Pacific Journal of Mathematics

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Volume 277 No. 1 September 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.119

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We use the notion of strong exceptional collections to give a construction of the global Chern–Simons functions for toric Calabi–Yau stacks of dimension three. Moduli spaces of sheaves on such stacks can be identified with critical loci of these functions. We give two applications of these functions. First, we prove Joyce's integrality conjecture of generalized DT invariants on local surfaces. Second, we prove a dimension reduction formula for virtual motives, which leads to a recursion formula for motivic Donaldson–Thomas invariants.

1. Introduction

Moduli spaces of sheaves (more generally, complexes of sheaves) on Calabi–Yau threefolds are examples of moduli problems with symmetric obstruction theories [Behrend 2009]. It is expected that such a moduli space is locally the critical set of a holomorphic function. Such functions are called Chern–Simons (CS) functions. Chern–Simons functions play an important role in Calabi–Yau (CY) geometry because Behrend proved that the Milnor number of a CS function is the microlocal version of the Donaldson–Thomas invariant [loc. cit.].

In a seminal work, Joyce and Song [2012] proved the existence of CS functions for moduli spaces of stable sheaves on compact CY 3-folds using analytic techniques in gauge theory. In this paper, we give a different construction of the CS functions on toric CY 3-folds. Our construction has a few new ingredients. First, the functions we construct are algebraic. Second, the moduli spaces of stable sheaves are, in fact, globally critical sets of these functions. Third, the construction is explicit; i.e., there is an algorithm to write down such functions starting with a toric CY 3-fold together with some extra data; see the end of Section 5.

The construction of CS function consists of three steps:

(1) Let Y be a complex CY 3-fold. Find a good t-structure in the derived category $\mathrm{D}^b(Y)$. The heart of this t-structure is the abelian category of representations of

MSC2010: 14F05, 14N35.
Keywords: algebraic geometry, derived category, Donaldson–Thomas theory.

a quiver with relations. Such an abelian category is good in the sense that it has enough projective modules and has finite projective dimension.

(2) On a moduli space of representations with fixed dimension vector, we find a *maximally degenerate* point, which corresponds to the semisimple representation. The tangent complex of the moduli space at this point is given by the well studied L_∞ (A_∞) Yoneda algebra in representation theory. We compute the L_∞ (A_∞) products and prove they are bounded. The Calabi–Yau condition defines a cyclic pairing on this L_∞ algebra, which together with the L_∞ products determines the CS function.

(3) Embed the moduli spaces of sheaves into the moduli spaces of representations as open substacks.

Step one is based on the existence of full, strong, exceptional collections of line bundles on toric Fano stacks of dimension two; see Theorem 3.3. This was proved in [Borisov and Hua 2009]. Passing from a strong exceptional collection to the associated quiver is a consequence of derived Morita equivalence. We will study this in Section 3.

Step two is based on the cyclic completion (see Theorem 4.2) and boundedness of L_∞ products (see Theorem 4.4). Theorem 4.2 was first proved by Aspinwall and Fidkowski [2006] and later reproved in a much more general setting by Segal [2008]. The terminology *cyclic completion* is due to Segal. The proofs of these two theorems are given in Section 4 just for our convenience.

In Section 5, we construct the CS functions and show that the moduli spaces of sheaves are open substacks of the critical sets modulo gauge groups. Several examples of CS functions are discussed in Section 6.

The language of L_∞ algebras and derived schemes (stacks) — developed in [Kontsevich and Soibelman 2009] — is extensively used in the paper. Each of the moduli spaces mentioned above is the zero locus of an odd vector field on a differential graded (dg) symplectic manifold and the CS functions we construct are essentially Hamiltonian functions associated to it. In Section 2, we give a short introduction to L_∞ algebras and dg schemes.

In the last three sections, we give two applications of the CS function. In Theorem 7.4, we prove that the L_∞ products vanish at semistable points of moduli space of sheaves on local surfaces, which leads to a proof of a special case of the integrality conjecture of Joyce and Song [2012]. In Theorem 8.3, we prove a dimension reduction formula of virtual motives for CS functions, which generalizes some results in [Behrend et al. 2013]. By manipulating this dimension reduction formula, we compute the generating series of moduli spaces of noncommutative Hilbert schemes on toric CY stacks; this is done in Section 9.

Notation. Three dimensional smooth toric Calabi–Yau stacks are in one to one correspondence with the set of 3-dimensional cones over convex lattice polygons Δ contained in an affine hyperplane, together with a triangulation of Δ. When the polygon Δ has at least one interior lattice point, we can consider the barycentric triangulation. (This means the triangulation has only one interior lattice point.) This gives a fan Σ on the affine hyperplane such that its supporting polygon is Δ. The fan Σ determines a 2-dimensional toric Fano stack X_Σ (X, for short). The cone over Σ determines a 3-dimensional toric CY stack Y_Σ (Y, for short), which is the total space of the canonical bundle over X_Σ. We call such a toric CY 3-stack a *local surface*. The CY 3-stacks associated to other triangulations of Δ are related to Y_Σ by a sequence of flops.

