natural homomorphism $S \rightarrow \text{Out}(E)$. We now distinguish two cases depending on p .

For $p > 2$ we have $\omega^2 \neq \mathbf{1}$, so $T_\epsilon(V)|_E$ is an isotypic multiple of V_{ω^2} . Then Mackey theory [Dornhoff 1971, Theorem 25.9] gives a tensor product decomposition $T_\epsilon(V) \cong U \otimes W_\epsilon$ where U and W_ϵ are projective representations of the group S such that $U|_E \cong V_{\omega^2}$ and such that every element of E acts trivially on W_ϵ . Via the nondegenerate alternating bilinear form defined by the commutator on $E/Z(E) \cong (\mathbb{F}_p)^{2n}$ we can identify the image of S in $\text{Out}(E)$ with a subgroup of the symplectic group $\text{Sp}_{2n}(\mathbb{F}_p)$. Looking at dimensions one then obtains from [Tiep and Zalesskii 1996, Theorem 5.2] that W_ϵ must be one of the two Weil representations of dimension $(p^n + \epsilon)/2$. Hence V is ϵ -small if and only if the image of S inside $\text{Sp}_{2n}(\mathbb{F}_p)$ acts irreducibly on this Weil representation.

For $p = 2$ on the other hand, $\omega^2 = \mathbf{1}$, so that the restriction $T_\epsilon(V)|_E$ is a sum of characters. By [Winter 1972] we can identify $\text{Out}(E)$ with an orthogonal group $O_{2n}^\pm(\mathbb{F}_2)$ where the type \pm of the quadratic form depends on E . Recall that a nondegenerate quadratic form on $(\mathbb{F}_2)^{2n}$ has type \pm if and only if there

are precisely $2^{n-1}(2^n \pm 1)$ isotropic vectors for this form. One then obtains the following identifications:

- If the quadratic form has $+$ type, the isotropic vectors in $(\mathbb{F}_2)^{2n}$ correspond precisely to the characters in $T_+(V)|_E$.
- If the quadratic form has $-$ type, the isotropic vectors in $(\mathbb{F}_2)^{2n}$ correspond precisely to the characters in $T_-(V)|_E$.

A similar interpretation holds for the anisotropic vectors. Hence it follows that V is small if and only if the image of S inside $O_{2n}^\pm(\mathbb{F}_2)$ acts transitively on the nonzero isotropic resp. anisotropic vectors. Note that the set of isotropic vectors always includes the zero vector as a single orbit, corresponding to the trivial summand $\mathbf{1} \hookrightarrow T_\epsilon(V)$.

4. Lie super algebras

It remains to determine all small $V \in \text{Rep}_k(G)$ when $H = [G^0, G^0]$ is quasisimple and $V|_H$ is irreducible. By the classification of reductive super groups in [Weissauer 2009], the Lie super algebra \mathfrak{g} of H must then either be an ordinary simple Lie algebra or an orthosymplectic Lie super algebra $\mathfrak{osp}_{1|2m}(k)$ with $m \in \mathbb{N}$. Note that $\text{Rep}_k(H)$ is a full subcategory of $\text{Rep}_k(\mathfrak{g})$, where the latter denotes the category of all Lie algebra representations of the Lie super algebra \mathfrak{g} on finite-dimensional super vector spaces over k . In particular $V|_H$ defines an irreducible representation of \mathfrak{g} .

The passage to representations of Lie algebras leads to a seemingly weaker notion of smallness. By the comments after [Theorem 1.1](#) we know that $G/(G^0 \cdot Z_G(G^0))$ is a subgroup of $\text{Out}(G^0)$ such that conjugation by any element φ of this subgroup fixes the isomorphism type of $V|_H$. For an irreducible summand $W \hookrightarrow T_\epsilon(V)$ in $\text{Rep}_k(G)$ it may happen that the restriction $W|_H$ splits into several irreducible summands, but all these summands must be conjugate via automorphisms φ as above. Abstracting from this situation, let us now denote by \mathfrak{g} any ordinary simple Lie algebra or $\mathfrak{osp}_{1|2m}(k)$ with $m \in \mathbb{N}$. We say that a representation $V \in \text{Rep}_k(\mathfrak{g})$ is ϵ -small if either $T_\epsilon(V) \cong W$ or $T_\epsilon(V) \cong W \oplus \mathbf{1}$, where W is a sum of irreducible representations which are all conjugate to each other via automorphisms $\varphi \in \text{Aut}(\mathfrak{g})$ fixing the isomorphism type of V . To finish the proof of [Theorem 1.1](#) we classify all irreducible small representations in this sense. For a uniform treatment the terms dimension, vector space, trace and Lie algebra will from now on be taken in the super sense for $\mathfrak{osp}_{1|2m}(k)$ but in the ordinary sense otherwise.

