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TOPOLOGY OF COMPLETE FINSLER MANIFOLDS ADMITTING CONVEX FUNCTIONS

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We investigate the topology of a complete Finsler manifold (M, F) admitting a locally nonconstant convex function.

1. Introduction

Let (M, F) be an n-dimensional Finsler manifold. The well-known Hopf–Rinow theorem (see for example [Bao et al. 2000]) states that M is complete if and only if the exponential map $\exp_p$ at some point $p \in M$ (and hence for every point on M) is defined on the whole tangent space T_pM to M at that point. This is equivalent to saying that (M, F) is geodesically complete with respect to forward geodesics at every point on M. Throughout this article we assume that (M, F) *is geodesically complete with respect to forward geodesics*.

A function $\varphi : (M, F) \to \mathbb{R}$ is said to be *convex* if and only if along every (forward and backward) geodesic $\gamma : [a, b] \to (M, F)$, the restriction $\varphi \circ \gamma : [a, b] \to \mathbb{R}$ is a convex function, that is,

$$\varphi \circ \gamma((1-\lambda)a + \lambda\, b) \leq (1-\lambda)\varphi \circ \gamma(a) + \lambda\varphi \circ \gamma(b), \quad 0 \leq \lambda \leq 1. \tag{1-1}$$

If the inequality in the above relation is strict for all γ and for all $\lambda \in (0, 1)$, then φ is called *strictly convex*. If the second order difference quotient, namely the quantity $\{\varphi \circ \gamma(h) - \varphi \circ \gamma(-h) - 2\varphi \circ \gamma(0)\}/h^2$ is bounded away from zero on every compact set on M along all γ, then φ is called *strongly convex*. In the case where φ is at least C^2, its convexity can be written in terms of the Finslerian Hessian of φ, but we do not need to do this in the present paper.

If $\varphi \circ \gamma$ is a convex function of one variable, then the function $\varphi \circ \bar{\gamma}$ is also convex, where $\bar{\gamma}$ is the reverse curve of γ. We recall that in general, if γ is a Finslerian geodesic, it does not mean that the inverse curve $\bar{\gamma}$ is also a geodesic.

Every noncompact manifold admits a complete (Riemannian or Finslerian) metric and a nontrivial smooth function which is convex with respect to this metric (see [Greene and Shiohama 1981b]).

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If a nontrivial convex function $\varphi : (M, F) \to \mathbb{R}$ is constant on an open set, then φ assumes its minimum on this open set and the number of components of *a level set* $M_a^a(\varphi) := \varphi^{-1}(\{a\})$, $a \geq \inf_M \varphi$ is equal to that of the boundary components of the minimum set of φ. Here we denote $\inf_M \varphi := \inf\{\varphi(x) \mid x \in M\}$.

A convex function φ is said to be *locally nonconstant* if it is not constant on any open set of M. From now on *we always assume that a convex function is locally nonconstant*.

The purpose of this article is to investigate the topology of complete Finsler manifolds admitting (locally nonconstant) convex functions $\varphi : (M, F) \to \mathbb{R}$. Convex functions on complete Riemannian manifolds have been fully discussed in [Greene and Shiohama 1981b] and elsewhere. Although the distance function on (M, F) is not symmetric and the backward geodesics do not necessarily coincide with the forward geodesics, we prove that most of the Riemannian results in [Greene and Shiohama 1981b] have Finsler extensions, as stated below.

We first discuss the topology of a Finsler manifold (M, F) admitting a convex function φ.

Theorem 1.1 (compare [Greene and Shiohama 1981b, Theorem F]). *Assume we have a convex function $\varphi : (M, F) \to \mathbb{R}$ all of whose level sets are compact.*

(1) *If $\inf_M \varphi$ is not attained, there exists a homeomorphism*

$$H : M_a^a(\varphi) \times (\inf_M \varphi, \infty) \to M,$$

for an arbitrary fixed number $a \in (\inf_M \varphi, \infty)$, such that

$$\varphi(H(y, t)) = t, \quad \forall y \in M_a^a(\varphi), \quad \forall t \in (\inf_M \varphi, \infty).$$

(2) *If $\lambda := \inf_M \varphi$ is attained, then M is homeomorphic to the normal bundle over $M_\lambda^\lambda(\varphi)$ in M.*

Next, we discuss the case where φ has a disconnected level.

Theorem 1.2 (compare [Greene and Shiohama 1981b, Theorem A]). *Assume the convex function $\varphi : (M, F) \to \mathbb{R}$ has a disconnected level set $M_c^c(\varphi)$ for some $c \in \varphi(M)$.*

(1) *The infimum $\inf_M \varphi$ is attained.*

(2) *If $\lambda := \inf_M \varphi$, then $M_\lambda^\lambda(\varphi)$ is a totally geodesic smooth hypersurface which is totally convex without boundary.*

(3) *The normal bundle of $M_\lambda^\lambda(\varphi)$ in M is trivial.*

(4) *If $b > \lambda$, then the boundary of the b-sublevel set $M^b(\varphi) := \{x \in M \mid \varphi(x) \leq b\}$ has exactly two components.*

The diameter function $\delta : \varphi(M) \to \mathbb{R}_+$ plays an important role in this article and is defined by

$$\delta(t) := \sup\{d(x, y) \mid x, y \in M_t^t(\varphi)\}. \tag{1-2}$$

Sharafutdinov [1978] had proved earlier the existence of a distance nonincreasing map $M_b^b(\varphi) \to M_a^a(\varphi)$, $b \geq a$, between two compact levels of a convex function φ on a complete Riemannian manifold (M, g).

It is known from [Sharafutdinov 1978] and [Greene and Shiohama 1981b] that the diameter function δ of a complete Riemannian manifold admitting a convex function is monotone nondecreasing. However it is not certain if it is monotone on a Finsler manifold. In Theorem 1.1, we do not use the monotone property but only the local Lipschitz property of δ which is proved in Proposition 3.3.

We finally discuss the number of ends of a Finsler manifold (M, F) admitting a convex function φ. As stated above, the diameter function δ, defined on the image of the convex function φ, may not be monotone. It might occur that a convex function defined on a Finsler manifold (M, F) may simultaneously admit both compact and noncompact levels. This fact makes it difficult to study the number of ends of the manifold (M, F). However, we shall discuss all the possible cases and prove:

Theorem 1.3 (compare [Greene and Shiohama 1981b, Theorems C, D and G]). *Let $\varphi : (M, F) \to \mathbb{R}$ be a convex function.*

(A) *Assume that φ admits a disconnected level.*
  - (A1) *If all the levels of φ are compact, then M has two ends.*
  - (A2) *If all the levels of φ are noncompact, then M has one end.*
  - (A3) *If both compact and noncompact levels of φ exist simultaneously, then M has at least three ends.*

(B) *Assume that all the levels of φ are connected and compact.*
  - (B1) *If $\inf_M \varphi$ is attained, then M has one end.*
  - (B2) *If $\inf_M \varphi$ is not attained, then M has two ends.*

(C) *If all the levels are connected and noncompact, then M has one end.*

(D) *Assume that all the levels of φ are connected and that φ admits both compact and noncompact levels simultaneously. Then we have:*
  - (D1) *If $\inf_M \varphi$ is not attained, then M has two ends.*
  - (D2) *If $\inf_M \varphi$ is attained, then M has at least two ends.*

(E) *Finally, if M has two ends, then all the levels of φ are compact.*

Remark 1.4. The supplementary condition that all of the levels of φ are simultaneously compact or noncompact in the hypothesis of Theorem 1.1 is necessary because we have not proved that the diameter function δ is monotone nondecreasing for a

Finsler manifold. If this property of monotonicity were true, then this assumption could be removed.

We summarize some historical background of convex and related functions on manifolds, G-spaces and Alexandrov spaces. Locally nonconstant convex functions, affine functions and peakless functions have been investigated on complete Riemannian manifolds and complete noncompact Busemann G-spaces and Alexandrov spaces in various ways. The topology of Riemannian manifolds admitting convex functions was investigated in [Bangert 1978; Greene and Shiohama 1981b; 1981a; 1987], and that of Busemann G-surfaces in [Innami 1982a; Mashiko 1999b]. It should be noted that convex functions on complete Alexandrov surfaces are *not continuous*.

A weaker notion than convex functions similar to quasiconvex functions, namely *peakless functions*, has been introduced by Busemann [1955], and studied later on in [Busemann and Phadke 1983] and [Innami 1983]. The topology of complete manifolds admitting locally geodesically (strictly) quasiconvex and uniformly locally convex filtrations have been investigated by Yamaguchi [1986a; 1986b; 1988]. The isometry groups of complete Riemannian manifolds (N, g) admitting strictly convex functions have been discussed in [Yamaguchi 1982] and other places. A well known classical theorem due to Cartan states that every compact isometry group on an Hadamard manifold H has a fixed point. This follows from the simple fact that the distance function to every point on H is strictly convex. Peakless functions and totally geodesic filtrations on complete manifolds have been discussed in [Innami 1983; Busemann and Phadke 1983; Yamaguchi 1986a; 1986b; 1988] and others.