- $\pi : Y \to X$ is the projection and $\iota : X \to Y$ is the inclusion of zero section;
- $\mathrm{D}^b(X)$ is the bounded derived category of coherent sheaves on X;
- $\mathrm{D}^b(Y)$ is the bounded derived category of coherent sheaves on Y;
- D_ω is the full subcategory of $\mathrm{D}^b(Y)$ of objects with cohomology sheaves supported on X.

2. L_∞ algebras and differential graded schemes

This is a short introduction to L_∞ algebras and differential graded schemes. A standard reference for this topic is [Kontsevich and Soibelman 2009]. The reader who is familiar with ∞-algebras can skip this section.

2A. L_∞ *algebras.* Let $\boldsymbol{k}$ be a field.

Definition 2.1. An L_∞ *algebra* is a graded $\boldsymbol{k}$-vector space L with a sequence $\mu_1, \dots, \mu_k, \dots$ of graded antisymmetric operations of degree 2, or equivalently, homogeneous multilinear maps

$$\mu_k : \bigwedge^k L \to L[2-k]$$

such that for each $n > 0$, the n-Jacobi rule holds:

$$\sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1<\cdots<i_k;\, j_1<\cdots<j_{n-k} \\ \{i_1,\dots,i_k\}\cup\{j_1,\dots,j_{n-k}\}=\{1,\dots,n\}}} (-1)^\epsilon \, \mu_n(\mu_k(x_{i_1}, \dots, x_{i_k}), x_{j_1}, \dots, x_{j_{n-k}}) = 0.$$

Here, the sign $(-1)^\epsilon$ equals the product of the sign $(-1)^\pi$ associated to the permutation

$$\pi = \left(\begin{smallmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{smallmatrix}\right)$$

with the sign associated by the Koszul sign convention to the action of π on the elements $(x_1, \dots, x_n)$ of L.

Definition 2.2. Let (L, μ_k) be an L_∞ algebra. An element $x \in L^1$ is called a *Maurer–Cartan element* if x satisfies the formal *Maurer–Cartan equation*:

$$\sum_{k=1}^{\infty} \frac{1}{k!}\mu_k(x, \dots, x) = 0.$$

If the above formal sum is convergent, then there is a map $Q : L^1 \to L^2$, defined by

$$x \mapsto \sum_{k=1}^{\infty} \frac{1}{k!}\mu_k(x, \dots, x).$$

called the *curvature map*. The set of elements in L^1 satisfying the Maurer–Cartan equation is denoted by $\mathrm{MC}(L)$.

Definition 2.3. Let L be an L_∞ algebra. We write δ for the first L_∞ product $\mu_1 : L \to L[1]$. It follows from the L_∞ relations that $\delta^2 = 0$. Let x be a Maurer–Cartan element of L. We define the *twisted* differential δ^x by the formula

$$\delta^x(y) = \delta(y) + \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\mu_k(x, \dots, x, y).$$

By manipulating the Maurer–Cartan equation and the L_∞ relations, one can check that $(\delta^x)^2 = 0$.

Given a homogeneous element $a \in L$, we denote its grading by $|a|$.

Definition 2.4. A finite dimensional L_∞ algebra (L, μ_k) is called *cyclic* if there exists a homogeneous bilinear map

$$\kappa : L \otimes L \longrightarrow \boldsymbol{k}[-3]$$

satisfies:

(1) $\kappa(a, b) = (-1)^{|a||b|}\kappa(b, a)$;

(2) $\kappa(\mu_k(a_1, \dots, a_k), a_{k+1}) = (-1)^{|a_1|(|a_2|+\cdots+|a_{k+1}|)}\kappa(\mu_k(a_2, \dots, a_{k+1}), a_1)$;

(3) κ is nondegenerate on $H^\bullet(L, \delta)$.

We call such a κ a *cyclic pairing* on L.

Definition 2.5. Let (L, μ_k, κ) be a cyclic L_∞ algebra. The *Chern–Simons function* associated to L is the formal function

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{\frac{k(k+1)}{2}}}{(k+1)!}\kappa(\mu_k(z, \dots, z), z).$$

2B. ***Differential graded schemes.***

Definition 2.6. A *differential graded scheme* X is a pair $(X^0, \mathcal{O}_X^\bullet)$, where X^0 is an ordinary scheme and $\mathcal{O}_X^\bullet$ is a sheaf of $\mathbb{Z}^-$-graded commutative dg algebras on X^0 such that:

(1) $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$;

(2) $\mathcal{O}_X^i$ are quasicoherent $\mathcal{O}_{X^0}$ modules.

