Pacific Journal of Mathematics

THE RAMIFICATION GROUP FILTRATIONS OF CERTAIN FUNCTION FIELD EXTENSIONS

JEFFREY A. CASTAÑEDA AND QINGQUAN WU

Volume 276 No. 2 August 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 276, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.276.309

THE RAMIFICATION GROUP FILTRATIONS OF CERTAIN FUNCTION FIELD EXTENSIONS

JEFFREY A. CASTAÑEDA AND QINGQUAN WU

We investigate the ramification group filtration of a Galois extension of function fields, if the Galois group satisfies a certain intersection property. For finite groups, this property is implied by having only elementary abelian Sylow p-subgroups. Note that such groups could be nonabelian. We show how the problem can be reduced to the totally wild ramified case on a p-extension. Our methodology is based on an intimate relationship between the ramification groups of the field extension and those of all degree-p subextensions. Not only do we confirm that the Hasse–Arf property holds in this setting, but we also prove that the Hasse–Arf divisibility result is the best possible by explicit calculations of the quotients, which are expressed in terms of the different exponents of all those degree-p subextensions.

1. Introduction

When investigating algebraic number fields and function fields, Hilbert ramification theory is a convenient tool, especially in the study of wild ramifications. Fix a function field K over a perfect constant field k with a place $\mathcal{P}$, and let L be a Galois extension of K with a place $\mathfrak{P}$ lying over $\mathcal{P}$. We investigate how the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ is related to the ramification group filtration of $\mathfrak{P}_{\mathfrak{m}}|\mathcal{P}$, where $\mathfrak{P}_{\mathfrak{m}}$ is a place of some intermediate field $K \subseteq M \subseteq L$, so that $\mathfrak{P}$ lies over $\mathfrak{P}_{\mathfrak{m}}$ and $[M : K] = p$ for some prime number p.

We first analyze how and why we can simplify the problem to the setting when $\mathfrak{P}|\mathcal{P}$ is totally wildly ramified, i.e., $[L : K] = p^n$, where $p > 0$ is the characteristic of k, n is some positive integer, and the ramification index $e(\mathfrak{P}|\mathcal{P}) = p^n$.

Next we study how the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ is closely related to the ramification group filtration of $\mathfrak{P}_{\mathfrak{m}}|\mathcal{P}$ for all those intermediate fields M such that $[M : K] = p$, for various degrees p. This relation is close if an intersection property (2-5) is assumed about $\mathrm{Gal}(L/K)$, which is satisfied by many abelian and nonabelian groups.

The second author is supported by the University Research Grant of the Texas A&M International University.
MSC2010: primary 11R58; secondary 11R32, 11S15.
Keywords: function fields, ramification groups, filtrations.

To prove such a relationship, we first prove a preliminary result, which states that the number of jumps in the ramification group filtration is equal to the number of pairwise distinct different exponents of the corresponding place extensions over all possible degree-p intermediate extensions M/K. This equality is significant since the degree-p intermediate extensions are considerably easier to investigate than the whole extension L/K. Nonetheless, we will show that these different exponents are closely related to those quotients given in the Hasse–Arf property, which we will show to be true.

We also study the relationship by applying the equality given by the transitivity of differents, where the different exponents are computed via Hilbert's different formulae applied to various field extension settings. These equalities lead to linear equations on the indices where the jumps of the ramification group filtration on L/K occur. With the intersection property assumption, we show that the number of such linear equations is equal to the number of such indices as the variables of these equations. Hence we can expect a unique solution. In fact, we can solve these linear equations explicitly to give closed-form formulae for the indices since the coefficient matrix of the linear equations is triangular.

The academic literature on ramification group filtration is extensive. A good introduction is [Serre 1979], where Herbrand's upper numbering is introduced. See [Fesenko and Vostokov 2002] for an introduction without the use of cohomologies. The ramification groups are studied in [Sen and Tate 1963] using class field theory. For an approach using Herbrand functions and without using class field theory, see [Wyman 1969]. Maus [1968] showed certain properties of a group filtration that are sufficient to guarantee it to be the ramification group filtration of a certain extension of complete discrete valuation fields. In [Maus 1972], the asymptotic behavior of quotients given by the Hasse–Arf property is studied. The paper [Maus 1971] is a collection of many results from Maus's Ph.D. thesis, without proofs.