We denote by $\varpi_1, \dots, \varpi_m$ the fundamental dominant weights of \mathfrak{g} with respect to some fixed system of simple positive roots; see [[Rittenberg and Scheunert 1982](#), Section 2.1] for the orthosymplectic Lie algebra $\mathfrak{g} = \mathfrak{osp}_{1|2m}(k)$ whose Dynkin type

we abbreviate by BC_m . Put

$$\beta_i = \begin{cases} 2\varpi_m & \text{if } \mathfrak{g} = \mathfrak{osp}_{1|2m}(k) \text{ and } i = m, \\ \varpi_i & \text{otherwise.} \end{cases}$$

The irreducible finite-dimensional representations of \mathfrak{g} are parametrized by highest weights $\lambda = \sum_{i=1}^m a_i \beta_i$ with $a_i \in \mathbb{N}_0$, see [Djoković 1976b, Theorem 6]. For any such λ we denote by V_λ the associated positive-dimensional irreducible representation. Note that, in the super case, negative-dimensional irreducible representations are obtained by the parity flip $W_\lambda = \Pi V_\lambda$ with $\dim(W_\lambda) = -\dim(V_\lambda)$ and $S^2(W_\lambda) \cong \Lambda^2(V_\lambda)$.

Theorem 4.1. *A positive-dimensional irreducible representation $V_\lambda \in \text{Rep}_k(\mathfrak{g})$ is ϵ -small if and only if its highest weight λ appears in the following table:*

		λ	$\epsilon = +1$	$\epsilon = -1$
A_m	$m \geq 1$	β_1, β_m	★	★
	$m = 1$	$2\beta_1$	○	★
		$3\beta_1$	—	○
	$m \geq 2$	$2\beta_1, 2\beta_m$	—	★
	$m = 3$	β_2	○	★
	$m \geq 4$	β_2, β_{m-1}	—	★
	$m = 5$	β_3	—	○
B_m	$m \geq 2$	β_1	○	★
	$m = 2$	β_2	★	○
	$m = 3$	β_3	○	—
C_m	$m \geq 3$	β_1	★	○
	$m = 3$	β_3	—	○
D_m	$m \geq 4$	β_1	○	★
	$m = 4$	β_3, β_4	○	★
	$m = 5$	β_4, β_5	—	★
	$m = 6$	β_5, β_6	—	○
BC_m	$m \geq 1$	β_1	★	○
E_6		β_1, β_6	—	★
E_7		β_7	—	○
G_2		β_1	○	—

Here the label ★ means that $T_\epsilon(V_\lambda)$ is irreducible, ○ means that $T_\epsilon(V_\lambda) = W \oplus 1$ with W irreducible, and — means that V_λ is not ϵ -small.

Note that for $\mathfrak{g} = \mathfrak{sl}_2(k)$ with its two-dimensional standard representation st , any irreducible representation is a symmetric power $V_\lambda = S^n(st)$ of weight $\lambda = n\beta_1$ for some $n \in \mathbb{N}$. In this case Theorem 4.1 holds by direct inspection. A similar

argument also works for $\mathfrak{g} = \mathfrak{osp}_{1|2}(k)$. Here we know from [Djoković 1976b, Theorems 7 and 11] that for $\lambda = n\beta_1$ the even subalgebra $\mathfrak{g}_0 = \mathfrak{sl}_2(k) \subset \mathfrak{g}$ acts on $V_\lambda = V_0 \oplus V_1$ via $V_0 = S^n(st)$ and $V_1 = S^{n-1}(st)$. A short computation yields the action on the even and odd parts of the tensor square $T_\epsilon(V)$ and Theorem 4.1 also holds in this case. Note that $\dim(V) = 1$ for all irreducible representations V of $\mathfrak{osp}_{1|2}(k)$. For all other cases we have:

Lemma 4.2. *For $\mathfrak{g} \neq \mathfrak{osp}_{1|2}(k)$ one has $\dim(V_\lambda) \leq \dim(\mathfrak{g})$ if and only if the highest weight λ appears among those listed in Tables 1 or 2.*

Proof. See [Andreev et al. 1967] for the ordinary case. For $\mathfrak{g} = \mathfrak{osp}_{1|2m}(k)$ with $m \geq 2$ we use the Kac–Weyl formula in [Tsohantjis and Cornwell 1990, Equation 11]. We embed the root system BC_m into a Euclidean space with standard basis $\epsilon_1, \dots, \epsilon_m$ such that $\beta_i = \epsilon_1 + \dots + \epsilon_i$ for all i . The irreducible representations of $\mathfrak{osp}_{1|2m}(k)$ are parametrized by weights which in our basis are written $\lambda = (\lambda_1, \dots, \lambda_m)$ with integers $\lambda_1 \geq \dots \geq \lambda_m \geq 0$. The Kac–Weyl formula gives

$$\dim(V_\lambda) = \prod_{1 \leq i < j \leq m} \left(\frac{\lambda_i - \lambda_j}{j - i} + 1 \right) \cdot \prod_{1 \leq i < j \leq m} \left(\frac{\lambda_i + \lambda_j}{2m + 1 - i - j} + 1 \right).$$