The cohomology sheaves of $\mathcal{O}_X^\bullet$, denoted by $\underline{H}^i(\mathcal{O}_X^\bullet)$ are $\mathcal{O}_{X^0}$ modules. In particular, $\underline{H}^0(\mathcal{O}_X^\bullet)$ is a quotient ring of $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$. We define the *"0-truncation"* of X to be the ordinary scheme

$$\pi_0(X) = \operatorname{Spec} \underline{H}^0(\mathcal{O}_X^\bullet).$$

It is a subscheme of X^0.

Definition 2.7. A *morphism of dg schemes* $f : X \to Y$ is a morphism of ordinary schemes $f_0 : X^0 \to Y^0$ together with a morphism of dg algebras $f_0^*\mathcal{O}_Y^\bullet \to \mathcal{O}_X^\bullet$. A morphism f is called a *quasi-isomorphism* if f induces isomorphisms between $\underline{H}^i(\mathcal{O}_X^\bullet)$ and $\underline{H}^i(\mathcal{O}_Y^\bullet)$ for all i.

Definition 2.8. A dg scheme X is called *smooth* (or a *dg manifold*) if the following conditions hold:

(a) X^0 is a smooth algebraic variety.

(b) Locally over the Zariski topology on X^0, we have an isomorphism of graded algebras $\mathcal{O}_X^\bullet \simeq \operatorname{Sym}_{\mathcal{O}_{X^0}} Q^{-1} \oplus Q^{-2} \oplus \cdots$, where Q^{-i} are vector bundles (of finite rank) on X^0.

Every L_∞ algebra defines a dg manifold.

Example 2.9. Let $L = L^{-k} \oplus \cdots \oplus L^0 \oplus L^1 \oplus \cdots$ be a finite dimensional L_∞ algebra and $\tau^{>0}L$ be the truncation of L in positive degrees. Let X^0 be the linear manifold L^1 and $\mathcal{O}_X^\bullet$ be the completed symmetric algebra $(\operatorname{Sym} \tau^{>0}L[1]^*)\widehat{\ }$, considered as a sheaf over L^1. It has the structure of differential graded algebra (dga). The L_∞ structure comprises the multilinear maps $\mu_k : \operatorname{Sym}^k L[1] \to L[2]$. The dual map of $\sum 1/k!\,\mu_k$ defines a derivation from $q : \mathcal{O}_X^\bullet \to \mathcal{O}_X^\bullet$ of degree one. The L_∞ relations are equivalent to the condition that $q^2 = 0$. It can be interpreted as an odd vector field on the dg manifold. The "0-truncation" $\pi_0(X)$ can be identified with the Maurer–Cartan locus $\mathrm{MC}(L)$. We call the dg manifold constructed in this way the *formal dg manifold associated to* L.

Given a cyclic L_∞ algebra (L, μ_k, κ), the formal dg manifold constructed in Example 2.9 is a formal symplectic dg manifold in the sense of [Kontsevich and Soibelman 2009]. The pairing κ can be viewed as an odd symplectic form.

On a formal dg manifold, we can define the analogue of the usual Cartan calculus [loc. cit.]. The CS function f is the Hamiltonian function of the odd vector field q on X with respect to the odd symplectic form κ. In particular, $\operatorname{crit}(f)$ coincides with the Maurer–Cartan locus of L.

Comments on A_∞ and L_∞ algebras. Given an A_∞ algebra (R, m_k), we can construct, in a canonical way, an L_∞ algebra (L, μ_k). This is done by replacing m_k by its antisymmetrizer. A lazy way to do that is to first construct a dg algebra quasi-isomorphic to R. Antisymmetrize it to form a dg Lie algebra and then take the cohomology. The Maurer–Cartan sets of R_ω and L_ω agree as sets. In the process of antisymmetrization, a cyclic A_∞ algebra goes to a cyclic L_∞ algebra. We will skip the formal definition of A_∞ algebra (it can be found in [loc. cit.]) although it is implicitly used in the later sections. Using L_∞ algebras has the advantage that one can make sense of the Maurer–Cartan set as a scheme instead of as a noncommutative scheme.

3. Derived categories of toric stacks and Morita equivalence

Definition 3.1. Let $\boldsymbol{k}$ be a field. Given a $\boldsymbol{k}$-linear triangulated category $\mathcal{T}$, an object $E \in \mathcal{T}$ is called *exceptional*, if $\operatorname{Ext}^i(E, E) = 0$ for all $i \neq 0$ and $\operatorname{Ext}^0(E, E) = \boldsymbol{k}$.

- A sequence of exceptional objects $E_1, \dots, E_n$ is called an *exceptional collection* if $\operatorname{Ext}^i(E_j, E_k) = 0$ for arbitrary i when $j > k$.
- An exceptional collection is called *strong* if $\operatorname{Ext}^i(E_j, E_k) = 0$ for any j and k unless $i = 0$.
- We say an exceptional collection is *full* if it generates $\mathcal{T}$.