The ramification group filtration is known to satisfy the Hasse–Arf property [Hasse 1930; 1934; Arf 1939] if the Galois group is abelian. However, the property may fail if the Galois group is not abelian. One such example is the Galois closure of a cyclotomic field over the rationals [Viviani 2004]. In [Doud 2003], it is shown that the ramification group filtration of a wildly ramified prime $\mathfrak{p}$ is uniquely determined by the $\mathfrak{p}$-adic valuation of the discriminant of the field extension L/K, when both the field extension degree and the residue characteristic of $\mathfrak{p}$ are equal to a prime number. When the Galois group is elementary abelian, the Galois module structure of certain ideals is related to the ramification group filtration, see [Byott and Elder 2002; 2005; 2009]. Such a relation is investigated when the Galois group is quaternion [Elder and Hooper 2007], and hence nonabelian.

For the function field extension setting, the wildly ramified case was studied in Artin–Schreier–Witt extensions, see [Thomas 2005]. The elementary abelian extension of Galois group $\mathbb{Z}_p \times \mathbb{Z}_p$ is investigated in [Anbar et al. 2009] and [Wu and Scheidler 2010]. It should be mentioned that the idea of utilizing transitivity of differents and Hilbert's different formula to investigate the ramification groups is used in [Garcia and Stichtenoth 2008], where the Hasse–Arf property for elementary abelian extensions of function fields is proved. In Roberts' review [2009] of the latter paper, it is shown that the proof can be very short if "some upper numbering system and its basic formalism" is applied. We take an approach similar to Garcia and Stichtenoth's, but we further explore the arithmetic and linear algebra provided by the application of transitivity of differents and Hilbert's different formula. Our objective in this paper is to generalize these results to function field extensions with Galois groups satisfying a certain intersection property which is true for elementary abelian groups.

2. Notation

A good introduction for the notation can be found in [Rosen 2002] or [Stichtenoth 2009]. Throughout this paper, we use the following notation:

- k is a perfect field of characteristic $p > 0$;
- K is a function field with constant field k;
- $\mathcal{P}$ is a place of K;
- $v_{\mathcal{P}} : K \to \mathbb{Z} \cup \{\infty\}$ is the (surjective) discrete valuation corresponding to $\mathcal{P}$;
- $\mathcal{O}_{\mathcal{P}} = \{\alpha \in K \mid v_{\mathcal{P}}(\alpha) \geq 0\}$ is the valuation ring corresponding to $\mathcal{P}$.

For any extension L of K and any place $\mathfrak{P}$ of L lying above $\mathcal{P}$, we write $\mathfrak{P}|\mathcal{P}$. Let $e(\mathfrak{P}|\mathcal{P})$ and $d(\mathfrak{P}|\mathcal{P})$ be the ramification index and different exponent of $\mathfrak{P}|\mathcal{P}$, respectively. If L/K is a Galois extension, the ramification groups of $\mathfrak{P}|\mathcal{P}$ are given by

$$G_i = G_i(\mathfrak{P}|\mathcal{P}) = \{\sigma \in \mathrm{Gal}(L/K) \mid v_{\mathfrak{P}}(t^\sigma - t) \geq i+1 \text{ for all } t \in \mathcal{O}_{\mathfrak{P}}\} \tag{2-1}$$

for $i \geq 0$. The connection between these groups and the different exponent is shown in Hilbert's different formula (see for example Theorem 3.8.7, p. 136, of [Stichtenoth 2009]):

$$d(\mathfrak{P}|\mathcal{P}) = \sum_{i=0}^{\infty} (\#G_i(\mathfrak{P}|\mathcal{P}) - 1). \tag{2-2}$$

We also recall the transitivity of the ramification index and the different exponent. If $K \subseteq F \subseteq L$ are function fields, $\mathfrak{P}$ a place of L, $\mathfrak{P}_{\mathfrak{F}} = \mathfrak{P} \cap F$, and $\mathcal{P} = \mathfrak{P}_{\mathfrak{F}} \cap K$, then

$$e(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}})e(\mathfrak{P}_{\mathfrak{F}}|\mathcal{P}), \tag{2-3}$$

and we have transitivity of differents:

$$d(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}})d(\mathfrak{P}_{\mathfrak{F}}|\mathcal{P}) + d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}}). \tag{2-4}$$

Henceforth, we assume that all nontrivial Sylow p-subgroups H_p of the Galois group $\mathrm{Gal}(L/K)$ satisfy the following intersection property.

Assume that $\#H_p = p^n > 1$. Then, for all proper subgroups $F \subsetneq H_p$, the intersection of all order p^{n-1} subgroups of H_p containing F is simply F. That is to say,

$$\bigcap_{\substack{H \supseteq F \\ \#H = p^{n-1}}} H = F. \tag{2-5}$$

It is easy to verify that all elementary abelian p-groups of order p^n satisfy this intersection property.

