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Perelman proved that an open 3-dimensional shrinking gradient Ricci soliton with bounded nonnegative sectional curvature is a quotient of $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$. We extend this result to higher dimensions with a decay condition on the Ricci tensor.

1. Introduction

A gradient Ricci soliton is a Riemannian manifold (M, g) together with a smooth function f such that

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g,$$

where λ is a constant. It is called shrinking, steady and expanding when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Gradient Ricci solitons are self-similar solutions of Hamilton's Ricci flow and play a vital role in the analysis of singularities of the flow. In dimension 2, Hamilton [1988] completely classified shrinking gradient Ricci solitons with bounded curvature and proved that they are the sphere, the projective space and the Euclidean space with constant curvature. In dimension 3, Ivey [1993] proved that compact shrinking gradient Ricci solitons have positive sectional curvature, and Perelman [2003] proved that shrinking gradient Ricci solitons with bounded nonnegative sectional curvature are quotients of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$.

In higher dimensions, there have been many results in the last several years. Chen [2009] showed that a complete shrinking gradient Ricci soliton has nonnegative scalar curvature. Ni and Wallach [2008] gave the classification of shrinking gradient Ricci solitons with nonnegative Ricci curvature and zero Weyl tensor. Petersen and Wylie [2010] and independently, Cao, Wang and Zhu [Cao et al. 2011], classified the shrinking gradient Ricci solitons with zero Weyl tensor. Fernández-López and García-Río [2011] considered solitons with harmonic Weyl tensor. In [Petersen and Wylie 2009], several natural curvature conditions are given that characterize gradient Ricci solitons of the flat vector bundle $N \times_\Gamma \mathbb{R}^m$, where N is an Einstein manifold,

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Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^m$, and $f = \frac{1}{4}d^2$ with d being the distance on the flat fiber to the base. In particular, it is shown in [Petersen and Wylie 2009] that a shrinking gradient Ricci soliton is rigid, i.e., of the form $N \times_\Gamma \mathbb{R}^m$, if the scalar curvature is constant and the sectional curvature of the plane containing ∇f is nonnegative. As a consequence of a theorem of Böhm and Wilking [2008], the gradient Ricci solitons with positive curvature operators are trivial. In view of this and the aforementioned result of Perelman, one naturally asks to what extend shrinking gradient Ricci solitons with nonnegative sectional curvature are rigid. Our first result in this paper is the rigidity under a decay condition on $|D\mathrm{Ric}|$, extending Perelman's result to higher dimensions. In all theorems we scale the metric so that $\lambda = \frac{1}{2}$.

Theorem 1.1. *Let (M, g, f) be a complete noncompact shrinking gradient Ricci soliton with bounded nonnegative sectional curvature. Assume that there exists $\delta > 0$ such that*

$$\int_M e^{\delta f} |D\mathrm{Ric}| \, d\mathrm{vol}_g < \infty.$$

Then (M^n, g) is isometric to $N \times_\Gamma \mathbb{R}^m$, where N is a compact Einstein manifold.

This is, to our knowledge, the first rigidity result in high dimensions without assumptions on the Weyl tensor. The potential function f is known to grow quadratically with respect to the distance from a fixed point, so our condition on $D\mathrm{Ric}$ says that it decays exponentially. Our proof also works under the assumption that $D\mathrm{Ric}$ decays polynomially with a degree depending on other geometric quantities.

The Cheeger–Gromoll soul theorem states that an open manifold with nonnegative sectional curvature is diffeomorphic to a vector bundle over a compact submanifold called a soul. The pull-back metric on the bundle can be highly twisted. However, if there exists a gradient soliton structure on such a bundle, then, by Theorem 1.1, the metric has to be locally trivial, provided that the decay condition is satisfied. The decay condition on $D\mathrm{Ric}$ in Theorem 1.1 is imposed in the region where f is large. Our next result deals with the rigidity under a condition on $D\mathrm{Ric}$ imposed in the region where f is small.

Theorem 1.2. *Let (M^n, g, f) be a complete shrinking gradient Ricci soliton with bounded nonnegative sectional curvature. Assume that the minima of f is a smooth compact nondegenerate critical submanifold and $D\mathrm{Ric}$ and $D^2\mathrm{Ric}$ vanish on the minima. Then (M^n, g) is noncompact and isometric to $N \times_\Gamma \mathbb{R}^m$, where N is a compact Einstein manifold.*

We derive some basic formulas in Section 2, and prove Theorems 1.1 and 1.2 in Sections 3 and 4 respectively.

2. Basic formulas

There are different conventions for the curvature tensor in the literature, so to avoid the confusion, we state ours as follows. The (3, 1) tensor $\mathrm{Rm}(X, Y, Z) = \mathrm{Rm}(X, Y)Z$ is defined as

$$\mathrm{Rm}(X, Y)Z = D_X D_Y Z - D_Y D_X X - D_{[X,Y]} Z$$

and the (4, 0) tensor as

$$\mathrm{Rm}(X, Y, Z, W) = \langle \mathrm{Rm}(X, Y)Z, W\rangle.$$

We use Ric to denote the Ricci tensor and R the scalar curvature. For a tangent vector X at p, we use $\mathrm{Ric}(X)$ to denote the vector such that

$$\langle \mathrm{Ric}(X), Y\rangle = \mathrm{Ric}(X, Y)$$

for any vector Y at p. For any smooth vector field V and any smooth function ϕ on manifold M, by $V(\phi)$, we mean $V(\phi) = d\phi(V) = \langle V, \nabla\phi\rangle$. In the remainder of the paper, we will rescale the metric and assume that our gradient Ricci soliton satisfies

$$\mathrm{Ric} + \mathrm{Hess}\, f = \tfrac{1}{2} g.$$

Since the curvature of (M, g) is assumed to be bounded, there exists a flow $\Phi_t : M \to M$ defined for all time with $\Phi_0 = \mathrm{Id}$ and $\partial\Phi/\partial t = \nabla f$ [Morgan and Tian 2007, p. 207]. For $t \in (\infty, 0)$, define $G(t) = |t|\Phi^*_{-\ln|t|} g$. Then $G(-1) = g$ and $G(t)$ satisfies

$$\mathrm{Ric}(G(t)) + \mathrm{Hess}\, f = \frac{1}{2\tau} G(t),$$

where Hess is taken with respect to the metric $G(t)$ and $\tau = |t| = -t$.

