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We present a family of weights on the unit disc for which the corresponding weighted Szegő projection operators are irregular on L^p spaces. We further investigate the dual spaces of weighted Hardy spaces corresponding to this family.

1. Introduction

1.1. ***Classical setting.*** Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle. Let $\mathcal{O}(\mathbb{D})$ denote the set of holomorphic functions on $\mathbb{D}$. For $1 \leq p < \infty$, the ordinary Hardy space is defined as

$$\mathcal{H}^p(\mathbb{T}) = \{f \in \mathcal{O}(\mathbb{D}) \text{ and } \|f\|_{\mathcal{H}^p} < \infty\},$$

where

$$\|f\|^p_{\mathcal{H}^p} = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

It is known (see [Duren 1970]) that functions in $\mathcal{H}^p(\mathbb{T})$ have boundary limits almost everywhere, i.e., for almost every $\theta \in [0, 2\pi]$

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

exists. Moreover,

$$\|f\|_{L^p(\mathbb{T})} = \|f\|_{\mathcal{H}^p(\mathbb{T})},$$

where $L^p(\mathbb{T})$ is defined using the standard Lebesgue measure (denoted by $d\theta$) on the unit circle. It is also known that $\mathcal{H}^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{T})$. In particular, for $p = 2$, the orthogonal projection operator, called the Szegő projection operator exists;

$$\boldsymbol{S} : L^2(\mathbb{T}) \longrightarrow \mathcal{H}^2(\mathbb{T}).$$

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The operator $\boldsymbol{S}$ is an integral operator with the kernel $S(z, w)$ (called the Szegő kernel), and for $f \in L^2(\mathbb{T})$,

$$\boldsymbol{S} f(z) = \int_{\mathbb{T}} S(z, w) f(w)\, d\theta.$$

It follows from the general theory of reproducing kernels that for any orthonormal basis $\{e_n(z)\}_{n=0}^{\infty}$ for $\mathcal{H}^2(\mathbb{T})$, the Szegő kernel is given by

$$S(z, w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

1.2. ***Weighted setting.*** Let $g(z)$ be a holomorphic function on $\mathbb{D}$ that is continuous on $\overline{\mathbb{D}}$ and has no zeros inside $\mathbb{D}$. We set $\mu(z) = |g(z)|^2$ and define weighted Hardy spaces and weighted Szegő projections using the function $\mu(z)$ as a weight on $\mathbb{T}$.

For $1 \leq p < \infty$, we define the weighted Lebesgue and Hardy spaces with respect to μ as

$$L^p(\mathbb{T}, \mu) = \{f \text{ measurable function on } \mathbb{D} \text{ and } \|f\|_{p,\mu} < \infty\},$$

where

$$\|f\|_{p,\mu}^p = \int_{\mathbb{T}} |f(w)|^p \mu(w) d\theta = \int_{\mathbb{T}} |f(w)(g(w))^{2/p}|^p\, d\theta,$$

and

$$\mathcal{H}^p(\mathbb{T}, \mu) = \{f \in \mathcal{O}(\mathbb{D}) \text{ such that } \|f\|_{\mathcal{H}^p,\mu} < \infty\},$$

where

$$\|f\|_{\mathcal{H}^p,\mu}^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})(g(re^{i\theta}))^{2/p}|^p\, d\theta.$$

Note that, $f \in \mathcal{H}^p(\mathbb{T}, \mu)$ implies $f(z)(g(z))^{2/p} \in \mathcal{H}^p(\mathbb{T})$, which in turn gives that $f(z)(g(z))^{2/p}$ has almost everywhere boundary limits. Hence so does $f(z)$. Additionally, $\|f\|_{\mathcal{H}^p,\mu} = \|f\|_{p,\mu}$. Furthermore, $L^p(\mathbb{T}, \mu)$ is a Banach space and $\mathcal{H}^p(\mathbb{T}, \mu)$ is a closed subspace of $L^p(\mathbb{T}, \mu)$.