For $\lambda_1 \geq 2$ the second product is ≥ 2 . Then the classical Weyl formula for the first product shows that $\dim(V_\lambda)$ is at least twice the dimension of the irreducible representation of $\mathfrak{sl}_m(k)$ with highest weight $\mu = (\lambda_1 - \lambda_m, \dots, \lambda_{m-1} - \lambda_m)$. Using that $\dim(\mathfrak{sl}_m(k)) \geq 2 \dim(\mathfrak{osp}_{1|2m}(k))$, it follows that μ is in the list for A_{m-1} in Table 1. Since $\lambda = \mu + \lambda_m \cdot \beta_m$ and since increasing the weight by β_m increases the dimension, this leaves only finitely many cases. For $\lambda_1 = 1$ we have $\lambda = \beta_r$

		λ	$S^2(V_\lambda)$	$\Lambda^2(V_\lambda)$
A_m	$m = 1$	β_1	$V_{2\beta_1}$	$\mathbf{1}$
	$m \geq 2$	β_1	$V_{2\beta_1}$	V_{β_2}
		β_m	$V_{2\beta_m}$	$V_{\beta_{m-1}}$
	$m \geq 2$	$2\beta_1$	$V_{4\beta_1} \oplus V_{2\beta_2}$	$V_{2\beta_1 + \beta_2}$
		$2\beta_m$	$V_{4\beta_m} \oplus V_{2\beta_{m-1}}$	$V_{2\beta_m + \beta_{m-1}}$
	$m \geq 4$	β_2	$V_{2\beta_2} \oplus V_{\beta_4}$	$V_{\beta_1 + \beta_3}$
		β_{m-1}	$V_{2\beta_{m-1}} \oplus V_{\beta_{m-3}}$	$V_{\beta_m + \beta_{m-2}}$
	$m = 3$	β_2	$V_{2\beta_2} \oplus \mathbf{1}$	$V_{\beta_1 + \beta_3}$
	$m = 5$	β_3	$V_{2\beta_3} \oplus V_{\beta_1 + \beta_5}$	$V_{\beta_2 + \beta_4} \oplus \mathbf{1}$
	$m =$	β_3	$V_{2\beta_3} \oplus V_{\beta_1 + \beta_5}$	$V_{\beta_2 + \beta_4} \oplus V_{\beta_6}$
$6, 7$	β_{m-2}	$V_{2\beta_{m-2}} \oplus V_{\beta_m + \beta_{m-4}}$	$V_{\beta_{m-1} + \beta_{m-3}} \oplus V_{\beta_{m-5}}$	

Table 1. All λ with $1 < \dim(V_\lambda) < \dim(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{osp}_{1|2m}(k)$ we denote by $W_\mu = \Pi V_\mu$ the parity shifts of the highest weight modules. (Continues on the next page.)