In the next lemma, we collect some well-known formulae.

Lemma 2.1. *On* $(M, G(t))$, *we have*

(1) $dR = 2\mathrm{Ric}(\nabla f, \cdot)$,

(2) $|\nabla f|^2 = f/\tau - R + \textit{constant}$,

(3) $R/\tau + \langle\nabla f, \nabla R\rangle = \Delta R + 2|\mathrm{Ric}|^2$,

(4) $\mathrm{div}\,\mathrm{Rm}(X, Y, Z) = \mathrm{Rm}(\nabla f, X, Y, Z)$,

(5) $D_X \mathrm{Ric}(Y, Z) = D_Y \mathrm{Ric}(X, Z) - \mathrm{Rm}(X, Y, \nabla f, Z)$,

where $\mathrm{div}\,\mathrm{Rm}(X, Y, Z) = \mathrm{trace}_{1,2}\, D\,\mathrm{Rm}(\cdot, \cdot, X, Y, Z)$.

Proof. The derivations of (1)–(3) can be found in [Hamilton 1995] and (4)–(5) in [Petersen and Wylie 2010]. □

Lemma 2.2. *On (M, g), we have*

$$\Delta|\mathrm{Ric}|^2 = 2|D\mathrm{Ric}|^2 + 2|\mathrm{Ric}|^2 + \nabla f(|\mathrm{Ric}|^2) - 4K_{ij}\lambda_i\lambda_j,$$

where λ_i are the eigenvalues of the Ricci tensor and K_{ij} is the sectional curvature of the plane spanned by the eigenvectors belonging to λ_i and λ_j respectively.

Proof. This follows from the formula derived in Lemma 2.1 in [Petersen and Wylie 2010]:

$$\Delta\mathrm{Ric} = D_{\nabla f}\mathrm{Ric} + \mathrm{Ric} - 2\sum_{k=1}^{n}\mathrm{Rm}(\,\cdot\,, e_k, \mathrm{Ric}(e_k), \cdot\,). \qquad \square$$

Throughout the computations in the paper, we assume $\{e_1, \dots, e_n\}$ is an orthonormal basis in a neighborhood of a fixed point x with $D_{e_i}e_j(x) = 0$ and further assume that each e_i is an eigenvector of Ric at x corresponding to the eigenvalue λ_i. Such a basis always exists. We also use the Einstein summation convention (unless otherwise specified).

Lemma 2.3. *On (M, g), we have*

$$\mathrm{div}(\mathrm{Ric}(\nabla R)) = \nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 - 2\langle Z, \nabla f\rangle + |\mathrm{Ric}|^2 - 2\sum_i \lambda_i^3,$$

where $Z = \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)$.

Proof. The following computations are done at x. From Lemma 2.1, we have

$$\begin{aligned} D_{e_i}\mathrm{Ric}(\nabla R, e_i) &= D_{\nabla R}\mathrm{Ric}(e_i, e_i) - \mathrm{Rm}(e_i, \nabla R, \nabla f, e_i) \\ &= |\nabla R|^2 - \mathrm{Ric}(\nabla R, \nabla f) = \tfrac{1}{2}|\nabla R|^2. \end{aligned}$$

We then obtain

$$\begin{aligned} \mathrm{div}(\mathrm{Ric}(\nabla R)) &= \langle D_{e_i}\mathrm{Ric}(\nabla R), e_i\rangle = e_i\mathrm{Ric}(\nabla R, e_i) \\ &= D_{e_i}\mathrm{Ric}(\nabla R, e_i) + \mathrm{Ric}(D_{e_i}\nabla R, e_i) \\ &= \tfrac{1}{2}|\nabla R|^2 + \mathrm{Ric}(e_i, e_j)\langle D_{e_i}\nabla R, e_j\rangle \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\langle D_{e_i}\mathrm{Ric}(\nabla f), e_j\rangle \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)e_i\mathrm{Ric}(\nabla f, e_j) \\ &= \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\big(D_{e_i}\mathrm{Ric}(\nabla f, e_j) + \mathrm{Ric}(D_{e_i}\nabla f, e_j)\big). \end{aligned}$$

That is,

$$\mathrm{div}(\mathrm{Ric}(\nabla R)) = \tfrac{1}{2}|\nabla R|^2 + 2\mathrm{Ric}(e_i, e_j)\big(D_{e_i}\mathrm{Ric}(\nabla f, e_j) + \mathrm{Ric}(D_{e_i}\nabla f, e_j)\big). \tag{2-1}$$

From the soliton equation

$$\mathrm{Ric} + \mathrm{Hess}\, f = \tfrac{1}{2}g,$$

it follows that

$$D_{e_i}\nabla f = \tfrac{1}{2}e_i - \mathrm{Ric}(e_i) = \tfrac{1}{2}e_i - \lambda_i e_i,$$

where we have used the assumption that e_i is an eigenvector of Ric at x belonging to the eigenvalue λ_i. Hence,

$$2\mathrm{Ric}(e_i, e_j)\mathrm{Ric}(D_{e_i}\nabla f, e_j) = 2\left(\tfrac{1}{2} - \lambda_i\right)(\mathrm{Ric}(e_i, e_j))^2 = 2\lambda_i^2\left(\tfrac{1}{2} - \lambda_i\right). \tag{2-2}$$

Lemma 2.1(5) implies that

$$D_{e_i}\mathrm{Ric}(\nabla f, e_j) = D_{\nabla f}\mathrm{Ric}(e_i, e_j) - \mathrm{Rm}(e_i, \nabla f, \nabla f, e_j).$$

It follows that

$$\begin{aligned} 2\mathrm{Ric}(e_i, e_j)D_{e_i}\mathrm{Ric}(\nabla f, e_j) &= 2\mathrm{Ric}(e_i, e_j)(D_{\nabla f}\mathrm{Ric}(e_i, e_j) - \mathrm{Rm}(e_i, \nabla f, \nabla f, e_j)) \\ &= 2\mathrm{Ric}(e_i, e_j)D_{\nabla f}\mathrm{Ric}(e_i, e_j) - 2\langle Z, \nabla f\rangle \\ &= \nabla f(|\mathrm{Ric}|^2) - 2\langle Z, \nabla f\rangle. \end{aligned} \tag{2-3}$$