In particular, again when $p = 2$, we obtain the weighted Szegő projection

$$\boldsymbol{S}_\mu : L^2(\mathbb{T}, \mu) \longrightarrow \mathcal{H}^2(\mathbb{T}, \mu).$$

Following the similar theory, we note that $\boldsymbol{S}_\mu$ is an integral operator

$$\boldsymbol{S}_\mu f(z) = \int_{\mathbb{T}} S_\mu(z, w) f(w) \mu(w)\, d\theta.$$

If $\{f_n(z)\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}^2(\mathbb{T},\mu)$ then

$$S_\mu(z,w)=\sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}.$$

We are interested in the action of $\boldsymbol{S}_\mu$ on $L^p(\mathbb{T},\mu)$. By definition, $\boldsymbol{S}_\mu$ is a bounded operator from $L^2(\mathbb{T},\mu)$ to $L^2(\mathbb{T},\mu)$. The problem we investigate is the boundedness of $\boldsymbol{S}_\mu$ from $L^p(\mathbb{T},\mu)$ to $L^p(\mathbb{T},\mu)$ for other values of $p\in(1,\infty)$. Note that for any given weight μ as above, we can associate an interval $I_\mu\subset(1,\infty)$ such that $\boldsymbol{S}_\mu$ is bounded from $L^p(\mathbb{T},\mu)$ to $L^p(\mathbb{T},\mu)$ if and only if $p\in I_\mu$. By definition, $2\in I_\mu$, and by duality and interpolation, I_μ is a conjugate symmetric interval around 2. Namely, if some $p_0>2$ is in I_μ, so is q_0 where $1/q_0+1/p_0=1$.

In the classical setting, i.e., $\mu\equiv 1$, the Szegő projection operator is bounded from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for any $1<p<\infty$, see [Zhu 2007, page 257].

The purpose of this note is to construct weights μ on $\mathbb{T}$ for which the corresponding interval I_μ can be any open interval larger than $\{2\}$ but smaller than $(1,\infty)$.

Theorem 1. *For any given $p_0>2$, there exists a weight μ on $\mathbb{T}$ such that $I_\mu=(q_0,p_0)$ where $1/q_0+1/p_0=1$, i.e., the weighted Szegő projection $\boldsymbol{S}_\mu$ is bounded on $L^p(\mathbb{T},\mu)$ if and only if $q_0<p<p_0$.*

Our proof of this theorem is similar to the proof of the analogous statement for weighted Bergman projections in [Zeytuncu 2013] with modifications from Bergman kernels to Szegő kernels. The main ingredient is the theory of A_p weights on $\mathbb{T}$.

When the weighted Szegő projection $\boldsymbol{S}_\mu$ is bounded on $L^p(\mathbb{T},\mu)$ for some p, one can identify the dual space of the weighted Hardy space $\mathcal{H}^p(\mathbb{T},\mu)$. However, when $\boldsymbol{S}_\mu$ fails to be bounded, a different approach is needed to identify the dual spaces. In the third section, we address this issue and describe the dual spaces of weighted Hardy spaces.

The following notation is used in the rest of the note. We write $f(z)\simeq g(z)$ when $c\cdot g(z)\le f(z)\le C\cdot g(z)$ for some positive constants c and C which are independent of z. Similarly we write $f(z)\lesssim g(z)$ when $f(z)\le C\cdot g(z)$ for some positive constant C. We use $d\theta$ for the Lebesgue measure on the unit circle $\mathbb{T}$. When we integrate functions (that are also defined on the unit disc) on $\mathbb{T}$, instead of writing $e^{i\theta}$, we keep z and w as the variables.

2. Proof of Theorem 1

2.1. ***Relation between weighted kernels.*** The particular choice of $\mu(z)$ indicates the following relation between the weighted Szegő kernels $S_\mu(z,w)$ and the ordinary Szegő kernel $S(z,w)$.

