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**SYSTEMS OF PARAMETERS AND HOLONOMICITY OF *A*-HYPERGEOMETRIC SYSTEMS**


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# SYSTEMS OF PARAMETERS AND HOLONOMICITY OF $A$-HYPERGEOMETRIC SYSTEMS

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**We give an elementary proof of holonomicity for $A$-hypergeometric systems, with no requirements on the behavior of their singularities, a result originally due to Adolphson (1994) after the regular singular case by Gelfand and Gelfand (1986). Our method yields a direct de novo proof that $A$-hypergeometric systems form holonomic families over their parameter spaces, as shown by Matusevich, Miller, and Walther (2005).**

***Dedication.*** Every now and then Andrei Zelevinsky had occasion to write a short and in many ways elementary paper with deep consequences. Particularly close to our hearts are his paper on graded nilpotent classes [Zelevinsky 1985] and his paper with Gelfand and Graev on hypergeometric systems [Gelfand et al. 1987]; both of these had enormous impact on our mathematical careers. It is in that spirit that we dedicate to Andrei this elementary perspective on topics that he influenced substantially for many years.

## Introduction

An $A$-hypergeometric system is the $D$-module counterpart of a toric ideal. Solutions to $A$-hypergeometric systems are functions, with a fixed infinitesimal homogeneity, on an affine toric variety. The solution space of an $A$-hypergeometric system behaves well in part because the system is holonomic, which in particular implies that the vector space of germs of analytic solutions at any nonsingular point has finite dimension.

This note provides an elementary proof of holonomicity for arbitrary $A$-hypergeometric systems, relying only on the statement that a module over the Weyl algebra in $n$ variables is holonomic precisely when its characteristic variety has dimension at most $n$ (see [Gabber 1981] or [Borel et al. 1987, Theorem 1.12]), along with standard facts about transversality of subvarieties and about Krull

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dimension. In particular, our proof requires no assumption about the singularities of the $A$-hypergeometric system; equivalently, the associated toric ideal need not be standard graded. Holonomicity was proved in the regular singular case by Gelfand and Gelfand [1986], and later by Adolphson [1994, §3] regardless of the behavior of the singularities of the system. Adolphson's proof relies on careful algebraic analysis of the coordinate rings of a collection of varieties whose union is the characteristic variety of the system. Another proof of the holonomicity of an $A$-hypergeometric system, by Schulze and Walther [2008], yields a more general result: for a weight vector $L$ from a large family of possibilities, the $L$-characteristic variety for the $L$-filtration is a union of conormal varieties and hence has dimension $n$. Holonomicity follows when $L = (0, \dots, 0, 1, \dots, 1)$ induces the order filtration on the Weyl algebra. The $L$-filtration method uses an explicit combinatorial interpretation of initial ideals of toric ideals, which requires a series of technical lemmas.

Holonomicity of $A$-hypergeometric systems forms part of the statement and proof, by Matusevich, Miller, and Walther [2005], that $A$-hypergeometric systems determine holonomic families over their parameter spaces. The new proof of that statement here serves as a model suitable for generalization to hypergeometric systems for reductive groups, in the sense of Kapranov [1998].

The main step (Theorem 1.2) in our proof is an easy geometric argument showing that the Euler operators corresponding to the rows of an integer matrix $A$ form part of a system of parameters on the product $\Bbbk^n \times X_A$, where $\Bbbk$ is any algebraically closed field and $X_A$ is the toric variety over $\Bbbk$ determined by $A$. This observation leads quickly in Section 2 to the conclusion that the characteristic variety of the associated $A$-hypergeometric system has dimension at most $n$, and hence that the system is holonomic. Since the algebraic part of the proof holds when the entries of the parameter $\beta$ are considered as independent variables that commute with all other variables, the desired stronger consequence is immediate: the $A$-hypergeometric system forms a holonomic family over its parameter space (Theorem 2.1).

## 1. Systems of parameters via transversality

Fix a field $\Bbbk$. Let $x = x_1, \dots, x_n$ and $\xi = \xi_1, \dots, \xi_n$ be sets of coordinates on $\Bbbk^n$ and let $x\xi$ denote the column vector with entries $x_1\xi_1, \dots, x_n\xi_n$. Given a rectangular matrix $L$ with $n$ columns, write $Lx\xi$ for the vector of bilinear forms given by multiplying $L$ times $x\xi$.