3. Reduction to the totally wildly ramified case

Let L be a Galois extension field of K, $\mathfrak{P}$ a place of L, and $\mathcal{P} = \mathfrak{P} \cap K$. Our goal in this section is to reduce the ramification group $G_i(\mathfrak{P}|\mathcal{P})$ calculation to the case that $\mathfrak{P}/\mathcal{P}$ is totally wildly ramified and the Galois group $\mathrm{Gal}\,(L/K)$ is a p-group for a certain prime number p.

Lemma 3.1. *Let L/K be a Galois extension of a function field, $\mathfrak{P}$ a place of L, $\mathcal{P} = \mathfrak{P} \cap K$, and $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{P} \cap M$, where M is the inertia field of $\mathcal{P}$ in L/K. Then, $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for every $i \geq 0$.*

Proof. Applying (2-4) to the field extension tower $L/M/K$, we have

$$d(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P}) + d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}). \tag{3-1}$$

However, $d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P}) = 0$ since $\mathcal{P}$ is unramified in M/K, so $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$. Now (2-2) yields

$$d(\mathfrak{P}|\mathcal{P}) = \sum_{i=0}^{\infty}(\#G_i(\mathfrak{P}|\mathcal{P}) - 1) \tag{3-2}$$

and

$$d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = \sum_{i=0}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) - 1). \tag{3-3}$$

By definition (2-1), it is easy to check that $G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ is the intersection of $G_i(\mathfrak{P}|\mathcal{P})$ and the Galois group of L/M. In particular, we have $\#G_i(\mathfrak{P}|\mathcal{P}) \geq \#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all i. Note that $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$, which implies that $\#G_i(\mathfrak{P}|\mathcal{P}) = \#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all $i \geq 0$ by (3-2) and (3-3). Thus, $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all $i \geq 0$. $\square$

Next, we want to reduce to the totally wildly ramified case, i.e., $[L : K] = e(\mathfrak{P}|\mathcal{P}) = p^m$, where p is the characteristic of K.

Proposition 3.2. *Let $p > 0$ be the characteristic of the Galois extension of function field L/K, $\mathfrak{P}$ a place of L, $\mathcal{P} = \mathfrak{P} \cap K$, N the inertia field of $\mathcal{P}$ in L/K, $\mathfrak{P}_{\mathfrak{n}} = \mathfrak{P} \cap N$, M the intermediate field of L/N corresponding to a Sylow p-subgroup of* $\mathrm{Gal}(L/N)$ *under Galois correspondence, and $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{P} \cap M$. Then, $\mathfrak{P}_{\mathfrak{m}}$ is totally wildly ramified in the p-extension L/M, and $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for every $i \geq 1$.*

Proof. By Lemma 3.1, we have $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}})$ for every $i \geq 0$. It suffices to show that $G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for every $i \geq 1$. Assume that $[L : N] = p^m q$ and $\gcd(p, q) = 1$. We have $d(\mathfrak{P}_{\mathfrak{m}}|\mathfrak{P}_{\mathfrak{n}}) = q - 1$ since $\mathfrak{P}_{\mathfrak{m}}/\mathfrak{P}_{\mathfrak{n}}$ is totally tamely ramified, and we also have $e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = p^m$. By (3-1), we have

$$d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) = p^m(q-1) + d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}). \tag{3-4}$$

Clearly $\#G_0(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) = p^m q$ and $\#G_0(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = p^m$. Hence,

$$d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) = p^m q - 1 + \sum_{i=1}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) - 1)$$

and

$$d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = p^m - 1 + \sum_{i=1}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) - 1)$$

by (3-2) and (3-3). Substituting these two equalities into (3-4), we have

$$\sum_{i=1}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) - 1) = \sum_{i=1}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) - 1),$$

which implies $G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for $i \geq 1$ since $G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{n}}) \geq G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all i. □

4. Main results

Henceforth, let L/K be a Galois extension whose Galois group is a p-group, where p is the characteristic of K. Set t to be the number of distinct $d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P})$, where M runs through all degree-p intermediate fields M of L/K, and $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{P} \cap M$. We assume that the ramification groups of $\mathfrak{P}|\mathcal{P}$ are

$$\begin{aligned} G_0 = \cdots = G_{m_0} \supsetneq G_{m_0+1} = \cdots = G_{m_1} \\ \supsetneq G_{m_1+1} = \cdots = G_{m_{l-1}} \supsetneq G_{m_{l-1}+1} = \{\mathrm{Id}\}. \end{aligned} \tag{4-1}$$

We let $\#G_{m_i} = p^{n_i}$ for $0 \leq i \leq l-1$. Then $p^{n_0} = p^n = [L : K]$. In order to investigate G_i, we need to know l, n_i, and m_i.