B_m	$m \geq 2$	β_1	$V_{2\beta_1} \oplus \mathbf{1}$	V_{β_2}
	$m = 2$	β_2	$V_{2\beta_2}$	$V_{\beta_1} \oplus \mathbf{1}$
	$m = 3$	β_3	$V_{2\beta_3} \oplus \mathbf{1}$	$V_{\beta_1} \oplus V_{\beta_2}$
	$m = 4$	β_4	$V_{2\beta_4} \oplus V_{\beta_1} \oplus \mathbf{1}$	$V_{\beta_2} \oplus V_{\beta_3}$
	$m = 5$	β_5	$V_{2\beta_5} \oplus V_{\beta_2} \oplus V_{\beta_1}$	$V_{\beta_3} \oplus V_{\beta_4} \oplus \mathbf{1}$
	$m = 6$	β_6	$V_{2\beta_6} \oplus V_{\beta_3} \oplus V_{\beta_2}$	$V_{\beta_1} \oplus V_{\beta_4} \oplus V_{\beta_5} \oplus \mathbf{1}$
C_m	$m \geq 3$	β_1	$V_{2\beta_1}$	$V_{\beta_2} \oplus \mathbf{1}$
	$m \geq 4$	β_2	$V_{\beta_4} \oplus V_{2\beta_2} \oplus V_{\beta_2} \oplus \mathbf{1}$	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3}$
	$m = 3$	β_2	$V_{2\beta_2} \oplus V_{\beta_2} \oplus \mathbf{1}$	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3}$
		β_3	$V_{2\beta_3} \oplus V_{2\beta_1}$	$V_{2\beta_2} \oplus \mathbf{1}$
D_m	$m \geq 4$	β_1	$V_{2\beta_1} \oplus \mathbf{1}$	V_{β_2}
	$m = 4$	β_3	$V_{2\beta_3} \oplus \mathbf{1}$	V_{β_2}
		β_4	$V_{2\beta_4} \oplus \mathbf{1}$	V_{β_2}
		β_4	$V_{2\beta_4} \oplus V_{\beta_1}$	V_{β_3}
	$m = 5$	β_5	$V_{2\beta_5} \oplus V_{\beta_1}$	V_{β_3}
		β_5	$V_{2\beta_5} \oplus V_{\beta_2}$	$V_{\beta_4} \oplus \mathbf{1}$
	$m = 6$	β_6	$V_{2\beta_6} \oplus V_{\beta_2}$	$V_{\beta_4} \oplus \mathbf{1}$
β_6		$V_{2\beta_6} \oplus V_{\beta_3}$	$V_{\beta_1} \oplus V_{\beta_5}$	
$m = 7$	β_7	$V_{2\beta_7} \oplus V_{\beta_3}$	$V_{\beta_1} \oplus V_{\beta_5}$	
BC_m	$m \geq 2$	β_1	$V_{2\beta_1}$	$V_{\beta_2} \oplus \mathbf{1}$
	$m \geq 4$	β_2	$V_{2\beta_2} \oplus V_{\beta_2} \oplus V_{\beta_4} \oplus \mathbf{1}$	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3}$
	$m = 2$	$\beta_1 + \beta_2$	$V_{2\beta_1+2\beta_2} \oplus 2V_{2\beta_1+\beta_2} \oplus 2V_{2\beta_1}$ $\oplus W_{3\beta_1} \oplus W_{\beta_1+2\beta_2}$ $\oplus 2W_{\beta_1+\beta_2}$	$V_{4\beta_1} \oplus V_{2\beta_1+\beta_2} \oplus V_{3\beta_2} \oplus 2V_{2\beta_2}$ $\oplus 2V_{\beta_2} \oplus W_{3\beta_1+\beta_2} \oplus W_{\beta_1+2\beta_2}$ $\oplus W_{\beta_1+\beta_2} \oplus W_{\beta_1} \oplus \mathbf{1}$
	$m = 3$	β_2	$V_{2\beta_2} \oplus V_{\beta_2} \oplus W_{\beta_1}$	$V_{2\beta_1} \oplus W_{\beta_1+\beta_2} \oplus \mathbf{1}$
		β_2	$V_{2\beta_2} \oplus V_{\beta_2} \oplus W_{\beta_3}$	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3} \oplus \mathbf{1}$
		β_3	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3} \oplus V_{2\beta_3}$ $\oplus W_{\beta_1+\beta_2}$	$V_{2\beta_2} \oplus V_{\beta_2}$ $\oplus W_{\beta_1} \oplus W_{\beta_2+\beta_3} \oplus W_{\beta_3} \oplus \mathbf{1}$
	$m = 4$	β_4	$V_{2\beta_1} \oplus V_{\beta_2+\beta_4} \oplus V_{\beta_2} \oplus V_{2\beta_4}$ $\oplus V_{\beta_4} \oplus W_{\beta_1} \oplus W_{\beta_2+\beta_3}$ $\oplus W_{\beta_3} \oplus \mathbf{1}$	$V_{2\beta_2} \oplus V_{\beta_1+\beta_3} \oplus V_{2\beta_3}$ $\oplus W_{\beta_1+\beta_2} \oplus W_{\beta_1+\beta_4}$ $\oplus W_{\beta_3+\beta_4}$
	$m = 5$	β_5	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3} \oplus V_{\beta_1+\beta_5}$ $\oplus V_{2\beta_3} \oplus V_{\beta_3+\beta_5} \oplus V_{2\beta_5}$ $\oplus W_{\beta_1+\beta_2} \oplus W_{\beta_1+\beta_4}$ $\oplus W_{\beta_3+\beta_4}$	$V_{2\beta_2} \oplus V_{\beta_2+\beta_4} \oplus V_{\beta_2} \oplus V_{2\beta_4}$ $\oplus V_{\beta_4} \oplus W_{\beta_1} \oplus W_{\beta_2+\beta_3}$ $\oplus W_{\beta_2+\beta_5} \oplus W_{\beta_3}$ $\oplus W_{\beta_4+\beta_5} \oplus W_{\beta_5} \oplus \mathbf{1}$
E_6	β_1	$V_{2\beta_1} \oplus V_{\beta_6}$	V_{β_3}	
	β_6	$V_{2\beta_6} \oplus V_{\beta_1}$	V_{β_5}	
E_7	β_7	$V_{2\beta_7} \oplus V_{\beta_1}$	$V_{\beta_6} \oplus \mathbf{1}$	
F_4	β_4	$V_{2\beta_4} \oplus V_{\beta_4} \oplus \mathbf{1}$	$V_{\beta_3} \oplus V_{\beta_1}$	
G_2	β_1	$V_{2\beta_1} \oplus \mathbf{1}$	$V_{\beta_1} \oplus V_{\beta_2}$	

Table 1 (continued).