**Lemma 1.1.** *Let $\Bbbk^{2n} = \Bbbk^n_x \times \Bbbk^n_\xi$ have coordinates $(x, \xi)$ and let $X \subseteq \Bbbk^n_\xi$ be a subvariety. If $L$ is an $\ell \times n$ matrix with entries in $\Bbbk$, then the variety* $\mathrm{Var}(Lx\xi)$ *of $Lx\xi$ in $\Bbbk^{2n}$ is transverse to $\Bbbk^n \times X$ at any smooth point of $\Bbbk^n \times X$ whose $\xi$-coordinates are all nonzero.*

*Proof.* It suffices to prove the statement after passing to the algebraic closure of $\Bbbk$, so assume $\Bbbk$ is algebraically closed. Let $(p, q)$ be a smooth point of $\Bbbk^n \times X$ that lies in $\mathrm{Var}(Lx\xi)$ and has all coordinates of $q$ nonzero. The tangent space to $\Bbbk^n \times X$ at $(p, q)$ contains $\Bbbk^n \times \{0\}$. The tangent space $T_{(p,q)}$ to $\mathrm{Var}(Lx\xi)$ is the kernel of the $\ell \times 2n$ matrix $[L(q)\ L(p)]$, where $L(p)$ (respectively, $L(q)$) is the $\ell \times n$ matrix that results after multiplying each column of $L$ by the corresponding coordinate of $p$ (respectively, $q$). Since the $q$ coordinates are all nonzero, $T_{(p,q)}$ projects surjectively onto the last $n$ coordinates; indeed, if $\eta \in \Bbbk^n_\xi$ is given, then taking $y_i = -p_i\eta_i/q_i$ yields $y \in \Bbbk^n_x$ with $L(q)y + L(p)\eta = 0$. Thus the tangent spaces at $(p, q)$ sum to the ambient space, so the intersection is transverse. □

The next result applies the lemma to an affine toric variety $X$. A fixed $d \times n$ integer matrix $A = [a_1\ a_2\ \cdots\ a_{n-1}\ a_n]$ defines an action of the algebraic torus $T = (\Bbbk^*)^d$ on $\Bbbk^n_\xi$ by

$$t \cdot \xi = (t^{a_1}\xi_1, \ldots, t^{a_n}\xi_n).$$

The orbit $\mathrm{Orb}(A)$ of the point $\mathbf{1} = (1, \ldots, 1) \in \Bbbk^n$ is the image of an algebraic map $T \to \Bbbk^n$ that sends $t \mapsto t \cdot \mathbf{1}$. The closure of $\mathrm{Orb}(A)$ in $\Bbbk^n$ is the affine toric variety $X_A = \mathrm{Var}(I_A)$ cut out by the *toric ideal*

$$I_A = \langle \xi^u - \xi^v \mid Au = Av\rangle \subseteq \Bbbk[\xi]$$

of $A$ in the polynomial ring $\Bbbk[\xi] = \Bbbk[\xi_1, \ldots, \xi_n]$. The $T$-action induces an $A$-*grading* on $\Bbbk[\xi]$ via $\deg(\xi_i) = a_i$, and the semigroup ring $S_A = \Bbbk[\xi]/I_A$ is $A$-graded [Miller and Sturmfels 2005, Chapters 7–8].

For any face $\tau$ of the real cone $\mathbb{R}_{\geq 0}A$ generated by the columns of $A$, write $\tau \preceq A$ and let $\mathbf{1}^\tau \in \{0, 1\}^n \subset \Bbbk^n$ be the vector with nonzero entry $\mathbf{1}^\tau_i = 1$ precisely when $A$ has a nonzero column $a_i \in \tau$. The variety $X_A$ decomposes as a finite disjoint union $X_A = \bigsqcup_{\tau \preceq A} \mathrm{Orb}(\tau)$ of orbits, where $\mathrm{Orb}(\tau) = T \cdot \mathbf{1}^\tau$. Each orbit has dimension $\dim \mathrm{Orb}(\tau) = \mathrm{rank}(A_\tau)$, where $A_\tau$ is the submatrix of $A$ consisting of those columns lying in $\tau$, and $\dim X_A = \mathrm{rank}(A)$.