First, we claim that the number of jumps in the ramification groups G_i is the number of distinct different exponents $d(\mathfrak{P}_\mathfrak{m}|\mathcal{P})$; this is, $l = t$. Note that a jump means an index where a group in the ramification filtration contains the next one properly. Before we prove the claim, we need a lemma.

Lemma 4.1. *Let G be a p-group of order $p^n > 1$, $H < G$ a subgroup of order p^{n-1}, and $H' < G$ a subgroup such that $H \not\supseteq H'$. Then $\#(H \cap H') = \#H'/p$.*

Proof. By Theorem 4.7, page 39 of [Hungerford 1974], we have $\#(HH') = \#H\,\#H'/\#(H \cap H')$. In particular, $\#(HH')$ is a power of p. Since $H \not\supseteq H'$, we know that $p^{n-1} < \#(HH') \le p^n$. Hence, $\#(HH') = p^n$. The result follows by substituting $\#(HH') = p^n$ and $\#H = p^{n-1}$ into the equality $\#(HH') = \#H\,\#H'/\#(H \cap H')$. □

Furthermore, by (2-4) we know that $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}_\mathfrak{m}|\mathcal{P})p^{n-1} + d(\mathfrak{P}|\mathfrak{P}_\mathfrak{m})$, where t distinct $d(\mathfrak{P}_\mathfrak{m}|\mathcal{P})$ values imply t distinct $d(\mathfrak{P}|\mathfrak{P}_\mathfrak{m})$. Let $d(\mathfrak{P}|\mathfrak{P}_\mathfrak{m}) = \sum_{i=0}^{\infty}(\#G_i(\mathfrak{P}|\mathfrak{P}_\mathfrak{m}) - 1)$, where $G_i(\mathfrak{P}|\mathfrak{P}_\mathfrak{m}) = H_m \cap G_i(\mathfrak{P}|\mathcal{P})$ such that $\#H_m = p^{n-1}$.

Now, for every jump $G_{m_i} \supsetneq G_{m_i+1}$ for $0 \le i \le l-1$, we want to know if there exists $H_m < G$ such that $\#H_m = p^{n-1}$ and $H_m \supseteq G_{m_i+1}$, but $H_m \not\supseteq G_{m_i}$. In other words, we want to know if there exists an order-p^{n-1} subgroup H_m which faithfully reveals a jump wherever it occurs in the ramification group filtrations. This is not a trivial question since a jump can be hidden if no such H_m can be found. The question is clarified by the following lemma.

Lemma 4.2. *Let G be a p-group satisfying property (2-5). Then for any two subgroups $F_1 \supsetneq F_2$ of G, there exists a subgroup H of G such that $\#H = p^{n-1}$ and $H \supseteq F_2$, but $H \not\supseteq F_1$.*

Proof. By way of contradiction, assume that the result is false. Then we can find subgroups $F_1 \supsetneq F_2$, and for all subgroups H of order p^{n-1}, $H \supseteq F_2$ implies $H \supseteq F_1$. Thus, the set $\{H < G \mid \#H = p^{n-1}, H \supseteq F_1\} = \{H < G \mid \#H = p^{n-1}, H \supseteq F_2\}$. Hence,

$$F_1 = \bigcap_{\substack{H \supseteq F_1 \\ \#H = p^{n-1}}} H = \bigcap_{\substack{H \supseteq F_2 \\ \#H = p^{n-1}}} H = F_2,$$

a contradiction by (2-5). □

Now we are ready to prove the result $l = t$.

Proposition 4.3. *Let L/K be a Galois extension of function fields whose Galois group is a p-group satisfying (2-5), where p is the characteristic of K. Let $\mathfrak{P} \mid \mathfrak{P}_\mathfrak{m} \mid \mathcal{P}$ be a tower of places, $\mathcal{P} \subseteq K$, $\mathfrak{P}_\mathfrak{m} \subseteq M$, $\mathfrak{P} \subseteq L$, where M is an intermediate field of degree p over K. Then the number of jumps in the ramification group filtration $G_i(\mathfrak{P}|\mathcal{P})$ is the number of distinct different exponents $d(\mathfrak{P}_\mathfrak{m}|\mathcal{P})$, where M runs through all intermediate fields of L/K of degree p over K.*

Proof. Fix $\mathfrak{P}$ and $\mathcal{P}$, and let M run through all possible degree-p intermediate fields of L/K. By (3-1), t is equal to the number of distinct values of $d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ since $d(\mathfrak{P}|\mathcal{P})$ and $e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = p^{n-1}$ are independent of the choice of M.