Proposition 2. *For $\mu(z) = |g(z)|^2$ as above, the following relation holds*

$$S(z, w) = g(z) S_\mu(z, w) \overline{g(w)}. \tag{1}$$

Proof. Let $\{e_n(z)\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}^2(\mathbb{T})$. Since $g(z)$ does not vanish inside $\mathbb{D}$, each $e_n(z)/g(z)$ is a holomorphic function on $\mathbb{D}$ and is in $\mathcal{H}^2(\mathbb{T}, |g|^2)$ by construction. Following the orthonormal properties of the $e_n(z)$ we have

$$\left\langle \frac{e_n(z)}{g(z)}, \frac{e_m(z)}{g(z)} \right\rangle_\mu = \langle e_n(z), e_m(z) \rangle = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta.

Also for any f in $\mathcal{H}^2(\mathbb{T}, |g|^2)$, $(f \cdot g)$ is in $\mathcal{H}^2(\mathbb{T})$ and hence can be written as a linear combination of the $e_n(z)$. Consequently so can f, using the quotients $e_n(z)/g(z)$. Hence, $\{e_n(z)/g(z)\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}^2(\mathbb{T}, |g|^2)$.

Therefore, using the basis representation of the Szegő kernels we obtain

$$\begin{aligned} S(z, w) &= \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)} = g(z) \left(\sum_{n=0}^{\infty} \frac{e_n(z)}{g(z)} \overline{\frac{e_n(w)}{g(w)}} \right) \overline{g(w)} \\ &= g(z) S_\mu(z, w) \overline{g(w)}. \end{aligned}$$

□

2.2. *A_p weights on $\mathbb{T}$.* For $p \in (1, \infty)$, a weight μ on $\mathbb{T}$ is said to be in $A_p(\mathbb{T})$ if

$$\sup_{\substack{I \\ I \subset \mathbb{T}}} \left(\frac{1}{|I|} \int_I \mu(\theta)\, d\theta \right) \left(\frac{1}{|I|} \int_I \mu(\theta)^{\frac{-1}{p-1}}\, d\theta \right)^{p-1} < \infty,$$

where I denotes intervals in $\mathbb{T}$.

These weights are used to characterize the L^p regularity of the ordinary Szegő projection on weighted spaces. The following result appears in [Garnett 1981] and is used in [Lanzani and Stein 2004, Equation (2.3)] in connection with a conformal map based approach to the investigation of the unweighted Szegő projection for a general domain.

Theorem 3. *The ordinary Szegő projection **S** is bounded from $L^p(\mathbb{T}, \mu)$ to $L^p(\mathbb{T}, \mu)$ if and only if $\mu \in A_p(\mathbb{T})$.*

Proof. This result is an immediate consequence of the fact that the Szegő kernel of the unit disc agrees with the Cauchy kernel (see [Kerzman and Stein 1978]) together with the classical weighted theory for the latter, see also [Garnett 1981]. □

The following theorem follows from Equation (1) and Theorem 3.

Proposition 4. *For $1 < p < \infty$ and $\mu(z) = |g(z)|^2$ as above, the following are equivalent.*

(1) *$\mathbf{S}_\mu$ is bounded from $L^p(\mathbb{T}, |g|^2)$ to $L^p(\mathbb{T}, |g|^2)$.*

(2) ***S** is bounded from $L^p(\mathbb{T}, |g|^{2-p})$ to $L^p(\mathbb{T}, |g|^{2-p})$.*

(3) $|g|^{2-p} \in A_p(\mathbb{T})$.

Proof. Theorem 3 gives the equivalence of (2) and (3). We show the equivalence of (1) and (2). Using the relation between the kernels from the previous proposition, we obtain the following relation between the corresponding operators:

$$g(z)\big(\boldsymbol{S}_\mu f\big)(z) = \big(\boldsymbol{S}(f \cdot g)\big)(z) \quad \text{for } f \in L^2(\mathbb{T}, |g|^2).$$

Indeed, suppose (2) is true. Then

$$\begin{aligned}
\|\boldsymbol{S}_\mu f\|^p_{p,|g|^2} &= \int_{\mathbb{T}} |(\boldsymbol{S}_\mu f)(w)|^p |g(w)|^2 \, d\theta \\
&= \int_{\mathbb{T}} |\big(\boldsymbol{S}(f \cdot g)\big)(w)|^p |g(w)|^{2-p} \, d\theta = \|\boldsymbol{S}(f \cdot g)\|^p_{p,|g|^{2-p}} \\
&\lesssim \|f \cdot g\|^p_{p,|g|^{2-p}} = \|f\|^p_{p,|g|^2},
\end{aligned}$$

which proves (1).