A convex function is said to be *affine* if and only if the equality in (1-1) holds for all γ and for all $\lambda \in (0, 1)$. A splitting theorem for Riemannian manifolds admitting affine functions has been investigated in [Innami 1982b], while Alexandrov spaces admitting affine functions have been studied in [Innami 1982b; Mashiko 1999a; Mashiko 2002]. An overview on the convexity of Riemannian manifolds can be found in [Burago and Zalgaller 1977].

The properties of isometry groups on Finsler manifolds admitting convex functions will be discussed elsewhere. For basic facts on Finsler and Riemannian geometry, we refer to [Bao et al. 2000; Chern et al. 1999; Cheeger and Ebin 2008; Sakai 1992].

2. Fundamental facts

We summarize some fundamental facts on convex sets and convex functions on a Finsler manifold (M, F). Most of these are trivial in the Riemannian case, but we consider it useful to formulate and prove them in the more general Finslerian setting.

Let (M, F) be a complete Finsler manifold. At each point $p \in M$, the indicatrix $\Sigma_p \subset T_pM$ at p is defined by $\Sigma_p := \{u \in T_pM \mid F(p, u) = 1\}$. The *reversibility function* $\lambda : (M, F) \to \mathbb{R}^+$ of (M, F) is given as

$$\lambda(p) := \sup\{F(p, -X) \mid X \in \Sigma_p\}.$$

Clearly, λ is continuous on M and

$$\lambda(p) = \max\left\{\frac{F(p, -X)}{F(p, X)} \mid X \in T_pM \setminus \{0\}\right\}.$$

Let $C \subset M$ be a compact set. There exists a constant $\lambda(C) > 0$ depending on C such that if $p \in C$ and $X \in \Sigma_p$, then

$$\frac{1}{\lambda(C)} F(p, X) \leq F(p, -X) \leq \lambda(C) \cdot F(p, X).$$

In particular, if $\sigma : [0, 1] \to C$ is a smooth curve, then the integral length

$$L(\sigma) := \int_0^1 F(\sigma(t), \dot{\sigma}(t))\, dt$$

of σ satisfies

$$\frac{1}{\lambda(C)} L(\sigma) \leq L(\sigma^{-1}) \leq \lambda(C) \cdot L(\sigma).$$

Here we set $\sigma^{-1}(t) := \sigma(1 - t)$, where $t \in [0, 1]$ is the reverse curve of σ.

It is well known that the topology of (M, F) as an inner metric space is equivalent to that of M as a manifold. For a compact set $C \subset M$, the inner metric d_F of (M, F) induced from the Finslerian fundamental function has the property

$$\frac{1}{\lambda(C)} d_F(p, q) \leq d_F(q, p) \leq \lambda(C) \cdot d_F(p, q), \quad \forall p, q \in C.$$

Let $\mathrm{inj} : (M, F) \to \mathbb{R}_+$ be the *injectivity radius function* of the exponential map. Namely, for a point $p \in M$, $\mathrm{inj}(p)$ is the maximal radius of a ball, centered at the origin of the tangent space T_pM at p, on which $\exp_p$ is injective.

A classical result due to J. H. C. Whitehead [1935] states that there exists a *convexity radius function* $r : (M, F) \to \mathbb{R}$ such that if

$$B(p, r) := \{x \in M \mid d(p, x) < r\}$$

is an r-ball centered at p, then for every $q \in B(p, r(p))$ and for every $r' \in (0, r(p))$, $B(q, r') \subset B(p, r)$ is *strongly convex*. Namely, the distance function to q is strongly convex along every geodesic in $B(q, r')$ with $r' \in (0, r(p))$ if its extension does not pass through q.

A closed set $U \subset M$ is called *locally convex* if and only if $U \cap B(p, r)$ is convex for every $x \in U$ and for some $r \in (0, r(p))$. Notice that this definition is stated only for closed sets, since every open set is obviously locally convex.

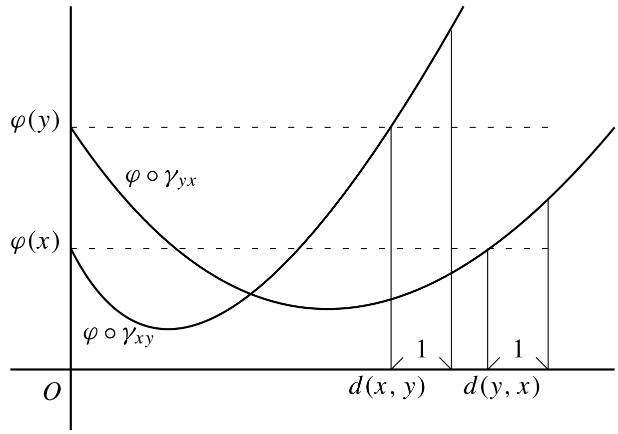

Figure 1. A convex function is locally Lipschitz.

A set $V \subset M$ is called *totally convex* if and only if every geodesic joining two points in V is contained entirely in V. A closed hemisphere in the standard sphere $\mathbb{S}^n$ is locally convex and an open hemisphere is strongly convex, while $\mathbb{S}^n$ itself is the only totally convex set in it. If it exists, the minimum set of a convex function on (M, F) is totally convex.

Proposition 2.1. *A convex function* $\varphi : (M, F) \to \mathbb{R}$ *defined as in* (1-1) *is locally Lipschitz.*

Proof. Let $C \subset M$ be an arbitrary fixed compact set and $C_1 := \{x \in M \mid d(C, x) \le 1\}$. Here we set $d(C, x) := \min\{d(y, x) \mid y \in C\}$. For points $x, y \in C_1$ we denote by

$$\gamma_{xy} : [0, d(x, y)] \to M, \quad \gamma_{yx} : [0, d(y, x)] \to M$$

minimizing geodesics with

$$\gamma_{xy}(0) = x, \quad \gamma_{xy}(d(x, y)) = y,$$
$$\gamma_{yx}(0) = y, \quad \gamma_{yx}(d(y, x)) = x.$$

The slope inequalities along the two convex functions $\varphi \circ \gamma_{xy}|_{[0,d(x,y)+1]}$ and $\varphi \circ \gamma_{yx}|_{[0,d(y,x)+1]}$ imply that, if $\Lambda := \sup_{C_1} \varphi$ and $\lambda := \inf_{C_1} \varphi$ (see Figure 1), then

$$\frac{\varphi(y) - \varphi(x)}{d(x, y)} \le \Lambda - \lambda, \quad \frac{\varphi(x) - \varphi(y)}{d(y, x)} \le \Lambda - \lambda.$$

It follows that there exists a constant $L = L(C) > 0$ such that

$$\sup\left\{\frac{d(x, y)}{d(y, x)} \mid x, y \in C\right\} \le L,$$

and therefore we have

$$\left|\frac{\varphi(x)-\varphi(y)}{d(x,y)}\right|,\ \left|\frac{\varphi(y)-\varphi(x)}{d(y,x)}\right| \le L(\Lambda-\lambda). \qquad \square$$

Proposition 2.2. *If $C \subset (M, F)$ is a closed locally convex set, then there exists a k-dimensional totally geodesic submanifold W of M contained in C, and its closure coincides with C.*

Proof. Let $r : (M, F) \to \mathbb{R}$ be the convexity radius function. For every point $p \in C$ there exists a $k(p)$-dimensional smooth submanifold of M which is contained entirely in C and such that $k(p)$ is the maximal dimension of all such submanifolds in C, where $0 \le k(p) \le n$. At least $\{p\}$ is such a submanifold, with dimension 0.

Let $K \subset M$ be a large compact set containing p and $r(K)$ the convexity radius of K, namely $r(K) := \min\{r(x) \mid x \in K\}$. We also put $k := \max\{k(p) \mid p \in C\}$.

Let $W(p) \subset C$ be a k-dimensional smooth submanifold of M. Suppose that $W(p) \cap B(p;r) \subsetneqq C \cap B(p;r)$ for a sufficiently small $r \in (0, r(K))$. Then there exists a point $q \in B(p;r) \cap (C \setminus W(p))$. Clearly $\dot\gamma_{pq}(0)$ is transversal to $T_pW(p)$, and hence a family of minimizing geodesics

$$\{\gamma_{xq} : [0, d(x,q)] \to B(p;r) \mid x \in W(p) \cap B(p;r)\}$$

with $\gamma_{xq}(0) = x$, $\gamma_{xq}(d(x,q)) = q$ has the property that every $\dot\gamma_{xq}(0)$ is transversal to $T_xW(p)$. Therefore, this family of geodesics forms a $(k+1)$-dimensional submanifold contained in C, a contradiction to the choice of k. This proves $W(p) \cap B(p;r) = C \cap B(p;r)$ for a sufficiently small $r \in (0, r(K))$. We then observe that $\bigcup_{p\in C} W(p) =: W \subset C$ forms a k-dimensional smooth submanifold which is totally geodesic. Indeed, for any tangent vector v to W, there exists $p \in C$ such that $v \in T_pW(p)$, and due to the convexity of C, the geodesic $\gamma_v : [0, \varepsilon] \to M$ cannot leave the submanifold W.