		λ	$S^2(V_\lambda)$	$\Lambda^2(V_\lambda)$
A_m	$m = 1$	$2\beta_1$	$V_{4\beta_1} \oplus \mathbf{1}$	$V_{2\beta_1}$
	$m = 2$	$\beta_1 + \beta_2$	$V_{2\beta_1+2\beta_2} \oplus V_{\beta_1+\beta_2} \oplus \mathbf{1}$	$V_{3\beta_1} \oplus V_{3\beta_2} \oplus V_{\beta_1+\beta_2}$
	$m \geq 2$	$\beta_1 + \beta_m$	$V_{2\beta_1+2\beta_m} \oplus V_{\beta_2+\beta_{m-1}} \oplus V_{\beta_1+\beta_m} \oplus \mathbf{1}$	$V_{\beta_2+2\beta_m} \oplus V_{2\beta_1+\beta_{m-1}} \oplus V_{\beta_1+\beta_m}$
B_m	$m = 2$	$2\beta_2$	$V_{\beta_1} \oplus V_{2\beta_1} \oplus V_{4\beta_2} \oplus \mathbf{1}$	$V_{\beta_1+2\beta_2} \oplus V_{2\beta_2}$
	$m = 3$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{2\beta_3} \oplus \mathbf{1}$	$V_{\beta_1+2\beta_3} \oplus V_{\beta_2}$
	$m = 4$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{2\beta_4} \oplus \mathbf{1}$	$V_{\beta_1+\beta_3} \oplus V_{\beta_2}$
	$m \geq 5$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{\beta_4} \oplus \mathbf{1}$	$V_{\beta_1+\beta_3} \oplus V_{\beta_2}$
C_m	$m \geq 3$	$2\beta_1$	$V_{4\beta_1} \oplus V_{2\beta_2} \oplus V_{\beta_2} \oplus \mathbf{1}$	$V_{2\beta_1} \oplus V_{2\beta_1+\beta_2}$
D_m	$m = 4$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{2\beta_3} \oplus V_{2\beta_4} \oplus \mathbf{1}$	$V_{\beta_2} \oplus V_{\beta_1+\beta_3+\beta_4}$
	$m = 5$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{\beta_4+\beta_5} \oplus \mathbf{1}$	$V_{\beta_2} \oplus V_{\beta_1+\beta_3}$
	$m \geq 6$	β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus V_{\beta_4} \oplus \mathbf{1}$	$V_{\beta_2} \oplus V_{\beta_1+\beta_3}$
BC_m	$m \geq 2$	$2\beta_1$	$V_{4\beta_1} \oplus V_{2\beta_2} \oplus V_{\beta_2} \oplus \mathbf{1}$	$V_{2\beta_1+\beta_2} \oplus V_{2\beta_1}$
	$m = 4$	β_3	$V_{2\beta_1} \oplus V_{\beta_1+\beta_3} \oplus V_{2\beta_3} \oplus W_{\beta_1+\beta_4}$	$V_{2\beta_2} \oplus V_{\beta_2} \oplus V_{\beta_2+\beta_4} \oplus V_{\beta_4} \oplus W_{\beta_3} \oplus \mathbf{1}$
E_6		β_2	$V_{2\beta_2} \oplus V_{\beta_1+\beta_6} \oplus \mathbf{1}$	$V_{\beta_2} \oplus V_{\beta_4}$
E_7		β_1	$V_{2\beta_1} \oplus V_{\beta_6} \oplus \mathbf{1}$	$V_{\beta_1} \oplus V_{\beta_3}$
E_8		β_8	$V_{\beta_1} \oplus V_{2\beta_8} \oplus \mathbf{1}$	$V_{\beta_7} \oplus V_{\beta_8}$
F_4		β_1	$V_{2\beta_1} \oplus V_{2\beta_4} \oplus \mathbf{1}$	$V_{\beta_1} \oplus V_{\beta_2}$
G_2		β_2	$V_{2\beta_1} \oplus V_{2\beta_2} \oplus \mathbf{1}$	$V_{3\beta_1} \oplus V_{\beta_2}$

Table 2. All λ with $1 < \dim(V_\lambda) = \dim(\mathfrak{g})$. For the ordinary simple Lie algebras precisely the adjoint representations occur by the result of [Andreev et al. 1967].

with $r \leq m$, and $\dim(V_\lambda) = \binom{2m}{r} - \binom{2m}{r-1}$ by the description in [Djoković 1976b, Section 5]. \square

Corollary 4.3. For $\mathfrak{g} \neq \mathfrak{sl}_2(k)$, $\mathfrak{osp}_{1|2}(k)$ and all weights λ one has $\dim(V_\lambda) \geq 2$, with equality holding only in the single case $(\mathfrak{g}, \lambda) = (\mathfrak{osp}_{1|4}(k), \beta_2)$.

5. Proof of Theorem 4.1

Recall that \mathfrak{g} admits a unique invariant nondegenerate bilinear form (\cdot, \cdot) up to multiplication by a scalar [Scheunert 1979, p. 94]. Fixing any such form, we associate to any root α a coroot $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Let $\alpha_1, \dots, \alpha_m$ be a system of simple positive roots so that the fundamental weights ϖ_i satisfy $(\alpha_i^\vee, \varpi_j) = \delta_{ij}$. Then $\rho = \varpi_1 + \dots + \varpi_m$ is half the sum of all positive roots, with the sign convention of [Tsohantjis and Cornwell 1990]. For the proof of Theorem 4.1 we consider the index of a representation $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, i.e., the scalar $l(V)$ defined by $\text{tr}(\varphi(X) \circ \varphi(Y)) = l(V) \cdot (X, Y)$.

