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REGULARITY AND ANALYTICITY OF SOLUTIONS IN A DIRECTION FOR ELLIPTIC EQUATIONS

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In this paper, we study the regularity and analyticity of solutions to linear elliptic equations with measurable or continuous coefficients. We prove that if the coefficients and inhomogeneous term are Hölder-continuous in a direction, then the second-order derivative in this direction of the solution is Hölder-continuous, with a different Hölder exponent. We also prove that if the coefficients and the inhomogeneous term are analytic in a direction, then the solution is analytic in that direction.

1. Introduction

We study the regularity and analyticity of solutions in a given direction to the elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = f(x) \quad \text{in } \Omega, \tag{1-1}$$

assuming that the coefficients a_{ij}, b_i, c and the inhomogeneous term f are smooth or analytic along the direction, where Ω is a bounded domain in the Euclidean space $\mathbb{R}^n$. We assume that the equation is uniformly elliptic, namely, that there exist positive constants $\Lambda > \lambda > 0$ such that

$$\lambda|\xi|^2 \le \sum a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for all } x \in \Omega. \tag{1-2}$$

We also assume that $b_i, c \in L^\infty(\Omega)$, and $f \in L^n(\Omega)$.

The regularity of solutions is a fundamental issue in the study of partial differential equations. Most regularity theories, such as the Schauder estimate and the $W^{2,p}$ estimate, are isotropic; namely, the solution is uniformly regular in all directions. An interesting question is whether the solution to (1-1) is smooth in a direction if the coefficients a_{ij}, b_i, c and the inhomogeneous term f are smooth in this direction only. This question can be asked for more general nonlinear elliptic and parabolic

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equations. One may also consider the regularity when the coefficients a_{ij}, b_i, c and the inhomogeneous term f are smooth in a submanifold of high codimensions.

This is a significant problem in partial differential equations as it is not only stronger than the Schauder estimate but also has applications in areas such as fluid mechanics, partial differential systems, manifolds with nonsmooth metric tensors, and other physical problems such as the propagation of singularities [Taylor 2000; Kukavica and Ziane 2007; Cao and Titi 2008; 2011]. For many PDE systems if one can first prove the regularity of solutions in a direction, one may be able to obtain the full regularity. At a first glance, one may feel that an affirmative answer would be too good to be true, even for an expert in the area. However in this paper we show that this is indeed true at least in dimension two, and also in higher dimensions if the coefficients are continuous. At the moment we are not aware of a counterexample without the continuity. This question is also open for most nonlinear equations and deserves further investigations.

The analyticity of solutions is also an important topic in the regularity theory of partial differential equations. For the linear elliptic equation (1-1), it is well known that if the coefficients a_{ij}, b_i, c and the inhomogeneous term f are analytic, then the solution is also analytic. A similar question is whether the solution is analytic in a direction if a_{ij}, b_i, c and f are analytic only in the given direction.

Let us first state our results on the analyticity of solutions in a given direction:

Theorem 1.1. *Let $u \in W^{2,n}(\Omega)$ be a strong solution to* (1-1). *Assume that the coefficients a_{ij}, b_i, c and the inhomogeneous term f are independent of the variable x_n. Then the solution u is analytic in x_n.*

The proof of Theorem 1.1 is based on the Krylov–Safonov Hölder-continuity of linear elliptic equations. Using the $W^{2,p}$ estimate, we also have:

Theorem 1.2. *Let $u \in W^{2,n}(\Omega)$ be a strong solution to* (1-1). *Assume that the coefficients a_{ij} are continuous, and a_{ij}, b_i, c and f are analytic in the variable x_n. Then the solution u is analytic in x_n.*

In Theorem 1.1, we do not assume the continuity of the coefficients a_{ij}, b_i, c but in Theorem 1.2 we do. An interesting question is whether one can remove the continuity of the a_{ij} in Theorem 1.2. An affirmative answer can be given in dimension two:

Theorem 1.3. *Let $u \in W^{2,2}(\Omega)$ be a strong solution to* (1-1). *Assume that $n = 2$ and a_{ij}, b_i, c and f are analytic in the variable x_2. Then the solution u is analytic in x_2.*

Our results are stronger than the classical results on the analyticity of solutions to linear elliptic equations. In the classical theory the coefficients a_{ij}, b_i, c and the inhomogeneous term f are assumed to be analytic in all directions.

When the coefficients are Hölder-continuous in a given direction, we have the following directional $C^{2,\alpha}$ regularity:

Theorem 1.4. *Let $u \in W^{2,n}(\Omega)$ be a strong solution to* (1-1). *Suppose that a_{ij}, b_i, c are C^α in the ξ-direction for some $0 < \alpha < 1$ and $a_{ij} \in C^0(\Omega)$ and satisfy* (1-2). *Suppose $f \in L^p(\Omega)$ for some $p > n/\alpha$. Then for any $0 < \beta < \alpha - n/p$ and any $y, z \in \Omega_\delta$, we have the estimate*

$$
\begin{aligned}
(1\text{-}3)\quad & |\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u(z)| \\
&\le C d^\beta \left[\sup\nolimits_\Omega |u| + \|f\|_{L^p(\Omega)} + \int_d^1 \frac{\omega_{f,\xi}(r)}{r^{1+\beta}} \right] + C \int_0^d \frac{\omega_{f,\xi}(r)}{r} \\
&\quad + C \|a_{ij}\|_{C^\alpha_\xi(\Omega)} (\|f\|_{L^p(\Omega)} + \sup\nolimits_\Omega |u|) d^{\alpha - n/p},
\end{aligned}
$$

where $\Omega_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$ and $d = |y - z|$. The constant C depends on $n, \alpha, \beta, \delta, p, \lambda, \Lambda$ and the modulus of continuity of a_{ij}.

In Theorem 1.4, ξ is a given unit vector, and the notation $\omega_{f,\xi}$ is defined at the beginning of Section 4. The continuity assumption of the a_{ij} is for the use of the $W^{2,p}$ estimate, hence it suffices to assume that the a_{ij} are in the VMO space [Chiarenza et al. 1993], or the a_{ij} are continuous in $n-1$ variables [Kim and Krylov 2007]. In particular, in dimension two, by the $W^{2,p}$ estimate in the latter reference, the continuity of the a_{ij} is not needed. Hence we have:

Corollary 1.5. *Let $u \in W^{2,2}(\Omega)$ be a strong solution to* (1-1). *Assume that $n = 2$ and a_{ij}, b_i, c and f are Hölder-continuous in direction ξ. Then $\partial_\xi \partial_x u$ is Hölder-continuous.*

Note that the Hölder-continuity of $\partial_\xi \partial_x u$ in Theorem 1.4 and Corollary 1.5 is uniform in all directions. But the Hölder exponent of the second derivative is smaller than that of the coefficients and we need to assume $f \in L^p$ for a large p.