We finally prove that the closure $\overline{W}$ of W coincides with C. Indeed, suppose that there exists a point $x \in C \setminus \overline{W}$. We then find a point $y \in \overline{W} \setminus W$ such that $d(x, y) = d(x, \overline{W}) < r(K)$. If

$$\dot\gamma_{xy}(d(x,y)) \in T_y\overline{W} := \lim_{y_j \to y} T_{y_j}M,$$

then $\gamma_{xy}(d(x,y)+\varepsilon) \in W$ for a sufficiently small $\varepsilon > 0$. Let $U \subset W \cap B(x, r(K))$ be an open set around $\gamma_{xy}(d(x,y)+\varepsilon)$.

Then a family of geodesics

$$\{\gamma_{xz} : [0, d(x,z)] \to B(x; r(K)) \mid z \in U\} \tag{2-1}$$

forms a k-dimensional submanifold contained in W and hence $y \in W$, a contradiction to $y \in \overline{W} \setminus W$. Therefore, $\dot\gamma_{xy}(d(x,y))$ does not belong to $T_y\overline{W}$, and (2-1) again

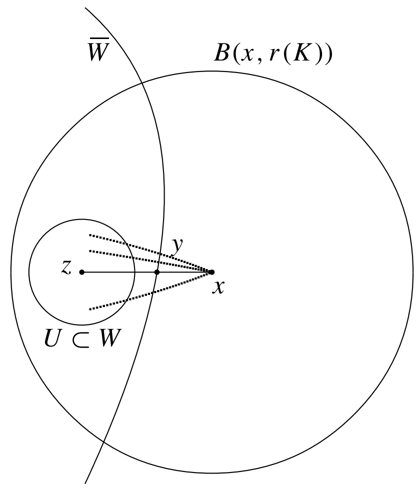

Figure 2. The closure $\overline{W}$ of W coincides with C.

forms a $(k+1)$-dimensional submanifold in C, a contradiction to the choice of k (see Figure 2). $\square$

Let $C \subset M$ be a closed locally convex set and $p \in C$. There exists a totally geodesic submanifold $W \subset C$ as stated in Proposition 2.2. We call W the *interior of* C and denote it by $\mathrm{Int}(C)$. The *boundary of* C is defined by $\partial C := C \setminus \mathrm{Int}(C)$, and *the dimension of* C is defined by $\dim C := \dim \mathrm{Int}(C)$. The *tangent cone* $\mathscr{C}_p(C) \subset T_pM$ of C at a point $p \in C$ is defined by

$$\mathscr{C}_p(C) := \{\xi \in T_pM \mid \exp_p t\xi \in \mathrm{Int}(C) \text{ for some } t > 0\}. \tag{2-2}$$

Clearly, $\mathscr{C}_p(C) = T_p \mathrm{Int}(C) \setminus \{0\}$ for $p \in \mathrm{Int}(C)$.

We also define the tangent space T_pC of C at a point $p \in \partial C$ as $\lim_{q\to p} T_q \mathrm{Int}(C)$. We claim that there exists for every point $p \in \partial C$ an open half space $T_pC_+ \subset T_pC$ containing $\mathscr{C}_p(C)$:

$$\mathscr{C}_p(C) \subset T_pC_+ \subset T_pC := \lim_{q\to p} T_q \mathrm{Int}(C),\ q \in \mathrm{Int}(C). \tag{2-3}$$

Indeed, for any points $p \in \partial C$ and $q \in B(p; r(K)) \cap \mathrm{Int}(C)$, consider a minimizing geodesic $\gamma_{qp} : [0, d(q, p)] \to B(p; r(K))$. Suppose that there is a point $q \in \mathrm{Int}(C)$ such that $z := \gamma_{qp}(d(q, p) + \varepsilon) \in C$ for a sufficiently small $\varepsilon > 0$. We then have $\dot{\gamma}_{qp}(d(q, p)) \in T_pC$, and hence the tangent cone $\mathscr{C}_p(C)$ as obtained in (2-2) is contained entirely in T_pC, a contradiction to the choice of $p \in \partial C$. From the above argument we observe that if $p \in \partial C$, then there exists a hyperplane $H_p \subset T_pC$ such that $\mathscr{C}_p(C)$ is contained in a half space $T_p(C)_+ \subset T_pC$ bounded by H_p.

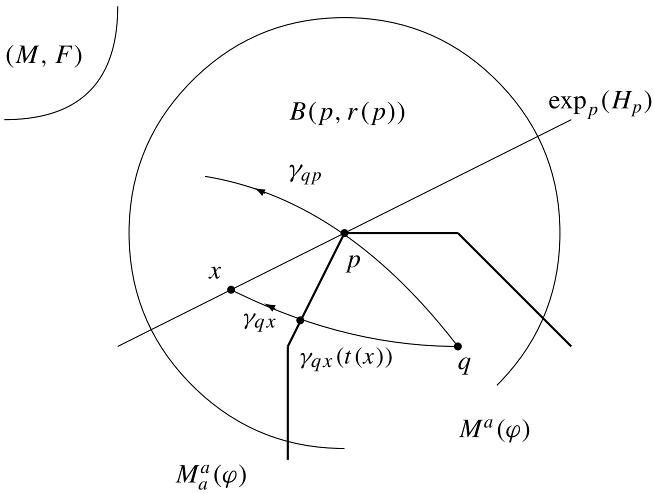

Figure 3. An atlas of local charts at an arbitrary point $p \in M_a^a(\varphi)$.

Proposition 2.3. *Let $\varphi : (M, F) \to \mathbb{R}$ be a convex function. Then, $M_a^a(\varphi)$ is an embedded topological submanifold of dimension $n - 1$ for every $a > \inf_M \varphi$.*

Proof. Let $p \in M_a^a(\varphi)$ and $q \in B(p; r(p)) \cap \mathrm{Int}(M^a(\varphi))$. There exists a hyperplane $H_p \subset T_pM$ such that

$$H_p = \partial T_p(M^a(\varphi))_+ \quad \text{and} \quad \mathscr{C}_p(M^a(\varphi)) \subset T_p(M^a(\varphi))_+.$$

Every point $x \in \exp_p(H_p) \cap B(p; r(p))$ is joined to q by a unique minimizing geodesic $\gamma_{qx} : [0, d(q, x)] \to M$ such that $\gamma_{qx}(0) = q$, $\gamma_{qx}(d(q, x)) = x$. Then there exists a unique parameter $t(x) \in (0, d(q, x)]$ such that $M_a^a(\varphi) \cap B(p; r(p))$ contains $\gamma_{qx}(t(x))$. Let $B_H(O; r(p))$ be the open $r(p)$-ball in H_p centered at the origin O of M_p. We then have a map $\alpha_p : B_H(O; r(p)) \to M_a^a(\varphi)$ such that

$$\alpha_p(u) := \gamma_{qx}(t(x)), \quad u \in B_H(O; r(p)), \quad \exp_p u = x.$$

Clearly, α_p gives a homeomorphism between $B_H(O; r(p))$ and its image in $M_a^a(\varphi)$. Thus the family of maps $\{(B_H(O; r(p)), \alpha_p) \mid p \in M_a^a(\varphi)\}$ forms an atlas of $M_a^a(\varphi)$ (see Figure 3). □

3. Level sets configuration

We shall give the proofs of Theorems 1.2 and 1.3. The following lemma is elementary and useful for our discussion.

Lemma 3.1. *Let $\varphi : (M, F) \to \mathbb{R}$ be a convex function. If $M_a^a(\varphi)$ is compact, then so is $M_b^b(\varphi)$ for all $b \geq a$. If $M_a^a(\varphi)$ is noncompact, then so is $M_b^b(\varphi)$ for all $b \leq a$.*

Proof. First of all we prove that if $M_a^a(\varphi)$ is compact, then so is $M_b^b(\varphi)$ for all $b \geq a$.

Suppose that $M_b^b(\varphi)$ is noncompact for some $b > a$. Take a point $p \in M_a^a(\varphi)$ and a divergent sequence $\{q_j\}_{j\geq 1}$ on $M_b^b(\varphi)$. Since $M_a^a(\varphi)$ is compact, there is a positive number L such that $d(p, x) < L$ for all $x \in M_a^a(\varphi)$. Let $\gamma_j : [0, d(p, q_j)] \to M$ be a minimizing geodesic with $\gamma_j(0) = p$, $\gamma_j(d(p, q_j)) = q$ for $j \geq 1$. Compactness of $M_a^a(\varphi)$ implies that each $\varphi \circ \gamma_j|_{[L,d(p,q_j)-L]}$ is monotone and nondecreasing for all large numbers j.