Let E, F be an exceptional collection of length 2 in $\mathcal{T}$. We define the left and right mutation, $L_E F$ and $R_F E$ respectively, using the distinguished triangles.

$$L_E F \longrightarrow \mathbf{R}\mathrm{Hom}(E, F) \otimes E \longrightarrow F$$
$$E \longrightarrow \mathbf{R}\mathrm{Hom}(E, F)^* \otimes F \longrightarrow R_F E$$

Mutations of exceptional collection are exceptional [Bondal 1990]. But mutations of strong exceptional collections are not necessary strong.

Given an exceptional collection $E_0, \dots, E_n$, we can define another exceptional collection $F_{-n}, F_{-n+1}, \dots, F_0$, called the *dual* exceptional collection to $E_0, \dots, E_n$. First let F_0 equal to E_0. Second, make $F_{-1} = L_{E_0} E_1$. Then define F_{-i} inductively by $L_{F_{-i+1}} L_{F_{-i+2}} \cdots L_{F_0} E_i$.

In our application, $\mathcal{T}$ will be the bounded derived category $\mathrm{D}^b(X)$ of a smooth algebraic variety (stack) X. The exceptional objects are always assumed to belong to the heart of a certain t-structure.

Given a full strong exceptional collection $E_0, \dots, E_n$, we denote the direct sum $\bigoplus_{i=0}^n E_i$ by T. It is called a tilting object.

Theorem 3.2 [Bondal 1990]. *The exact functor* $\mathbf{R}\mathrm{Hom}(T, -)$ *induces an equivalence between triangulated categories* $\mathrm{D}^b(X)$ *and* $\mathrm{D}^b(\text{mod-}A)$, *where* $A = \mathrm{End}(T)$. *This equivalence is usually referred to as derived Morita equivalence.*

Let $\mathcal{E}$ be an object in $\mathrm{D}^b(X)$, the right A-module structure on $\mathbf{R}\mathrm{Hom}(T, \mathcal{E})$ is given by precomposition. The quasi-inverse functor of $\mathbf{R}\mathrm{Hom}(T, -)$ is $- \otimes_A^{\boldsymbol{L}} T$.

We can define a quiver with relations from a strong exceptional collection by the following recipe. First, define the set of nodes of $\mathcal{Q}$, denoted by $\mathcal{Q}_0$ to be the ordered set $\{0, 1, \dots, n\}$. The i-th node corresponds to the generator of $\mathrm{Hom}(E_i, E_i)$. The set of arrows of $\mathcal{Q}$, denoted by $\mathcal{Q}_1$ is double graded by source and target. The graded piece $\mathcal{Q}_1^{i,j}$ is a set with cardinality $\dim_{\mathbb{C}} \mathrm{Hom}(E_i, E_j)$. With a choice of basis on $\mathrm{Hom}(E_i, E_j)$, the elements of $\mathcal{Q}_1^{i,j}$ are in one-to-one correspondence with such a basis. The exceptional condition guarantees that there is no arrow that decreases the indices of nodes. The relations of $\mathcal{Q}$ are determined by the commutativity of composition of morphisms. The nodes and arrows generate the free path algebra $\mathbb{C}\mathcal{Q}$, which is spanned as a vector space by all the possible paths. Multiplication in $\mathbb{C}\mathcal{Q}$ is defined by concatenation of paths. The relations in $\mathcal{Q}$ form a two-side ideal $\mathcal{I}$ of $\mathbb{C}\mathcal{Q}$. We call $\mathbb{C}\mathcal{Q}/\mathcal{I}$ the path algebra of $(\mathcal{Q}, \mathcal{I})$. In some situations, we omit $\mathcal{I}$ and write just $\mathcal{Q}$. It follows from the construction that $\mathbb{C}\mathcal{Q}/\mathcal{I} \simeq A$.

A representation of $(\mathcal{Q}, \mathcal{I})$ is given by the following pieces of data:

- a finite dimensional vector space V_i associated to each node i;
- a matrix $a^{i,j}$ associated to each arrow from nodes i to j such that the matrix associated to any element in $\mathcal{I}$ is zero.

Denote the category of finite dimensional representations of $(\mathcal{Q}, \mathcal{I})$ by $\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}, \mathcal{I})$. There are equivalences of abelian categories:

$$\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}, \mathcal{I}) \cong \mathbb{C}\mathcal{Q}/\mathcal{I}\text{-mod} \cong A\text{-mod}.$$

The abelian category mod-A is Noetherian and Artinian. Its simple objects are exactly those representations S_i that have a one-dimensional vector space over node i and 0 over all other nodes. Under the functor $\mathbf{R}\mathrm{Hom}(T, -)$, the exceptional objects E_i are mapped to projective right A-modules, and the objects F_{-i} are mapped to shifts of simple modules $S_i[-i]$.