Now when (1) is true,

$$\begin{aligned}
\|\boldsymbol{S} f\|^p_{p,|g|^{2-p}} &= \int_{\mathbb{T}} |(\boldsymbol{S} f)(w)|^p |g(w)|^{2-p} d\theta \\
&= \int_{\mathbb{T}} |(\boldsymbol{S}_\mu (f/g))(w)|^p |g(w)|^2 d\theta = \|\boldsymbol{S}_\mu (f/g)\|^p_{p,|g|^2} \\
&\lesssim \|f/g\|^p_{p,|g|^2} = \|f\|^p_{p,|g|^{2-p}}
\end{aligned}$$

and hence (2) is true. □

We can now present a family of weights that behave as claimed in Theorem 1.

Theorem 5. *For $\alpha \geq 0$, let $g_\alpha(z) = (z-1)^\alpha$ and $\mu_\alpha(z) = |g_\alpha(z)|^2$. Then the weighted Szegő projection operator $\boldsymbol{S}_{\mu_\alpha}$ is bounded on $L^p(\mathbb{T}, \mu_\alpha)$ if and only if $p \in \big(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\big)$.*

Remark 6. Theorem 5 is a quantitative version of Theorem 1 and therefore we obtain a proof of Theorem 1 when we prove Theorem 5.

Remark 7. Note that as $\alpha \to 0^+$ the interval $\big(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\big)$ approaches $(1, \infty)$ and as $\alpha \to \infty$ the interval $\big(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\big)$ approaches $\{2\}$. Hence, any conjugate symmetric interval around 2 can be achieved as the boundedness range of a weighted Szegő projection.

Proof of Theorem 5. First note that on intervals I with $\theta = 0 \notin I$, the weight $|g_\alpha(z)|^{2-p} = |z-1|^{\alpha(2-p)} \simeq C$. Therefore, both integrals in the $A_p(\mathbb{T})$ condition are finite and hence so is the supremum over all such intervals when p is in the given range. On intervals that contain $z = 0$ we have the following.

Step 1. We show that for the weights $\omega(z) = |g_\alpha(z)|^{2-p} = |z-1|^{\alpha(2-p)}$, the second integral in the $A_p(\mathbb{T})$ condition diverges for arcs $I = (-\epsilon, \epsilon)$ if and only if p is outside the given region.

For intervals $I = (-\epsilon, \epsilon)$ with small ϵ and $p \le \frac{2\alpha+1}{\alpha+1}$,

$$\begin{aligned}\int_I \omega(z)^{\frac{1}{1-p}}\, d\theta &= \int_{-\epsilon}^{\epsilon} |e^{i\theta}-1|^{\frac{\alpha(2-p)}{(1-p)}}\, d\theta\\ &= \int_{-\epsilon}^{\epsilon} \big(\sqrt{2(1-\cos(\theta))}\big)^{\frac{\alpha(2-p)}{(1-p)}}\, d\theta\\ &\simeq \int_{-\epsilon}^{\epsilon} \theta^{\frac{\alpha(2-p)}{(1-p)}}\, d\theta = \infty,\end{aligned}$$

because $\alpha(2-p)/(1-p) \le -1$. Hence $\omega \notin A_p(\mathbb{T})$ for such p.

Also, when $p \ge \frac{2\alpha+1}{\alpha}$,

$$\int_I \omega(z)\, d\theta \simeq \int_{-\epsilon}^{\epsilon} \theta^{\alpha(2-p)}\, d\theta = \infty,$$

because $\alpha(2-p) \le -1$. Hence $\omega \notin A_p(\mathbb{T})$ for $p \ge \frac{2\alpha+1}{\alpha}$ either.

The same calculations show convergence of all integrals for p in the desired range.

Step 2. We show that for $p \in \big(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\big)$ the integrals in the A_p condition are finite over any (general) interval $I = (\theta_0 - R, \theta_0 + R)$ with $\theta_0 \ne 0$. We consider two cases.

