Pacific Journal of Mathematics

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Volume 276 No. 2 August 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 276, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.276.369

THE COMPLEX MONGE–AMPÈRE EQUATION ON SOME COMPACT HERMITIAN MANIFOLDS

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We consider the complex Monge–Ampère equation on compact manifolds when the background metric is a Hermitian metric (in complex dimension 2) or a Hermitian metric satisfying an additional condition (in higher dimensions). We prove that the Laplacian estimate holds when F is in W^{1,q_0} for any $q_0 > 2n$. As an application, we show that, up to scaling, there exists a unique classical solution in W^{3,q_0} for the complex Monge–Ampère equation when F is in W^{1,q_0}.

1. Introduction

We consider the regularity problem of the complex Monge–Ampère equation on some compact Hermitian manifolds. Let (M, g) be a compact Hermitian manifold of complex dimension $n \geq 2$. For a real-valued function F on M, we consider the Monge–Ampère equation

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = e^F \det(g_{ij}),$$

with $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$, for a real-valued function ϕ such that $\sup_M \phi = -1$. We write

$$\omega = \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \quad \text{and} \quad \tilde{\omega} = \sqrt{-1}\tilde{g}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j,$$

where $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$. Thus, the Monge–Ampère equation can be written as

$$\text{(1-1)} \qquad \begin{cases} \tilde{\omega}^n = e^F \omega^n, \\ \tilde{\omega} = \omega + \sqrt{-1}\,\partial\bar{\partial}\phi > 0, \\ \sup_M \phi = -1. \end{cases}$$

For functions f, h and a holomorphic coordinate $z = (z^1, \dots, z^n)$ we write

$$f_{i\bar{j}} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}, \qquad \Delta f = g^{i\bar{j}} f_{i\bar{j}}, \qquad \tilde{\Delta} f = \tilde{g}^{i\bar{j}} f_{i\bar{j}},$$
$$|\nabla f|^2 = g^{i\bar{j}} f_i f_{\bar{j}}, \quad |\tilde{\nabla} f|^2 = \tilde{g}^{i\bar{j}} f_i f_{\bar{j}}, \quad \langle \nabla f, \nabla h \rangle = g^{i\bar{j}} f_i h_{\bar{j}}.$$

MSC2010: 35J96, 53C55.
Keywords: complex Monge–Ampère equation, compact Hermitian manifold.

We use $\|f\|_{L^p(M,\omega)}$ and $\|\nabla^m f\|_{L^p(M,\omega)}$ to denote the corresponding norms with respect to (M, ω).

When ω is Kähler, the complex Monge–Ampère equation is very important. Calabi [1957] presented his famous conjecture and transformed that problem into (1-1). Yau [1978] proved the existence of the classical solution of (1-1) by using the continuity method and solved Calabi's conjecture.

The Dirichlet problem for the complex Monge–Ampère equation is also very important. Bedford and Taylor [1976; 1982] studied the weak solution. After their work, weak solutions of the complex Monge–Ampère equation have been studied extensively. There are many existence, uniqueness and regularity results of the complex Monge–Ampère equation under different conditions, and we refer the reader to [Błocki 2005; Demailly and Pali 2010; Dinew 2009; Eyssidieux et al. 2009; Guedj and Zeriahi 2007; Kołodziej 1998; 2008; Zhang 2006].

On the other hand, the classical solvability of the Dirichlet problem was established by Caffarelli, Kohn, Nirenberg and Spruck [1985] for strongly pseudoconvex domains in $\mathbb{C}^n$. The reader can also see [Krylov 1989; Krylov 1994]. For further information, we refer the reader to [Phong et al. 2012], which is a survey of some recent developments in the theory of the complex Monge–Ampère equation.

When ω is not Kähler, the existence of the solution of the complex Monge–Ampère equation has been studied under some assumptions on ω (see [Cherrier 1987; Guan and Li 2009; Hanani 1996; Tosatti and Weinkove 2010b]). For a general ω, Tosatti and Weinkove [2010a] obtained the key C^0-estimate. As an application, they showed that, up to scaling, the complex Monge–Ampère equation on a compact Hermitian manifold admits a smooth solution when the right hand side F is smooth.

Chen and He [2012] have proved that, on a compact Kähler manifold of complex dimension n, the Laplacian estimate and the gradient estimate hold and there exists a classical solution in W^{3,q_0} for the complex Monge–Ampère equation when the right-hand side F is in W^{1,q_0} for any $q_0 > 2n$.

In this paper, we generalize the work of Chen and He. We use a different method (we don't need the gradient estimate to get the Laplacian estimate) to consider the regularity problem of (1-1) on some compact Hermitian manifolds (including compact Kähler manifolds).

Definition 1.1. A compact Hermitian manifold (M, ω) of complex dimension n satisfies condition $(*)$ if, for any $\phi \in C^2(M)$ such that

$$\tilde{\omega} = \omega + \sqrt{-1}\,\partial\bar{\partial}\phi > 0, \quad \|\phi\|_{L^\infty(M,\omega)} \le \Lambda_1 \quad \text{and} \quad \Lambda_2^{-1}\omega^n \le \tilde{\omega}^n \le \Lambda_2\omega^n,$$

there exists a constant $C = C(\Lambda_1, \Lambda_2, M, \omega)$ such that

$$-C\omega^n \le \sqrt{-1}\partial\bar{\partial}\tilde{\omega}^{n-1} \le C\omega^n.$$

Remark 1.2. When $n = 2$, condition $(*)$ is trivial. Since

$$\partial\bar{\partial}\tilde{\omega} = \partial\bar{\partial}\omega,$$

all compact Hermitian manifolds of complex dimension 2 satisfy condition $(*)$.

Remark 1.3. When $n = 3$, if (M, ω) is a compact Hermitian manifold satisfying

$$\partial\bar{\partial}\omega = 0,$$

then we have

$$\partial\bar{\partial}\tilde{\omega}^2 = 2\partial\omega \wedge \bar{\partial}\omega,$$

which implies this Hermitian manifold (M, ω) satisfies condition $(*)$.

Remark 1.4. When $n \geq 4$, condition $(*)$ is not a very strong restricted condition. For example, if (M, ω) is a compact Hermitian manifold satisfying

$$\partial\bar{\partial}\omega = 0 \text{ and } \partial\bar{\partial}\omega^2 = 0, \tag{1-2}$$

then we can conclude that $\partial\bar{\partial}\omega^k = 0$ for all $1 \leq k \leq n-1$ (see, for example, [Fino and Tomassini 2011]), which implies that $\partial\bar{\partial}\tilde{\omega}^k = 0$ for all $1 \leq k \leq n-1$. Thus, such a Hermitian manifold (satisfying (1-2)) satisfies condition $(*)$. For example, the products of a complex curve with a Kähler metric and a complex surface with a non-Kähler Gauduchon metric satisfy (1-2). More examples are constructed in [Fino and Tomassini 2011].