The *Yoneda algebra* R of A is defined to be $\mathrm{Ext}_A^\bullet(\bigoplus_{i=0}^n S_i, \bigoplus_{i=0}^n S_i)$. It has a canonical A_∞ algebra structure.

Theorem 3.2 builds up a link between the geometry and the representation theory of a quiver, assuming that one can find a full strong exceptional collection in $\mathrm{D}^b(X)$. In general, there is no reason why such a collection (even a single exceptional object) should exist. However, the existence result can be proved for toric Fano stacks of dimension two.

Recall that a two dimensional convex lattice polygon Δ with a distinguished interior lattice point determines a fan Σ associated to the barycentric triangulation. This uniquely determines a toric stack, which is denoted by X_Σ. The Fano condition is equivalent to the convexity of Δ. We refer the reader to [Borisov and Hua 2009, Section 3] for an introduction to toric Deligne–Mumford (DM) stacks.

Theorem 3.3 [Borisov and Hua 2009]. *Let X_Σ be a complete toric Fano DM stack of dimension two. The bounded derived category of coherent sheaves $\mathrm{D}^b(X_\Sigma)$ has a full strong exceptional collection consisting of line bundles. The length of the strong exceptional collection is always equal to the integral volume of Δ, which is also equal to the Euler characteristic $\chi(X_\Sigma)$.*

We will try to extend the derived Morita equivalence to the study of the CY stack Y. Consider the exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$, where $B = \mathrm{Hom}^\bullet(\pi^*T, \pi^*T)$. It turns out that this is still an equivalence of triangulated categories if we define the right-hand side appropriately. The algebra B (called the roll-up helix algebra by Bridgeland), in general, carries a nontrivial dg algebra structure. However, in order to apply the quiver techniques, we need to find a strong exceptional collection such that the differential of B vanishes; this is an additional condition on a strong exceptional collection.

The following proposition generalizes [Bridgeland 2005, Proposition 4.1], which was originally proved for $\mathbb{P}^2$.

Proposition 3.4. *Let $\mathcal{L}_0, \dots, \mathcal{L}_n$ be a full strong exceptional collection of line bundles on a toric Fano stack of dimension two. The roll-up (dg)-helix algebra B is in fact an algebra, i.e., $\mathrm{Ext}^{>0}(\pi^*T, \pi^*T) = 0$. Therefore, the exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ induces an equivalence from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$.*

Proof. We need a technical lemma from [Borisov and Hua 2009] about cohomology of line bundles on toric stacks.

For every $\boldsymbol{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$ we denote by $\mathrm{Supp}(\boldsymbol{r})$ the simplicial complex on the vertices $\{1, \dots, n\}$ which consists of all subsets $J \subseteq \{1, \dots, n\}$ such that $r_i \geq 0$ for all $i \in J$ and there exists a cone of Σ that contains all v_i, $i \in J$. For example, if all coordinates r_i are negative then the simplicial complex $\mathrm{Supp}(\boldsymbol{r})$ consists of the empty set only, and its geometric realization is the zero cone of Σ. In the other extreme case, if all r_i are nonnegative then the simplicial complex $\mathrm{Supp}(\boldsymbol{r})$ encodes the fan Σ, which is its geometric realization.

Lemma 3.5 [Borisov and Hua 2009, Proposition 4.1]. *Let N be an integral lattice, Σ a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$, and X_Σ the toric stack associated to Σ. The cohomology*

$H^p(X_\Sigma, \mathcal{L})$ is isomorphic to the direct sum over all $\boldsymbol{r} = (r_i)_{i=1}^n$ such that

$$\bigoplus\Big(\sum_{i=1}^{n} r_i E_i\Big) \cong \mathcal{L}$$

with E_i being toric invariant divisors of the $(\mathrm{rk}(N) - p)$*-th reduced homology of the simplicial complex* $\mathrm{Supp}(\boldsymbol{r})$.

By adjunction,

$$\mathrm{Hom}^d(\pi^*T, \pi^*T) = \bigoplus_{k\geq 0} \mathrm{Hom}^d_X(T, T\otimes\omega_X^{-k}).$$

In order to prove the proposition, it suffices to show that $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j\otimes\omega_X^{-1}) = 0$ for $d = 1, 2$. Since $\mathcal{L}_0, \ldots, \mathcal{L}_j$ is strong exceptional, we have $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j) = 0$ for $d = 1, 2$. Consider all the possible integral linear combinations $\sum_{i=1}^m r_i E_i$ such that $\mathcal{O}\big(\sum_{i=1}^m r_i E_i\big) = \mathcal{L}_i^{-1}\otimes\mathcal{L}_j$. By Lemma 3.5, $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j) = 0$ for $d = 1, 2$ means $\mathrm{Supp}(\boldsymbol{r})$ is contractible. Notice that if $\mathrm{Supp}(\boldsymbol{r})$ is contractible then $\mathrm{Supp}(\boldsymbol{r}+1)$ is also contractible. Again by Lemma 3.5, $H^d(X, \mathcal{L}_i^{-1}\otimes\mathcal{L}_j\otimes\omega_X^{-1}) = 0$ for $d = 1, 2$. $\square$