Lemma 5.1. *The index has the following properties.*

- (a) *For the symmetric or alternating square of a representation V it is given by the formula $l(T_\epsilon(V)) = (\dim(V) + 2\epsilon) \cdot l(V)$.*
- (b) *There exists a constant $\kappa \neq 0$ such that $\kappa \cdot l(V_\mu) = \dim(V_\mu) \cdot c(\mu)$ for the scalar $c(\mu) = (\mu, \mu) + 2(\mu, \rho) > 0$ and for any highest weight $\mu \neq 0$.*
- (c) *The index satisfies $l(\mathbf{1}) = 0$, and it is invariant under automorphisms and additive for direct sums in the sense that $l(V \oplus V') = l(V) + l(V')$.*

Proof. For (a) note that upon applying any tensor construction to V the index is multiplied by a constant depending only on $n = \dim(V)$. To compute this constant for $T_\epsilon(V)$, recall from [Scheunert 1979, p. 128] that $\mathfrak{sl}(V)$ is simple for $n \neq 0$. It then only remains to check that $\text{tr}((T_\epsilon(X))^2) = (n + 2\epsilon) \text{tr}(X^2)$ for a suitably chosen elementary matrix $X \in \mathfrak{sl}(V)$. For (b) one checks, by looking at the action on a highest weight vector, that the Casimir operator acts on V_μ by some fixed multiple of $c(\mu)$. The setting for $\mathfrak{osp}_{1|2m}(k)$ is described in [Djoković 1976b, p. 28; 1976a, p. 223]. One then has $\kappa = \dim(\text{Ad}) \cdot c(\text{Ad})$ for the adjoint representation Ad . Part (c) is obvious. \square

Via these index computations, we may now complete the classification of ϵ -small representations for $\mathfrak{g} \neq \mathfrak{sl}_2(k)$, $\mathfrak{osp}_{1|2}(k)$ as follows.

Proof of Theorem 4.1. Suppose that V_λ is ϵ -small. By Corollary 4.3 we may assume that $n = \dim(V_\lambda) > 2$. Put $T_\epsilon(V_\lambda) = W \oplus \mathbf{1}^\delta$ where $\delta \in \{0, 1\}$ denotes the multiplicity with which the trivial representation enters. Note that by smallness all highest weights μ occurring in W are conjugate to each other. For any such μ Lemma 5.1(b)–(c) hence imply that $\kappa \cdot l(W) = \dim(W) \cdot c(\mu) = (n(n + \epsilon)/2 - \delta) \cdot c(\mu)$ and $\kappa \cdot l(V_\lambda) = n \cdot c(\lambda)$. So Lemma 5.1(a) shows

$$(\star) \quad (n + 2\epsilon) \cdot n \cdot c(\lambda) = \frac{1}{2}(n(n + \epsilon) - 2\delta) \cdot c(\mu).$$

Now we distinguish between the symmetric and the alternating square. For $\epsilon = +1$ we may take $\mu = 2\lambda$. Then $c(\mu) = 4|\lambda|^2 + 4(\lambda, \rho)$. Since $c(\lambda) = |\lambda|^2 + 2(\lambda, \rho)$, Equation (\star) easily gives

$$(n - 2\delta) \cdot |\lambda|^2 = 2(\lambda, \rho) \quad \text{and hence} \quad |\lambda| \leq \frac{2|\rho|}{n - 2\delta}$$

by the Cauchy–Schwartz inequality. Let Δ_0 be the set of simple positive roots of the even subalgebra \mathfrak{g}_0 . Then

$$|(\lambda, \alpha^\vee)| \leq |\lambda| \cdot |\alpha^\vee| \leq \frac{2|\rho| |\alpha^\vee|}{n - 2\delta} < \frac{\dim(\mathfrak{g}) - 1}{n - 2\delta} \quad \text{for any } \alpha \in \Delta_0,$$

where for the last inequality we have used the numerical values of $|\rho|^2$ and R in Table 3 and our assumption $\mathfrak{g} \neq \mathfrak{sl}_2(k)$, $\mathfrak{osp}_{1|2}(k)$. On the other hand $(\lambda, \alpha^\vee) \in \mathbb{Z}$

	$ \rho ^2$	R	$\dim(\mathfrak{g})$	r_i for $i = 1, \dots, m$	$ \text{Out}(\mathfrak{g}) $
A_m	$\frac{m(m+1)(m+2)}{12}$	$\sqrt{2}$	$m(m+2)$	$\frac{2(m+1)}{i(m+1-i)}$	2
B_m	$\frac{m(2m-1)(2m+1)}{12}$	2	$m(2m+1)$	$\frac{2}{i}(1 + \delta_{im})$	1
C_m	$\frac{m(m+1)(2m+1)}{6}$	$\sqrt{2}$	$m(2m+1)$	$\frac{2}{i}(1 + \delta_{im})$	1
D_m	$\frac{m(m-1)(2m-1)}{6}$	$\sqrt{2}$	$m(2m-1)$	$\frac{2}{i}$ if $i < m-1$ $\frac{8}{m}$ if $i \in \{m-1, m\}$	2 if $m \neq 4$ 6 if $m = 4$
BC_m	$\frac{m(2m-1)(2m+1)}{12}$	$\sqrt{2}$	$m(2m-1)$	$\frac{2}{i}(1 + \delta_{im})$	1
E_6	78	$\sqrt{2}$	78	$\frac{3}{2}, 1, \frac{3}{5}, \frac{1}{3}, \frac{3}{5}, \frac{3}{2}$	2
E_7	$\frac{399}{2}$	$\sqrt{2}$	133	$1, \frac{4}{7}, \frac{1}{3}, \frac{1}{6}, \frac{4}{15}, \frac{1}{2}, \frac{4}{3}$	1
E_8	620	$\sqrt{2}$	248	$\frac{1}{2}, \frac{1}{4}, \frac{1}{7}, \frac{1}{15}, \frac{1}{10}, \frac{1}{6}, \frac{1}{3}, 1$	1
F_4	39	2	52	$1, \frac{1}{3}, \frac{1}{3}, 1$	1
G_2	14	$\sqrt{2}$	14	1	1