Theorem 1.4 improves [Tian and Wang 2010, Theorem 3.2], where the coefficients a_{ij} were assumed to be Lipschitz in ξ, and the directional $C^{2,\alpha}$ regularity was obtained by differentiating (1-1). We point out that Corollary 1.5 was also obtained in [Dong 2012, Section 6]. By the $W^{2,p}$ estimate [Kim and Krylov 2007], related result holds in higher dimension too. That is, if u is a strong solution to (1-1) and if a_{ij}, b_i, c and f are Hölder-continuous in $x' = (x_1, \ldots, x_{n-1})$, then $\partial_{x'} \partial_x u$ is Hölder-continuous. The $C^{2,\alpha}$ regularity of solutions in a given direction was also investigated in [Dong and Kim 2011]. See also [Tian and Wang 2010] for discussions.

To prove Theorems 1.1–1.3, we introduce appropriate function spaces and establish related interpolation inequalities. We will prove Theorem 1.1 in Section 2, Theorems 1.2 and 1.3 in Section 3, and Theorem 1.4 in Section 4. In Section 5, we give a brief discussion on equations of divergence form.

2. Proof of Theorem 1.1

For simplicity we assume $b_i = c = 0$; namely, we consider the equation

$$L[u] := \sum_{i,j=1}^{n} a_{i,j}(x')u_{ij} = f(x) \quad \text{in } \Omega, \tag{2-1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $x' = (x_1, \dots, x_{n-1})$, and $u_{ij} = u_{x_i x_j}$. The proof is similar if $b_i \neq 0$ and $c \neq 0$, provided they satisfy the conditions specified in the introduction. We assume that the coefficients a_{ij} are measurable and satisfy the uniformly elliptic condition (1-2), $f \in L^n(\Omega)$, and the a_{ij} and f are analytic in the x_n variable.

Set $u' = u_{x_n}$, $u'' = u_{x_n x_n}$,

$$u^{(k)} = \frac{\partial^k u}{\partial x_n^k}, \quad k = 1, 2, \dots,$$

$$\langle u\rangle_{\alpha,\Omega} = \sup_{x,y\in\Omega}\left\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} \;\Big|\; (y-x)//e_n \right\},$$

and

$$|u|_{k+\alpha,\Omega} = \sup_\Omega |u| + \langle u^{(k)}\rangle_{\alpha,\Omega}, \quad k = 0, 1, 2, \dots, \tag{2-2}$$

$$\|u\|_{k+\alpha,\Omega} = \sup_\Omega |u| + \sup_{x,y\in\Omega} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha},$$

where $0 < \alpha \le 1$ and $(y-x)//e_n$ means the vector $y-x$ is parallel to the vector $e_n = (0, \dots, 0, 1)$. We also set

$$\langle u^{(k)}\rangle^{(\beta)}_{\alpha,\Omega} = \sup_{Q_{2r}(x)\subset\Omega} r^{k+\alpha+\beta} \langle u^{(k)}\rangle_{\alpha,Q_r(x)}, \quad \beta \in \mathbb{R},$$

$$|u|^{(\beta)}_{k+\alpha,\Omega} = \sup_{Q_{2r}(x)\subset\Omega} [r^\beta \|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\beta} \langle u^{(k)}\rangle_{\alpha,Q_r(x)}], \tag{2-3}$$

and

$$\|u\|^{(\beta)}_{k+\alpha,\Omega} = \sup_{Q_{2r}(x)\subset\Omega}\left[r^\beta \|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\beta} \sup_{y,z\in Q_r(x)} \frac{|D^k u(y) - D^k u(z)|}{|y-z|^\alpha} \right],$$

where $Q_r(x)$ denotes the open cube with center x and side-length $2r$. We can extend the above definition to $\alpha = 0$ by letting

$$|u|^{(\beta)}_{k,\Omega} = \sup_{Q_{2r}(x)\subset\Omega} [r^\beta \|u\|_{L^\infty(Q_r(x))} + r^{k+\beta} \langle u^{(k-1)}\rangle_{1,Q_r(x)}] \quad \text{if } k > 0,$$

$$|u|^{(\beta)}_{0,\Omega} = \sup_{Q_{2r}(x)\subset\Omega} r^\beta \|u\|_{L^\infty(Q_r(x))} \quad \text{if } k = 0.$$

We point out the equivalence of the norm $|u|^{(\beta)}_{k+\alpha,\Omega}$ given in (2-3) and the norm

$$[u]^{(\beta)}_{k+\alpha,\Omega} := \sup_{Q_{(1+\sigma)r}(x)\subset\Omega}[r^{\beta}\|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\beta}\langle u^{(k)}\rangle_{\alpha,Q_r(x)}],$$

where $\sigma > 0$ is a constant. Namely,

$$C^{-1}|u|^{(\beta)}_{k+\alpha,\Omega} \le [u]^{(\beta)}_{k+\alpha,\Omega} \le C|u|^{(\beta)}_{k+\alpha,\Omega},$$

for some constant C depending only on n, k, α, β and σ. To prove the above inequalities, it suffices to divide the cube $Q_{3r/2}$ into 2^n disjoint smaller cubes if $\sigma \in \left[\frac{1}{2}, 2\right]$, and divide into more, smaller cubes for other σ. Note that if $\beta = -k$, the constant C is independent of k.

We also point out three differences between our definition of the norms $|u|^{(\beta)}_{k+\alpha,\Omega}$ and the usual one [Gilbarg and Trudinger 1998]. That is, (i) the derivative in the former one is taken only on the x_n-direction; (ii) in the Hölder seminorm (2-2) we assume that $(y-x)//e_n$; and (iii) the supremum in (2-3) is taken among all cubes $Q_r(x)$ satisfying the condition $Q_{2r}(x) \subset \Omega$. The reason of choosing the cubes with the property $Q_{2r}(x) \subset \Omega$ is that the norm is homogeneous under rescaling.