Now we can write B simply by $\mathrm{End}(\pi^*T)$. It is also the path algebra of a quiver with relations. This quiver can be constructed by the same recipe as in the previous section. Let's denote it by $\mathcal{Q}_\omega$. Notice that $\mathcal{Q}_\omega$ will have cyclic paths because the pull back of exceptional objects will have homomorphisms in both directions. Again, we have an equivalence of abelian categories

$$\mathrm{Rep}_{\boldsymbol{k}}(\mathcal{Q}_\omega, \mathcal{I}) \cong \text{mod-}B.$$

The path algebra B is naturally graded by path length. A B-module M is called *nilpotent* if there exists $k \gg 0$ such that $B_k M = 0$. The exact functor $\mathbf{R}\mathrm{Hom}(\pi^*T, -)$ maps D_ω to the derived category of nilpotent B-modules $D^b(\mathrm{mod}_0\text{-}B)$.

The pushforward ι_* defines an exact functor from $\mathrm{D}^b(X)$ to D_ω. Under Morita equivalence, the modules $\iota_*(F_{-i}[i])$ are the simple modules in $\mathrm{D}^b(\mathrm{mod}_0\text{-}B)$ corresponding to those one dimensional representations associated to each of the vertices of $\mathcal{Q}_\omega$.

Similarly, we call the self-extension algebra

$$\mathrm{Ext}^\bullet_B\Big(\bigoplus_{i=0}^n \iota_* S_i, \bigoplus_{i=0}^n \iota_* S_i\Big)$$

the Yoneda algebra, denoted by R_ω. It carries a natural A_∞ structure as well.

We now give the example of derived Morita equivalence on $\mathbb{P}^2$ and local $\mathbb{P}^2$.

Example 3.6. Let X be $\mathbb{P}^2$. The line bundles $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ form a full strong exceptional collection. Take the tilting bundle to be $T = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$. The quiver $\mathcal{Q}$ is

(1)
$$\begin{tikzcd} 0 \arrow[r, bend left, "x"] \arrow[r, "y"] \arrow[r, bend right, "z"'] & 1 \arrow[r, bend left, "x'"] \arrow[r, "y'"] \arrow[r, bend right, "z'"'] & 2 \end{tikzcd}$$

with the ideal of relations generated by

$$x'y - y'x, \quad y'z - z'y, \quad z'x - x'z.$$

The dual collection to $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ is $\Omega^2(2), \Omega^1(1), \mathcal{O}$. They map to simple modules $S_2[-2], S_1[-1], S_0$ under $\mathbf{R}\mathrm{Hom}(T, -)$.

The roll-up helix algebra $B = \mathrm{End}(\pi^* T)$ is the path algebra of the quiver $\mathcal{Q}_\omega$ given by

(2)
$$\begin{tikzcd} & 0 \arrow[dr, bend left, "x"] \arrow[dr, "y"] \arrow[dr, bend right, "z"'] & \\ 2 \arrow[ur, bend left, "x''"] \arrow[ur, "y''"] \arrow[ur, bend right, "z''"'] & & 1 \arrow[ll, bend right, "z'"'] \arrow[ll, "y'"] \arrow[ll, bend left, "x'"] \end{tikzcd}$$

with relations

$$\begin{aligned} &x'y - y'x, && y'z - z'y, && z'x - x'z; \\ &x''y' - y''x', && y''z' - z''y', && z''x' - x''z'; \\ &xy'' - yx'', && yz'' - zy'', && zx'' - xz''. \end{aligned}$$

4. The cyclic completion of the Yoneda algebra

Two technical results are proved in this section.

- We show the Yoneda algebra L_ω is the cyclic completion of the Yoneda algebra L. This is the algebraic counterpart of the cotangent bundle construction.
- We show that the operations μ_k on L vanish when $k > \chi(X)$. Then, by the cyclic completion construction, the same is true for L_ω.

Theorem 4.2 was proved first by Aspinwall and Fidkowski [2006, Section 4.3] and reproved in a more general form by Segal [2008, Theorem 4.2]. For our own convenience, we give a slightly different proof here since some techniques in the proof are used in the later sections. But the ideas are quite similar to the ones given in those two references.

These two results, together with the existence theorem of strong exceptional collections (Definition 3.1) and Proposition 3.4, guarantee the existence of global algebraic CS functions. In fact, they provide a recipe to construct CS functions, starting from a strong exceptional collection satisfying Proposition 3.4.