Remark 1.5. All compact Kähler manifolds satisfy condition $(*)$.

Now, we state our Laplacian estimate as follows.

Theorem 1.6. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. Assume that either*

(1) *$n = 2$, or*

(2) *$n \geq 3$ and (M, ω) satisfies condition $(*)$.*

For any $q_0 > 2n$, if ϕ is a smooth solution of (1-1), then

$$\|n + \Delta\phi\|_{L^\infty(M,\omega)} \leq C(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

Usually, we need the gradient estimate to derive the Laplacian estimate. However, the computation on Hermitian manifolds is more complicated due to the existence of torsion terms. As a result, the gradient estimate is very difficult to obtain. In order to solve this problem, we introduce a new method to obtain the Laplacian estimate directly. By using Moser's iteration [1960], L^p estimates (for example, see [Gilbarg and Trudinger 1977]) and some interpolation inequalities, we can obtain the Laplacian estimate without doing any calculations involving the gradient, which makes the argument simpler and clearer. Therefore, we believe that our ideas can be applied to other nonlinear equations on compact manifolds.

As an application of Theorem 1.6, we have the following theorem:

Theorem 1.7. *Assume that (M, ω) satisfies condition (1) or (2) of Theorem 1.6. Let F be a function in W^{1,q_0} for any $q_0 > 2n$. Then there exist a function $\phi \in W^{3,q_0}$ and a constant b such that*

$$\begin{cases} \tilde{\omega}^n = e^{F+b}\omega^n, \\ \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0, \\ \sup_M \phi = -1. \end{cases}$$

2. Some preliminary computations

We need the following C^0-estimate from [Tosatti and Weinkove 2010a]:

Theorem 2.1. *For any compact Hermitian manifold (M, ω), if ϕ is a smooth solution of (1-1), then we have*

$$\|\phi\|_{L^\infty(M,\omega)} \leq C,$$

where $C = C(\sup_M F, M, \omega)$.

We need the following lemma from [Tosatti and Weinkove 2015]:

Lemma 2.2. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. If ϕ is a smooth solution of (1-1), then, for any $\epsilon > 0$, we have*

$$\tilde{\Delta}(\Delta\phi) + (\epsilon - 1)\frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n+\Delta\phi)} \geq \Delta F - A(1+1/\epsilon)(n+\Delta\phi)(n-\tilde{\Delta}\phi), \tag{2-1}$$

where $A = A(M, \omega, \|F\|_{L^\infty(M,\omega)})$.

Proof. We need the following equation, which is [Tosatti and Weinkove 2015, (9.5)]:

$$\tilde{\Delta}(\log(\mathrm{tr}_g \tilde{g})) \geq \frac{2}{(\mathrm{tr}_g \tilde{g})^2}\operatorname{Re}(\tilde{g}^{k\bar{l}}T^i_{ik}(\mathrm{tr}_g \tilde{g})_{\bar{l}}) + \frac{\Delta F}{\mathrm{tr}_g \tilde{g}} - C_1 \mathrm{tr}_{\tilde{g}}\, g - C_1,$$

where the tensor T is the torsion of (M, ω) and $C_1 = C_1(M, \omega, \|F\|_{L^\infty(M,\omega)})$. After some calculations, we have

$$\tilde{\Delta}(\Delta\phi) - \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n+\Delta\phi)} \geq \frac{2}{(n+\Delta\phi)}\operatorname{Re}(\tilde{g}^{k\bar{l}}T^i_{ik}(\Delta\phi)_{\bar{l}}) + \Delta F - C_2(n+\Delta\phi)(n-\tilde{\Delta}\phi),$$

where $C_2 = C_2(M, \omega, \|F\|_{L^\infty(M,\omega)})$; we have used that $\operatorname{tr}_{\tilde g} g = (n - \tilde\Delta\phi) \ge ne^{-F/n}$. By the Cauchy–Schwarz inequality, for any $\epsilon > 0$, we have that

$$\begin{aligned}\tilde\Delta(\Delta\phi) - \frac{|\tilde\nabla(\Delta\phi)|^2}{(n+\Delta\phi)}&\\ &\ge -\epsilon\frac{|\tilde\nabla(\Delta\phi)|^2}{(n+\Delta\phi)} - \frac{A}{\epsilon}(n+\Delta\phi)(n-\tilde\Delta\phi) + \Delta F - A(n+\Delta\phi)(n-\tilde\Delta\phi),\end{aligned}$$

where $A = A(M, \omega, \|F\|_{L^\infty(M,\omega)})$ and we have used that $(n+\Delta\phi) \ge ne^{F/n}$. □

Lemma 2.3. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. If ϕ is a smooth solution of (1-1), then, for any $p \ge 1$, we have*

$$\begin{aligned}\tilde\Delta(e^{f_p(\phi)}(n+\Delta\phi)^p)&\\ &\ge C_1(p)(n+\Delta\phi)^{p+\frac{1}{n-1}} - C_2(p)(n+\Delta\phi)^p + pe^{f_p(\phi)}(n+\Delta\phi)^{p-1}\Delta F,\end{aligned}$$

where

$$\begin{aligned} f_p(\phi) &= e^{-A(p+3)\phi}, & A &= A(\|F\|_{L^\infty(M,\omega)}, M, \omega),\\ C_1(p) &= C_1(p, \|F\|_{L^\infty(M,\omega)}, M, \omega), & C_2(p) &= C_2(p, \|F\|_{L^\infty(M,\omega)}, M, \omega).\end{aligned}$$

Proof. By direct calculation, we have

$$\begin{aligned}\tilde\Delta(e^{f_p(\phi)}(n+\Delta\phi)^p)& \\ = f_p' e^{f_p(\phi)}(\tilde\Delta\phi)(n+\Delta\phi)^p &+ (f_p'^2 + f_p'')e^{f_p(\phi)}|\tilde\nabla\phi|^2(n+\Delta\phi)^p \\ &+ pe^{f_p(\phi)}\tilde\Delta(\Delta\phi)(n+\Delta\phi)^{p-1} + p(p-1)e^{f_p(\phi)}|\tilde\nabla(\Delta\phi)|^2(n+\Delta\phi)^{p-2}\\ &+2pf_p' e^{f_p(\phi)}(n+\Delta\phi)^{p-1}\operatorname{Re}(\tilde g^{k\bar l}\phi_k(\Delta\phi)_{\bar l}).\end{aligned} \tag{2-2}$$