Combining (2-2) and (2-3), we obtain that

$$\begin{aligned} &2\mathrm{Ric}(e_i, e_j)(D_{e_i}\mathrm{Ric}(\nabla f, e_j) + \mathrm{Ric}(D_{e_i}\nabla f, e_j)) \\ &\qquad = \nabla f(|\mathrm{Ric}|^2) - 2\langle Z, \nabla f\rangle + 2\lambda_i^2(\tfrac{1}{2} - \lambda_i). \end{aligned}$$

Substituting the above into (2-1) gives

$$\begin{aligned} \mathrm{div}(\mathrm{Ric}(\nabla R)) &= \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\mathrm{Ric}|^2) - 2\langle Z, \nabla f\rangle + 2\lambda_i^2(\tfrac{1}{2} - \lambda_i) \\ &= \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\mathrm{Ric}|^2) - 2\langle Z, \nabla f\rangle + |\mathrm{Ric}|^2 - 2\sum_i \lambda_i^3. \qquad \square \end{aligned}$$

Remark 2.4. We have $\langle Z, \nabla f\rangle \geq 0$ when the sectional curvature of (M, g) is nonnegative. In fact, at x, $\langle Z, \nabla f\rangle = \lambda_i \,\mathrm{Rm}(\nabla f, e_i, e_i, \nabla f)$.

The next lemma is a slight variation of Lemma 2.3.

Lemma 2.5. *On* (M, g), *we have*

$$\nabla f(|\mathrm{Ric}|^2) = 2\left(\langle Z, \nabla f\rangle + \sum_{i=1}^n \lambda_i\left(\lambda_i - \tfrac{1}{2}\right)^2\right) + \tfrac{1}{2}\langle\nabla f, \nabla R\rangle - \tfrac{1}{2}|\nabla R|^2 - \mathrm{div}(D_{\nabla R}\nabla f).$$

Proof. It follows from Lemma 2.3 that

$$\mathrm{div}(\mathrm{Ric}(\nabla R)) = \tfrac{1}{2}|\nabla R|^2 + \nabla f(|\mathrm{Ric}|^2) - 2\langle Z, \nabla f\rangle + |\mathrm{Ric}|^2 - 2\sum_i \lambda_i^3.$$

Using $\mathrm{Ric}(\nabla R) = \frac{1}{2}\nabla R - D_{\nabla R}\nabla f$ and Lemma 2.1(3), we have

$$\begin{aligned} \nabla f(|\mathrm{Ric}|^2) = \frac{R}{2} - 2|\mathrm{Ric}|^2 + 2\sum_i \lambda_i^3 + 2\langle Z, \nabla f\rangle \\ + \tfrac{1}{2}\langle\nabla f, \nabla R\rangle - \tfrac{1}{2}|\nabla R|^2 - \mathrm{div}(D_{\nabla R}\nabla f). \end{aligned}$$

The lemma now follows as $R/2 - 2|\mathrm{Ric}|^2 + 2\sum_i \lambda_i^3 = 2\sum_{i=1}^n \lambda_i\left(\lambda_i - \frac{1}{2}\right)^2$. $\square$

Combining Lemmas 2.2 and 2.3 gives the following proposition.

Proposition 2.6. *On* (M, g),

$$P = \tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2 - \mathrm{Ric}(\nabla R)\big),$$

where $P = K_{ij}(\lambda_i - \lambda_j)^2 + |D\mathrm{Ric}|^2 + 2\langle Z, \nabla f\rangle$.

Proof. Lemma 2.2 implies that

$$-2K_{ij}\lambda_i\lambda_j + |D\mathrm{Ric}|^2 = -\tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) - |\mathrm{Ric}|^2 + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2\big),$$

while Lemma 2.3 implies that

$$2\sum_i \lambda_i^3 + 2\langle Z, \nabla f\rangle = \nabla f(|\mathrm{Ric}|^2) + |\mathrm{Ric}|^2 + \tfrac{1}{2}|\nabla R|^2 - \mathrm{div}(\mathrm{Ric}(\nabla R)).$$

Adding the corresponding sides of the last two equations and noting that $2\sum_i \lambda_i^3 - 2\sum_{i,j} K_{ij}\lambda_i\lambda_j = \sum_{i,j} K_{ij}(\lambda_i - \lambda_j)^2$, we obtain Proposition 2.6. □

Remark 2.7. Clearly, $P \geq 0$ when the sectional curvature of (M, g) is nonnegative.

The proof of Theorem 1.1 will use an alternative form of Proposition 2.6 in which the term $|D\mathrm{Ric}|^2$ is replaced by $|\,\mathrm{div}\,\mathrm{Rm}\,|^2$. An integral from of the next lemma is proved in [Cao 2007].

Lemma 2.8. *On* (M, g),

$$|D\mathrm{Ric}|^2 = |\,\mathrm{div}\,\mathrm{Rm}\,|^2 + 2\langle Z, \nabla f\rangle - \tfrac{1}{2}\nabla f(|\mathrm{Ric}|^2) + \mathrm{div}\big(\tfrac{1}{2}\nabla|\mathrm{Ric}|^2 - 2Z\big).$$