For $0 \le i \le l-1$, the i-th jump occurs at index m_i; that is, $G_{m_i} \supsetneq G_{m_i+1}$. By Lemma 4.2, there exists an order p^{n-1} subgroup H_i of G_0, so that $H_i \supseteq G_{m_i+1}$, but $H_i \not\supseteq G_{m_i}$. By Lemma 4.1, we have $\#(H_i \cap G_{m_i}) = \#G_{m_i}/p$. Similarly, $\#(H_i \cap G_j) = \#G_j/p$ for all $j \le m_i$ since G_j is decreasing. Hence, $\#(H_i \cap G_j) = \#G_j/p$ for $0 \le j \le m_i$, and $\#(H_i \cap G_j) = \#G_j$ for $j > m_i$.

Let M_i be the intermediate field of L/K corresponding to H_i under Galois correspondence, and set $\mathfrak{P}_i = \mathfrak{P} \cap M_i$. Since $G_j(\mathfrak{P}|\mathfrak{P}_{\mathrm{i}}) = H_i \cap G_j(\mathfrak{P}|\mathcal{P}) = H_i \cap G_j$ for all j, we have

$$d(\mathfrak{P}|\mathfrak{P}_i) = \sum_{j=0}^{\infty}(\#G_j(\mathfrak{P}|\mathfrak{P}_i) - 1) = \sum_{j=0}^{\infty}(\#(H_i \cap G_j) - 1) = \sum_{j=0}^{m_i}\Big(\frac{\#G_j}{p} - 1\Big) + \sum_{j=m_i+1}^{\infty}(\#G_j - 1). \tag{4-2}$$

The right-hand side of (4-2) is strictly decreasing with i. Hence, we find l pairwise distinct values of $d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$. Note that any order p^{n-1} subgroup H of G_0 contains $G_{m_l+1} = \{\mathrm{Id}\}$ but not $G_{m_0} = G_0$. Hence, for any such H, there exists an i such that $H \supseteq G_{m_i+1}$, but $H \not\supseteq G_{m_i}$. In other words, H is one of the H_i by the previous analysis of the choice of H_i. Thus, there are exactly l pairwise distinct values of $d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ when M runs over all possible degree-p intermediate fields of L/K. Hence, there are exactly l pairwise distinct values of $d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P})$, i.e., $l = t$. □

By Proposition 3.7.8, p. 127 of [Stichtenoth 2009], $d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P})$ is a multiple of $p-1$ for any M. Hence,

$$d_i = \frac{d(\mathfrak{P}_i|\mathcal{P})}{p-1} - 1 \tag{4-3}$$

is an integer for all $0 \le i \le l-1$. By (4-2), $d(\mathfrak{P}|\mathfrak{P}_i)$ is strictly decreasing with i. Hence, d_i is strictly increasing with i by (4-3). Now we are ready for the main result.

Theorem 4.4. *Let L/K be a Galois extension of a function fields whose Galois group is a p-group satisfying (2-5), where p is the characteristic of K. Let $\mathfrak{P}|\mathcal{P}$ be places, $\mathcal{P} \subseteq K$, $\mathfrak{P} \subseteq L$, m_i as in (4-1) for $0 \le i$, and d_j as in (4-3) for $j \le l-1$. Then*

$$m_i = d_0 + \sum_{j=1}^{i} p^{n-n_j}(d_j - d_{j-1}) \quad \textit{for } 0 \le i \le l-1.$$

Proof. By applying (3-2) to L/K, we have

$$d(\mathfrak{P}|\mathcal{P}) = \sum_{j=0}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}), \tag{4-4}$$

where $m_{-1} = -1$.

For $0 \leq i \leq l-1$ and $\#G_j = p^{n_i}$ for $m_{i-1} < j \leq m_i$, then (4-2) yields

$$d(\mathfrak{P}|\mathfrak{P}_i) = \sum_{j=0}^{i} (p^{n_j - 1} - 1)(m_j - m_{j-1}) + \sum_{j=i+1}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}). \tag{4-5}$$

Now substituting (4-4), (4-5), $e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}}) = p^{n-1}$, and (4-3) into (3-1) for the case $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{P}_i$, we have

$$\begin{aligned}
&\sum_{j=0}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}) \\
&= p^{n-1}(d_i + 1)(p-1) + \sum_{j=0}^{i} (p^{n_j - 1} - 1)(m_j - m_{j-1}) + \sum_{j=i+1}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}).
\end{aligned}$$

Hence, we have

$$\sum_{j=0}^{i} (p^{n_j} - 1)(m_j - m_{j-1}) = (p^n - p^{n-1})(d_i + 1) + \sum_{j=0}^{i} (p^{n_j - 1} - 1)(m_j - m_{j-1}),$$

which implies

$$\sum_{j=0}^{i} (p^{n_j} - p^{n_j - 1})(m_j - m_{j-1}) = (p^n - p^{n-1})(d_i + 1). \tag{4-6}$$