Choosing a subsequence $\{\gamma_i\}$ of $\{\gamma_j\}$ if necessary, we find a ray $\gamma_\infty : [0, \infty) \to M$ emanating from p such that $\varphi \circ \gamma_\infty$ is monotone, nondecreasing and bounded above, and hence is identically equal to a. This contradicts the assumption that $M_a^a(\varphi)$ is compact. $\square$

The following Proposition 3.2 is the basic piece in the proof of Theorem 1.1. Under the assumptions in Theorem 1.1, we divide M into countable compact sets such that

$$M = \bigcup_{j=-\infty}^{\infty} \varphi^{-1}[t_{j-1}, t_j],$$

where $\{t_j\}$ is monotone increasing and $\lim_{j\to-\infty} t_j = \inf_M \varphi$ (if $\inf_M \varphi$ is not attained) and $\lim_{j\to\infty} t_j = \infty$. In applying Proposition 3.2 to each $\varphi^{-1}[t_{j-1}, t_j]$, the undefined numbers b_{k+1} and b_0 appearing in the proof of the proposition play the role of margins to be pasted with $\varphi^{-1}[t_j, t_{j+1}]$ (using b_{k+1}) and with $\varphi^{-1}[t_{j-2}, t_{j-1}]$ (using b_{-1}), respectively.

Proposition 3.2. *Let $M_a^a(\varphi) \subset M$ be a connected and compact level set and $b > a$ a fixed value. Then there exists a homeomorphism $\Phi_a^b : M_b^b(\varphi) \times [a, b] \to M_a^b(\varphi)$ such that*

$$\varphi \circ \Phi_a^b(x, t) = t, \quad (x, t) \in M_b^b(\varphi) \times [a, b]. \tag{3-1}$$

Proof. Let $K \subset M$ be a compact set with $M_a^b(\varphi) \subset \mathrm{Int}(K)$ and $r := r(K)$ the convexity radius over K. We define two divisions as follows. Let $a = a_0 < a_1 < \cdots < a_k = b$ and $b_{-1} < b_0 < \cdots < b_k$ be given such that $\varphi^{-1}[b_{-1}, b_k] \subset \mathrm{Int}(K)$ and

(1) $b_{-1} < a_0 < b_0 < a_1 < \cdots < a_{k-1} < b_{k-1} < a_k = b < b_k$,

(2) $b_j := \frac{1}{2}(a_j + a_{j+1}),\ j = 0, 1, \ldots, k-1$,

(3) $\varphi^{-1}(\{a_{j-1}\}) \subset \bigcup\{B(x, r) \mid x \in \varphi^{-1}(\{a_{j+1}\})\},\ j = 1, \ldots, k-1$,

(4) $\varphi^{-1}(\{b_{-1}\}) \subset \bigcup\{B(y, r) \mid y \in \varphi^{-1}(\{a_1\})\}$,

(5) $\varphi^{-1}(\{a_{k-1}\}) \subset \bigcup\{B(z, r) \mid z \in \varphi^{-1}(\{b_k\})\}$.

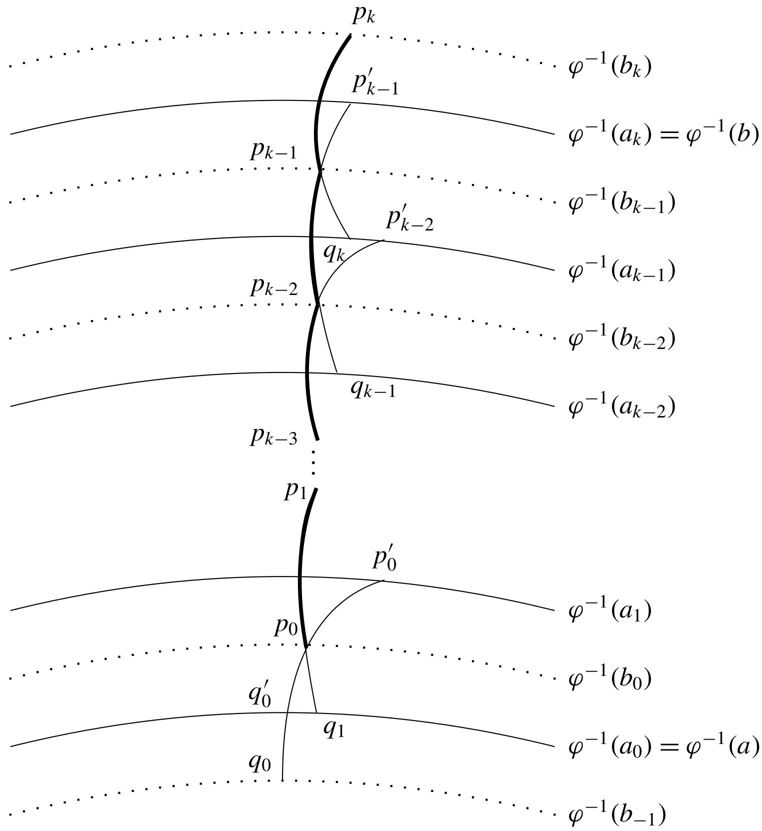

Figure 4. The broken geodesic $T(p_k)$.

Obviously we have $[a, b] \subset (b_{-1}, b_k)$.

For an arbitrary fixed point $p'_j \in \varphi^{-1}(\{a_{j+1}\})$, we have a minimizing geodesic $T(p'_j, q_j)$ realizing the distance $d(p'_j, \varphi^{-1}(-\infty, a_{j-1}])$ and q_j the foot of p'_j on $\varphi^{-1}(-\infty, a_{j-1}]$. Then the family of all such minimizing geodesics emanating from all the points on $\varphi^{-1}(\{a_{j+1}\})$ to the points on $\varphi^{-1}(\{a_{j-1}\})$ simply covers the set $\varphi^{-1}[b_{j-1}, b_j]$, $j = 1, 2, \ldots, k$. We define $p_j := T(p'_j, q_j) \cap \varphi^{-1}(\{b_j\})$ and $p_{j-1} := T(p'_j, q_j) \cap \varphi^{-1}(\{b_{j-1}\})$. With this point p_{j-1}, we then choose $p'_{j-1} \in \varphi^{-1}(\{a_j\})$ and $q_{j-1} \in \varphi^{-1}(\{a_{j-2}\})$ in such a way that $T(p'_{j-1}, q_{j-1})$ realizes the distance $d(p'_{j-1}, q_{j-1}) = d(p_{j-1}, \varphi^{-1}(-\infty, a_{j-2}])$ and contains p_{j-1} in its interior. We thus obtain the inductive construction of a sequence $\{T(p'_j, q_j) \mid j = 1, \ldots, k\}$ of minimizing geodesics.

We finally choose a point $p'_0 \in \varphi^{-1}(\{a_1\})$ and $q_0 \in \varphi^{-1}(\{b_{-1}\})$ such that $T(p'_0, q_0)$ is a unique minimizing geodesic, with q_0 being the foot of p'_0 on $\varphi^{-1}(-\infty, b_{-1}]$. If we set $q'_0 := \varphi^{-1}(\{a_0\}) \cap T(p'_0, q_0)$, then $d(p_0, q_1) \le d(p_0, q'_0)$ follows from the fact that q_1 is the foot of p_0 on $\varphi^{-1}(-\infty, a_0]$. Therefore the slope inequality along

$T(p'_0, q_0)$ implies

$$\frac{a_0 - b_0}{d(p_0, q_1)} \leq \frac{a_0 - b_0}{d(p_0, q'_0)} \leq \frac{b_{-1} - a_0}{d(q'_0, q_0)},$$

and hence there exists a positive number

$$\Delta_a^b(K) := \min\left\{ \frac{a - b_{-1}}{d(q'_0, \varphi^{-1}(-\infty, b_{-1}])} \;\middle|\; q'_0 \in \varphi^{-1}(\{a\}) \right\}$$

with the property that all the slopes of $\varphi \circ T(p'_j, q_j)$ for every $j = 0, 1, \dots, k$ are negative and bounded above by $-\Delta_a^b(K)$.