First we prove an interpolation inequality for the norm $\|u\|^{(\beta)}_{k+\alpha,\Omega}$:

Lemma 2.1. *Suppose that $j+\beta < k+\alpha$, where $j, k = 0, 1, 2, \dots$ and $0 \le \alpha, \beta \le 1$. Assume that $u \in C^{k,\alpha}(\Omega)$. Then there exists a positive constant C depending on j, k, α, β, such that*

$$\|u\|^{(\gamma)}_{j+\beta,\Omega} \le C[\|u\|^{(\gamma)}_{k+\alpha,\Omega}]^{(j+\beta)/(k+\alpha)}[\|u\|^{(\gamma)}_{0,\Omega}]^{1-(j+\beta)/(k+\alpha)}. \tag{2-4}$$

Proof. It is well known [Hörmander 1976] that there is a positive constant $C = C(j, k, \alpha, \beta)$ such that

$$\|u\|_{j+\beta,Q_1(0)} \le C(\|u\|_{k+\alpha,Q_1(0)})^{(j+\beta)/(k+\alpha)}(\|u\|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}. \tag{2-5}$$

For any $Q_r(x) \subset \Omega$, by rescaling, we obtain

$$\begin{aligned}\|u\|_{L^\infty(Q_r(x))} + r^{j+\beta}\langle D^j u\rangle_{\beta,Q_r(x)} &\le C(\|u\|_{L^\infty(Q_r(x))})^{1-(j+\beta)/(k+\alpha)} \\ &\quad\times \left(\|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha}\langle D^k u\rangle_{\alpha,Q_r(x)}\right)^{(j+\beta)/(k+\alpha)}.\end{aligned} \tag{2-6}$$

That is,

$$\begin{aligned}r^{\gamma}\|u\|_{L^\infty(Q_r(x))} + r^{j+\beta+\gamma}\langle D^j u\rangle_{\beta,Q_r(x)} &\le C(r^{\gamma}\|u\|_{L^\infty(Q_r(x))})^{1-(j+\beta)/(k+\alpha)} \\ &\quad\times \left(r^{\gamma}\|u\|_{L^\infty(Q_r(x))} + r^{k+\alpha+\gamma}\langle D^k u\rangle_{\alpha,Q_r(x)}\right)^{(j+\beta)/(k+\alpha)}.\end{aligned}$$

Taking the supremum of all cubes $Q_r(x)$ with $Q_{2r}(x) \subset \Omega$, we obtain (2-4). □

Next we extend the inequality (2-4) to the norm $|u|^{(\beta)}_{k+\alpha,\Omega}$:

Lemma 2.2. *Suppose that $j+\beta<k+\alpha$, where $j, k=0, 1, 2, \dots$ and $0\le\alpha, \beta\le 1$. Assume that $u\in L^\infty(\Omega)$ and $u^{(k)}\in C^\alpha(\Omega)$. Then there exists a positive constant C depending on j, k, α, β, such that*

$$|u|^{(\gamma)}_{j+\beta,\Omega}\le C[|u|^{(\gamma)}_{k+\alpha,\Omega}]^{(j+\beta)/(k+\alpha)}[|u|^{(\gamma)}_{0,\Omega}]^{1-(j+\beta)/(k+\alpha)}.$$

Proof. By the rescaling argument in the proof of Lemma 2.1, it suffices to prove

$$|u|_{j+\beta,Q_1(0)}\le C(|u|_{k+\alpha,Q_1(0)})^{(j+\beta)/(k+\alpha)}(\|u\|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}. \tag{2-7}$$

By the definition (2-3), it suffices to prove

$$\langle u^{(j)}\rangle_{\beta,Q_1(0)}\le C(|u|_{k+\alpha,Q_1(0)})^{(j+\beta)/(k+\alpha)}(\|u\|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}. \tag{2-8}$$

Again, by the definition of (2-3), there exists x_0' such that

$$\begin{aligned}\langle u^{(j)}\rangle_{\beta,Q_1(0)}&\le 2\sup\left\{\frac{|u^{(j)}(x_0',x_n)-u^{(j)}(x_0',y_n)|}{|x_n-y_n|^\beta}\;\Big|\;-1<x_n,y_n<1\right\}\\&=2\langle u^{(j)}(x_0',\cdot)\rangle_{\beta,I},\end{aligned}$$

where $I=(-1,1)\subset\mathbb{R}^1$ is the unit interval. By (2-5) in the one-dimensional case, the right-hand side is bounded by

$$\begin{aligned}\langle u^{(j)}(x_0',\cdot)\rangle_{\beta,I}&\le(\|u(x_0',\cdot)\|_{k+\alpha,I})^{(j+\beta)/(k+\alpha)}(\|u(x_0',\cdot)\|_{L^\infty(I)})^{1-(j+\beta)/(k+\alpha)}\\&\le(|u|_{k+\alpha,Q_1(0)})^{(j+\beta)/(k+\alpha)}(\|u\|_{L^\infty(Q_1(0))})^{1-(j+\beta)/(k+\alpha)}.\end{aligned}$$ □

Theorem 2.3. *Let $u\in W^{2,n}(\Omega)$ be a strong solution of* (2-1), *where the coefficients a_{ij} are measurable and independent of x_n and satisfy the uniformly elliptic condition* (1-2). *Assume that f is analytic in x_n. Then there exists a constant $C=C(n,\lambda,\Lambda)$ such that, for any $Q_R(x_0)\subset\Omega$, the following inequality holds*:

$$|u^{(k)}(x_0)|\le\left(\frac{Ck}{R}\right)^k(\|u\|_{L^\infty(Q_R(x_0))}+1). \tag{2-9}$$

Proof. As the coefficients a_{ij} are independent of x_n and u is a strong solution, one sees that

$$u_\delta':=\frac{1}{\delta}(u(x+\delta e_n)-u(x))$$

is a strong solution to $L[u]=f_\delta'$, where L is the elliptic operator in (2-1). Hence the Krylov–Safonov Hölder estimate holds for u_δ', uniformly in δ. Similarly,

$$u_\delta'':=\frac{1}{\delta^2}(u(x+\delta e_n)+u(x-\delta e_n)-2u(x))$$

is a strong solution to $L[u]=f_\delta''$, and is uniformly Hölder-continuous as $\delta\to 0$. Sending $\delta\to 0$, we see that u'' is Hölder-continuous. By induction, we see that for any $k > 0$, $u^{(k)}$ is Hölder-continuous, and

$$\langle u^{(k)} \rangle_{\alpha, Q_{1/4}(x)} \le C\big(\|u^{(k)}\|_{L^\infty(Q_{1/2}(x))} + \|f^{(k)}\|_{L^\infty(Q_{1/2}(x))}\big) \tag{2-10}$$

for all $k = 1, 2, \ldots$, and the constant C is independent of k.