Table 3. Some numerical values. We put $r_i = |\alpha_i|^2/|\beta_i|^2$ and $R = \max_{\alpha \in \Delta_0} |\alpha^\vee|$ for the set Δ_0 of simple positive roots of \mathfrak{g}_0 .

for all $\alpha \in \Delta_0$, and for $\lambda \neq 0$ at least one of these scalar products is nonzero. Thus we can find $\alpha \in \Delta_0$ with $|(\lambda, \alpha^\vee)| \geq 1$. This implies $n - 2\delta < \dim(\mathfrak{g}) - 1$. Hence λ is one of the highest weights in tables 1 and 2 by Lemma 4.2.

It remains to discuss the case $\epsilon = -1$. By smallness all highest weights in $\Lambda^2(V_\lambda)$ are conjugate to each other via automorphisms fixing λ . Hence Remark 5.2 below implies

$$(\star\star) \quad \lambda = r \cdot (\beta_{i_1} + \dots + \beta_{i_s}) \quad \text{for some } r \in \mathbb{N} \text{ and } i_1 < i_2 < \dots < i_s,$$

and that for all $i \in \{i_1, \dots, i_s\}$ the weight $\mu = 2\lambda - \alpha_i$ occurs as a highest weight in $\Lambda^2(V_\lambda)$. In what follows we fix $i \in \{i_1, \dots, i_s\}$ with the smallest norm $|\beta_i|$. Since the norm of any simple positive root is given by the formula $|\alpha_i|^2 = 2(\alpha_i, \rho)$, we have $c(\mu)/2 = c(\lambda) + |\lambda|^2 - 2(\lambda, \alpha_i)$ so that (\star) becomes

$$(n + 2\epsilon) \cdot n \cdot c(\lambda) = (n(n + \epsilon) - 2\delta) \cdot (c(\lambda) + |\lambda|^2 - 2(\lambda, \alpha_i)).$$

Now for $\epsilon = -1$ the first of the two factors on the right is $> (n + 2\epsilon) \cdot n$ since by assumption $n > 2$ and $\delta \in \{0, 1\}$. Hence

$$c(\lambda) + |\lambda|^2 - 2(\lambda, \alpha_i) < c(\lambda)$$

and therefore $2(\lambda, \alpha_i) > |\lambda|^2 \geq r^2 \cdot |\beta_i|^2 \cdot s$, where the second inequality comes from $(\star\star)$ together with the fact that all scalar products between $\beta_{i_1}, \dots, \beta_{i_s}$ are nonnegative and β_i has the smallest norm among all these weights. On the other

hand $2(\lambda, \alpha_i) = r \cdot 2(\beta_i, \alpha_i) = r \cdot |\alpha_i|^2$ by **(★★)**. Hence $r_i := |\alpha_i|^2/|\beta_i|^2 > r \cdot s \geq 1$, which leaves only finitely many cases in view of [Table 3](#). Note that for the Dynkin type A_m we may by duality assume $i < (m + 1)/2$ so that $r_i \leq 4/i$. \square

For the convenience of the reader we include a proof of the following basic fact used in the above argument; see also [\[Aslaksen 1994, Theorem 5\]](#).

Remark 5.2. *Let $\lambda = \sum_{i=1}^m a_i \beta_i$ with $a_i \in \mathbb{N}_0$. If $a_i > 0$, then the weight $2\lambda - \alpha_i$ appears as a highest weight in the alternating tensor square $\Lambda^2(V_\lambda)$.*

Proof. Let v be a highest weight vector of V_λ . For $a_i > 0$ let $X_\pm \in \mathfrak{g}_{\pm\alpha_i}$ be generators for the root spaces of the roots $\pm\alpha_i$ of \mathfrak{g} and put $H = [X_+, X_-]$. It then follows from $X_+v = 0$ that $X_+X_-v = Hv = (\alpha_i, \lambda) \cdot v \neq 0$. Since v and X_-v have different weights (λ and $\lambda - \alpha_i$ respectively), this implies that $v \wedge X_-v \in \Lambda^2(V_\lambda)$ is a nonzero highest weight vector of weight $2\lambda - \alpha_i$. \square