Next, we define a broken geodesic $T(p_k) := T(p_k, p_{k-1}) \cup \cdots \cup T(p_1, p_0)$ for $p_k \in \varphi^{-1}(\{b_k\})$ with its break points at $p_j \in \varphi^{-1}(\{b_j\})$, $j = 0, 1, \dots, k-1$ in such a way that each $T(p_j, p_{j-1})$ is a proper subarc of a unique minimizing geodesic $T(p'_j, q_j)$, where q_j is the foot of p'_j on $\varphi^{-1}(-\infty, a_{j-1}]$ (see Figure 4). Then $T(p_{j-1}, p_{j-2})$ is a proper subarc of $T(p'_{j-1}, q_{j-1})$. Clearly, the convex function along $T(p_k)$ is monotone strictly decreasing, since the slopes along $\varphi \circ T(p_k)$ are all bounded above by $-\Delta_a^b(K)$. We then observe from the construction that the family of all the broken geodesics emanating from all points on $\varphi^{-1}(\{b_k\})$ and ending at points on $\varphi^{-1}(\{b_{-1}\})$ simply covers $\varphi^{-1}[a, b]$. The desired homeomorphism Φ_a^b is now obtained by defining $\Phi_a^b(x, t)$ as the intersection of a $T(x)$ emanating from x:

$$\Phi_a^b(x, t) = T(x) \cap \varphi^{-1}(\{t\}). \qquad \square$$

Proposition 3.3. *Assume that all the levels of φ are compact. Then the diameter function $\delta : \varphi(M) \to \mathbb{R}$ defined by*

$$\delta(a) := \sup\{d(x, y) \mid x, y \in \varphi^{-1}(\{a\}),\ a \in \varphi(M)\}$$

is locally Lipschitz.

Proof. Let $\inf_M \varphi < a < b < \infty$, and let $r = r(M_a^b(\varphi))$ be the convexity radius over $M_a^b(\varphi)$. Let $x, y \in \varphi^{-1}(\{s\})$ for $s \in [a, b)$ be such that $d(x, y) = \delta(s)$. Proposition 3.2 then implies that there are points $x', y' \in M_b^b(\varphi)$ such that $\Phi_a^b(x', s) = x$ and $\Phi_a^b(y', s) = y$. Moreover, we have $\Phi_a^b(x', t) = T(x') \cap M_t^t(\varphi)$ and $\Phi_a^b(y', t) = T(y') \cap M_t^t(\varphi)$, and the length $L(T(p)|_{[s,t]})$ of $T(p)|_{[s,t]}$, for $p \in M_b^b(\varphi)$ and for every $a \leq s < t \leq b$, is bounded above by

$$L(T(p)|_{[s,t]}) \leq |t - s| / \Delta_a^b(M_a^b(\varphi)). \tag{3-2}$$

Therefore, by setting $\lambda = \lambda(M_a^b(\varphi))$ the reversibility constant on $M_a^b(\varphi)$, we have

$$\begin{aligned}\delta(s) = d(x, y) &\le d(x, \Phi_a^b(x', t)) + d(\Phi_a^b(x', t), \Phi_a^b(y', t)) + d(\Phi_a^b(y', t), y)\\ &\le \lambda|t-s|/\Delta_a^b(M_a^b(\varphi)) + \delta(t) + |t-s|/\Delta_a^b(M_a^b(\varphi))\\ &= (1+\lambda)|t-s|/\Delta_a^b(M_a^b(\varphi)) + \delta(t).\end{aligned}$$

Similarly, by choosing $x, y \in \varphi^{-1}(\{t\})$, $d(x, y) = \delta(t)$, we obtain

$$\begin{aligned}\delta(t) = d(x, y) &\le d(x, \Phi_a^b(x', s)) + d(\Phi_a^b(x', s), \Phi_a^b(y', s)) + d(\Phi_a^b(y', s), y)\\ &\le (1+\lambda)|t-s|/\Delta_a^b(M_a^b(\varphi)) + \delta(s),\end{aligned}$$

and hence,

$$|\delta(t) - \delta(s)| \le (1+\lambda)|t-s|/\Delta_a^b(M_a^b(\varphi)). \qquad \square$$

Proof of Theorem 1.1. We first assume that $\inf_M \varphi$ is not attained. Let $\{a_j\}_{j\in\mathbb{Z}}$ be a monotone increasing sequence of real numbers with $\lim_{j\to-\infty} a_j = \inf_M \varphi$ and $\lim_{j\to\infty} a_j = \infty$. We then apply Proposition 3.2 to each integer j and obtain a homeomorphism $\Phi_j^{j+1} : \varphi^{-1}(\{a_{j+1}\}) \times (a_j, a_{j+1}] \to M_{a_j}^{a_{j+1}}$ such that

$$\varphi \circ \Phi_j^{j+1}(x, t) = t, \quad x \in \varphi^{-1}(\{a_{j+1}\}), \quad t \in (a_j, a_{j+1}].$$

The composition of these homeomorphisms gives the desired homeomorphism $\varphi : \varphi^{-1}(\{a\}) \times (\inf_M \varphi, \infty) \to M$.

If $\lambda := \inf_M \varphi$ is attained, then $M_\lambda^\lambda(\varphi)$ is a k-dimensional totally geodesic submanifold which is totally convex with $0 \le k \le \dim M - 1$. A tubular neighborhood $B\big(M_\lambda^\lambda(\varphi), r(M_\lambda^\lambda(\varphi))\big)$ around the minimum set is a normal bundle over $M_\lambda^\lambda(\varphi)$ in M and its boundary $\partial B\big(M_\lambda^\lambda(\varphi), r(M_\lambda^\lambda(\varphi))\big)$ is homeomorphic to a level of φ. Therefore M is homeomorphic to the normal bundle over the minimum set in M. This proves Theorem 1.1. $\square$

Remark 3.4. Under the assumption in Theorem 1.1, it is not certain whether or not $\lim_{t\to\inf_M \varphi} \delta(t) = \infty$. It might happen that every level set above the infimum is compact but the minimum set is noncompact. However, we do not know such an example on a Finsler manifold.

Remark 3.5. The basic difference of treatments of convex functions between Riemannian and Finsler geometry can be interpreted as follows.

In the case where $\varphi : (M, g) \to \mathbb{R}$ is a convex function with noncompact levels, on a Riemannian manifold, the homeomorphism $\Phi_a^b : M_b^b(\varphi) \times [a, b] \to M_a^b(\varphi)$ is obtained as follows. Fix a point $p \in M_a^a(\varphi)$ and a sequence of R_j-balls centered at p, $\{B(p, R_j)\}_{j\ge1}$ with $\lim_{j\to\infty} R_j = \infty$. Setting K_j to be the closure of $B(p, R_j)$, for $j \ge 1$, we find a sequence of constants $\Delta_j := \Delta_a^b(K_j)$. If

$x \in K_j \cap M_b^b(\varphi)$ is a fixed point, we then have a broken geodesic $T(x) := T(x_k, x_{k-1}) \cup \cdots \cup T(x_1, x_0)$ as obtained in the proof of Proposition 3.2, where $x_0 \in M_a^a(\varphi)$. The properties of the Riemannian distance function now apply to $T(x_j, x_{j-1}) : [0, d(x_j, x_{j-1})] \to (M, g)$. Consequently, the distance function from $p \in M_a^a(\varphi)$, namely $t \mapsto d(p, T(x_j, x_{j-1}))(t)$, is strictly monotone decreasing. Here $T(x_j, x_{j-1})$ is parameterized by arc-length such that $T(x_j, x_{j-1})(0) = x_j$ and $T(x_j, x_{j-1})(d(x_j, x_{j-1})) = x_{j-1}$. Therefore, $T(x)$ is contained entirely in K_j, and moreover, the length $L(T(x))$ of $T(x)$ satisfies

$$L(T(x)) \leq (b-a)/\Delta_j, \quad \forall x \in K_j \cap M_a^b(\varphi).$$

If $y_0 \in M_a^a(\varphi) \cap K_j$ is an arbitrary fixed point, then Proposition 3.2 again implies that there exists a point $y = y_m \in M_b^b(\varphi)$ such that $T(y) = T(y_k, y_{k-1}) \cup \cdots \cup T(y_1, y_0)$ has length at most $(b-a)/\Delta_j$. Hence we have

$$d(p, y) < R_j + (b-a)/\Delta_j + 1.$$

We therefore observe that the correspondence $x \mapsto x_0$ between $M_b^b(\varphi)$ and $M_a^a(\varphi)$ through $T(x)$ is bijective, and the desired homeomorphism is constructed.

However, in the Finslerian case where all the levels of a convex function $\varphi : (M, F) \to \mathbb{R}$ are noncompact, the correspondence $x \mapsto x_0$ between $M_b^b(\varphi)$ and $M_a^a(\varphi)$ through $T(x) = T(x_k, x_{k-1}) \cup \cdots \cup T(x_1, x_0)$ may not be obtained. In fact, the monotone decreasing property of $t \mapsto d(p, T(x_j, x_{j-1}))$ might not hold for a Finsler metric. Therefore, for a point $x \in K_j \cap M_b^b(\varphi)$, $T(x)$ may not necessarily be contained in K_j. Hence, we may fail in controlling the length of $T(x)$ in terms of Δ_j. By the same reason, we cannot prove the monotone nondecreasing property of the diameter function for compact levels of a convex function $\varphi : (M, F) \to \mathbb{R}$.