By the definition of $f_p(\phi)$, we have

$$\begin{cases} f_p'(\phi) = -A(p+3)e^{-A(p+3)\phi} < 0,\\ f_p''(\phi) = A^2(p+3)^2 e^{-A(p+3)\phi} > 0.\end{cases} \tag{2-3}$$

Thus, by the Cauchy–Schwarz inequality, we have

$$2\operatorname{Re}(\tilde g^{k\bar l}\phi_k(\Delta\phi)_{\bar l}) \le \frac{(f_p'^2 + f_p'')(n+\Delta\phi)}{-pf_p'}|\tilde\nabla\phi|^2 + \frac{-pf_p'}{(f_p'^2+f_p'')(n+\Delta\phi)}|\tilde\nabla(\Delta\phi)|^2,$$

which implies that

$$\begin{aligned} 2pf_p' e^{f_p(\phi)}(n+\Delta\phi)^{p-1}\operatorname{Re}(\tilde g^{k\bar l}\phi_k(\Delta\phi)_{\bar l})& \\ \ge -(f_p'^2 + f_p'')e^{f_p(\phi)}&|\tilde\nabla\phi|^2(n+\Delta\phi)^p \\ &- \frac{p^2 f_p'^2}{f_p'^2 + f_p''}e^{f_p(\phi)}(n+\Delta\phi)^{p-2}|\tilde\nabla(\Delta\phi)|^2.\end{aligned} \tag{2-4}$$

Combining (2-2) and (2-4), we have

$$
\begin{aligned}
&\tilde{\Delta}(e^{f_p(\phi)}(n+\Delta\phi)^p)\\
&\qquad \geq f_p' e^{f_p(\phi)}(n+\Delta\phi)^p\tilde{\Delta}\phi + pe^{f_p(\phi)}\tilde{\Delta}(\Delta\phi)(n+\Delta\phi)^{p-1}\\
&\qquad\qquad + |\tilde{\nabla}(\Delta\phi)|^2(n+\Delta\phi)^{p-2}e^{f_p(\phi)}\left(p(p-1)-\frac{p^2f_p'^2}{f_p'^2+f_p''}\right)\\
&\qquad \geq pe^{f_p(\phi)}(n+\Delta\phi)^{p-1}\left(\tilde{\Delta}(\Delta\phi)+\left(\frac{pf_p''}{(f_p')^2+f_p''}-1\right)\frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n+\Delta\phi)}\right)\\
&\qquad\qquad + f_p' e^{f_p(\phi)}(n+\Delta\phi)^p\tilde{\Delta}\phi.
\end{aligned}
$$

By Lemma 2.2 (take $\epsilon = pf_p''/((f_p')^2+f_p'')$), we obtain

$$
\begin{aligned}
&\tilde{\Delta}(e^{f_p(\phi)}(n+\Delta\phi)^p)\\
&\qquad \geq f_p' e^{f_p(\phi)}(n+\Delta\phi)^p\tilde{\Delta}\phi + pe^{f_p(\phi)}(n+\Delta\phi)^{p-1}\Delta F\\
&\qquad\qquad - Ape^{f_p(\phi)}(n+\Delta\phi)^p(n-\tilde{\Delta}\phi)\left(1+\frac{(f_p')^2+f_p''}{pf_p''}\right)\\
&\qquad = nf_p' e^{f_p(\phi)}(n+\Delta\phi)^p + pe^{f_p(\phi)}(n+\Delta\phi)^{p-1}\Delta F\\
&\qquad\qquad + e^{f_p(\phi)}(n+\Delta\phi)^p(n-\tilde{\Delta}\phi)\left(-f_p' - Ap\left(1+\frac{(f_p')^2+f_p''}{pf_p''}\right)\right)\\
&\qquad \geq nf_p' e^{f_p(\phi)}(n+\Delta\phi)^p + pe^{f_p(\phi)}(n+\Delta\phi)^{p-1}\Delta F\\
&\qquad\qquad + Ae^{f_p(\phi)}(n+\Delta\phi)^p(n-\tilde{\Delta}\phi),
\end{aligned}
\tag{2-5}
$$

where we have used that $\sup_M \phi = -1$ and (2-3). It is clear that

$$
\operatorname{tr}_g \tilde{g} \leq (\operatorname{tr}_{\tilde{g}} g)^{n-1}\frac{\det\tilde{g}}{\det g},
$$

which implies that

$$
(n+\Delta\phi) \leq (n-\tilde{\Delta}\phi)^{n-1}e^F. \tag{2-6}
$$

Combining this with (2-5) and (2-6), the proof is complete. □

For convenience, we introduce some notation here: we set

$$
u = e^{f_1(\phi)}(n+\Delta\phi). \tag{2-7}
$$

Thus, by Young's inequality and Lemma 2.3, we have

$$
\tilde{\Delta}u \geq e^{f_1(\phi)}\Delta F - \tilde{C}, \tag{2-8}
$$

where $\tilde{C} = \tilde{C}(\|F\|_{L^\infty(M,\omega)}, M, \omega)$.

3. The Laplacian estimate

We remark that in this section our constants may differ from line to line.

Lemma 3.1. *Let (M, ω) be a compact Hermitian manifold. If ϕ is a smooth solution of (1-1), then, for any $f \in C^\infty(M)$, we have*

$$|\nabla f|^2 \leq Cu|\tilde{\nabla} f|^2,$$

where u is defined in (2-7) and $C = C(\|F\|_{L^\infty(M,\omega)}, M, \omega)$.

Proof. By direct calculation, we have

$$|\nabla f|^2 \leq (n + \Delta\phi)|\tilde{\nabla} f|^2.$$

Combining this with (2-7) and Theorem 2.1, the proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem 1.6, for any $p \geq 0$, we have*

$$\int_M |\nabla(u^{\frac{p}{2}})|^2\omega^n \leq C(p^2+1)\int_M u^p(1+|\nabla F|^2)\omega^n + Cp\int_M u^p|\nabla\phi||\nabla F|\omega^n + C\int_M u^{p+1}\omega^n,$$

where u is defined in (2-7) and $C = C(\|F\|_{L^\infty(M,\omega)}, M, \omega)$.