Set $Q_0 = Q_R(x_0)$. Let $Q_{2r}(\hat{x}) \subset Q_R(x_0)$ be any given cube. Then there exist $x_1, x_2 \in Q_r(\hat{x})$ with $(x_2 - x_1)//e_n$ such that

$$r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} \le 2r^{1+\alpha} \frac{|u'(x_2) - u'(x_1)|}{|x_2 - x_1|^\alpha}.$$

If $|x_2 - x_1| \ge \frac{1}{4} r$, then, by Lemma 2.2 with $j = 1$, $\beta = 0$, $k = 1$,

$$\begin{aligned} r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} &\le 2 \cdot 4^\alpha r |u'(x_1) - u'(x_2)| \\ &\le 4^{1+\alpha} r \|u'\|_{L^\infty(Q_r(\hat{x}))} \\ &\le C\big(r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} + \|u\|_{L^\infty(Q_r(\hat{x}))}\big)^{1/(1+\alpha)} (\|u\|_{L^\infty(Q_r(\hat{x}))})^{\alpha/(1+\alpha)} \\ &\le C\big[(r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})})^{1/(1+\alpha)} (\|u\|_{L^\infty(Q_r(\hat{x}))})^{\alpha/(1+\alpha)} + \|u\|_{L^\infty(Q_r(\hat{x}))}\big]. \end{aligned} \tag{2-11}$$

If $|x_2 - x_1| < \frac{1}{4} r$, then, by (2-10) and Lemma 2.2,

$$\begin{aligned} r^{1+\alpha} \langle u' \rangle_{\alpha, Q_r(\hat{x})} &\le 2 \cdot r^{1+\alpha} \langle u' \rangle_{\alpha, Q_{r/4}(x_1)} \\ &\le C[r \|u'\|_{L^\infty(Q_{r/2}(x_1))} + r \|f'\|_{L^\infty(Q_{r/2}(x_1))}] \\ &\le C\big\{(r^{1+\alpha} \langle u' \rangle_{\alpha, Q_{r/2}(x_1)})^{1/(1+\alpha)} (\|u\|_{L^\infty(Q_{r/2}(x_1))})^{\alpha/(1+\alpha)} \\ &\qquad + \|u\|_{L^\infty(Q_{r/2}(x_1))} + r \|f'\|_{L^\infty(Q_{r/2}(x_1))}\big\}. \end{aligned} \tag{2-12}$$

Taking the supremum among all the cubes $Q_r(\hat{x})$ with $Q_{2r}(\hat{x}) \subset Q_R(x_0)$, we obtain from the above estimates (2-11) and (2-12) that

$$\langle u' \rangle^{(0)}_{\alpha, Q_0} \le C\big\{(\langle u' \rangle^{(0)}_{\alpha, Q_0})^{1/(1+\alpha)} (\|u\|_{L^\infty(Q_0)})^{\alpha/(1+\alpha)} + \|u\|_{L^\infty(Q_0)} + R \|f'\|_{L^\infty(Q_0)}\big\},$$

which implies

$$|u|^{(0)}_{1+\alpha, Q_0} \le C(\|u\|_{L^\infty(Q_0)} + R \|f'\|_{L^\infty(Q_0)}).$$

By Lemma 2.2 it follows that

$$\|u'\|_{L^\infty(Q_{R/2}(x_0))} \le \frac{C}{R} (\|u\|_{L^\infty(Q_0)} + R \|f'\|_{L^\infty(Q_0)}).$$

Hence we obtain

$$\begin{aligned} |u'(x_0)| &\le \frac{C}{R} (\|u\|_{L^\infty(Q_0)} + R \|f'\|_{L^\infty(Q_0)}) \\ &\le \frac{C}{R} (\|u\|_{L^\infty(Q_0)} + 1), \end{aligned} \tag{2-13}$$

where we used the analyticity of f in x_n.

Next we estimate higher derivatives of u at x_0. Suppose by induction that

$$|u^{(k)}(x_0)| \le \left(\frac{C}{R}\right)^k k^k(\|u\|_{L^\infty(Q_0)}+1). \tag{2-14}$$

By (2-13), (2-14), and observing that for any $x \in Q_{R/(k+1)}(x_0)$, $Q_{kR/(k+1)}(x) \subset Q_R(x_0)$, we have

$$\begin{aligned}
|u^{(k+1)}(x_0)| &= |(u^{(k)})'(x_0)| \\
&\le \frac{C}{\frac{R}{k+1}}\left(\|u^{(k)}\|_{L^\infty(Q_{R/(k+1)}(x_0))} + \frac{R}{k+1}\|f^{(k+1)}\|_{L^\infty(Q_{R/(k+1)}(x_0))}\right) \\
&\le \frac{C(k+1)}{R}\left\{\left(\frac{C}{\frac{k}{k+1}R}\right)^k k^k(\|u\|_{L^\infty(Q_0)}+1) + \frac{R}{k+1}\|f^{(k+1)}\|_{L^\infty(Q_0)}\right\} \\
&\le \left(\frac{C}{R}\right)^{k+1}(k+1)^{k+1}(\|u\|_{L^\infty(Q_0)}+1).
\end{aligned}$$

In the last inequality we used the analyticity of f in x_n. □

Theorem 2.4. *Let $u \in W^{2,n}(\Omega)$ be a strong solution to* (2-1). *Assume that the coefficients a_{ij} are measurable and independent of x_n and satisfy* (1-2). *Assume that f is analytic in x_n. Then the solution u is analytic in x_n.*

Proof. For any given point $x_0 = (x_0', x_{0,n})$ in Ω, let $r_0 = \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)$. Consider the Taylor expansion of u in $Q_{r_0}(x_0)$

$$u(x_0', x_n) = \sum_{k=0}^{n} \frac{u^{(k)}(x_0)}{k!}(x_n - x_{0,n})^k + \frac{u^{(n+1)}(x_0', \xi)}{(n+1)!}(x_n - x_{0,n})^{n+1}, \tag{2-15}$$

where $\xi = tx_{0,n} + (1-t)x_n$ for some $t \in (0,1)$. By Theorem 2.3, we know that

$$\begin{aligned}
|u^{(k)}(x_0)| &\le \left(\frac{Ck}{r_0}\right)^k M, \\
|u^{(k+1)}(x_0', \xi)| &\le \left(\frac{C(k+1)}{r_0}\right)^{k+1} M,
\end{aligned}$$

where $M := \|u\|_{L^\infty(Q_{2r_0}(x_0))} + 1$. By Stirling's formula we have

$$(k+1)^{(k+1)} < e^{k+1}(k+1)!.$$

Hence when $|x - x_0| \le r_0/2Ce$ we have

$$\frac{|u^{(k)}(x_0)|}{k!}|x_n - x_{0,n}|^k \le \frac{M}{2^k} \to 0 \quad \text{as } k \to \infty.$$

Hence u is analytic in the x_n direction. □

3. Proof of Theorem 1.2

In this section we prove the analyticity of solutions in x_n to the equation

$$L[u] := \sum_{i,j=1}^{n} a_{ij}(x) u_{ij} = f(x) \quad \text{in } \Omega, \tag{3-1}$$

where the coefficients a_{ij} also depend on x_n. We assume that the a_{ij} are in $C^0(\Omega)$ and satisfy (1-2) and $f \in L^p(\Omega)$ $(p \geq n)$. We also assume that a_{ij} and f are analytic in x_n and satisfy

$$|\partial_{x_n}^k a_{ij}| + |\partial_{x_n}^k f| \leq B^k k! \tag{3-2}$$

for all $k \geq 1$, where $B > 0$ is a constant.