4. Proof of Theorem 1.2

We take a minimizing geodesic $\sigma : [0, \ell] \to M$ such that $\sigma(0)$ and $\sigma(\ell)$ belong to distinct components of $M_c^c(\varphi)$.

For the proof of (1), we assert that $\inf_M \varphi = \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t)$. Suppose that $b := \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t) > \inf_M \varphi$. Since φ is locally nonconstant, we may assume without loss of generality that $b := \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t)$ is attained at a unique point, say, $q = \sigma(\ell_0)$.

Setting $r = r(\sigma(\ell_0))$, we find a number $a \in (\inf_M \varphi, b)$ such that there is a unique foot $p \in M_a^a(\varphi)$ of q on $M_a^a(\varphi)$, namely $d(\sigma(\ell_0), M_a^a(\varphi)) = d(\sigma(\ell_0), p)$.

Let $\alpha : [0, d(q, p)] \to M$ be the unique minimizing geodesic with $\alpha(0) = q$, $\alpha(d(q, p)) = p$. The points on $\alpha(t)$, for $0 \leq t \leq d(q, p)$, can be joined to $q_\pm := \sigma(\ell_0 \pm r)$ by a unique minimizing geodesic $\gamma_{\alpha(t)q_\pm} : [0, d(\alpha(t), q_\pm)] \to B(q; r)$ with $\gamma_{\alpha(t)q_\pm}(0) = \alpha(t)$, $\gamma_{\alpha(t)q_\pm}(d(\alpha(t)), q_\pm) = q_\pm$.

Since $\varphi(q_\pm) > b$, the right-hand derivative of $\varphi \circ \gamma_{\alpha(t)q_\pm}$ at $d(\alpha(t), q_\pm)$ is bounded below by

$$(\varphi \circ \gamma_{\alpha(t)q_\pm})'_+(\varphi \circ \gamma_{\alpha(t)q_\pm}(d(\alpha(t)), q_\pm)) > \frac{\varphi(q_\pm) - b}{2r} > 0.$$

Thus, for every $t \in [0, d(q, p)]$, $\gamma_{\alpha(t)q_\pm}$ meets $M_c^c(\varphi)$ at $\gamma_{\alpha(t)q_\pm}(u^\pm(t))$ with

$$u^\pm(t) \leq \frac{2r(c-a)}{\varphi(q_\pm) - b} + 2r,$$

and hence there are curves $C_0^\pm : [0, d(q, p)] \to M_c^c(\varphi)$ with

$$C_0^+(0) = \sigma(\ell), \quad C_0^-(0) = \sigma(0),$$
$$C_0^+(d(q, p)) = \gamma_{pq_+}(u^+(d(q, p))), \quad C_0^-(d(q, p)) = \gamma_{pq_-}(u^-(d(q, p))).$$

Let $\tau_t : [0, d(p, \sigma(t))] \to M$ for $t \in [\ell_0 - r, \ell_0 + r]$ be a minimizing geodesic with $\tau_t(0) = p$, $\tau_t(d(p, \sigma(t))) = \sigma(t)$. Every τ_t meets $M_c^c(\varphi)$ at a parameter value $\leq 2rc/(b-a)$, and hence we have a curve $C_1 : [\ell_0 - r, \ell_0 + r] \to M_c^c(\varphi)$ such that

$$C_1(t) = \tau_t[0, 2rc/(b-a)] \cap M_c^c(\varphi).$$

Thus, considering the union $C_0^- \cup C_1 \cup (C_0^+)^{-1}$, it follows that $\sigma(0)$ can be joined to $\sigma(\ell)$ in $M_c^c(\varphi)$, a contradiction. This proves (1) (see Figure 5).

We next prove (2). Let $\lambda := \inf_M \varphi$. Clearly $M_\lambda^\lambda(\varphi)$ is totally convex, and hence Proposition 2.2 implies that $M_\lambda^\lambda(\varphi)$ carries the structure of a smooth totally geodesic submanifold.

Suppose that $\dim M_\lambda^\lambda(\varphi) < n - 1$. Then the normal bundle is connected, and at each point $p \in M_\lambda^\lambda(\varphi)$ the indicatrix $\Sigma_p \subset T_pM$ has the property that $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda\varphi))$ is arcwise connected. Here, $\Sigma_p(M_\lambda^\lambda(\varphi)) \subset \Sigma_p$ is the indicatrix at p of $M_\lambda^\lambda(\varphi)$. Choose points q_0 and q_1 on distinct components of $M_c^c(\varphi)$, and an interior point $p \in M_\lambda^\lambda(\varphi)$. If $\gamma_i : [0, d(p, q_i)] \to M$ for $i = 0, 1$ is a minimizing geodesic with $\gamma_i(0) = p$, $\gamma_i(d(p, q_i)) = q_i$, then $\dot\gamma_0(0)$ and $\dot\gamma_1(0)$ are joined by a curve $\Gamma : [0, 1] \to \Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$ such that $\Gamma(0) = \dot\gamma_0(0)$, $\Gamma(1) = \dot\gamma_1(0)$. The same method as developed in the proof of (1) yields a continuous 1-parameter family of geodesics $\gamma_t : [0, \ell_t] \to M$ with $\gamma_t(0) = p$, $\dot\gamma_t(0) = \Gamma(t)$ and $\gamma_t(\ell_t) \in M_c^c(\varphi)$ for all $t \in [0, 1]$. Thus we have a curve $t \mapsto \gamma_t(\ell_t)$ in $M_c^c(\varphi)$ joining q_0 to q_1, a contradiction. This proves $\dim M_\lambda^\lambda(\varphi) = n - 1$.

We use the same idea to prove that $M_\lambda^\lambda(\varphi)$ has no boundary. In fact, supposing that the boundary is nonempty, the tangent cone of $M_\lambda^\lambda(\varphi)$ at a boundary point x is contained entirely in a closed half space of $T_xM_\lambda^\lambda(\varphi)$, and hence $\Sigma_x \setminus \Sigma_x(M_\lambda^\lambda(\varphi))$ is arcwise connected. A contradiction is derived by constructing a curve in $M_c^c(\varphi)$ joining q_0 to q_1. This proves (2).

The triviality of the normal bundle over $M_\lambda^\lambda(\varphi)$ in M is now clear, giving (3).

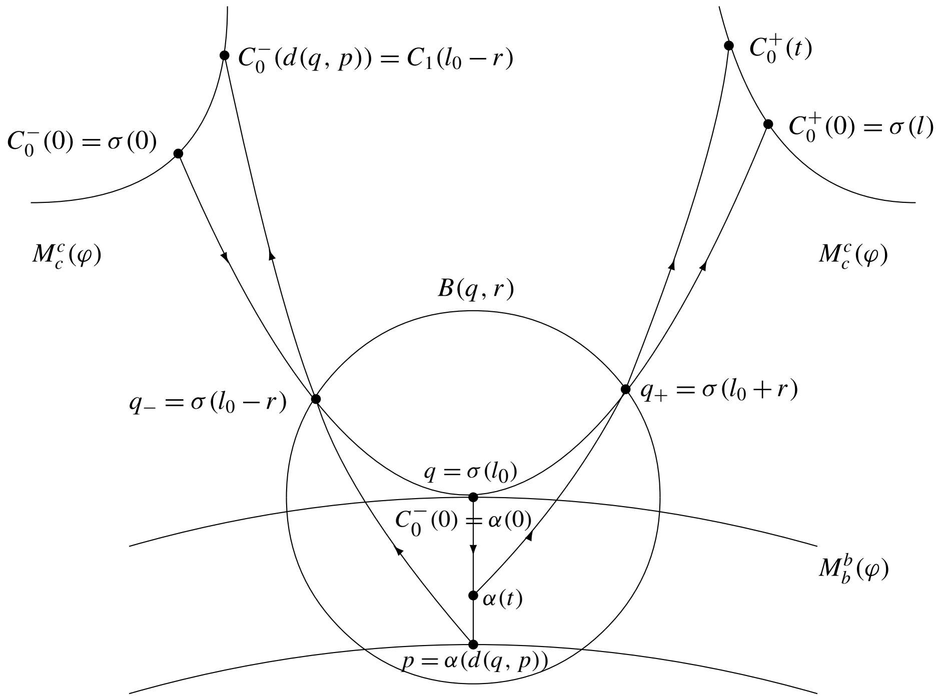

Figure 5. The proof of Theorem 1.2.