Proof. As before, we fix an orthonormal basis, $\{e_1, \dots, e_n\}$, in a neighborhood of a fixed point x and assume that $D_{e_i}e_j(x) = 0$ and that each e_i is an eigenvector of Ric at x corresponding to the eigenvalue λ_i. Recall that $Z = \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)$, so at x,

$$\begin{aligned}
\mathrm{div}(Z) &= \langle D_{e_k}Z, e_k\rangle = \langle D_{e_k}(\mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j)), e_k\rangle \\
&= e_k(\mathrm{Ric}(e_i, e_j))\,\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Ric}(e_i, e_j)\langle D_{e_k}(\mathrm{Rm}(\nabla f, e_i, e_j)), e_k\rangle \\
&= D_{e_k}\mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Ric}(e_i, e_j)e_k(\mathrm{Rm}(\nabla f, e_i, e_j, e_k)) \\
&= D_{e_k}\mathrm{Ric}(e_i, e_j)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\big(D_{e_k}\mathrm{Rm}(\nabla f, e_i, e_j, e_k) + \mathrm{Rm}(D_{e_k}\nabla f, e_i, e_j, e_k)\big) \\
&= (D_{e_i}\mathrm{Ric}(e_j, e_k) - \mathrm{Rm}(e_k, e_i, \nabla f, e_j))\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\,\mathrm{div}\,\mathrm{Rm}(e_j, e_i, \nabla f) + \lambda_i\,\mathrm{Rm}\big(\big(\tfrac{1}{2} - \lambda_k\big)e_k, e_i, e_i, e_k\big) \\
&= D_{e_i}\mathrm{Ric}(e_j, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) + \mathrm{div}\,\mathrm{Rm}(e_j, e_i, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) \\
&\qquad + \mathrm{Ric}(e_i, e_j)\,\mathrm{Rm}(\nabla f, e_j, e_i, \nabla f) + K_{ij}\lambda_i\big(\tfrac{1}{2} - \lambda_j\big).
\end{aligned}$$

In the above calculation, we have repeatedly used Lemma 2.1. The lemma now follows from Lemma 2.2 and the following two identities, whose proofs are easy:

$$D_{e_i}\mathrm{Ric}(e_j, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) = 0,$$
$$\mathrm{div}\,\mathrm{Rm}(e_j, e_i, e_k)\,\mathrm{div}\,\mathrm{Rm}(e_i, e_j, e_k) = \tfrac{1}{2}|\mathrm{div}\,\mathrm{Rm}|^2. \qquad \square$$

Lemma 2.8, together with Proposition 2.6, implies the following:

Lemma 2.9. *On* (M, g),

$$Q = \nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}|\nabla R|^2 + \mathrm{div}(2Z - \mathrm{Ric}(\nabla R)),$$

where $Q = K_{ij}(\lambda_i - \lambda_j)^2 + |\mathrm{div}\,\mathrm{Rm}|^2 + 4\langle Z, \nabla f\rangle$.

Remark 2.10. We note that $Q \geq 0$ when the sectional curvature of (M, g) is nonnegative.

The next lemma deals with the term $\nabla f(|\mathrm{Ric}|^2)$ in Lemma 2.9.

Lemma 2.11. *On* (M, g),

$$\nabla f(|\mathrm{Ric}|^2) = \tfrac{1}{2}|\nabla R|^2 + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle \nabla f, \nabla R\rangle) + \mathrm{div}\big(D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\big). \tag{2-4}$$

Proof. It follows from Lemma 2.1(1) and (3) that

$$\tfrac{1}{2}\nabla f(\Delta R) = -\nabla f(|\mathrm{Ric}|^2) + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle \nabla f, \nabla R\rangle).$$

The Bochner–Weitzenböck formula implies that

$$\begin{aligned}\mathrm{div}\big(\tfrac{1}{2}\nabla\langle \nabla f, \nabla R\rangle\big) &= \tfrac{1}{2}\Delta\langle \nabla f, \nabla R\rangle \\ &= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R) + \tfrac{1}{2}\nabla R(\Delta f) + \mathrm{Ric}(\nabla f, \nabla R) \\ &= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R) + \tfrac{1}{2}\nabla R\big(\tfrac{n}{2} - R\big) + \tfrac{1}{2}|\nabla R|^2 \\ &= \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle + \tfrac{1}{2}\nabla f(\Delta R).\end{aligned}$$

But,

$$\begin{aligned}\mathrm{div}(D_{\nabla R}\nabla f) &= \langle D_{e_i} D_{\nabla R}\nabla f, e_i\rangle = e_i\langle D_{\nabla R}\nabla f, e_i\rangle = e_i\langle D_{e_i}\nabla f, \nabla R\rangle \\ &= \big\langle D_{e_i}\big(\tfrac{1}{2}e_i - \mathrm{Ric}(e_i)\big), \nabla R\big\rangle + \big\langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\big\rangle \\ &= -D_{e_i}\mathrm{Ric}(e_i, \nabla R) + \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle \\ &= -\tfrac{1}{2}|\nabla R|^2 + \langle \mathrm{Hess}\, f, \mathrm{Hess}\, R\rangle.\end{aligned}$$

The lemma follows. $\square$

We now have the following proposition which will be used in the proof of Theorem 1.1.

Proposition 2.12. *On* (M, g),

$$Q = |\nabla R|^2 + \tfrac{1}{2}\langle\nabla f, \nabla R\rangle + \tfrac{1}{2}\nabla f(\langle\nabla f, \nabla R\rangle) + \operatorname{div}\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle\nabla f, \nabla R\rangle\bigr).$$

Proof. This is merely a consequence of Lemmas 2.9 and 2.11. □

3. Proof of Theorem 1.1

We will use ϕ to denote a real-valued nonnegative C^4 function on $\mathbb{R}$ and write $\phi \circ f$ as $\phi(f)$. We will show that R is a constant function and then appeal to [Petersen and Wylie 2009] to complete the proof. We begin with the following proposition.

Proposition 3.1. *On* (M, g),

$$\phi(f)Q = \tfrac{1}{2}\langle\nabla f, \nabla R\rangle\bigl((\phi-\phi')(f) - (\phi+\phi')(f)\Delta f - (\phi''+\phi')(f)|\nabla f|^2\bigr) + (\phi+\phi')(f)|\nabla R|^2 - 2\phi'\langle Z, \nabla f\rangle + \operatorname{div}(X), \tag{3-1}$$

where

$$X = \tfrac{1}{2}\langle\nabla f, \nabla R\rangle(\phi'+\phi)(f)\nabla f + \phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle\nabla f, \nabla R\rangle\bigr).$$