Proof. By Lemma 3.1 and direct calculation, we have

$$\begin{aligned}
\int_M |\nabla(u^{\frac{p}{2}})|^2\omega^n &\leq C_1\int_M u|\tilde{\nabla}(u^{\frac{p}{2}})|^2\tilde{\omega}^n \\
&= C_1 np\sqrt{-1}\int_M \partial u^p \wedge \bar{\partial} u \wedge \tilde{\omega}^{n-1} \\
&= -C_1 np\sqrt{-1}\int_M u^p\,\partial\bar{\partial} u \wedge \tilde{\omega}^{n-1} + \frac{C_1 np}{p+1}\sqrt{-1}\int_M \bar{\partial} u^{p+1} \wedge \partial\tilde{\omega}^{n-1} \\
&= -C_1 p\int_M u^p(\tilde{\Delta} u)\tilde{\omega}^n - \frac{C_1 np}{p+1}\sqrt{-1}\int_M u^{p+1}\,\partial\bar{\partial}\tilde{\omega}^{n-1},
\end{aligned}$$

where $C_1 = C_1(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Since M satisfies condition $(*)$ (when $n = 2$, all Hermitian manifolds satisfy condition $(*)$), we have

$$-\frac{C_1 np}{p+1}\sqrt{-1}\int_M u^{p+1}\,\partial\bar{\partial}\tilde{\omega}^{n-1} \leq C_2\int_M u^{p+1}\omega^n,$$

where $C_2 = C_2(\|F\|_{L^\infty(M,\omega)}, M, \omega)$ (Since $n = \dim_{\mathbb{C}} M$, we can absorb it into the constant C_2). By (2-8) and $\tilde{\omega}^n = e^F\omega^n$, we obtain

$$\begin{aligned}-C_1 p\int_M u^p(\tilde{\Delta}u)\tilde{\omega}^n &\le C_3 p\int_M u^p(\tilde{C}-e^{f_1(\phi)}\Delta F)\tilde{\omega}^n\\ &\le C_3\tilde{C}p\int_M u^p\tilde{\omega}^n - C_3 p\int_M e^{f_1(\phi)}u^p(\Delta(e^F)-e^F|\nabla F|^2)\omega^n\\ &\le C_4 p\int_M u^p(1+|\nabla F|^2)\omega^n + C_3 p\int_M \langle\nabla(e^{f_1(\phi)}u^p),\nabla(e^F)\rangle\omega^n\\ &\qquad -\sqrt{-1}C_3 np\int_M e^{f_1(\phi)}u^p\bar{\partial}e^F\wedge\partial\omega^{n-1},\end{aligned}$$

where $C_3 = C_3(\|F\|_{L^\infty(M,\omega)}, M, \omega)$, $C_4 = C_4(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. It is clear that

$$\begin{aligned}&C_3 p\int_M \langle\nabla(e^{f_1(\phi)}u^p),\nabla(e^F)\rangle\omega^n\\ &\quad = C_3 p\int_M u^p\langle\nabla(e^{f_1(\phi)}),\nabla(e^F)\rangle\omega^n + C_3 p\int_M e^{f_1(\phi)}\langle\nabla(u^p),\nabla(e^F)\rangle\omega^n\\ &\quad \le C_5 p\int_M u^p|\nabla F||\nabla\phi|\omega^n + \frac{1}{2}\int_M |\nabla u^{\frac{p}{2}}|^2\omega^n + C_5 p^2\int_M u^p|\nabla F|^2\omega^n,\end{aligned}$$

where $C_5 = C_5(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Here we have used the Cauchy–Schwarz inequality. We notice that

$$-\sqrt{-1}C_3 np\int_M e^{f_1(\phi)}u^p\,\bar{\partial}e^F\wedge\partial\omega^{n-1} \le C_6 p\int_M u^p|\nabla F|\omega^n,$$

where $C_6 = C_6(\|F\|_{L^\infty(M,\omega)}, M, \omega)$ (Since $n = \dim_{\mathbb{C}} M$, we can absorb it into the constant C_6). Combining the above inequalities, we complete the proof. □

Theorem 3.3. *Under the assumptions of Theorem 1.6, we have*

$$\|u\|_{L^\infty(M,\omega)} \le C\big(\|u\|_{L^{\frac{q_0}{2}}(M,\omega)}, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega\big).$$

Proof. Without loss of generality, we can assume that $q_0 < \infty$. We use the iteration method (see [Moser 1960]). By the Sobolev inequality (Corollary A.2) and Lemma 3.2, for $p \ge 1$ we have

$$\begin{aligned}&\left(\int_M u^{p\beta}\omega^n\right)^{\frac{1}{\beta}}\\ &\quad\le C_1\int_M u^p\omega^n + C_1\int_M |\nabla(u^{\frac{p}{2}})|^2\omega^n\\ &\quad\le C_1\int_M u^p\omega^n + C_1 p^2\int_M u^p(1+|\nabla F|^2)\omega^n\\ &\qquad + C_1 p\int_M u^p|\nabla\phi||\nabla F|\omega^n + C_1\int_M u^{p+1}\omega^n\\ &\quad\le C_1 p^2\int_M u^{p+1}\omega^n + C_1 p^2\int_M u^p|\nabla F|^2\omega^n + C_1 p^2\int_M u^p|\nabla\phi||\nabla F|\omega^n,\end{aligned}$$

where $\beta = n/(n-1)$ and $C_1 = C_1(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Here we have used Young's inequality and the inequality $p \le p^2$. By the Hölder inequality, we have

$$\int_M u^p |\nabla F|^2 \omega^n \le \left(\int_M u^{pr_0}\omega^n\right)^{\frac{1}{r_0}} \left(\int_M |\nabla F|^{q_0}\omega^n\right)^{\frac{2}{q_0}}$$

and

$$\int_M u^p |\nabla\phi||\nabla F|\omega^n \le \left(\int_M u^{pr_0}\omega^n\right)^{\frac{1}{r_0}} \left(\int_M |\nabla\phi|^{q_0}\omega^n\right)^{\frac{1}{q_0}} \left(\int_M |\nabla F|^{q_0}\omega^n\right)^{\frac{1}{q_0}},$$

where $1/r_0 + 2/q_0 = 1$. Combining the above inequalities, when $pr_0 \ge p+1$ (that is, $p \ge (q_0-2)/2$), we obtain

$$\begin{aligned}\|u\|_{L^{p\beta}(M,\omega)} &\le (C_2 p^2(\|\nabla\phi\|_{L^{q_0}(M,\omega)}+1))^{\frac{1}{p}} \big(\|u\|^{\frac{p+1}{p}}_{L^{p+1}(M,\omega)} + \|u\|_{L^{pr_0}(M,\omega)}\big)\\ &\le (C_2 p^2(\|\nabla\phi\|_{L^{q_0}(M,\omega)}+1))^{\frac{1}{p}} \|u\|^{\frac{p+1}{p}}_{L^{pr_0}(M,\omega)},\end{aligned}$$

where $C_2 = C_2(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega)$. By Lemma A.6, we have

$$\begin{aligned}\|\nabla\phi\|_{L^{q_0}(M,\omega)} &\le C_3\|u\|_{L^{\frac{2nq_0}{2n+q_0}}(M,\omega)} + C_3\\ &\le C_3\|u\|_{L^{\frac{q_0}{2}}(M,\omega)} + C_3,\end{aligned}$$

where $C_3 = C_3(q_0, \|F\|_\infty, M, \omega)$. Thus, for any $k \ge 0$, we have