Case 1. $I \cap \mathrm{Arc}(0, 2R) = \varnothing$.

On such intervals I, $3R < \theta_0$ and so $2\theta_0/3 \le \theta_0 - R \le \theta \le \theta_0 + R \le 4\theta_0/3$ giving $\theta \simeq \theta_0$. So, $\omega = |z-1|^{\alpha(2-p)} \simeq \theta_0^{\alpha(2-p)}$. Therefore,

$$\frac{1}{|I|}\int_I \omega(z)\, d\theta \lesssim \frac{1}{2R}\int_I \theta_0^{\alpha(2-p)}\, d\theta = \theta_0^{\alpha(2-p)}.$$

and

$$\begin{aligned}\left(\frac{1}{|I|}\int_I \omega(z)^{\frac{1}{1-p}}\, d\theta\right)^{p-1} &\lesssim \left(\frac{1}{2R}\int_I \theta_0^{\frac{\alpha(2-p)}{1-p}}\, d\theta\right)^{p-1}\\ &= \theta_0^{-\alpha(2-p)}.\end{aligned}$$

Hence the supremum over all such intervals is finite.

Case 2. $I \cap \mathrm{Arc}(0, 2R) \ne \varnothing$.

In this case, since $I \subset \mathrm{Arc}(0, 4R)$ and $\alpha(2-p)+1 > 0$ when $2\alpha + 1/\alpha > p$, we have

$$\frac{1}{|I|}\int_I \omega(z)\, d\theta \simeq \frac{2}{8R}\int_0^{4R} \theta^{\alpha(2-p)}\, d\theta = \frac{1}{4R}\frac{\theta^{\alpha(2-p)+1}}{\alpha(2-p)+1}\Big|_0^{4R} = \frac{4R^{\alpha(2-p)}}{\alpha(2-p)+1}.$$

Also since $\alpha(2-p)/(1-p)+1>0$ when $2\alpha+1/(\alpha+1)<p$,

$$\left(\frac{1}{|I|}\int_I \omega(z)^{\frac{1}{1-p}}\,d\theta\right)^{p-1} \simeq \left(\frac{2}{8R}\int_0^{4R}\theta^{\frac{\alpha(2-p)}{1-p}}\,d\theta\right)^{p-1} = \left(\frac{1}{4R}\frac{\theta^{\frac{\alpha(2-p)}{1-p}+1}}{\frac{\alpha(2-p)}{1-p}+1}\right)^{p-1}$$
$$\simeq \left(\frac{R^{\frac{\alpha(2-p)}{1-p}}}{\frac{\alpha(2-p)}{1-p}+1}\right)^{p-1} = \frac{2R^{-\alpha(2-p)}}{\left(\frac{\alpha(2-p)}{1-p}+1\right)^{p-1}}.$$

Therefore, the supremum over all intervals of the type in case 2 are also finite and $\omega=|g|^{2-p}\in A^p(\mathbb{T})$ if and only if $p\in\left(\frac{2\alpha+1}{\alpha+1},\frac{2\alpha+1}{\alpha}\right)$. □

Remark 8. The analog of Theorem 1 for domains in $\mathbb{C}^n$ ($n\geq 2$) is an open problem. See [Békollé and Bonami 1995] for a partial result. Also see [Lanzani and Stein 2013] for the regularity on strongly pseudoconvex domains.

3. Duality

In this section, we investigate the duals of Hardy spaces corresponding to weights from the previous section. For $\alpha\geq 0$ and $\mu_\alpha(z)=|z-1|^{2\alpha}$, a consequence of Theorem 5 is the following.