Proof. We multiply each side of the equation in Proposition 2.12 by $\phi(f)$ to get

$$\begin{aligned}\phi(f)Q = \phi(f)|\nabla R|^2 &+ \frac{\phi(f)}{2}\langle\nabla f, \nabla R\rangle + \frac{\phi(f)}{2}\nabla f(\langle\nabla f, \nabla R\rangle)\\ &- \phi'(f)\langle 2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle\nabla f, \nabla R\rangle, \nabla f\rangle\\ &+ \operatorname{div}\bigl(\phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle\nabla f, \nabla R\rangle\bigr)\bigr).\end{aligned}$$

It follows from the soliton equation and Lemma 2.1(1) that

$$\begin{aligned}\langle -\operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f, \nabla f\rangle &= \bigl\langle \tfrac{1}{2}\nabla R - 2\operatorname{Ric}(\nabla R), \nabla f\bigr\rangle\\ &= \tfrac{1}{2}\langle\nabla f, \nabla R\rangle - |\nabla R|^2.\end{aligned}$$

We thus obtain

$$\begin{aligned}\phi(f)Q = (\phi+\phi')(f)|\nabla R|^2 &+ \frac{\phi-\phi'}{2}(f)\langle\nabla f, \nabla R\rangle\\ &- 2\phi'\langle Z, \nabla f\rangle + \frac{\phi+\phi'}{2}(f)\nabla f(\langle\nabla f, \nabla R\rangle)\\ &+ \operatorname{div}\bigl(\phi(f)\bigl(2Z - \operatorname{Ric}(\nabla R) + D_{\nabla R}\nabla f - \tfrac{1}{2}\nabla\langle\nabla f, \nabla R\rangle\bigr)\bigr).\end{aligned} \tag{3-2}$$

Now, we observe that

$$\begin{aligned}(\phi+\phi')(f)\nabla f(\langle\nabla f, \nabla R\rangle) &= \langle\nabla\langle\nabla f, \nabla R\rangle, (\phi'+\phi)(f)\nabla f\rangle\\ &= -\langle\nabla f, \nabla R\rangle\bigl((\phi'+\phi)(f)\Delta f + (\phi''+\phi')(f)|\nabla f|^2\bigr)\\ &\quad + \operatorname{div}\bigl(\langle\nabla f, \nabla R\rangle(\phi'+\phi)(f)\nabla f\bigr).\end{aligned}$$

Substituting the above into (3-2), we obtain (3-1). Proposition 3.1 is thus proved. □

The idea now is to choose an appropriate function ϕ and integrate (3-1) over M. The divergence term, after integration, vanishes because of the fall-off condition we impose. The right-hand side will then be nonpositive while the left is always nonnegative, and consequently, R is a constant. Theorem 1.1 follows from [Petersen and Wylie 2009].

Proof of Theorem 1.1. We normalize f by adding a constant so that Lemma 2.1(2) takes the form $|\nabla f|^2 = f - R$. Since $R \geq 0$, we always have $|\nabla f|^2 \leq f$. On the other hand, since R is assumed to be bounded and f grows quadratically with respect to the distance from a fixed point [Cao and Zhou 2010; Naber 2006], we have $|\nabla f|^2 \geq \frac{1}{2} f$, when f is sufficiently large. Thus, there exists $T > 2$ so that when $f \geq T$,

$$\tfrac{1}{2} f \leq |\nabla f|^2 \leq f. \tag{3-3}$$

Fix $0 < \eta < \delta$ and define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = 0$ for $t \leq T$, and $\phi(t) = (t-T)^k e^{\eta t}$ for $t \geq T$, where k is a sufficiently large number to be determined. Throughout this section, we will use this ϕ in (3-1). By our fall-off assumption, there exists a sequence $t_i \to \infty$ such that

$$\int_{f=t_i} e^{\delta f} \frac{1}{|\nabla f|} |D\,\mathrm{Ric}| \to 0, \quad \text{as } i \to \infty.$$

From this, we now deduce that

$$\int_{f \leq t_i} \mathrm{div}(X) = \int_{f=t_i} \frac{\langle X, \nabla f \rangle}{|\nabla f|} \to 0, \quad \text{as } i \to \infty. \tag{3-4}$$

To this end, we look at each of the five terms in X and denote by X_i the i-th term. Then, when $f > T$,

$$\frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} = \tfrac{1}{2} |\langle \nabla f, \nabla R\rangle| (\phi' + \phi)(f) |\nabla f| \leq C_1 f^{k+1} e^{\eta f} |\nabla R|,$$

where C_1 is a constant depending only on k and η. Now by the Cauchy–Schwarz inequality,

$$|D\,\mathrm{Ric}|^2 = \sum_{i,j,k} (D_{e_i} \mathrm{Ric}(e_j, e_k))^2 \geq \frac{1}{n} \sum_i \Big(\sum_j D_{e_i} \mathrm{Ric}(e_j, e_j) \Big)^2 = \frac{1}{n} |\nabla R|^2.$$

Thus,

$$|\nabla R| \leq \sqrt{n} |D\,\mathrm{Ric}|.$$

Hence,

$$\frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} \leq C_1 \sqrt{n} f^{k+1} e^{\eta f} |D\,\mathrm{Ric}|.$$

Integrating the above over $\{f = t_i\}$ and noting that

$$C_1\sqrt{n} f^{k+1} e^{\eta f} |D\mathrm{Ric}| \le e^{\delta f} \frac{|D\mathrm{Ric}|}{|\nabla f|},$$

when f is sufficiently large, we conclude that

$$\int_{f=t_i} \frac{|\langle X_1, \nabla f\rangle|}{|\nabla f|} \to 0, \quad \text{as } i \to \infty.$$

Now note that $\langle X_2, \nabla f\rangle = 2\phi\langle Z, \nabla f\rangle = 2\phi \sum_i \lambda_i \,\mathrm{Rm}(\nabla f, e_i, e_i, \nabla f)$. Since Ric is assumed to be bounded and since the sectional curvature is nonnegative,

$$\frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \le C_2 f^{k-1/2} e^{\eta f} \mathrm{Ric}(\nabla f, \nabla f) = C_2 f^{k-1/2} e^{\eta f} \tfrac{1}{2}\langle \nabla f, \nabla R\rangle,$$

where C_2 is a constant dependent only on the bound of Ric, and the last equality follows from Lemma 2.1. Hence, when f is sufficiently large,

$$\frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \le \tfrac{1}{2} C_2 f^k e^{\eta f} |\nabla R| \le e^{\delta f} \frac{|D\mathrm{Ric}|}{|\nabla f|}.$$