**Theorem 1.2.** *The ring $\Bbbk[x, \xi]/(I_A + \langle Ax\xi\rangle)$ has Krull dimension $n$. In particular, if $A$ has rank $d$ then the forms $Ax\xi$ are part of a system of parameters for $\Bbbk[x] \otimes_\Bbbk S_A$.*

*Proof.* Let $\Bbbk^\tau \subseteq \Bbbk^n$ be the subspace consisting of vectors with 0 in coordinate $i$ if $a_i \notin \tau$, and let $|\tau|$ be its dimension. Since $\Bbbk[x, \xi]/I_A = \Bbbk[x] \otimes_\Bbbk S_A$ has dimension $n + \mathrm{rank}(A)$ and the number of $\Bbbk$-linearly independent generators of $\langle Ax\xi\rangle$ is $\mathrm{rank}(A)$, the Krull dimension in question is at least $n$. Hence it suffices to prove that $(\Bbbk^n \times \mathrm{Orb}(\tau)) \cap \mathrm{Var}(Ax\xi) \subseteq \Bbbk^n \times \Bbbk^\tau$ has dimension at most $n$. Let $x_\tau$ and $\xi_\tau$ denote the subsets corresponding to $\tau$ in the variable sets $x$ and $\xi$, respectively. The projection of the intersection onto the subspace $\Bbbk^\tau \times \Bbbk^\tau$ has image contained in

$$(\Bbbk^\tau \times \mathrm{Orb}(\tau)) \cap \mathrm{Var}(A_\tau x_\tau \xi_\tau) \subseteq \Bbbk^\tau \times \Bbbk^\tau.$$

It therefore suffices to show that the dimension of this latter intersection is at most $|\tau|$. By Lemma 1.1, the intersection is transverse in $\Bbbk^\tau \times \Bbbk^\tau$. But the dimension of $\mathrm{Orb}(\tau)$ is the codimension of $\mathrm{Var}(A_\tau x_\tau \xi_\tau)$ in $\Bbbk^\tau \times \Bbbk^\tau$, which completes the proof. □

## 2. Hypergeometric holonomicity

In this section, the matrix $A$ is a $d \times n$ integer matrix of full rank $d$. Let

$$D = \mathbb{C}\langle x, \partial \mid [\partial_i, x_j] = \delta_{ij} \text{ and } [x_i, x_j] = 0 = [\partial_i, \partial_j]\rangle$$

denote the Weyl algebra over the complex numbers $\mathbb{C}$, where $x = x_1, \dots, x_n$ and $\partial_i$ corresponds to $\partial/\partial x_i$. This is the ring of $\mathbb{C}$-linear differential operators on $\mathbb{C}[x]$.

For $\beta \in \mathbb{C}^d$, the $A$-*hypergeometric system* with parameter $\beta$ is the left $D$-module

$$M_A(\beta) = D/D \cdot (I_A^\partial, \{E_i - \beta_i\}_{i=1}^d),$$

where $I_A^\partial = \langle \partial^u - \partial^v \mid Au = Av\rangle \subseteq \mathbb{C}[\partial]$ is the toric ideal associated to $A$ and

$$E_i - \beta_i = \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$$

are *Euler operators* associated to $A$.

The *order filtration* $F$ filters $D$ by order of differential operators. The *symbol* of an operator $P$ is its image $\mathrm{in}(P) \in \mathrm{gr}^F D$. Writing $\xi_i = \mathrm{in}(\partial_i)$, this means $\mathrm{gr}^F D$ is the commutative polynomial ring $\mathbb{C}[x, \xi]$. The *characteristic variety* of a left $D$-module $M$ is the variety in $\mathbb{A}^{2n}$ of the associated graded ideal $\mathrm{gr}^F \mathrm{ann}(M)$ of the annihilator of $M$. A nonzero $D$-module is *holonomic* if its characteristic variety has dimension $n$; this is equivalent to requiring that the dimension be at most $n$ (see [Gabber 1981] or [Borel et al. 1987, Theorem 1.12]). The *rank* of a holonomic $D$-module $M$ is the (always finite) dimension of $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} M$ as a vector space over $\mathbb{C}(x)$; this number equals the dimension of the vector space of germs of analytic solutions of $M$ at any nonsingular point in $\mathbb{C}^n$ [Saito et al. 2000, Theorem 1.4.9].