As before, we set $u' = u_{x_n}$, $u'' = u_{x_n x_n}$ and $u^{(k)} = \dfrac{\partial^k u}{\partial x_n^k}$ for all integer $k \geq 1$. In this section we also set

$$\begin{aligned} [u]_{W^{2,p}(\Omega)} &= \sum_{|s|=2} \|D^s u\|_{L^p(\Omega)}, \\ [u^{(\ell)}]^{(\beta)}_{W^{2,p}(\Omega)} &= \sup\nolimits_{Q_r(x)\subset\Omega} d_{Q_r(x)}^{\ell+2-n/p+\beta} [u^{(\ell)}]_{W^{2,p}(Q_r(x))}, \\ \|u^{(\ell)}\|^{(\beta)}_{L^p(\Omega)} &= \sup\nolimits_{Q_r(x)\subset\Omega} d_{Q_r(x)}^{\ell-n/p+\beta} \|u^{(\ell)}\|_{L^p(Q_r(x))}, \end{aligned} \tag{3-3}$$

for $\ell = 0, 1, 2, \ldots$ and $\beta \in \mathbb{R}$, where $d_{Q_r(x)} = \operatorname{dist}(Q_r(x), \partial\Omega)$.

By the $W^{2,p}$ estimate, we have:

Lemma 3.1. *Let $u \in W^{2,n}(\Omega)$ be a strong solution to* (3-1). *Assume that the a_{ij} are in $C^0(\Omega)$ and satisfy* (1-2), *$f \in L^p(\Omega)$ $(p \geq 1)$, and $Q_R(x_0) \subset \Omega$. There exists a constant C such that*, *if* $0 < r < r + \delta < R$, *then*

$$\|u\|_{W^{2,p}(Q_r(x_0))} \leq C \left\{ \frac{1}{\delta^2} \|u\|_{L^p(Q_{r+\delta}(x_0))} + \|f\|_{L^p(Q_{r+\delta}(x_0))} \right\}, \tag{3-4}$$

where C depends only on n, p, λ, Λ and the moduli of the continuity of the coefficients a_{ij}.

Proof. When $r \leq \delta$, by the $W^{2,p}$ estimate for elliptic equations [Gilbarg and Trudinger 1998] and a rescaling argument, we have

$$\begin{aligned} \|D^2 u\|_{L^p(Q_r(x_0))} &\leq \frac{C}{\delta^2} (\|u\|_{L^p(Q_{r+\delta}(x_0))} + (r+\delta)^2 \|f\|_{L^p(Q_{r+\delta}(x_0))}) \\ &\leq C \left(\frac{1}{\delta^2} \|u\|_{L^p(Q_{r+\delta}(x_0))} + \|f\|_{L^p(Q_{r+\delta}(x_0))} \right). \end{aligned} \tag{3-5}$$

When $\delta < r$, we choose $m \ge 2$ such that $r/m \le \delta < r/(m-1)$, and equally divide the cube $Q_r(x_0)$ into smaller cubes with side-length r/m. Then

$$\|D^2u\|^p_{L^p(Q_r(x_0))} = \sum_i \|D^2u\|^p_{L^p(Q_{r/m}(x_i))}.$$

By (3-5),

$$\|D^2u\|^p_{L^p(Q_{r/m}(x_i))} \le C\left(\frac{1}{\delta^{2p}}\|u\|^p_{L^p(Q_{2r/m}(x_i))} + \|f\|^p_{L^p(Q_{2r/m}(x_i))}\right). \tag{3-6}$$

Note that for each $Q_{r/m}(x_i)$ there are at most 3^n cubes of the form $Q_{2r/m}(x_j)$ intersecting with it. Hence, summing up, we obtain

$$\|D^2u\|^p_{L^p(Q_r(x_0))} \le C\left(\frac{1}{\delta^{2p}}\|u\|^p_{L^p(Q_{r+\delta}(x_0))} + \|f\|^p_{L^p(Q_{r+\delta}(x_0))}\right). \tag{3-7}$$

We obtain (3-4). □

We remark that in Lemma 3.1 the assumption $u \in W^{2,n}(\Omega)$ implies that $f \in L^n(\Omega)$. But the inequality (3-4) holds for all $p \ge 1$.

Theorem 3.2. *Let $u \in W^{2,n}(\Omega)$ be a solution to* (3-1). *Assume that the a_{ij} are in $C^0(\Omega)$ and satisfy* (1-2). *Assume also that the a_{ij} and f are analytic in x_n and satisfy* (3-2). *Then u is analytic in x_n.*

Proof. By (3-1), we have

$$\sum a_{ij}(x+\delta e_n)[u'_\delta]_{ij} = -\sum [a_{ij}]'_\delta u_{ij} + f'_\delta, \tag{3-8}$$

where $u'_\delta = (1/\delta)[u(x+\delta e_n) - u(x)]$, $[a_{ij}]'_\delta = (1/\delta)[a_{ij}(x+\delta e_n) - a_{ij}(x)]$, and $e_n = (0, \dots, 0, 1)$ is the unit vector on the x_n-axis. Since the a_{ij} are continuous, by the $W^{2,p}$ estimate, we see that $u'_\delta \in W^{2,p}(\Omega')$ $(p = n)$ for any $\Omega' \subset \Omega$. Sending $\delta \to 0$, we obtain that $u' \in W^{2,p}_{\text{loc}}(\Omega)$ and is a solution to $L[u'] = f' - a'_{ij}u_{ij}$. Similarly $u^{(k)} \in W^{2,p}_{\text{loc}}(\Omega)$ and is a solution to

$$L[u^{(k)}] = f^{(k)} - \sum_{\ell=1}^{k} \binom{\ell}{k} a^{(\ell)}_{ij} u^{(k-\ell)}_{ij} := f^{(k)} - \phi \quad \text{in } \Omega, \tag{3-9}$$

where $\binom{\ell}{k} = k!/(\ell!(k-\ell)!)$.

We will prove Theorem 3.2 by induction. There is no loss of generality in assuming that $\Omega = Q_0$ is the cube of side-length two centered at the origin. By the definition of $[u]^{(n/p)}_{W^{2,p}(Q_0)}$, there exists a cube $Q_{r_0}(x_0) \subset Q_0$ such that

$$[u]^{(n/p)}_{W^{2,p}(Q_0)} \le 2d_0^2\,[u]_{W^{2,p}(Q_{r_0}(x_0))},$$

where $d_0 = \operatorname{dist}(Q_{r_0}(x_0), \partial Q_0)$. We may assume that the center of Q_{r_0} is the origin; otherwise we may replace $Q_{r_0}(x_0)$ by the larger cube $Q_{1-d_0}(0)$. Therefore the last inequality becomes

$$[u]^{(n/p)}_{W^{2,p}(Q_0)} \le 2(1-r_0)^2[u]_{W^{2,p}(Q_{r_0})}, \tag{3-10}$$

where Q_{r_0} is centered at the origin. Thanks to Lemma 3.1, there is a constant C independent of r_0 such that

$$\begin{aligned}[u]_{W^{2,p}(Q_{r_0})} &\le C\{4(1-r_0)^{-2}\|u\|_{L^p(Q'_{r_0})} + \|f\|_{L^p(Q'_{r_0})}\}\\ &\le C\{4(1-r_0)^{-2}\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)}\},\end{aligned}$$

where $Q'_{r_0} = Q_{r_0+(1-r_0)/2} \subset Q_0$. Hence we obtain

$$[u]^{(n/p)}_{W^{2,p}(Q_0)} \le C(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)}). \tag{3-11}$$