Theorem 9. *Let $\alpha\geq 0$ and $\mu_\alpha(z)=|z-1|^{2\alpha}$. For any $p\in\left(\frac{2\alpha+1}{\alpha+1},\frac{2\alpha+1}{\alpha}\right)$, the dual space of the weighted Hardy space $\mathcal{H}^p(\mathbb{T},|z-1|^{2\alpha})$ can be identified with $\mathcal{H}^q(\mathbb{T},|z-1|^{2\alpha})$, where $1/p+1/q=1$, under the pairing*

$$\langle f,h\rangle=\int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2\alpha}\,d\theta.$$

Proof. This is a standard argument; however, we present a proof here for completeness. For a given function $h\in\mathcal{H}^q(\mathbb{T},|z-1|^{2\alpha})$, we define a linear functional on $\mathcal{H}^p(\mathbb{T},|z-1|^{2\alpha})$ by

$$\boldsymbol{G}(f)=\int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2\alpha}\,d\theta.$$

It is clear that, by Hölder's inequality, $\boldsymbol{G}$ is a bounded functional with operator norm less than $\|h\|_{\mathcal{H}^q(\mathbb{T},|z-1|^{2\alpha})}$.

Conversely, let $\boldsymbol{G}$ be a bounded linear functional on $\mathcal{H}^p(\mathbb{T},|z-1|^{2\alpha})$. By the Hahn–Banach theorem, $\boldsymbol{G}$ extends to a bounded linear functional on $L^p(\mathbb{T},|z-1|^{2\alpha})$. Now using the duality of L^p spaces, we find a function $h\in L^q(\mathbb{T},|z-1|^{2\alpha})$ such that

$$\boldsymbol{G}(f)=\int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2\alpha}dz \quad \text{for } f\in L^p(\mathbb{T},|z-1|^{2\alpha}).$$

When we restrict $\boldsymbol{G}$ to $L^p(\mathbb{T}, |z-1|^{2\alpha}) \cap \mathcal{H}^2(\mathbb{T}, |z-1|^{2\alpha})$ and use self-adjointness of $\boldsymbol{S}_{\mu_\alpha}$ we get the following.

$$\begin{aligned}\boldsymbol{G}(f) &= \int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2\alpha}\,d\theta \\ &= \int_{\mathbb{T}} (\boldsymbol{S}_{\mu_\alpha} f)(z)\overline{h(z)}|z-1|^{2\alpha}\,d\theta \\ &= \int_{\mathbb{T}} f(z)\overline{(\boldsymbol{S}_{\mu_\alpha} h)(z)}|z-1|^{2\alpha}\,d\theta\end{aligned}$$

for $f \in L^p(\mathbb{T}, |z-1|^{2\alpha}) \cap \mathcal{H}^2(\mathbb{T}, |z-1|^{2\alpha})$.

Since the intersection of these two spaces is dense in $\mathcal{H}^p(\mathbb{T}, |z-1|^{2\alpha})$, we note that $\boldsymbol{G}$ is represented by the function $(\boldsymbol{S}_{\mu_\alpha} h)(z)$ and $\boldsymbol{S}_{\mu_\alpha} h \in \mathcal{H}^q(\mathbb{T}, |z-1|^{2\alpha})$ by Theorem 5. □

A natural question arises after this statement. How can we identify the dual space of the weighted Hardy space, $\mathcal{H}^p(\mathbb{T}, |z-1|^{2\alpha})$, when $p \notin \left(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\right)$?

The answer to this question follows from the following result on the boundedness of the weighted Szegő projection, $\boldsymbol{S}_{\mu_\alpha}$. Similar results for weighted Bergman projections have been presented recently in [Arroussi and Pau 2014] and [Constantin and Peláez 2015].

Proposition 10. *Let $\alpha \geq 0$ and $\mu_\alpha = |z-1|^{2\alpha}$. For any $1 < p < \infty$, the weighted Szegő projection $\boldsymbol{S}_{\mu_\alpha}$ is bounded on $L^p(\mathbb{T}, |z-1|^{\alpha p})$.*

Remark 11. Note that as p varies, changes occur not only in the integrability scale but also in the measure.

Proof. The proof follows from the relation between the kernels in Proposition 2 and the fact that the unweighted Szegő projection $\boldsymbol{S}$ is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$.