To prove (4), suppose that $M_a^a(\varphi)$ for some $a \in \varphi(M)$ has more than two components. Let $q_1, q_2, q_3 \in M_a^a(\varphi)$ be in distinct components, and take $p \in M_\lambda^\lambda(\varphi)$. Let $\gamma_i : [0, d(p, q_i)] \to M$ for $i = 1, 2, 3$ be minimizing geodesics with $\gamma_i(0) = p$, $\gamma_i(d(p, q_i)) = q_i$. Since the normal bundle over $M_\lambda^\lambda(\varphi)$ in M is trivial by (3), it follows that $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$ has exactly two components. Two of the three initial vectors, say $\dot\gamma_1(0)$ and $\dot\gamma_2(0)$, belong to the same component of $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$. Then the same technique as developed in the proof that $\dim M_\lambda^\lambda(\varphi) = n - 1$ applies, and q_1 is joined to q_2 by a curve in $M_a^a(\varphi)$. This contradiction proves (4). $\square$

5. Ends of (M, F)

An *end* ε of a noncompact manifold X is an assignment to each compact set $K \subset X$ a component $\varepsilon(K)$ of $X \setminus K$ such that $\varepsilon(K_1) \supset \varepsilon(K_2)$ if $K_1 \subset K_2$. Every noncompact manifold has at least one end. For instance, $\mathbb{R}^n$ has one end if $n > 1$ and two ends if $n = 1$.

In the present section we discuss the number of ends of (M, F) admitting a convex function, namely we will prove Theorem 1.3. As is seen in the previous section, it may happen that a convex function $\varphi : (M, F) \to \mathbb{R}$ has both compact

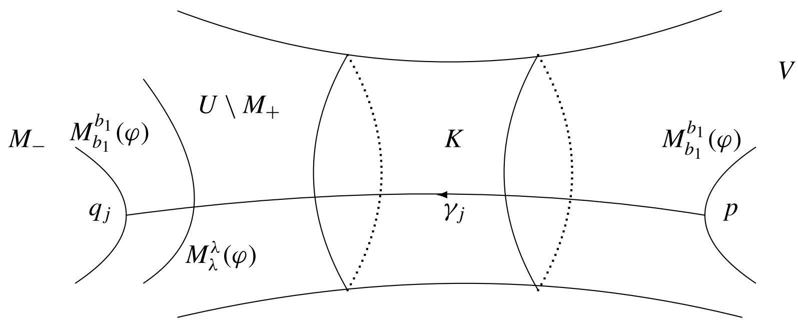

Figure 6. The proof of Theorem 1.3 (A2).

and noncompact levels simultaneously. In this section let $\{K_j\}_{j\geq 1}$ be an increasing sequence of compact sets such that $\lim_{j\to\infty} K_j = M$.

Proof of Theorem 1.3. We first prove (A1).

Theorem 1.2 (1) implies that φ attains its infimum $\lambda := \inf_M \varphi$. Given an arbitrary compact set $A \subset M$, there exists a number $a \in \varphi(M)$ such that $M_a^a(\varphi)$ has two components and $A \subset \varphi^{-1}[\lambda, a]$. Then $M \setminus A$ contains two unbounded open sets $\varphi^{-1}(a, \infty)$, proving (A1).

We next prove (A2). Suppose that M has more than one end. There is a compact set $K \subset M$ such that $M \setminus K$ has at least two unbounded components, say U and V. Setting $a := \min_K \varphi$ and $b := \max_K \varphi$, we have

$$\lambda \leq a < b < \infty.$$

We assert that

$$M_\lambda^\lambda(\varphi) \cap U \neq \varnothing, \quad M_\lambda^\lambda(\varphi) \cap V \neq \varnothing, \quad M_\lambda^\lambda(\varphi) \cap K \neq \varnothing.$$

In order to prove that $M_\lambda^\lambda(\varphi) \cap K \neq \varnothing$, we suppose that $\lambda < a$. Once $M_\lambda^\lambda(\varphi) \cap K \neq \varnothing$ has been established, it will turn out that $M_\lambda^\lambda(\varphi)$ intersects all the unbounded components of $M \setminus K$.

Suppose the contrary, namely $M_\lambda^\lambda(\varphi) \cap K = \varnothing$. Without loss of generality we may assume $M_\lambda^\lambda(\varphi) \subset U$. From Theorem 1.2 (3) it follows that $M \setminus M_\lambda^\lambda(\varphi) = M_- \cup M_+$ (a disjoint union with $\partial M_+ = \partial M_- = M_\lambda^\lambda(\varphi)$).

Setting $M_- \subset U$, we observe that $K \cup V \subset M_+$.

If $b_1 > b$, then M_- contains a component of $M_{b_1}^{b_1}(\varphi)$ and another component of $M_{b_1}^{b_1}(\varphi)$ is contained entirely in V. We then observe that if $\sup_{U\setminus M_-} \varphi = \infty$, then $U \setminus M_-$ contains a component of $M_{b_1}^{b_1}(\varphi)$, for φ takes values $\leq b$ on $\partial(U \setminus M_-)$ and $M_{b_1}^{b_1}(\varphi)$ does not meet the boundary of $U \setminus M_-$. This contradicts Theorem 1.2 (4), for $\partial M_{b_1}^{b_1}(\varphi)$ has at least three components. Therefore we have $\sup_{U\setminus M_-} \varphi < \infty$.

Let $\{q_j\} \subset M_{b_1}^{b_1}(\varphi)$ be a divergent sequence of points, and fix $p \in M_{b_1}^{b_1}(\varphi) \subset V$. Let $\gamma_j : [0, d(p, q_j)] \to M \setminus M_-$ be a minimizing geodesic with $\gamma_j(0) = p$, $\gamma_j(d(p, q_j)) = q_j$ for $j = 1, 2, \ldots$. Clearly γ_j passes through a point on K and $\varphi \circ \gamma_j$ is bounded above by b_1. If $\gamma : [0, \infty) \to M \setminus M_-$ is a ray with $\dot\gamma(0) = \lim_{j\to\infty} \dot\gamma_j(0)$, then $\varphi \circ \gamma$ is constant on $[0, \infty)$ and $\varphi \circ \gamma(t) = b_1$ for all $t > b_1$. This is a contradiction to the choice of $b = \max_K \varphi$, for γ passes through a point on K at which φ takes the value b_1. This proves the assertion (see Figure 6).

We next assert that if $b_1 > b$ is fixed, then $M_{b_1}^{b_1}(\varphi)$ has at least four components. In fact, we observe from $M_{b_1}^{b_1}(\varphi) \cap K = \varnothing$ that each unbounded component of $(M \setminus M_\lambda^\lambda(\varphi)) \cap (M \setminus K)$ contains a component of $M_{b_1}^{b_1}(\varphi)$. This contradicts Theorem 1.2 (4), and (A2) is proved.

The proof of (A3) is a consequence of (D2), and given after the proof of (D2).

We now prove (B1). From the assumption that $\inf_M \varphi$ is attained, it follows from Lemma 3.1 that $\varphi^{-1}[\inf_M \varphi, b_j]$ is compact for all j, where $\{b_j\}$ is a monotone divergent sequence. Then $K_j := \varphi^{-1}[\lambda, b_j]$ is monotone increasing and $\lim_{j\to\infty} K_j = M$. Clearly $M \setminus K_j$ contains a unique unbounded domain $\varphi^{-1}(b_j, \infty)$ for every j. This proves Theorem 1.3 (B1).

For (B2), if $\inf_M \varphi$ is not attained, we have monotone sequences $\{a_j\}$ and $\{b_j\}$ such that

$$\lim_{j\to\infty} a_j = \inf_M \varphi, \quad \lim_{j\to\infty} b_j = \infty,$$
$$[a_j, b_j] = \varphi(K_j), \quad j = 1, 2, \ldots.$$

Then for all large numbers j, $M \setminus K_j$ contains two unbounded domains

$$M \setminus K_j \supset \varphi^{-1}(b_j, \infty) \cup \varphi^{-1}(\inf_M \varphi, a_j).$$

This proves that M has exactly two ends.

We first prove (C) under an additional assumption that $\lambda := \inf_M \varphi$ is attained. Suppose that M has more than one end. Using the same notation as in the proof of (A2),

$$\lambda := \inf_M \varphi, \quad a := \min_K \varphi, \quad b := \max_K \varphi,$$

where $K \subset M$ is a compact set such that $M \setminus K$ has at least two unbounded components U and V.

We first assert that $K \cap M_\lambda^\lambda(\varphi) \neq \varnothing$. In fact, supposing that $K \cap M_\lambda^\lambda(\varphi) = \varnothing$ we find a component V of $M \setminus K$ such that if $b' > b$ then $M_{b'}^{b'}(\varphi) \subset V$ and $M_\lambda^\lambda \subset U$. Here the assumption that all the levels of φ are connected is essential. As is seen in the proof of (A2), there exist at least two components of $M_{b'}^{b'}$ for $b' > b$ such that one component lies in U and another in V. This contradicts the assumption in (C), and the first assertion is proved.