Definition 4.1. [Segal 2008] Let $L = \bigoplus_{i=0}^{d} L^i$ be a finite dimensional L_∞ algebra over $\boldsymbol{k}$, with its L_∞ products denoted by μ_k. Define $\bar{L}$ to be the graded vector space $L \oplus L[-d-1]$, i.e., $\bar{L}^i = L^i \oplus (L^{d+1-i})^*$. Define the cyclic pairing and L_∞ products $\bar{\mu}_k : \bigwedge^k \bar{L} \to \bar{L}[2-k]$ according to the following rules:

(1) Define the bilinear form κ on $\bar{L}$ by the natural pairing between L and L^*.

(2) If the inputs of $\bar{\mu}_k$ all belong to L, then define $\bar{\mu}_k = \mu_k$.

(3) If more than one input belongs to L^*, then define $\bar{\mu}_k = 0$.

(4) If there is exactly one input $a_i^* \in L^*$, then define $\bar{\mu}_k$ by

$$\kappa(\bar{\mu}_k(a_1, \ldots, a_i^*, \ldots, a_k), b) = (-1)^{\epsilon} \kappa(\mu_k(a_{i+1}, \ldots, a_k, b, a_1, \ldots, a_{i-1}), a_i^*)$$

for arbitrary $b \in L$, where $\epsilon = |a_1|(|a_2| + \cdots + |b|) + \cdots + |a_i^*|(|a_{i+1}| + \cdots + |b|)$;

It is easy to check that $(\bar{L}, \bar{\mu}_k, \kappa)$ forms a cyclic L_∞ algebra. We call $\bar{L}$ the *cyclic completion* of L.

We have defined the Yoneda algebras $R = \mathrm{Ext}_A^\bullet(\bigoplus_{i=0}^n S_i, \bigoplus_{i=0}^n S_i)$ and $R_\omega = \mathrm{Ext}^\bullet(\bigoplus_{i=0}^n \iota_* S_i, \bigoplus_{i=0}^n \iota_* S_i)$ in previous section. Take the associated L_∞ algebras and denote them by L and L_ω. Since X is a surface, $d = 2$ in Definition 4.1.

The following theorem will play a central role in this paper.

Theorem 4.2 [Aspinwall and Fidkowski 2006; Segal 2008]. *The Yoneda algebra L_ω is the cyclic completion of the Yoneda algebra L.*

Proof. This can be done in three steps. First, we need to verify that L_ω and $\bar{L}$ coincide as graded vector spaces. Second, we will show the pairing on $\bar{L}$ defined by (1) of Definition 4.1 coincides with the Serre pairing on L_ω. Finally, we need to check that the L_∞ products on L_ω satisfy properties (2)–(4) in Definition 4.1.

Given an object $E \in \mathrm{D}^b(\text{mod-}B) \simeq \mathrm{D}^b(Y)$ that is scheme theoretically supported on X, one can view E as a complex of finitely generated A-modules. There is a projective A resolution $P^\bullet$ for E:

$$P^\bullet \longrightarrow E \longrightarrow 0$$

such that each P^i is a direct sum of copies of $E_0, \ldots, E_n$.

Because Y is the total space of canonical bundle over X, there is a tautological short exact sequence of sheaves:

$$0 \longrightarrow \pi^*(\omega_X^{-1}) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Tensor it with $\pi^* E$ to obtain

$$0 \longrightarrow \pi^* E(\omega_X^{-1}) \longrightarrow \pi^* E \longrightarrow \iota_* E \longrightarrow 0.$$

Since π^* preserves the projective modules, by replacing E with $P^\bullet$ we obtain a projective B resolutions of $\iota_* E$ as total complex of the following double complex

$$\begin{tikzcd}
\cdots \arrow[r] & \pi^* P^{-2} \arrow[r] & \pi^* P^{-1} \arrow[r] & \pi^* P^0 \arrow[r] & 0 \\
\cdots \arrow[r] & \pi^* P^{-2} \otimes \pi^* \omega_X^{-1} \arrow[r] \arrow[u] & \pi^* P^{-1} \otimes \pi^* \omega_X^{-1} \arrow[r] \arrow[u] & \pi^* P^0 \otimes \pi^* \omega_X^{-1} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{3}$$

We denote this resolution of $\iota_* E$ by $P^\bullet_\omega$.

As a graded vector space, L_ω is computed as the cohomology of $\mathrm{Hom}^\bullet_Y(P^\bullet_\omega, \iota_* E)$. Because $P^\bullet_\omega$ is the total complex of the above double complex, $\mathrm{Hom}^\bullet_Y(P^\bullet_\omega, \iota_* E)$ is quasi-isomorphic with the total complex of the following double complex:

$$\begin{tikzcd}
\cdots & \mathrm{Hom}(\pi^* P^{-1}, \iota_* E) \arrow[l] \arrow[d] & \mathrm{Hom}(\pi^* P^0, \iota_* E) \arrow[l] \arrow[d] & 0 \arrow[l] \\
\cdots & \mathrm{Hom}(\pi^* P^{-1} \otimes \pi^* \omega_X^{-1}, \iota_* E) \arrow[l] & \mathrm{Hom}(\pi^* P^0 \otimes \pi^* \omega_X^{-1}, \iota_* E) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The spectral sequence associated to this double complex degenerates at E_1 page. Using adjunction together with Serre duality, we obtain