When $i = 0$, (4-6) yields

$$(p^n - p^{n-1})(m_0 + 1) = (p^{n_0} - p^{n_0 - 1})(m_0 + 1) = (p^n - p^{n-1})(d_0 + 1),$$

which implies $m_0 = d_0$. Thus, the formula in Theorem 4.4 is true when $i = 0$. Now we induct on i. By (4-6), we have

$$\begin{aligned}
(p^{n_i} - p^{n_i - 1})m_i - (p^{n_i} - p^{n_i - 1})m_{i-1} + \sum_{j=0}^{i-1} (p^{n_j} - p^{n_j - 1})(m_j - m_{j-1}) \\
= (p^n - p^{n-1})(d_i + 1).
\end{aligned}$$

It follows that

$$(p^{n_i} - p^{n_i - 1})m_i - (p^{n_i} - p^{n_i - 1})m_{i-1} + (p^n - p^{n-1})(d_{i-1} + 1) = (p^n - p^{n-1})(d_i + 1)$$

by applying (4-6) to the case $i-1$, which implies

$$(p^{n_i}-p^{n_i-1})m_i=(p^n-p^{n-1})(d_i-d_{i-1})+(p^{n_i}-p^{n_i-1})m_{i-1}.$$

By the induction hypothesis, it follows that

$$(p^{n_i}-p^{n_i-1})m_i \\ =(p^n-p^{n-1})(d_i-d_{i-1})+(p^{n_i}-p^{n_i-1})\Bigl(d_0+\sum_{j=1}^{i-1}p^{n-n_j}(d_j-d_{j-1})\Bigr).$$

Dividing both sides by $p^{n_i}-p^{n_i-1}$, we obtain

$$m_i=d_0+p^{n-n_i}(d_i-d_{i-1})+\sum_{j=1}^{i-1}p^{n-n_j}(d_j-d_{j-1})=d_0+\sum_{j=1}^{i}p^{n-n_j}(d_j-d_{j-1}).$$

Our result follows by induction. □

The formula in Theorem 4.4 can be reformulated to be easily compared to the Hasse–Arf property.

Corollary 4.5. *With notation as in Theorem 4.4, and setting $m_{-1}=-1$, we have $m_i-m_{i-1}=p^{n-n_i}(d_i-d_{i-1})$ for $0\le i\le l-1$.*

Proof. This is immediate by applying the formula in Theorem 4.4 to the cases i and $i-1$. □

5. The Hasse–Arf property

The formula in Corollary 4.5 is expected due to the well-known Hasse–Arf property, see [Arf 1939]. It claims that the distance between two consecutive jumps in a ramification group filtration is divisible by the index of the group at the jump in the first group of the filtration. The Hasse–Arf property is true when the Galois group is abelian yet not always true otherwise.

In our setting, the Hasse–Arf property translates to $p^{n-n_j}\mid m_j-m_{j-1}$. So, according to Corollary 4.5, not only do we verify that it is true, we also know the quotient to be d_i-d_{i-1}. An advantage of knowing the quotient explicitly is that we can discuss whether the Hasse–Arf property can be improved or not. In fact, we can construct an example where the group index is a power of p, and no higher power of p can divide m_j-m_{j-1} than the power guaranteed by the Hasse–Arf Property. Notice that the strictly increasing property and $d_i\not\equiv 0 \pmod p$ are the only two restrictions on the sequence d_i of positive integers. See [Anbar et al. 2009] or [Wu and Scheidler 2010] for a discussion of the type of the extension L/K. Although an explicit construction is not given there, the extension L/K herein is of the same type as described in those two papers. Actually, we can construct Artin–Schreier extensions M_i over the same base field K with any prescribed different exponent $(d_i+1)(p-1)$; then we can construct L to be the composite of those M_i. That is to say, for any strictly increasing sequence of nonnegative integers d_i of length l, we can construct a Galois extension L/K of function fields and corresponding extension of places $\mathfrak{P}$ lying over $\mathcal{P}$, so that there are exactly l jumps in the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ and exactly l pairwise distinct values of the different exponents $(d_0+1)(p-1) < (d_1+1)(p-1) < \cdots < (d_{l-1}+1)(p-1)$ for $d(\mathfrak{P}_{\mathfrak{m}}|\mathcal{P})$ when M ranges over all degree-p intermediate fields of L/K. In particular, we can require that $d_i = d_{i-1}+1$ for all $1 \le i \le l-1$. With this example, we know that there is no way to improve the Hasse–Arf divisibility result. On the other hand, $d_i - d_{i-1}$ can be any prescribed positive integer, so it is possible to strengthen the Hasse–Arf divisibility result arbitrarily under certain circumstances.