Viewing the $A$-hypergeometric system $M_A(\beta)$ as having a varying parameter $\beta \in \mathbb{C}^d$, the rank of $M_A(\beta)$ is upper semicontinuous as a function of $\beta$ [Matusevich et al. 2005, Theorem 2.6]. This follows by viewing $M_A(\beta)$ as a *holonomic family* [ibid., Definition 2.1] parametrized by $\beta \in \mathbb{C}^d$. By definition, this means not only that $M_A(\beta)$ is holonomic for each $\beta$, but also that it satisfies a coherence condition over $\mathbb{C}^d$: after replacing $\beta$ with variables $b = b_1, \dots, b_d$, the module $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} M_A(b)$ is finitely generated over $\mathbb{C}(x)[b]$. (The definition of holonomic family cited above allows sheaves of $D$-modules over arbitrary complex base schemes, but that generality is not needed here.)

The derivation of the holonomic family property for $M_A(b)$ from the holonomicity of the $A$-hypergeometric system is more or less the same as [ibid., Theorem 7.5], which was phrased in the generality of Euler–Koszul homology of toric modules. The brief deduction here isolates the steps necessary for $A$-hypergeometric systems; its brevity stems from the special status of affine semigroup rings among all toric modules [ibid., Definition 4.5]. Note further that this proof does not require technical combinatorial arguments using standard pairs, as in [Saito et al. 2000]; indeed, $\mathrm{in}(I_A)$ need not be a monomial ideal.

**Theorem 2.1.** *The module $M_A(b)$ forms a holonomic family over $\mathbb{C}^d$ with coordinates $b$. In more detail, as a $D[b]$-module the parametric $A$-hypergeometric system $M_A(b)$ satisfies*:

(1) *the fiber $M_A(\beta) = M_A(b) \otimes_{\mathbb{C}[b]} \mathbb{C}[b]/\langle b-\beta\rangle$ is holonomic for all $\beta$; and*

(2) *the module $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} M_A(b)$ is finitely generated over $\mathbb{C}(x)[b]$.*

*Proof.* Since $R = \mathbb{C}[x,\xi]/\langle \mathrm{in}(I_A), Ax\xi\rangle$ surjects onto $\mathrm{gr}^F M_A(\beta)$, it is enough to show that the ring $R$ has dimension $n$. If $M_A(\beta)$ is standard $\mathbb{Z}$-graded (equivalently, the rowspan of $A$ over the rational numbers contains the row $[1\ 1\ \cdots\ 1\ 1]$ of length $n$), then $\mathrm{in}(I_A) = I_A \subseteq \mathbb{C}[\xi]$, and the result follows from Theorem 1.2.

When $M_A(\beta)$ is not standard $\mathbb{Z}$-graded, let $\hat{A}$ be the $(d+1)\times(n+1)$ matrix obtained by adding a row of 1's across the top of $A$ and then adding as the leftmost column $(1,0,\dots,0)$. If $\xi_0$ denotes a new variable corresponding to the leftmost column of $\hat{A}$, and $\hat{\xi} = \{\xi_0\}\cup\xi$, then $\mathbb{C}[\xi]/\mathrm{in}(I_A) \cong \mathbb{C}[\hat{\xi}]/\langle I_{\hat{A}}, \xi_0\rangle$. In particular,

$$\frac{\mathbb{C}[\hat{x},\xi]}{\langle \mathrm{in}(I_A), Ax\xi\rangle} \cong \frac{\mathbb{C}[\hat{x},\hat{\xi}]}{\langle I_{\hat{A}}, \xi_0, \hat{A}\hat{x}\hat{\xi}\rangle},$$

where $\hat{x} = \{x_0\}\cup x$. Since $\langle I_{\hat{A}}, \xi_0\rangle$ is $\hat{A}$-graded and $\hat{A}$ has a row $[1\ 1\ \cdots\ 1\ 1]$, we have reduced to the case where $M_A(\beta)$ is $\mathbb{Z}$-graded, completing part (1).

With $R$ as in part (1), the ring $R[b]$ surjects onto $\mathrm{gr}^F M_A(b)$, so it suffices for part (2) to show that $R[b]$ becomes finitely generated over $\mathbb{C}(x)[b]$ upon inverting all nonzero polynomials in $x$. Since the ideal $\langle \mathrm{in}(I_A), Ax\xi\rangle$ has no generators involving $b$ variables, it suffices to show that $R(x)$ itself has finite dimension over $\mathbb{C}(x)$. The desired result follows from the statement proved for part (1): any scheme of dimension $n$ has finite degree over $\mathbb{C}^n_x$. $\square$

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