Next we consider the $W^{2,p}$ estimate for u'. Similarly to (3-10), there exists a cube Q_{r_1}, centered at the origin, such that

$$[u']^{(n/p)}_{W^{2,p}(Q_0)} \le 2(1-r_1)^3\,[u']_{W^{2,p}(Q_{r_1})}.$$

By (3-9) and Lemma 3.1,

$$[u']_{W^{2,p}(Q_{r_1})} \le C\left\{\frac{9}{(1-r_1)^2}\|u'\|_{L^p(Q'_{r_1})} + \|f'\|_{L^p(Q'_{r_1})} + \sum_{i,j=1}^{n}\|a'_{ij}u_{ij}\|_{L^p(Q'_{r_1})}\right\},$$

where $Q'_{r_1} = Q_{r_1+(1-r_1)/3}$ is a cube centered at the origin. By the interpolation inequality, the right-hand side of the above formula is

$$\begin{aligned}&\le C\big\{(1-r_1)^{-3}\|u\|_{L^p(Q'_{r_1})} + (1-r_1)^{-1}\|D^2u\|_{L^p(Q'_{r_1})} + \|f'\|_{L^p(Q'_{r_1})} + B\|D^2u\|_{L^p(Q'_{r_1})}\big\}\\ &\le CB(1-r_1)^{-3}\{\|u\|_{L^p(Q_0)} + \|f'\|_{L^p(Q_0)} + [u]^{(n/p)}_{W^{2,p}(Q_0)}\}.\end{aligned}$$

Therefore we obtain

$$[u']^{(n/p)}_{W^{2,p}(Q_0)} \le CB(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1),$$

where the number 1 arises in $\|f'\|_{L^p(Q_0)}$.

By induction, let us assume for $\ell = 0, 1, 2, \dots, k$ that

$$[u^{(\ell)}]^{(n/p)}_{W^{2,p}(Q_0)} \le A^{\ell}\ell!(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1). \tag{3-12}$$

Then, similarly to (3-10), there exists a cube $Q_{r_{k+1}} \subset Q_0$, centered at the origin, such that

$$[u^{(k+1)}]^{(n/p)}_{W^{2,p}(Q_0)} \le 2(1-r_{k+1})^{k+3}[u^{(k+1)}]_{W^{2,p}(Q_{r_{k+1}})}, \tag{3-13}$$

where $Q_{r_{k+1}}$ is a cube with center at the origin. By Lemma 3.1, with $\delta = \dfrac{1-r_{k+1}}{k+3}$,

$$
\begin{aligned}
&(1-r_{k+1})^{k+3}[u^{(k+1)}]_{W^{2,p}(Q_{r_{k+1}})} \\
&\qquad \le C(1-r_{k+1})^{k+3}\bigg\{\frac{(k+3)^2}{(1-r_{k+1})^2}\|u^{(k+1)}\|_{L^p(Q'_{r_{k+1}})} + \|f^{(k+1)}\|_{L^\infty(Q'_{r_{k+1}})} \\
&\qquad\qquad + \sum_{i,j=1}^{n}\sum_{m=0}^{k}\binom{m}{k+1}\|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})}\bigg\},
\end{aligned}
$$

where $Q'_{r_{k+1}} := Q_{r_{k+1}+(1-r_{k+1})/(k+3)}$. Note that $\operatorname{dist}(Q'_{r_{k+1}}, \partial Q_0) = \dfrac{k+2}{k+3}(1-r_{k+1})$. We have

$$
\begin{aligned}
&(k+3)^2(1-r_{k+1})^{k+1}\|u^{(k+1)}\|_{L^p(Q'_{r_{k+1}})} \\
&\qquad \le (k+3)^2\Big(\frac{k+3}{k+2}\Big)^{k+1}\Big(\frac{k+2}{k+3}(1-r_{k+1})\Big)^{k+1}[u^{(k-1)}]_{W^{2,p}(Q'_{r_{k+1}})} \\
&\qquad \le 4(k+3)^2[u^{(k-1)}]^{(n/p)}_{W^{2,p}(Q_0)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(1-r_{k+1})^{k+3}\|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})} \\
&\qquad\qquad \le \|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}(1-r_{k+1})^{m+2}[u^{(m)}]_{W^{2,p}(Q'_{r_{k+1}})} \\
&\qquad\qquad \le 4\|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}[u^{(m)}]^{(n/p)}_{W^{2,p}(Q_0)}.
\end{aligned}
$$

Hence for fixed i, j, by the induction assumptions,

$$
\begin{aligned}
&(1-r_{k+1})^{k+3}\sum_{m=0}^{k}\binom{m}{k+1}\|a_{ij}^{(k+1-m)}u_{ij}^{(m)}\|_{L^p(Q'_{r_{k+1}})} \\
&\qquad\qquad \le 4\sum_{m=0}^{k}\binom{m}{k+1}\|a_{ij}^{(k+1-m)}\|_{L^\infty(Q_0)}[u^{(m)}]^{(n/p)}_{W^{2,p}(Q_0)} \\
&\qquad\qquad \le 4(k+1)!A^m B^{k+1-m}(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1) \\
&\qquad\qquad \le 4(k+1)!A^k B(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).
\end{aligned}
$$

Hence by (3-13) we obtain

$$
\begin{aligned}
[u^{(k+1)}]^{(n/p)}_{W^{2,p}(Q_0)} &\le C\big\{(k+3)^2[u^{(k-1)}]^{(n/p)}_{W^{2,p}(Q_0)} + \|f^{(k+1)}\|_{L^\infty(Q_0)} \\
&\qquad + (k+1)!A^k B(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1)\big\}.
\end{aligned}
$$

By (3-2) and the induction assumption (3-12), we then obtain

$$
[u^{(k+1)}]^{(n/p)}_{W^{2,p}(Q_0)} \le C(k+1)!(A^{k-1} + A^k B + B^{k+1})(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).
$$

Choosing $A \gg B$, we obtain (3-12) for $k+1$.