Now we want to analyze the Hasse–Arf property under a more general assumption; that is to say, to remove the totally wildly ramified assumption. First, we consider the not totally ramified case. With notation as in Theorem 4.4 and M as the inertia field of the place extension $\mathfrak{P}|\mathcal{P}$, we know that $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all $i \ge 0$ by Lemma 3.1. Hence, the Hasse–Arf property is true for the partially ramified case with identical parameters and formulae to those in the totally ramified case.

However, the situation changes when we move to the tamely ramified case. For that purpose, let $[L:K] = p^m q$ such that $p \nmid q$, M the intermediate field of L/K corresponding to a Sylow p-subgroup of $\mathrm{Gal}(L/K)$ under Galois correspondence, and $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{P} \cap M$. By Corollary 4.5, we have $m_i - m_{i-1} = p^{n-n_i}(d_i - d_{i-1})$ for $0 \le i \le l-1$, where m_i and d_i are defined for the ramification filtrations $G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$. By Proposition 3.2, we know $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_{\mathfrak{m}})$ for all $i \ge 1$. Hence, the i-th jump in the ramification filtrations of $G_i(\mathfrak{P}|\mathcal{P})$ is equal to $p^{n-n_i}(d_i - d_{i-1})$ for $1 \le i \le l-1$. Therefore, the tamely ramified case does not satisfy the Hasse–Arf property in general since the distance needs to be divisible by $p^{n-n_i}q$, not p^{n-n_i}. Noticeably, it violates the Hasse–Arf property simply because it has an unexpected leading element G_0 in the ramification filtration. Hence, this is a removable violation. An easy way to address this is to manually modify the group index assumption from G_0 to a Sylow p-subgroup of $\mathrm{Gal}(L/K)$ under Galois correspondence. As a consequence, the Hasse–Arf property is true in the case that L/K is not necessarily assumed to be totally wildly ramified.

6. Conclusion

We analyzed the ramification group filtrations of a Galois function field extension, and reduced the investigation to the totally wildly ramified case. It turns out that the result is explicit. An explanation of why we can obtain such an explicit formula as in Theorem 4.4 is as follows. From (4-6), we have exactly l linear equations for the l variables m_i for $0 \le l \le l-1$. Since the coefficient matrix is triangular in addition to being nonsingular, we can expect that the solution not only exists and is unique but also could be expressed explicitly.

Due to the explicit nature of Corollary 4.5, we can discuss the Hasse–Arf property of such extensions and explore whether it can be strengthened or not. The general answer is no, and there exist examples to show that Hasse–Arf is the best possible divisibility result. Although we can discuss the ramification groups under the totally wild ramified assumption without loss of generality, we discussed whether or not the Hasse–Arf property is true under the general assumptions. The answer is yes, but we have to slightly modify the formulation of the Hasse–Arf property to apply it to the tamely ramified case.

Acknowledgments

We thank Rachel Pries for indicating a result similar to Corollary 4.5. It is likely that G. Griffith Elder anticipated it too, judging by some personal communications with him. We also thank the anonymous referee for her or his comments, which greatly improved the quality of the paper.

References

[Anbar et al. 2009] N. Anbar, H. Stichtenoth, and S. Tutdere, "On ramification in the compositum of function fields", *Bull. Braz. Math. Soc.* (*N.S.*) **40**:4 (2009), 539–552. MR 2011c:14081 Zbl 1187.14029

[Arf 1939] C. Arf, "Untersuchungen über reinverzweigte Erweiterungen diskret bewerteter perfekter Körper", *J. Reine Angew. Math.* **181** (1939), 1–44. MR 1,4a Zbl 0021.20201

[Byott and Elder 2002] N. P. Byott and G. G. Elder, "Biquadratic extensions with one break", *Canad. Math. Bull.* **45**:2 (2002), 168–179. MR 2003a:11149 Zbl 1033.11054

[Byott and Elder 2005] N. P. Byott and G. G. Elder, "New ramification breaks and additive Galois structure", *J. Théor. Nombres Bordeaux* **17**:1 (2005), 87–107. MR 2006b:11149 Zbl 1162.11394

[Byott and Elder 2009] N. P. Byott and G. G. Elder, "On the necessity of new ramification breaks", *J. Number Theory* **129**:1 (2009), 84–101. MR 2009j:11186 Zbl 1196.11156

[Doud 2003] D. Doud, "Wild ramification in number field extensions of prime degree", *Arch. Math.* (*Basel*) **81**:6 (2003), 646–649. MR 2004j:11130 Zbl 1056.11066

[Elder and Hooper 2007] G. G. Elder and J. J. Hooper, "On wild ramification in quaternion extensions", *J. Théor. Nombres Bordeaux* **19**:1 (2007), 101–124. MR 2008m:11230 Zbl 1123.11037