(3-1) $$\|u\|_{L^{p_k\beta}(M,\omega)} \le a_k \|u\|^{b_k}_{L^{p_k r_0}(M,\omega)},$$

where

$$\begin{aligned} a_k &= \big(C_4 p_k^2\big(\|u\|_{L^{\frac{q_0}{2}}(M,\omega)}+1\big)\big)^{\frac{1}{p_k}}, & C_4 &= C_4(\|F\|_{W1,q_0(M,\omega)}, q_0, M, \omega),\\ b_k &= \frac{p_k+1}{p_k}, & p_k &= \frac{q_0-2}{2}\left(\frac{\beta}{r_0}\right)^k.\end{aligned}$$

Here we point out that $q_0 > 2n$ implies that $\beta/r_0 > 1$. By (3-1), we have

(3-2) $$\|u\|_{L^{p_k\beta}(M,\omega)} \le a_k a_{k-1}^{b_k} \cdots a_0^{b_k\cdots b_1} \|u\|^{b_k\cdots b_0}_{L^{p_0 r_0}(M,\omega)}.$$

Without loss of generality, we can assume that $a_k \ge 1$ for $k \ge 0$. We observe that $\prod_{i=0}^\infty b_k$ and $\prod_{i=0}^\infty a_k$ are convergent. In (3-2), letting $k \to \infty$, we obtain

$$\|u\|_{L^\infty(M,\omega)} \le C\big(\|u\|_{L^{\frac{q_0}{2}}(M,\omega)}, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega\big). \qquad \square$$

Lemma 3.4. *Under the assumptions of Theorem 1.6, for any $p \ge 1$, we have*

$$\int_M u^{p+\frac{1}{n-1}}\omega^n \le C(p)\int_M u^{p-1}|\nabla\phi||\nabla F|\omega^n + C(p)\int_M u^{p-1}|\nabla F|^2\omega^n + C(p),$$

where $C(p) = C(p, \|F\|_{L^\infty(M,\omega)}, M, \omega)$.

Proof. Starting with Lemma 2.3 and then integrating over $(M,\tilde{\omega})$, for any $p\geq 1$ we obtain

$$\begin{aligned}\int_M &\tilde{\Delta}(e^{f_p(\phi)}(n+\Delta\phi)^p)\tilde{\omega}^n\\ &\geq C_1(p)\int_M u^{p+\frac{1}{n-1}}\tilde{\omega}^n - C_2(p)\int_M u^p\tilde{\omega}^n + p\int_M e^{f_p(\phi)}(n+\Delta\phi)^{p-1}\Delta F e^F\omega^n,\end{aligned}$$

where $C_1(p)=C_1(p,\|F\|_{L^\infty(M,\omega)},M,\omega)$ and $C_2(p)=C_2(p,\|F\|_{L^\infty(M,\omega)},M,\omega)$. Here we have used (2-7) and Theorem 2.1. Since M satisfies condition $(*)$, we have

$$\begin{aligned}\int_M \tilde{\Delta}(e^{f_p(\phi)}(n+\Delta\phi)^p)\tilde{\omega}^n &= n\sqrt{-1}\int_M \partial\bar{\partial}(e^{f_p(\phi)}(n+\Delta\phi)^p)\wedge\tilde{\omega}^{n-1}\\ &= n\sqrt{-1}\int_M e^{f_p(\phi)}(n+\Delta\phi)^p\,\partial\bar{\partial}\tilde{\omega}^{n-1}\\ &\leq C_3(p)\int_M u^p\omega^n,\end{aligned}$$

where $C_3(p)=C_3(p,\|F\|_{L^\infty(M,\omega)},M,\omega)$ (Since $n=\dim_{\mathbb{C}}M$, we can absorb it into the constant C_3). Combining the above inequalities, we compute that

$$\begin{aligned}\int_M &u^{p+\frac{1}{n-1}}\omega^n\\ &\leq C_4(p)\int_M e^{f_p(\phi)}(n+\Delta\phi)^{p-1}(|\nabla F|^2e^F-\Delta(e^F))\omega^n + C_5(p)\int_M u^p\omega^n\\ &\leq C_5(p)\int_M u^{p-1}|\nabla F|^2\omega^n + C_4(p)\int_M\langle\nabla(e^{f_p(\phi)}(n+\Delta\phi)^{p-1}),\nabla e^F\rangle\omega^n\\ &\qquad - C_4(p)n\sqrt{-1}\int_M e^{f_p}(n+\Delta\phi)^{p-1}\,\bar{\partial}e^F\wedge\partial\omega^{n-1} + C_5(p)\int_M u^p\omega^n\\ &\leq C_5(p)\int_M u^p\omega^n + C_5(p)\int_M u^{p-1}|\nabla F|^2\omega^n + C_5(p)\int_M u^{p-1}|\nabla F|\omega^n\\ &\qquad + C_5(p)\int_M|\nabla(u^{p-1})||\nabla F|\omega^n + C_5(p)\int_M u^{p-1}|\nabla\phi||\nabla F|\omega^n,\end{aligned}$$

where $C_4(p)=C_4(p,\|F\|_{L^\infty(M,\omega)},M,\omega)$ and $C_5(p)=C_5(p,\|F\|_{L^\infty(M,\omega)},M,\omega)$ (Since $n=\dim_{\mathbb{C}}M$, we can absorb it into the constant C_5). By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}C_5(p)\int_M|\nabla(u^{p-1})||\nabla F|\omega^n &= C_5(p)\int_M|\nabla(u^{\frac{p-1}{2}})|u^{\frac{p-1}{2}}|\nabla F|\omega^n\\ &\leq C_5(p)\int_M|\nabla(u^{\frac{p-1}{2}})|^2\omega^n + C_5(p)\int_M u^{p-1}|\nabla F|^2\omega^n.\end{aligned}$$

Combining this with the above inequalities and Lemma 3.2, we get

$$\begin{aligned}\int_M & u^{p+\frac{1}{n-1}}\omega^n \\ &\le C_6(p)\int_M u^p\omega^n + C_6(p)\int_M u^{p-1}|\nabla\phi||\nabla F|\omega^n + C_6(p)\int_M u^{p-1}|\nabla F|^2\omega^n,\end{aligned}$$

where $C_6(p) = C_6(p, \|F\|_{L^\infty(M,\omega)}, M, \omega)$. Using Young's inequality, we complete the proof. □