It then follows that

$$\int_{f=t_i} \frac{|\langle X_2, \nabla f\rangle|}{|\nabla f|} \to 0, \quad \text{as } i \to \infty.$$

The arguments for the other X_i are similar; we will skip X_3 and X_4. Now look at X_5. Repeatedly using Lemma 2.1(2), we see that

$$\begin{aligned}
\langle X_5, \nabla f\rangle &= -\tfrac{1}{2}\phi \nabla f(\langle \nabla f, \nabla R\rangle) = -\phi \nabla f(\mathrm{Ric}(\nabla f, \nabla f)) \\
&= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + 2\mathrm{Ric}(D_{\nabla f}\nabla f, \nabla f)\big) \\
&= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + \mathrm{Ric}(\nabla f - \nabla R, \nabla f)\big) \\
&= -\phi\big(D_{\nabla f}\mathrm{Ric}(\nabla f, \nabla f) + \tfrac{1}{2}\langle \nabla f, \nabla R\rangle - \mathrm{Ric}(\nabla R, \nabla f)\big).
\end{aligned}$$

Since $|\nabla R|$ can be bounded by $|D\mathrm{Ric}|$, we have $|\langle X_5, \nabla f\rangle| \le C_5 e^{\eta f} f^{k+3} |D\mathrm{Ric}|$. Equation (3-4) then follows.

To simplify notations, we put

$$\begin{aligned}
F = \tfrac{1}{2}\langle \nabla f, \nabla R\rangle\big((\phi - \phi')(f) - (\phi + \phi')(f)\Delta f - (\phi'' + \phi')(f)|\nabla f|^2\big) \\
+ (\phi + \phi')(f)|\nabla R|^2 - 2\phi'\langle Z, \nabla f\rangle.
\end{aligned}$$

Then,

$$\phi(f) Q = F + \mathrm{div}(X).$$

It follows easily from the arguments in the proof of (3-4) that $\int_M F\, d\mathrm{vol}_g < \infty$. We thus have

$$\int_M \phi(f) Q = \int_M F. \tag{3-5}$$

We now show that $\int_M F\, d\mathrm{vol}_g \le 0$. First, we note that $-\Delta f = R - n/2 \le \Lambda$, where Λ is an upper bound of R; hence $-(\phi+\phi')(f)\Delta f \le \Lambda(\phi+\phi')$, as ϕ and ϕ' are both nonnegative. Next, we observe that, by Lemma 2.1,

$$|\nabla R|^2 = 2\mathrm{Ric}(\nabla f, \nabla R) = 2\sum_i \lambda_i e_i(f) e_i(R)$$

and $e_i(R) = \langle \nabla R, e_i\rangle = 2\mathrm{Ric}(\nabla f, e_i) = 2\lambda_i e_i(f)$. So for each i, $e_i(f)e_i(R) \ge 0$. Hence $|\nabla R|^2 \le 2\Lambda\langle\nabla f, \nabla R\rangle$. Finally, we recall that $\langle Z, \nabla f\rangle \ge 0$ (Remark 2.4). We thus conclude, from (3-3), that

$$F \le \tfrac{1}{2}\langle \nabla f, \nabla R\rangle F_1, \tag{3-6}$$

where

$$F_1 = (\phi-\phi')(f) + \Lambda(\phi+\phi')(f) + 4\Lambda(\phi+\phi') - \tfrac{1}{2} f(\phi''+\phi')(f).$$

It follows from (3-5) and (3-6) that

$$\int_M \phi(f) Q \le \frac{1}{2}\int_M \langle \nabla f, \nabla R\rangle F_1. \tag{3-7}$$

A direct computation leads to

$$\begin{aligned} F_1 &= (\phi-\phi')(t)+\Lambda(\phi+\phi')(t)+4\Lambda(\phi+\phi')(t)-\tfrac{1}{2}t(\phi''+\phi')(t)\\ &= -\tfrac{1}{2}\delta(1+\delta)(t-T)^{k+1}e^{\delta t} - \Big(\tfrac{1}{2}(1+2\delta)k - 5(1+\delta)\Lambda - 1 + \frac{T-2}{2}\delta\Big)(t-T)^k e^{\delta t}\\ &\qquad -k\big(\tfrac{1}{2}(k-1) - 5\Lambda + \tfrac{1}{2}T+\big)1(t-T)^{k-1}e^{\delta t} - \tfrac{1}{2}T\phi''. \end{aligned}$$

If we choose $k > 10\Lambda + 2$, the above expression will clearly be negative for $t > T$. We have therefore shown that $F_1 \le 0$ everywhere and $F_1 < 0$ where $f > T$. Since $Q \ge 0$ (Remark 2.10) and $\langle\nabla f, \nabla R\rangle = 2\mathrm{Ric}(\nabla f, \nabla f) \ge 0$ (Lemma 2.1), we conclude from (3-7) that $\langle \nabla f, \nabla R\rangle = 0$ in the region $\{f > T\}$. But as we noted earlier in the proof, $|\nabla R|^2 \le 2\Lambda\langle\nabla f, \nabla R\rangle$. Hence $\nabla R = 0$ in the region $\{f > T\}$. The analyticity of the metric [Bando 1987; Kotschwar 2013] then implies that R is a constant function. Theorem 1.1 then follows from [Petersen and Wylie 2009]. □

4. Proof of Theorem 1.2

We first show that the Ricci tensor has a zero eigenvalue at any point p in C, then show that the soliton splits in a neighborhood of p, which, in turn, implies that the scalar curvature is a constant.

Let C be the critical manifold of minima of f. Since C is assumed to be nondegenerate, the Morse–Bott lemma implies that for any point $p \in C$, there exists an open neighborhood U of p and a diffeomorphism $\phi : U \to \mathbb{R}^n$ such that $\phi(U \cap C) = \{(0, \dots, 0, x_{m+1}, \dots, x_n)\}$, $\phi(p) = 0$ and $f\circ\phi^{-1}(x_1, \dots, x_n) = c + \frac{1}{4}(x_1^2 + \cdots + x_m^2)$.