6. An application to Brill–Noether sheaves

In this independent section we briefly discuss an application of [Theorem 1.1](#) to algebraic geometry. Let A be a complex abelian variety, and let $D(A) = D_c^b(A, \mathbb{C})$ denote the derived category of bounded constructible sheaf complexes on A in the sense of [\[Hotta et al. 1995\]](#). For any sheaf complexes $K, L \in D(A)$ we may consider the exterior tensor product

$$K \boxtimes L = p_1^*(K) \otimes_{\mathbb{C}} p_2^*(L) \in D(A \times A),$$

where $p_1, p_2 : A \times A \rightarrow A$ denote the projections onto the two factors and where the tensor product on the right has to be taken in the derived sense. Passing to the direct image under the group law $a : A \times A \rightarrow A$ we then define the convolution product by

$$K * L = Ra_*(K \boxtimes L) \in D(A).$$

It has been shown in [\[Weissauer 2007; 2011\]](#) that with respect to this convolution product the category $D(A)$ is a rigid symmetric monoidal \mathbb{C} -linear category, though it is not abelian but only triangulated. Now for any perverse sheaf $K \in D(A)$ in the sense of [\[Hotta et al. 1995\]](#), the convolution powers of K generate an algebraic tensor category inside a certain natural symmetric monoidal quotient category $\bar{D}(A)$ of $D(A)$; see [\[Krämer and Weissauer 2015\]](#) for details. The Tannaka super group of this tensor category is an ordinary complex algebraic group $G(K)$ which is reductive if the perverse sheaf K is semisimple.

Now consider the special case where $A = \text{Jac}(X)$ is the Albanese variety of a smooth complex projective curve X of genus $g \geq 1$. Fix an embedding $X \hookrightarrow A$, and denote by \mathbb{C}_X the constant sheaf with support on the image curve. It will be more convenient to replace this constant sheaf by the sheaf complex $K = \mathbb{C}_X[1]$ placed

in degree -1 since the degree shift by one leads to a complex which is a perverse sheaf. The group $G(K)$ depends on the chosen embedding $X \hookrightarrow A$, though one may show its commutator group does not. In what follows we choose the embedding so that the highest alternating convolution power $\Lambda^{*(2g-2)}(K)$ is represented in $\overline{D}(A)$ by the skyscraper sheaf $\mathbf{1}$ of rank one supported in the origin. We can achieve this via a suitable translation since by [loc. cit., Proposition 10.1] this alternating power is given in $\overline{D}(A)$ by a skyscraper sheaf of rank one. With this normalization of the embedding, the group $G_X = G(K)$ becomes an intrinsic invariant of X , and for $g > 2$ the classification in Theorem 1.1 leads to a very easy proof of the following result from [Weissauer 2007].

Theorem 6.1. *Let X be a smooth complex projective curve of genus $g \geq 1$ which is embedded into its Jacobian variety $A = \text{Jac}(X)$ as above. Then*

$$G_X = \begin{cases} \text{Sp}_{2g-2}(\mathbb{C}) & \text{if } X \text{ is hyperelliptic,} \\ \text{SL}_{2g-2}(\mathbb{C}) & \text{otherwise.} \end{cases}$$

Proof for $g > 2$. For hyperelliptic curves X the Abel–Jacobi map $f : X^2 \rightarrow A$ is generically finite of degree two over its image, but blows down the hyperelliptic linear series g_2^1 to a point $a \in A(\mathbb{C})$. By our choice of the embedding $X \hookrightarrow A$ we can assume $a = 0$. Then one easily checks that the convolution square of the constant perverse sheaf $K = \mathbb{C}_X[1]$ has the form

$$K * K = Rf_*(\mathbb{C}_{X \times X}[2]) = \delta_+ \oplus \delta_- \oplus \mathbf{1}$$

for certain simple perverse sheaves δ_\pm and the rank one skyscraper sheaf $\mathbf{1}$ with support in the origin. The definition of the symmetry constraint in [Weissauer 2007] shows that $\mathbf{1}$ lies in the alternating convolution square of K . If $G = G_X$ denotes our Tannaka group and if $V \in \text{Rep}_k(G)$ denotes the representation corresponding to the perverse sheaf K , it follows that the symmetric square $T_+(V)$ is irreducible and that $T_-(V)$ decomposes into an irreducible plus a trivial representation.

The ϵ -smallness of V for $\epsilon = +1$ rules out case (b) in Theorem 1.1. Case (d) is ruled out for the same reason because, by [Krämer and Weissauer 2015], the dimension of any representation of G is the Euler characteristic of the underlying perverse sheaf, which in our situation is $d = \dim_{\mathbb{C}}(V) = 2g - 2 > 2$ for $g > 2$. Since $T_+(V)$ is irreducible whereas the symmetric square of the standard representation of the orthogonal group is not, case (c) is impossible. Case (e) is impossible since the group of connected components of the Tannaka group of a perverse sheaf is abelian [Weissauer 2012]. So case (a) occurs, and we look for entries in Theorem 4.1 with a \star for $\epsilon = +1$ and a \circ for $\epsilon = -1$. As we are dealing with ordinary groups, the only case is the standard representation of $\text{Sp}_{2m}(\mathbb{C})$ where $2m = d = 2g - 2$; for $g = 3$ notice $B_2 \cong C_2$. The nonhyperelliptic case is similar but here no summand $\mathbf{1}$ occurs. □

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Received March 5, 2014. Revised November 6, 2014.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY



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Volume 276 No. 1 July 2015

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