From (3-12), we obtain that

$$[u^{(k+1)}]_{W^{2,p}(Q_{1/2}(0))} \le 2^{k+1}A^{k+1}(k+1)!(\|u\|_{L^p(Q_0)} + \|f\|_{L^p(Q_0)} + 1).$$

By the Sobolev embedding and since $p > n$, we have

$$|u^{(k+1)}(0)| \le C2^{k+1}A^{k+1}(k+1)!.$$

Hence u is analytic in x_n at the origin. □

As we remarked in Section 1, the continuity assumption on the a_{ij} can be relaxed. The continuity is used for the $W^{2,p}$ estimate; it suffices to assume that the a_{ij} are continuous in any $n-1$ variables [Kim and Krylov 2007]. In particular, in the dimension-two case, we can remove the continuity of a_{ij} in Theorem 1.2, as the analyticity of a_{ij} automatically implies that they are continuous in one variable. Therefore, for the equation

$$\sum_{i,j=1}^{2} a_{ij}(x)u_{ij} = f(x) \quad \text{in } \Omega, \tag{3-14}$$

where the coefficients a_{ij} satisfy the uniformly elliptic condition (1-2), we have:

Theorem 3.3. *Let $u \in W^{2,2}(\Omega)$ be a strong solution to* (3-14). *Assume that the a_{ij} satisfy* (1-2) *and assume that a_{ij} and f are analytic in x_2. Then under the above conditions, u is analytic in x_2.*

4. Proof of Theorem 1.4

Let Ω be a bounded domain in $\mathbb{R}^n$. Let ξ be a unit vector in $\mathbb{R}^n$ and ϕ a function defined in Ω. Set

$$\omega_{\phi,\xi}(r) = \sup\{|\phi(x) - \phi(x+t\xi)| \mid x, x+t\xi \in \Omega,\ |t| \le r\}.$$

We say ϕ is Hölder-continuous in the ξ direction with Hölder exponent α if $\omega_{\phi,\xi} \in C^\alpha$, and write $\phi \in C^\alpha_\xi(\Omega)$, with the norm

$$\|\phi\|_{C^\alpha_\xi(\Omega)} = \sup_{x\in\Omega} |\phi(x)| + \sup_{t>0} \frac{\omega_{\phi,\xi}(t)}{t^\alpha}.$$

To prove Theorem 1.4, we assume for simplicity that $b_i = c = 0$ and consider the equation

$$L[u] := \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = f(x) \quad \text{in } \Omega, \tag{4-1}$$

where the coefficients a_{ij} satisfies the uniformly elliptic condition (1-2). The proof below is based on a perturbation argument and follows closely that of [Wang 2006].

Proof of Theorem 1.4. Without loss of generality we assume $\xi = e_1 = (1, 0, \dots, 0)$ and $\Omega = B_1(0)$, the unit ball. We set

$$B_k = B_{2^{-k}}(0), \quad \hat{a}_{ij}(x) = a_{ij}(0, x_2, \dots, x_n), \quad \hat{f}(x) = f(0, x_2, \dots, x_n).$$

For $k = 0, 1, 2, \dots$, let u_k be the solution of

$$\begin{aligned} \sum_{i,j=1}^{n} \hat{a}_{ij}(x)(u_k)_{x_i x_j} &= \hat{f}(x) \quad \text{in } B_k, \\ u_k &= u \qquad \text{on } \partial B_k. \end{aligned} \tag{4-2}$$

Then

$$\begin{aligned} \sum_{i,j=1}^{n} \hat{a}_{ij}(x)(u_k - u)_{x_i x_j} &= \sum_{i,j=1}^{n} (a_{ij}(x) - \hat{a}_{ij}(x)) u_{x_i x_j} + \hat{f}(x) - f(x) \quad \text{in } B_k, \\ u_k - u &= 0 \qquad \text{on } \partial B_k. \end{aligned} \tag{4-3}$$

Hence, by the Alexandrov maximum principle, for $k \geq 1$,

$$\begin{aligned} &\sup_{B_k} |u - u_k| \\ &\leq C 2^{-k} \left[\int_{B_k} |(a_{ij}(x) - \hat{a}_{ij}(x)) u_{x_i x_j}|^n \, dx \right]^{1/n} + C 2^{-2k} \omega_{f,\xi}(2^{-k}) \\ &\leq C 2^{-k} \|a_{ij}\|_{C^\alpha_\xi(B_k)} \left[\int_{B_k} |x|^{n\alpha} |u_{x_i x_j}|^n \, dx \right]^{1/n} + C 2^{-2k} \omega_{f,\xi}(2^{-k}) \\ &\leq C 2^{-k} \|a_{ij}\|_{C^\alpha_\xi(B_k)} \left[\left(\int_{B_k} |x|^{n\alpha p/(p-n)} \, dx \right)^{(p-n)/p} \left(\int_{B_k} |u_{x_i x_j}|^p \, dx \right)^{n/p} \right]^{1/n} \\ &\qquad + C 2^{-2k} \omega_{f,\xi}(2^{-k}) \\ &\leq C 2^{-k} \|a_{ij}\|_{C^\alpha_\xi(B_k)} (2^{-k})^{\alpha + 1 - n/p} \|u\|_{W^{2,p}(B_k)} + C 2^{-2k} \omega_{f,\xi}(2^{-k}) \\ &\leq C(A \cdot (2^{-k})^{2 + \alpha - n/p} + 2^{-2k} \omega_{f,\xi}(2^{-k})), \end{aligned} \tag{4-4}$$

where

$$A = \|u\|_{W^{2,p}(B_1)} \|a_{ij}\|_{C^\alpha_\xi(\Omega)}.$$

Since the a_{ij} are continuous and satisfy the uniformly elliptic condition, by the $W^{2,p}$ estimate,

$$A \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}) \|a_{ij}\|_{C^\alpha_\xi(\Omega)}.$$

Hence

$$\begin{aligned} \|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} &\leq C\{A \cdot (2^{-k})^{2 + \alpha - n/p} + 2^{-2k} \omega_{f,\xi}(2^{-k})\} \\ &= C 2^{-2k} \{A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k})\}. \end{aligned} \tag{4-5}$$

Since $w_k := u_{k+1} - u_k$ satisfies

$$\hat{a}_{ij}(x) w_{x_i x_j} = 0$$

in B_{k+1}, where the coefficients $\hat{a}_{ij}(x)$ are independent of x_1, by differentiating the equation and by the $W^{2,p}$ estimate, we have

$$\|\partial_\xi w_k\|_{W^{2,p}(B_{k+2})} \le C 2^{3k} \|w_k\|_{L^\infty(B_{k+1})} \quad \text{for all } p > 1.$$

Hence by the Sobolev embedding theorem,

$$\|\partial_\xi w_k\|_{C^{1,\beta}(B_{k+2})} \le C 2^{2k+2\beta} \|w_k\|_{L^\infty(B_{k+1})} \quad \text{for all } \beta \in (0, 1).$$

Therefore by rescaling,

$$\|\partial_\xi \partial_x w_k\|_{L^\infty(B_{k+2})} \le C[A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k})],$$

$$\|\partial_\xi \partial_x w\|_{C^\beta(B_{k+2})} \le C 2^{k\beta} [A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k})]. \tag{4-6}$$