Let us take $f(z) \in L^p(\mathbb{T}, |z-1|^{\alpha p})$ and set

$$\tilde{f}(z) = f(z)\frac{|z-1|^{2\alpha}}{(z-1)^\alpha},$$

then we have $\tilde{f} \in L^p(\mathbb{T})$. Using this notation, we notice

$$\begin{aligned}\boldsymbol{S}_{\mu_\alpha} f(z) &= \int_{\mathbb{T}} S_{\mu_\alpha}(z,w) f(w)|w-1|^{2\alpha}\,d\theta \\ &= \frac{(z-1)^\alpha}{(z-1)^\alpha}\int_{\mathbb{T}} S_{\mu_\alpha}(z,w)(w-1)^\alpha f(w)\frac{|w-1|^{2\alpha}}{(w-1)^\alpha}\,d\theta \\ &= \frac{1}{(z-1)^\alpha}\int_{\mathbb{T}} S(z,w)\tilde{f}(w)\,d\theta \\ &= \frac{1}{(z-1)^\alpha}\boldsymbol{S}\big(\tilde{f}(w)\big)(z),\end{aligned}$$

where we invoke Proposition 2 when we pass from the second to the third line. Next by using the fact that the unweighted Szegő projection operator $\boldsymbol{S}$ is bounded on $L^p(\mathbb{T})$, we obtain the following.

$$\begin{aligned}\|\boldsymbol{S}_{\mu_\alpha} f\|_{L^p(\mathbb{T},|z-1|^{\alpha p})} &= \int_{\mathbb{T}} |z-1|^{\alpha p} \frac{1}{|z-1|^{\alpha p}} \big|\boldsymbol{S}\big(\tilde{f}(w)\big)(z)\big|^p \, d\theta \\ &= \big\|\boldsymbol{S}\big(\tilde{f}(w)\big)\big\|^p_{L^p(\mathbb{T})} \lesssim \|\tilde{f}(w)\|^p_{L^p(\mathbb{T})} \\ &= \left\| f(w) \frac{|w-1|^{2\alpha}}{(w-1)^\alpha} \right\|^p_{L^p(\mathbb{T})} \\ &= \|f\|^p_{L^p(\mathbb{T},|z-1|^{\alpha p})}.\end{aligned}$$

This finishes the proof of the proposition. □

Now we can answer the duality question by using Proposition 10. Following the same argument as in the proof of Theorem 9, we obtain the following statement.

Theorem 12. *Let $\alpha \geq 0$ and $\mu_\alpha = |z-1|^{2\alpha}$. Then for any $p \in (1, \infty)$, the dual space of the weighted Hardy space $\mathcal{H}^p(\mathbb{T}, |z-1|^{\alpha p})$ can be identified with $\mathcal{H}^q(\mathbb{T}, |z-1|^{\alpha q})$, where $1/p + 1/q = 1$, under the pairing*

$$\langle f, h\rangle = \int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2\alpha} \, d\theta.$$

At first, the two duality results in Theorem 9 and Theorem 12 may seem confusing for $p \in \left(\frac{2\alpha+1}{\alpha+1}, \frac{2\alpha+1}{\alpha}\right)$. However, the main point is to note the difference in the exponents of the weights and the way the pairing is defined. We illustrate these two results in the following example.

Example 13. Let us take $\alpha = 1/2$. Then $\boldsymbol{S}_{|z-1|}$ is bounded on $L^p(\mathbb{T}, |z-1|)$ for $p \in (4/3, 4)$. In particular, for any $p \in (4/3, 4)$, the dual space of $\mathcal{H}^p(\mathbb{T}, |z-1|)$ can be identified with $\mathcal{H}^q(\mathbb{T}, |z-1|)$, where $1/p + 1/q = 1$, under the pairing

$$\langle f, h\rangle_{|z-1|} = \int_{\mathbb{T}} f(z)\overline{h(z)}|z-1| \, d\theta.$$

On the other hand, using the second duality result for any $p > 1$, the dual space of $\mathcal{H}^p(\mathbb{T}, |z-1|)$ can be identified with $\mathcal{H}^q(\mathbb{T}, |z-1|^{q/p})$ when $1/p + 1/q = 1$, under the pairing

$$\langle f, h\rangle_{|z-1|^{2/p}} = \int_{\mathbb{T}} f(z)\overline{h(z)}|z-1|^{2/p} \, d\theta.$$

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