Now, we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. Without loss of generality, we assume that $q_0 < \infty$. By Lemma 3.4 and $F \in W^{1,q_0}$, for any $p \ge 1$, we have

$$\begin{aligned}\int_M u^{p+\frac{1}{n-1}}\omega^n &\le C_1(p)\int_M u^{p-1}|\nabla\phi||\nabla F|\omega^n + C_1(p)\int_M u^{p-1}|\nabla F|^2\omega^n + C_1(p) \\ &\le C_1(p)\int_M u^{p-1}|\nabla\phi|^2\omega^n + C_2(p)\int_M u^{(p-1)\frac{q_0}{q_0-2}}\omega^n + C_2(p),\end{aligned}$$

where $C_1(p) = C_1(p, \|F\|_{L^\infty(M,\omega)}, M, \omega)$, $C_2(p) = C_2(p, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega)$ and we have used the Hölder inequality in the last line. When $p \ge 1$ satisfies that

$$p + \frac{1}{n-1} > (p-1)\frac{q_0}{q_0-2}, \quad \text{or equivalently} \quad p < \frac{q_0-2}{2n-2} + \frac{q_0}{2},$$

we can use Young's inequality to get the inequality

$$\int_M u^{p+\frac{1}{n-1}}\omega^n \le C_3(p)\int_M u^{p-1}|\nabla\phi|^2\omega^n + C_3(p),$$

where $C_3(p) = C_3(p, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega)$. Now, we take $p = q_0/2 - 1/(n-1)$, we obtain

$$\begin{aligned}\int_M u^{\frac{q_0}{2}}\omega^n &\le C_4\int_M u^{\frac{q_0}{2}-\beta}|\nabla\phi|^2\omega^n + C_4 \\ &\le \frac{1}{2}\int_M u^{(\frac{q_0}{2}-\beta)\frac{q_0}{q_0-2\beta}}\omega^n + C_4\int_M |\nabla\phi|^{\frac{q_0}{\beta}}\omega^n + C_4,\end{aligned}$$

where $C_4 = C_4(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega)$ and $\beta = n/(n-1)$. It then follows that

$$\|u\|_{L^{\frac{q_0}{2}}(M,\omega)} \le C_4\|\nabla\phi\|^{\frac{2}{\beta}}_{L^{\frac{q_0}{\beta}}(M,\omega)} + C_4. \tag{3-3}$$

By Lemma A.7, we have

$$\|\nabla\phi\|_{L^{\frac{q_0}{\beta}}(M,\omega)} \le C_5\|u\|^{\frac{1}{2}}_{L^{\frac{q_0}{2\beta}}(M,\omega)} + C_5, \tag{3-4}$$

where $C_5 = C_5(q_0, \|F\|_{L^\infty(M,\omega)}, M, \omega)$. Combining (3-3), (3-4) and $\beta > 1$, we get

$$\|u\|_{L^{\frac{q_0}{2}}(M,\omega)} \leq C_6(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

By Theorem 3.3, we complete the proof. $\square$

4. The Hölder estimate of second order, and solving the equation

We note that when F is in W^{1,q_0}, for any $q_0 > 2n$, Sobolev embedding implies that $F \in C^{\alpha_0}$, where $\alpha_0 = 1 - 2n/q_0$. By Theorem 1.1 of [Tosatti et al. 2014], we have the following theorem:

Theorem 4.1. *Let (M, ω) be a compact Hermitian manifold. If ϕ is a smooth solution of (1-1) and $F \in C^{\alpha_0}$, then there exists a constant $\alpha \in (0, 1)$ such that*

$$\|\phi\|_{C^{2,\alpha}(M,\omega)} \leq C,$$

where α and C depend only on $\|\phi\|_{L^\infty(M,\omega)}$, $\|\Delta\phi\|_{L^\infty(M,\omega)}$, α_0, $\|F\|_{C^{\alpha_0}(M,\omega)}$, q_0, M and ω.

Now we are in a position to prove Theorem 1.7.

Proof of Theorem 1.7. Our argument here is similar to the argument in [Chen and He 2012]. Let $F \in W^{1,q_0}$ on M such that $\|F\|_{W^{1,q_0}(M,\omega)} \leq \Lambda$ for some positive constant Λ. Let $\{F_k\}$ be a sequence of smooth functions such that $F_k \to F$ in W^{1,q_0}. In particular, we can assume that $\|F_k\|_{W^{1,q_0}(M,\omega)} \leq \Lambda + 1$ for any k. By [Tosatti and Weinkove 2010a], there is a unique smooth solution ϕ_k and constant b_k such that

$$\det(g_{i\bar{j}} + (\phi_k)_{i\bar{j}}) = e^{F+b_k} \det(g_{i\bar{j}}),$$

and such that $(g_{i\bar{j}} + (\phi_k)_{i\bar{j}}) > 0$ with normalized condition $\sup_M \phi_k = -1$. By the maximum principle, we have

$$|b_k| \leq C_1(\|F_k\|_{L^\infty(M,\omega)}, M, \omega). \tag{4-1}$$

By Theorem 1.6, Theorem 2.1 and Theorem 4.1, there exists a constant $\alpha \in (0, 1)$ such that

$$\|\phi_k\|_{C^{2,\alpha}(M,\omega)} \leq C_2(\|F_k\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

To get a W^{3,q_0}-estimate, we can localize the estimate as follows. Let ∂ denote an arbitrary first-order differential operator in a domain $\Omega \subset M$. Since we have a $C^{2,\alpha}$-estimate, we compute that

$$\tilde{\Delta}_{g_k}(\partial\phi_k) = \partial(F_k + \log(\det(g_{i\bar{j}}))) - (g_k)^{i\bar{j}}\partial g_{i\bar{j}}$$

in Ω, where $(g_k)_{i\bar{j}} = g_{i\bar{j}} + (\phi_k)_{i\bar{j}}$. Since $\tilde{\Delta}_{g_k}$ is a uniform elliptic operator, by L^p estimates (for example, see [Gilbarg and Trudinger 1977]), for any $\Omega' \subset \Omega$ we have

$$\|\partial\phi_k\|_{W^{2,q_0}(\Omega',\omega)} \le C_3(\Omega, \Omega', q_0, \Lambda, \omega),$$

which implies

$$\|\phi_k\|_{W^{3,q_0}(M,\omega)} \le C_4(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, \Lambda, M, \omega). \tag{4-2}$$

By (4-1) and (4-2), we know that there is a subsequence $\{(\phi_{k_l}, b_{k_l})\}$ of $\{(\phi_k, b_k)\}$ such that $\{b_{k_l}\}$ converges to b and $\{\phi_{k_l}\}$ weakly converges to $\phi \in W^{3,q_0}$ such that $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$, which defines a W^{1,q_0} Hermitian metric. Since the Sobolev embedding $W^{3,q_0} \hookrightarrow C^2$ is compact, the subsequence $\{\phi_{k_l}\}$ converges to ϕ in C^2. Hence ϕ with constant b is a classical solution of the complex Monge–Ampère equation. The uniqueness follows from Remark 5.1 in [Tosatti and Weinkove 2010b]. □

Appendix

Let $g_{\mathbb{R}}$ denote the Riemannian metric induced by g; thus $(M, g_{\mathbb{R}})$ is a Riemannian manifold of real dimension $2n$. In this appendix, we deduce some interpolation inequalities on the Hermitian manifold (M, ω) by using some fundamental inequalities on the Riemannian manifold $(M, g_{\mathbb{R}})$.