In what follows in this section, unless specified otherwise, the range for the Greek letters $\alpha, \beta, \ldots$ is 1 to m while that for the Latin letters $i, j, \ldots$ is $m+1$ to n.

We observe that we may assume that for all α and i, $g^{\alpha i}(p)=0$. In fact, by making a change of variables, $y_\alpha = x_\alpha$ and $y_i = x_i - \sum_{\beta=1}^m g^{i\beta}(p)x_\beta$, we see that in the new coordinates, at p, $g^{\alpha i} = \langle \nabla y_\alpha, \nabla y_i\rangle = 0$ for α and i. Moreover, $f(y_1, \ldots, y_m, y_{m+1}, \ldots, y_n) = c + \frac{1}{4}(y_1^2 + \cdots + y_m^2)$. From now on, we assume in the original coordinates $(x_1, \ldots, x_n)$ that $g^{\alpha i}(p) = 0$ for all α and i. As a consequence, we also have $g_{\alpha i}(p) = 0$.

Next lemma computes the Ricci tensor at p.

Lemma 4.1. *At p, we have*

$$\operatorname{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \tfrac{1}{2}(g_{\alpha\beta}(p) - \delta_{\alpha\beta}), \quad \operatorname{Ric}(p)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \tfrac{1}{2} g_{ij},$$

$$\operatorname{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_i}\right) = 0.$$

Proof. Since

$$\nabla f = \tfrac{1}{2} g^{\alpha\beta} x_\alpha \frac{\partial}{\partial x_\beta} + \tfrac{1}{2} g^{\alpha i} x_\alpha \frac{\partial}{\partial x_i},$$

we have at p,

$$\operatorname{Hess}(f)(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \tfrac{1}{2}\delta_{\alpha\beta},$$

$$\operatorname{Hess}(f)(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_i}\right) = \operatorname{Hess}(f)(p)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0.$$

The lemma follows from the soliton equation. □

Let μ_γ^{-1} $(\gamma = 1, \ldots, m)$ denote the eigenvalues of the positive definite symmetric matrix $g_{\alpha\beta}(p)$. Then there exists $(v_{1\gamma}, \ldots, v_{m\gamma}) \neq 0$ such that $\sum_\beta g_{\alpha\beta}(p) v_{\beta\gamma} = \mu_\gamma^{-1} v_{\alpha\gamma}$. Let $v_\gamma = \sum_\alpha v_{\alpha\gamma}(\partial/\partial x_\alpha)$. The first part of Lemma 4.1 implies that

$$\begin{aligned}
\operatorname{Ric}(p)(v_\gamma, v_\gamma) &= \sum_{\alpha,\beta} v_{\alpha\gamma} v_{\beta\gamma} \operatorname{Ric}(p)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) \\
&= \tfrac{1}{2}(\mu_\gamma^{-1} - 1) \sum_\alpha (v_{\alpha\gamma})^2 \\
&= \tfrac{1}{2}(\mu_\gamma^{-1} - 1)\mu_\gamma g(p)(v_\gamma, v_\gamma) \\
&= \tfrac{1}{2}(1 - \mu_\gamma) g(p)(v_\gamma, v_\gamma).
\end{aligned}$$

We conclude from this and the rest of Lemma 4.1 that the eigenvalues of the Ricci tensor at p are $(1-\mu_\alpha)/2$, with $\alpha = 1, \ldots, m$, and $\frac{1}{2}$ with multiplicity $n-m$. Since the Ricci tensor is assumed to be semipositive definite, $\mu_\alpha \leq 1$ for each α. Of course, $\mu_\alpha > 0$. Our goal is to show that $\mu_\alpha = 1$.

Now assume $\{e_1, \dots, e_n\}$ is an orthonormal basis in a neighborhood of a fixed point $p \in C$ with $D_{e_i} e_j(p) = 0$ for $1 \le i, j \le n$. We may assume that each e_α is an eigenvector of Ric at p corresponding to the eigenvalue $(1-\mu_\alpha)/2$ for $1 \le \alpha \le m$ and e_i an eigenvector corresponding to $\frac{1}{2}$ for $m+1 \le i \le n$.

By our assumption, $D\,\mathrm{Ric} = D^2\,\mathrm{Ric} = 0$ at p. Hence, for each $1 \le s \le n$, in the neighborhood of p,

$$\mathrm{Ric}(e_s, e_s) = r_s + \sum_{i,j,k=1}^{n} r_{sijk} x_i x_j x_k + \text{higher-order terms},$$

where r_s and r_{sijk} are constants. We make the following observation.

Lemma 4.2. *Given that $K_{s\alpha}$ is the sectional curvature of the section spanned by e_s and e_α, we have*

$$r_\alpha = \frac{1-\mu_\alpha}{2}, \quad \alpha = 1, \dots, m, \qquad r_i = \tfrac{1}{2}, \quad i = m+1, \dots, n, \qquad \sum_{\alpha=1}^{m} K_{s\alpha}\mu_\alpha = 0,$$

Proof. We only need to prove the second line. At p,

$$(\Delta \mathrm{Ric})(e_s, e_s) = \Delta(\mathrm{Ric}(e_s, e_s)) = 0.$$

On the other hand, we have $\Delta \mathrm{Ric} = D_{\nabla f}\mathrm{Ric} + \mathrm{Ric} - 2\sum_{l=1}^{n} \mathrm{Rm}(\,\cdot\,, e_l, \mathrm{Ric}(e_l), \cdot\,)$ (Lemma 2.1 in [Petersen and Wylie 2010], see also the proof of Lemma 2.2). Hence,

$$\begin{aligned}
0 &= \mathrm{Ric}(e_s, e_s) - 2\sum_{l=1}^{n} \mathrm{Rm}(e_s, e_l, \mathrm{Ric}(e_l), e_s) \\
&= r_s - 2\sum_{\alpha=1}^{m} \mathrm{Rm}(e_s, e_\alpha, \mathrm{Ric}(e_\alpha), e_s) - 2\sum_{i=m+1}^{n} \mathrm{Rm}(e_s, e_i, \mathrm{Ric}(e_i), e_s) \\
&= r_s - \sum_{\alpha=1}^{m} (1-\mu_\alpha)\,\mathrm{Rm}(e_s, e_\alpha, e_\alpha, e_s) - \sum_{i=m+1}^{n} \mathrm{Rm}(e_s, e_i, e_i, e_s) \\
&= \sum_{\alpha=1}^{m} K_{s\alpha}\mu_\alpha.
\end{aligned}$$

□

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from Lemma 4.2 and the assumption of nonnegative sectional curvature that $K_{s\alpha}(p) = 0$ for all $1 \le s \le n$. So, $\mathrm{Ric}(p)$ vanishes on the subspace spanned by $\{\partial/\partial x_\alpha \mid \alpha = 1, \dots, m\}$.