The same proof technique as developed in (A2) implies that $M_\lambda^\lambda(\varphi)$ passes through points on K, U and V. Fix a point $p \in V \cap M_\lambda^\lambda(\varphi)$, a divergent sequence $\{q_j\}$ of points in $U \setminus M_\lambda^\lambda(\varphi)$ and $\gamma_j : [0, d(p, q_j)] \to M$ a minimizing geodesic with $\gamma_j(0) = p$, $\gamma_j(d(p, q_j)) = q_j$.

From the construction of γ_j, we observe that $\varphi \circ \gamma$ is strictly increasing, and hence we find a number $t_j > 0$ such that $\gamma_j(t_j) \in M_{b'}^{b'}(\varphi) \cap U$. More precisely, $\gamma_j[0, d(p, q_j)]$ meets $M_\lambda^\lambda(\varphi)$ only at the origin, for $M_\lambda^\lambda(\varphi)$ is totally convex and hence if $\gamma(t_0) \in M_\lambda^\lambda(\varphi)$, for some $t_0 \in [0, d(p, q_j))$, then $\gamma_j[0, d(p, q_j)]$ is contained entirely in $M_\lambda^\lambda(\varphi)$.

Therefore $M_{b'}^{b'}(\varphi)$ has more than one component (one in U and another in V), a contradiction to the assumption in (C). This concludes the proof of (C) in this case.

We next prove (C) in the case where $\inf_M \varphi$ is not attained. Assume again that M has more than one end. We then have

$$\inf_M \varphi < a < b < \infty, \quad a := \min_K \varphi, \quad b := \max_K \varphi.$$

Since all the levels are connected, we find $\inf_M \varphi < a' < a$ and $b < b'$ such that $M_{a'}^{a'}(\varphi) \subset U$ and $M_{b'}^{b'}(\varphi) \subset V$. Let $\{y_j\} \subset M_{b'}^{b'}(\varphi)$ be a divergent sequence of points and fix a point $x \in M_{a'}^{a'}(\varphi)$. Let $\gamma_j : [0, d(x, y_j)] \to M$ for $j = 1, 2, \dots$ be a minimizing geodesic with $\gamma_j(0) = x$ and $\gamma_j(d(x, y_j)) = y_j$. There exists a ray $\gamma : [0, \infty) \to M$ emanating from x such that $\dot\gamma(0) = \lim_{j\to\infty} \dot\gamma_j(0)$. Clearly, every γ_j passes through a point on K and hence, so does γ. From construction, $\varphi \circ \gamma : [0, \infty) \to \mathbb{R}$ is bounded from above by b', and therefore is constant. However it is impossible, for $\varphi(x) = a'$ and $\varphi \circ \gamma(t_0) \geq a > a'$ at a point $\gamma(t_0) \in K$. This completes the proof of (C).

For the proof of (D1), suppose that M has more than two ends.

Let $K \subset M$ be a connected compact subset such that $M \setminus K$ contains at least three unbounded components, say U, V and W. We may consider that U contains $\varphi^{-1}[b', \infty)$, for all $b' > b$. Since all the levels of φ are connected, we have

$$\sup_{M\setminus U} \varphi \leq b.$$

In fact, suppose that there exists a point $x \in M \setminus U$ such that $\varphi(x) = b'$ for some $b' > b$. Then $M_{b'}^{b'}(\varphi) \cap K = \varnothing$ and hence $M_{b'}^{b'}(\varphi)$ is disconnected, a contradiction to the assumption of (D).

Let $\{x_j\} \subset V$ and $\{y_j\} \subset W$ be two divergent sequences of points, and let $\gamma_j : [0, d(x_j, y_j)] \to M \setminus U$ be a minimizing geodesic joining x_j to y_j. Since γ_j passes through a point on K, there exists a straight line $\gamma : \mathbb{R} \to M \setminus U$ such that $\dot\gamma(0)$ is obtained as the limit of a converging sequence of vectors $\dot\gamma_j(t_j) \in K$ for $j = 1, 2, \dots$. Clearly, $\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}$ is bounded above, and hence constant taking

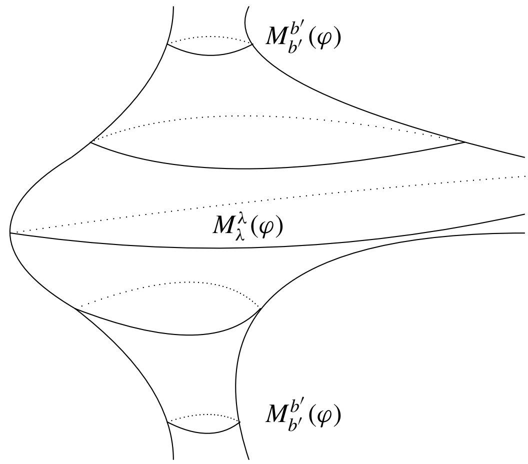

Figure 7. The proof of Theorem 1.3 (D1).

a value $\mu = \varphi \circ \gamma(0) \in [a, b]$. We therefore observe that

$$M_{\mu}^{\mu}(\varphi) \cap K \neq \varnothing, \quad M_{\mu}^{\mu}(\varphi) \cap W \neq \varnothing \quad \text{and} \quad M_{\mu}^{\mu}(\varphi) \cap V \neq \varnothing.$$

We next choose a value $a' \in (\inf_M \varphi, a)$. We may assume without loss of generality that $M_{a'}^{a'}(\varphi) \subset V$. Let $\{z_j\} \subset M_{\mu}^{\mu}(\varphi) \cap W$ be a divergent sequence of points and $x \in M_{a'}^{a'}(\varphi)$ an arbitrary fixed point. Let $\sigma_j : [0, d(x, z_j)] \to M \setminus U$ be a minimizing geodesic with $\sigma_j(0) = x$, $\sigma(d(x, z_j)) = z_j$ for all $j = 1, 2, \ldots$. Clearly, $\varphi \circ \sigma_j$ is monotone increasing in W. Let $\sigma : [0, \infty) \to M$ be a ray such that $\dot{\sigma}(0) = \lim_{j\to\infty} \dot{\sigma}_j(0)$. We then observe that $\varphi \circ \sigma$ is monotone increasing on an unbounded interval $[\bar{b}, \infty)$ for some $\bar{b} > 0$, and bounded above by μ. Thus, it is identically equal to a'. Recall that $\varphi \circ \sigma(0) = \varphi(x) = a'$. However this is impossible since $a' < \min_K \varphi = a$ and $\sigma[0, \infty)$ passes through a point on K. We therefore observe that $M \setminus (K \cup U)$ has exactly one end. This proves (D1).

The proof of (D2) is now clear and omitted.

The proof of (A3) is now a straightforward consequence of (D2). See Figure 8. If $M_b^b(\varphi)$ is compact for some $b \in \varphi(M)$, then $\varphi^{-1}[b, \infty)$ has two ends. From the assumption and Theorem 1.2 (1), we observe that $M_{\lambda}^{\lambda}(\varphi)$ is noncompact. Therefore $M^b(\varphi) = \varphi^{-1}[\lambda, b]$ is noncompact and so has at least one end. This proves (A3).

Finally, we prove (E). Suppose that φ admits both compact and noncompact levels simultaneously. The same notation as in the proof of (D) will be used. If φ admits a disconnected level, then $\varphi^{-1}[b', \infty)$ consists of two unbounded components for all $b' > b$.

Then Theorem 1.2 (1) and Lemma 3.1 imply that $\lambda := \inf_M \varphi$ is attained and $M_{\lambda}^{\lambda}(\varphi)$ is connected and noncompact. Therefore, every compact set K containing

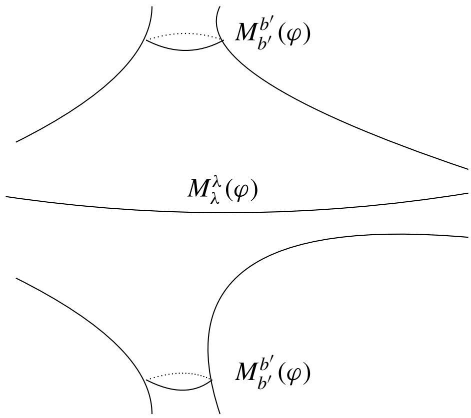

Figure 8. The proof of Theorem 1.3 (A3).

$M_{b'}^{b'}(\varphi)$ has the property that $M \setminus K$ has more than two unbounded components. In fact, two components of $M \setminus K$ contain $\varphi^{-1}[b', \infty)$ and the other component intersects with $M_{\lambda}^{\lambda}(\varphi)$ outside K. This proves that M has at least three ends, a contradiction to the assumption of (E).

If all the levels of φ are connected and noncompact, then M has one end by (C), a contradiction to the assumption of (E). This completes the proof of (E) and hence of Theorem 1.3. $\square$

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