Let us recall the definition of $g_{\mathbb{R}}$ first. For any local holomorphic coordinates $(z^1, \dots, z^n)$ with $z^i = x^i + \sqrt{-1}y^i$, $(x^1, \dots, x^n, y^1, \dots, y^n)$ forms a smooth local coordinate system. We define

$$g_{\mathbb{R}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{\mathbb{R}}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\,\mathrm{Re}(g_{i\bar{j}}),$$

while

$$g_{\mathbb{R}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = 2\,\mathrm{Im}(g_{i\bar{j}}).$$

For the Riemannian metric $g_{\mathbb{R}}$, let $\nabla_{\mathbb{R}}$ and $dV_{\mathbb{R}}$ denote the Levi-Civita connection and the volume form, respectively. By direct calculation, we have

$$dV_{\mathbb{R}} = \frac{1}{n!}\omega^n. \tag{A-1}$$

For convenience, we introduce some notation. For any function $f \in C^\infty(M)$, let $\nabla^m_{\mathbb{R}} f$ and $\Delta_{\mathbb{R}} f$ denote the m-th covariant derivative and the Laplacian of f with respect to $g_{\mathbb{R}}$. Let $\|f\|_{L^p(M,g_{\mathbb{R}})}$ and $\|\nabla^m_{\mathbb{R}} f\|_{L^p(M,g_{\mathbb{R}})}$ denote the corresponding norms with respect to $(M, g_{\mathbb{R}})$.

Thus, by (A-1) and some calculation, we have the following lemma:

Lemma A.1. *For any $f \in C^\infty(M)$, we have*

$$\|f\|_{L^p(M,g_{\mathbb{R}})} = C_1(p)\|f\|_{L^p(M,\omega)} \quad \textit{and} \quad \|\nabla_{\mathbb{R}} f\|_{L^p(M,g_{\mathbb{R}})} = C_2(p)\|\nabla f\|_{L^p(M,\omega)},$$

where $C_1(p) = C_1(p,n)$ and $C_2(p) = C_2(p,n)$.

Corollary A.2. *For any $f \in C^\infty(M)$, we have the Sobolev inequality*

$$\left(\int_M f^{2\beta}\omega^n\right)^{\frac{1}{\beta}} \le C\int_M f^2\omega^n + C\int_M |\nabla f|^2\omega^n,$$

where $\beta = n/(n-1)$ and $C = C(M,\omega)$.

Proof. By the Sobolev embedding $W^{1,2}(M,g_{\mathbb{R}}) \hookrightarrow L^{2\beta}(M,g_{\mathbb{R}})$, we have

$$\left(\int_M f^{2\beta}\,dV_{\mathbb{R}}\right)^{\frac{1}{\beta}} \le C_s\int_M f^2\,dV_{\mathbb{R}} + C_s\int_M |\nabla_{\mathbb{R}} f|^2\,dV_{\mathbb{R}},$$

where $C_s = C_s(M,g_{\mathbb{R}})$. Thus, combining this with Lemma A.1, we complete the proof. □

Since $(M,g_{\mathbb{R}})$ is a Riemannian manifold of real dimension $2n$, we have the following interpolation inequality (for example, see [Aubin 1998]):

Theorem A.3. *Let q, r be real numbers such that $1 \le q, r \le +\infty$ and j, m integers such that $0 \le j < m$. Then there exists a constant*

$$C = C(M, g_{\mathbb{R}}, m, j, q, r, \alpha)$$

such that, for all $f \in C^\infty(M)$ with $\int_M f\,dV_{\mathbb{R}} = 0$, we have

$$\|\nabla_{\mathbb{R}}^j f\|_{L^p(M,g_{\mathbb{R}})} \le C\|\nabla^m f\|^{\alpha}_{L^r(M,g_{\mathbb{R}})}\|f\|^{1-\alpha}_{L^q(M,g_{\mathbb{R}})}, \tag{A-2}$$

where

$$\frac{1}{p} = \frac{j}{2n} + \alpha\left(\frac{1}{r} - \frac{m}{2n}\right) + (1-\alpha)\frac{1}{q}$$

for all α in the interval $j/m \le \alpha \le 1$, for which p is nonnegative. If $r = 2n/(m-j) \ne 1$, then (A-2) is not valid for $\alpha = 1$.

Corollary A.4. *Let $f \in C^\infty(M)$; for any $\epsilon > 0$ and $1 \le p < \infty$, we have*

$$\|\nabla_{\mathbb{R}} f\|_{L^p(M,g_{\mathbb{R}})} \le \epsilon\|\nabla_{\mathbb{R}}^2 f\|_{L^p(M,g_{\mathbb{R}})} + C(\epsilon,p)\|f\|_{L^p(M,g_{\mathbb{R}})},$$

where $C(\epsilon,p) = C(\epsilon,p,M,\omega)$.

Proof. Set $\tilde{f} = f - 1/\mathrm{Vol}(M,g_{\mathbb{R}})\int_M f\,dV_{\mathbb{R}}$; then $\int_M \tilde{f}\,dV_{\mathbb{R}} = 0$. By Theorem A.3 we have

$$\|\nabla_{\mathbb{R}}\tilde{f}\|_{L^p(M,g_{\mathbb{R}})} \le C_1(p)\|\nabla_{\mathbb{R}}^2\tilde{f}\|^{\frac{1}{2}}_{L^p(M,g_{\mathbb{R}})}\|\tilde{f}\|^{\frac{1}{2}}_{L^p(M,g_{\mathbb{R}})},$$

where $C_1(p) = C_1(p, M, g_{\mathbb{R}})$. Thus, by the Cauchy–Schwarz inequality, for any $\epsilon > 0$ we obtain

$$\|\nabla_{\mathbb{R}} \tilde{f}\|_{L^p(M,g_{\mathbb{R}})} \leq \epsilon \|\nabla^2_{\mathbb{R}} \tilde{f}\|_{L^p(M,g_{\mathbb{R}})} + C_2(\epsilon, p)\|\tilde{f}\|_{L^p(M,g_{\mathbb{R}})},$$

where $C_2(\epsilon, p) = C_2(\epsilon, p, M, g_{\mathbb{R}})$. By the definition of $\tilde{f}$, the proof is complete. □

Lemma A.5. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. If ϕ is a smooth solution of (1-1), then, for any $1 < p < \infty$, we have*

$$\|\Delta_{\mathbb{R}}\phi\|_{L^p(M,\omega)} \leq C_1(p)\|\Delta\phi\|_{L^p(M,\omega)} + C_2(p),$$

where $C_1 = C_1(p, n)$ and $C_2(p) = C_2(p, \|F\|_{L^\infty(M,\omega)}, M, \omega)$.