We first prove that a neighborhood of p splits isometrically as $U \times V$, where U is at least m-dimensional and $\mathrm{Ric} \equiv 0$ on U. We have shown that $\mathrm{Ric}_{\alpha\beta}(p) = 0$. The rest of the argument is along the lines of the proof of Lemma 8.2 in [Hamilton

1986] and that of Corollary 2.1 in [Ni and Tam 2003]. Denote by $K(x,t)$ the null space of $\mathrm{Ric}(x,t)$, i.e.,

$$K(x,t)=\{w\in T_xM \mid \mathrm{Ric}(x,t)(w)=0\}.$$

Let $w_0\in K(p,-1)$ and $\gamma(s)$ a smooth curve starting from p. Parallel translating w_0 along γ gives a vector field w along γ. Denote the extension of w to a neighborhood of γ still by w. Now we project w onto $K(x,t)$ to get a vector field $v(x,t)$. Then $v(\gamma(s),t)\in K(\gamma(s),t)$. We first show that $D_{\gamma'}v$ is also in $K(\gamma(s),t)$. We fix an orthonormal basis in $g(t)$, $\{e_1,\dots,e_n\}$, in a neighborhood of a fixed point $\gamma(s)$ and assume that $e_i(\gamma(s))$ are the eigenvectors of Ric. For simplicity, we denote $e_i(\gamma(s))$ by $e_i(s)$. Since $\mathrm{Ric}(v)=0$, we have $((\partial/\partial t)\mathrm{Ric})(v,v)=0$. The evolution equation for Ricci tensor then implies that at $\gamma(s)$,

$$(\Delta\mathrm{Ric})(v,v)-2\langle\mathrm{Ric}(v),\mathrm{Ric}(v)\rangle+2\mathrm{Ric}(e_i,e_i)K(e_i,v)=0,$$

where the repeated indices are being summed over. Since the sectional curvature $K(e_i,v)$ is nonnegative and since $\mathrm{Ric}(v)=0$, we deduce that $(\Delta\mathrm{Ric})(v,v)\le 0$. Direct computations give

$$\begin{aligned}(\Delta\mathrm{Ric})(v,v)=\Delta(\mathrm{Ric}(v,v))-4e_i(\mathrm{Ric}(v,D_{e_i}v))+2\mathrm{Ric}(v,D_{e_i}D_{e_i}v)\\ +2\mathrm{Ric}(v,D_{D_{e_i}e_i}v)+2\mathrm{Ric}(D_{e_i}v,D_{e_i}v).\end{aligned}$$

Using $(\Delta\mathrm{Ric})(v,v)\le 0$ and $\mathrm{Ric}(v)=0$, we obtain $\mathrm{Ric}(D_{e_i}v,D_{e_i}v)\le 0$. Since Ric is positive semidefinite, we conclude that $\mathrm{Ric}(D_{e_i}v)=0$ for each i, and hence $D_{\gamma'}v\in K(\gamma(s),t)$. As in the proof of Corollary 2.1 in [Ni and Tam 2003], we conclude that $w\in K(x,t)$. Since parallel translation preserves inner product, for each fixed t, the dimension of $K(x,t)$ is independent of x. We then use the de Rham decomposition theorem to conclude that a neighborhood of p splits.

Note that $|\nabla f|^2\ge f$ on $U\times V$. In fact, for any $q\in V$, the restriction of g and f on $U\times\{q\}$ gives a soliton on $U\times\{q\}$ with zero Ric tensor. Lemma 2.1(2) implies that $|\nabla_{U\times\{q\}}f|^2=f|_{U\times\{q\}}$, where $\nabla_{U\times\{q\}}f$ is the gradient of $f|_{U\times\{q\}}$ with respect to the metric $g|_{U\times\{q\}}$. Since $|\nabla f|^2\ge|\nabla_{U\times\{q\}}f|^2$, we infer that $|\nabla f|^2(x,q)\ge f(x,q)$ for all $x\in U$. Since q is an arbitrary point in V, it follows that $|\nabla f|^2\ge f$ on $U\times V$.

We now prove that $|\nabla f|^2\le f$ on $U\times V$. Given any point $y\in U\times V$, denote by $\gamma(s)$ the integral curve of $\nabla f/|\nabla f|^2$ such that $\gamma(0)=y$. Then $f(\gamma(s))=s+f(\gamma(0))$. On the other hand, using Lemma 2.1(1) and (2), we have

$$\begin{aligned}\frac{d}{ds}|\nabla f|^2(\gamma(s))&=\frac{1}{|\nabla f|^2}\nabla f(|\nabla f|^2)=\frac{1}{|\nabla f|^2}(|\nabla f|^2-\langle\nabla f,\nabla R\rangle)\\ &=\frac{1}{|\nabla f|^2}(|\nabla f|^2-2\mathrm{Ric}(\nabla f,\nabla f)).\end{aligned}$$

Since $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$, we obtain $(d/ds)|\nabla f|^2(\gamma(s)) \leq 1$. Integrating this inequality from $-f(\gamma(0))$ to s and noting that $\nabla f(\gamma(s)) = 0$ at $s = -f(\gamma(0))$ give us the desired inequality $|\nabla f|^2 \leq f$.

We have thus proved that $|\nabla f|^2 = f$, which, when combined with Lemma 2.1(2), implies that R is constant in a neighborhood of p. Hence R is constant on the entire M. The proof of Theorem 1.2 is therefore completed. □

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