[Fesenko and Vostokov 2002] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, 2nd ed., Translations of Mathematical Monographs **121**, Amer. Math. Soc., Providence, RI, 2002. MR 2003c:11150 Zbl 1156.11046

[Garcia and Stichtenoth 2008] A. Garcia and H. Stichtenoth, "Some remarks on the Hasse–Arf theorem", pp. 141–146 in *Finite fields and applications*, edited by G. L. Mullen et al., Contemp. Math. **461**, Amer. Math. Soc., Providence, RI, 2008. MR 2010a:11229 Zbl 1236.11100

[Hasse 1930] H. Hasse, "Führer, Diskriminante und Verzweigungskörper relativ–Abelscher Zahlkörper", *J. Reine Angew. Math.* **162** (1930), 169–184. JFM 56.0166.03

[Hasse 1934] H. Hasse, "Normenresttheorie galoisscher Zahlkörper mit Anwendungen auf Führer und Diskriminante abelscher Zahlkörper", *J. Fac. Sci., Univ. Tokyo, Sect. I* **2** (1934), 477–498. Zbl 0009.04906

[Hungerford 1974] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics **73**, Springer, New York, 1974. MR 82a:00006 Zbl 0442.00002

[Maus 1968] E. Maus, "Die gruppentheoretische Struktur der Verzweigungsgruppenreihen", *J. Reine Angew. Math.* **230** (1968), 1–28. MR 37 #1356 Zbl 0165.35703

[Maus 1971] E. Maus, "On the jumps in the series of ramifications groups", pp. 127–133 in *Colloque de Théorie des Nombres* (Univ. Bordeaux, Bordeaux, 1969), Mém. Soc. Math. France **25**, Soc. Math. France, Paris, 1971. MR 51 #449 Zbl 0245.12014

[Maus 1972] E. Maus, "Über die Verteilung der Grundverzweigungszahlen von wild verzweigten Erweiterungen $\mathfrak{p}$-adischer Zahlkörper", *J. Reine Angew. Math.* **257** (1972), 47–79. MR 50 #7099 Zbl 0263.12007

[Roberts 2009] D. P. Roberts, "Review of [Garcia and Stichtenoth 2008]", review, MathSciNet, 2009, Available at http://www.ams.org/mathscinet-getitem?mr=2436331.

[Rosen 2002] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer, New York, 2002. MR 2003d:11171 Zbl 1043.11079

[Sen and Tate 1963] S. Sen and J. Tate, "Ramification groups of local fields", *J. Indian Math. Soc.* (*N.S.*) **27** (1963), 197–202. MR 34 #1303 Zbl 0136.02702

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, New York-Berlin, 1979. MR 82e:12016 Zbl 0423.12016

[Stichtenoth 2009] H. Stichtenoth, *Algebraic function fields and codes*, 2nd ed., Graduate Texts in Mathematics **254**, Springer, Berlin, 2009. MR 2010d:14034 Zbl 1155.14022

[Thomas 2005] L. Thomas, "Ramification groups in Artin–Schreier–Witt extensions", *J. Théor. Nombres Bordeaux* **17**:2 (2005), 689–720. MR 2007a:11157 Zbl 1207.11109

[Viviani 2004] F. Viviani, "Ramification groups and Artin conductors of radical extensions of $\mathbb{Q}$", *J. Théor. Nombres Bordeaux* **16**:3 (2004), 779–816. MR 2006j:11148 Zbl 1075.11073

[Wu and Scheidler 2010] Q. Wu and R. Scheidler, "The ramification groups and different of a compositum of Artin–Schreier extensions", *Int. J. Number Theory* **6**:7 (2010), 1541–1564. MR 2011k:11161 Zbl 1225.11148

[Wyman 1969] B. F. Wyman, "Wildly ramified gamma extensions", *Amer. J. Math.* **91** (1969), 135–152. MR 39 #2726 Zbl 0188.11003

Received May 22, 2014. Revised December 10, 2014.

JEFFREY A. CASTAÑEDA
DEPARTMENT OF MATHEMATICS AND PHYSICS
TEXAS A&M INTERNATIONAL UNIVERSITY
LAREDO, TX 78041-1900
UNITED STATES
jcasta1@dusty.tamiu.edu

QINGQUAN WU
DEPARTMENT OF MATHEMATICS AND PHYSICS
TEXAS A&M INTERNATIONAL UNIVERSITY
LAREDO, TX 78041-1900
UNITED STATES
qingquan.wu@tamiu.edu

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US \$420/year for the electronic version, and \$570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 276 No. 2 August 2015