Proof. After some calculations, we have

$$\|\Delta_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} \leq 2\|\Delta\phi\|_{L^p(M,g_{\mathbb{R}})} + C_3(p)\|\nabla_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})}, \tag{A-3}$$

where $C_3 = C_3(p, M, \omega)$. For (A-3), one can find more details in [Tosatti 2007] (Lemma 3.2 there shows the exact relation between $\Delta_{\mathbb{R}}$ and 2Δ). By Corollary A.4 we obtain

$$C_3(p)\|\nabla_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} \leq \tfrac{1}{2}\|\Delta_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} + C_4(p)\|\phi\|_{L^p(M,g_{\mathbb{R}})}, \tag{A-4}$$

where $C_4 = C_4(p, M, \omega)$. Combining this with (A-3) and (A-4), we obtain

$$\|\Delta_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} \leq 4\|\Delta\phi\|_{L^p(M,g_{\mathbb{R}})} + C_5(p)\|\phi\|_{L^p(M,g_{\mathbb{R}})},$$

where $C_5 = C_5(p, M, \omega)$. By Theorem 2.1 and Lemma A.1, the proof is complete. □

Lemma A.6. *Under the assumptions of Theorem 1.6, for any $1 < p < 2n$ we have*

$$\|\nabla\phi\|_{L^{\frac{2np}{2n-p}}(M,\omega)} \leq C(p)\|u\|_{L^p(M,\omega)} + C(p),$$

where u is defined in (2-7) and $C(p) = C(p, \|F\|_{L^\infty(M,\omega)}, M, \omega)$.

Proof. By the Sobolev embedding $W^{2,p}(M, g_{\mathbb{R}}) \hookrightarrow W^{1,\frac{2np}{2n-p}}(M, g_{\mathbb{R}})$, we have

$$\|\nabla_{\mathbb{R}}\phi\|_{L^{\frac{2np}{2n-p}}(M,g_{\mathbb{R}})} \leq C_1(p)\|\nabla^2_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} + C_1(p)\|\nabla_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} + C_1(p)\|\phi\|_{L^p(M,g_{\mathbb{R}})},$$

where $C_1(p) = C_1(p, M, g_{\mathbb{R}})$. Combining this with Corollary A.4, we have

$$\|\nabla\phi\|_{L^{\frac{2np}{2n-p}}(M,g_{\mathbb{R}})} \leq C_2(p)\|\nabla^2_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} + C_2(p)\|\phi\|_{L^p(M,g_{\mathbb{R}})},$$

where $C_2(p) = C_2(p, M, g_{\mathbb{R}})$. By Theorem 2.1 and L^p estimates, we have

$$\|\nabla\phi\|_{L^{\frac{2np}{2n-p}}(M,g_{\mathbb{R}})} \leq C_3(p)\|\Delta_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} + C_3(p),$$

where $C_3(p) = C_3(p, \|F\|_{L^\infty(M,\omega)}, M, g_{\mathbb{R}})$. By Lemma A.1 and Lemma A.5, we have

$$\|\nabla\phi\|_{L^{\frac{2np}{2n-p}}(M,\omega)} \leq C_4(p)\|\Delta\phi\|_{L^p(M,\omega)} + C_4(p),$$

where $C_4(p) = C_4(p, \|F\|_{L^\infty(M,\omega)}, M, g_{\mathbb{R}})$. By (2-7) and Theorem 2.1, the proof is complete. □

Lemma A.7. *Let p, r be real numbers such that $1 < p, r < \infty$. Under the assumptions of Theorem 1.6, we have*

$$\|\nabla\phi\|_{L^p(M,\omega)} \leq C(p,r)\|u\|_{L^r}^{\alpha} + C(p,r),$$

where $C(p,r) = C(p, r, \|F\|_{L^\infty(M,\omega)}, M, \omega)$ and

$$\frac{1}{p} = \frac{1}{2n} + \alpha\Big(\frac{1}{r} - \frac{1}{n}\Big)$$

for α in the interval $\frac{1}{2} \leq \alpha < 1$.

Proof. Set $\tilde{\phi} = \phi - 1/\mathrm{Vol}(M, g_R)\int_M \phi\, dV_{\mathbb{R}}$; then $\int_M \tilde{\phi}\, dV_{\mathbb{R}} = 0$. By Theorem 2.1, Lemma A.1 and Theorem A.3, we have

$$\|\nabla_{\mathbb{R}}\tilde{\phi}\|_{L^p(M,g_{\mathbb{R}})} \leq C_1(p,r)\|\nabla_{\mathbb{R}}^2\tilde{\phi}\|_{L^r(M,g_{\mathbb{R}})}^{\alpha},$$

which implies that

$$\|\nabla_{\mathbb{R}}\phi\|_{L^p(M,g_{\mathbb{R}})} \leq C_1(p,r)\|\nabla_{\mathbb{R}}^2\phi\|_{L^r(M,g_{\mathbb{R}})}^{\alpha},$$

where $C_1(p,r) = C_1(p, r, \|F\|_{L^\infty(M,\omega)}, M, \omega)$ and

$$\alpha = \frac{(2n-p)r}{(2n-2r)p}.$$

Combining Lemma A.1, Lemma A.5, (2-7) and L^p estimates, the proof is complete. □

Acknowledgements

The author would like to thank his adviser Gang Tian for leading him to study the complex Monge–Ampère equation, constant encouragement and several useful comments on an earlier version of this paper. The author would also like to thank Valentino Tosatti for his helpful comments and suggestions, especially for pointing out that Lemma 2.2 holds when the background metric is not balanced when $n \geq 3$, which helped the author remove the balanced condition assumption for $n \geq 3$ in an earlier version of this paper. The author would also like to thank Wenshuai Jiang and Feng Wang for many helpful conversations.

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Received May 15, 2014. Revised October 22, 2014.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 276 No. 2 August 2015