As the coefficients a_{ij} are continuous, the solution can be approximated by smooth solutions. Hence, to prove Theorem 1.4, we may assume that u is smooth, so that

$$D^2 u_k(0) \to D^2 u(0).$$

For y near 0, let $m \ge 1$ be such that

$$2^{-m-4} \le |y| < 2^{-m-3}.$$

Then

$$\begin{aligned} |\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u(0)| \le |\partial_\xi \partial_x u_m(y) - \partial_\xi \partial_x u_m(0)| + |\partial_\xi \partial_x u_m(0) - \partial_\xi \partial_x u(0)| \\ + |\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u_m(y)|. \end{aligned} \tag{4-7}$$

We have

$$\begin{aligned} |\partial_\xi \partial_x u_m(0) - \partial_\xi \partial_x u(0)| &\le \sum_{k=m}^{\infty} |\partial_\xi \partial_x u_k(0) - \partial_\xi \partial_x u_{k+1}(0)| \\ &\le C \sum_{k=m}^{\infty} [A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k})] \\ &\le C \left\{ A \cdot (2^{-m})^{\alpha - n/p} + \int_0^{|y|} \frac{\omega_{f,\xi}(r)}{r} \right\} \\ &\le C \left\{ A \cdot |y|^{\alpha - n/p} + \int_0^{|y|} \frac{\omega_{f,\xi}(r)}{r} \right\}. \end{aligned} \tag{4-8}$$

Similarly,

$$|\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u_m(y)| \le C \left\{ A \cdot |y|^{\alpha - n/p} + \int_0^{|y|} \frac{\omega_{f,\xi}(r)}{r} \right\}.$$

By (4-6) we have

$$|\partial_\xi \partial_x w_k(y) - \partial_\xi \partial_x w_k(0)| \le \|\partial_\xi \partial_x w_k\|_{C^\beta(B_{k+2})} |y|^\beta \le C|y|^\beta 2^{k\beta} [A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k})]. \tag{4-9}$$

Write

$$u_m = u_1 + \sum_{k=1}^{m-1} w_k.$$

We have, for $\beta < \alpha - n/p$,

$$\begin{aligned} &|\partial_\xi \partial_x u_m(y) - \partial_\xi \partial_x u_m(0)| \\ &\quad \le |\partial_\xi \partial_x u_1(y) - \partial_\xi \partial_x u_1(0)| + \sum_{k=1}^{m-1} |\partial_\xi \partial_x w_k(y) - \partial_\xi \partial_x w_k(0)| \\ &\quad \le C|y|^\beta \biggl(\|u_1\|_{L^\infty(\Omega)} + \sum_{k=1}^{m-1} 2^{k\beta} \bigl(A \cdot (2^{-k})^{\alpha - n/p} + \omega_{f,\xi}(2^{-k}) \bigr) \biggr) \\ &\quad \le C|y|^\beta \biggl(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \int_{|y|}^1 \frac{\omega_{f,\xi}(r)}{r^{1+\beta}} \biggr). \end{aligned} \tag{4-10}$$

This completes the proof of Theorem 1.4. □

5. Equation of divergence form

We consider the following linear elliptic equation of divergence form:

$$Lu = \operatorname{div}(A(x)\nabla u(x)) = \operatorname{div} f(x) \quad \text{in } \Omega, \tag{5-1}$$

where the coefficient matrix $A(x) = (a_{ij}(x))_{n\times n}$ satisfies the uniformly elliptic condition (1-2) and $f(x) = (f_1(x), f_2(x), \dots, f_n(x)) \in [L^p(\Omega)]^n$ for $p > 1$. We assume also that a_{ij} and f are analytic in x_n, and that there exists a constant $B > 0$ such that

$$|\partial_{x_n}^k a_{ij}| + |\partial_{x_n}^k f| \le B^k k! \tag{5-2}$$

for all $k \ge 1$.

Definition 5.1. Let $1 < p < \infty$. We say that u is a solution to (5-1) if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\int_\Omega a_{ij}(x) u_{x_j} \phi_{x_i}\, dx = \int_\Omega f(x) \phi_{x_i}\, dx$$

for all $\phi \in C_0^\infty(\Omega)$.

As before, we set $u' = u_{x_n}$, $u'' = u_{x_n x_n}$ and $u^{(k)} = \dfrac{\partial^k u}{\partial x_n^k}$ for all integers $k \geq 1$. We also define

$$
\begin{aligned}
[u]_{W^{1,p}(\Omega)} &= \|Du\|_{L^p(\Omega)},\\
\|u^{(k)}\|^{(\beta)}_{W^{1,p}(\Omega)} &= \sup_{Q_r(x)\subset\Omega} d_{Q_r(x)}^{k+1-n/p+\beta}[u^{(k)}]_{W^{1,p}(Q_r(x))},\\
\|u^{(k)}\|^{(\beta)}_{L^p(\Omega)} &= \sup_{Q_r(x)\subset\Omega} d_{Q_r(x)}^{k-n/p+\beta}\|u^{(k)}\|_{L^p(Q_r(x))},
\end{aligned}
\tag{5-3}
$$

where $d_{Q_r(x)} = \operatorname{dist}(Q_r(x), \partial\Omega)$, k is a nonnegative integer, and $p > 1$ is a constant.

By the $W^{1,p}$ estimate for the divergence form (5-1) in [Di Fazio 1996], we have:

Lemma 5.2. *Let u be a solution to* (5-1). *Assume that the a_{ij} satisfy* (1-2), *$f \in [L^p(\Omega)]^n$ $(p > 1)$ and $Q_R(x_0) \subset \Omega$. There exists a constant C such that, if $0 < r < r + \delta < R$, then*

$$
[u]_{W^{1,p}(Q_r(x_0))} \leq C\left\{\frac{1}{\delta}\|u\|_{L^p(Q_{r+\delta}(x_0))} + \sum_{i=1}^{n}\|f_i\|_{L^p(Q_{r+\delta}(x_0))}\right\},
\tag{5-4}
$$

where the constant C depends only on n, p, λ, Λ.

By Lemma 5.2 we then have:

Theorem 5.3. *Let u be a solution to* (5-1). *Assume that the a_{ij} satisfy* (1-2) *and $f \in [L^p(\Omega)]^n$ $(p > n)$. Assume that the a_{ij} and f are analytic in the variable x_n. Then u is analytic in x_n.*

The proofs of Lemma 5.2 and Theorem 5.3 are similar to those in Section 3 and are omitted here. Note that the assumption $p > n$ in Theorem 5.3 is for the use of Sobolev embedding; namely, by the estimate $\|u^{(k)}\|_{W^{1,p}(Q_r(0))} \leq C$ one infers that $|u^{(k)}(0)| \leq C_1$.

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