$$\begin{aligned}
\mathrm{Hom}(\iota_* E, \iota_* E) &= \mathrm{Hom}_X(E, E), \\
\mathrm{Ext}^1(\iota_* E, \iota_* E) &= \mathrm{Ext}^1_X(E, E) \oplus \mathrm{Ext}^2_X(E, E)^*, \\
\mathrm{Ext}^2(\iota_* E, \iota_* E) &= \mathrm{Ext}^2_X(E, E) \oplus \mathrm{Ext}^1_X(E, E)^*, \\
\mathrm{Ext}^3(\iota_* E, \iota_* E) &= \mathrm{Hom}_X(E, E)^*.
\end{aligned}$$

The above fact holds for any object E with scheme theoretic support on X. We are particularly interested in the case when E is $\oplus_{i=0}^n F_{-i}[i]$, i.e., the direct sum of the simple objects in mod-A. This identifies L_ω and $\bar{L}$ as graded vector spaces since both will be equal to $L \oplus L[-3]^*$.

In order to verify property (1), we need to write down a bilinear pairing κ on $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$ such that its restriction on cohomology gives the obvious duality between L and L^*. By adjunction, $\mathrm{Hom}^3(P^\bullet_\omega, P^\bullet_\omega)$ has a direct summand $\mathrm{Hom}^2(\pi^* P^\bullet \otimes \omega_X^{-1}, \pi^* P^\bullet)$, which is isomorphic to $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes (\bigoplus_{k \leq 1} \omega_X^k))$. It contains the finite dimensional graded piece $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes \omega)$, which has a trace map to $H^2(X, \omega_X) \simeq \mathbb{C}$. Given any two elements x and y in $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$, we define the bilinear pairing $\kappa(x, y)$ to be the projection of $x \circ y$ to the graded piece $\mathrm{Hom}^2_X(P^\bullet, P^\bullet \otimes \omega)$ followed by the trace map. Clearly, the restriction of κ on cohomology satisfies property (1).

Now we need to verify properties (2) to (4) for L_ω. For dimension reasons, it suffices to check the case when all the inputs of the L_∞ products μ_k lie in L^1_ω. Since L_ω is constructed as the cohomology of $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$, the element in L^1_ω can be represented by either the vertical or horizontal arrows in diagram (3). More specifically, a class in $\mathrm{Ext}^1_X(E, E)$ is represented by a horizontal arrow and a class in $\mathrm{Ext}^2_X(E, E)^*$ is represented by a vertical arrow. Then property (2) follows immediately since the rows of the double complex are simply the pullback of $P^\bullet$ (up to $\otimes\, \omega_X^{-1}$), which is the projective resolution of E.

If we write $\mathrm{Ext}^2(E, E)^*$ as $\mathrm{Ext}^0(E, E \otimes \omega_X)$, then we can see that

$$\bar{\mu}_2 : \mathrm{Ext}^1(E, E) \otimes \mathrm{Ext}^0(E, E \otimes \omega_X) \longrightarrow \mathrm{Ext}^1(E, E \otimes \omega_X) \simeq \mathrm{Ext}^1(E, E)^*$$

is the only nonzero term that can involve $\mathrm{Ext}^2(E, E)^*$. For example, if both inputs of μ_2 belong to $\mathrm{Ext}^0(E, E \otimes \omega_X)$, then the output is $\mathrm{Ext}^0(E, E \otimes \omega_X^2)$, which is not in L^2_ω. Similarly, this argument shows that any nonzero term of μ_k of L_ω can involve at most one $\mathrm{Ext}^2(E, E)^*$ term. This proves property (3).

Property (4) is essentially the cyclic symmetry of μ_k. Since the κ on cohomology is a restriction of a bilinear form (also denoted by κ) on the dga $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$ with differential d, property (4) will follow from the following cyclic symmetry properties on $\mathrm{Hom}^\bullet(P^\bullet_\omega, P^\bullet_\omega)$. For arbitrary elements x, y, and z:

- ◇ $\kappa(x, y) = \pm\kappa(y, x)$
- ◇ $\kappa(dx, y) = \pm\kappa(dy, x)$;
- ◇ $\kappa(x \circ y, z) = \pm\kappa(y \circ z, x)$.

The first property is clear since the commutator is trace-free. The trace map will factor through the morphism

$$\mathrm{Hom}^2(P^\bullet, P^\bullet \otimes \omega) \longrightarrow L^3_\omega = \mathrm{Ext}^2(E, E \otimes \omega) \simeq \mathrm{Hom}(E, E)^*.$$

Therefore, the trace of a coboundary is zero, so the second property follows from the Leibniz rule. The third property follows from the first and associativity of the product. □

Remark 4.3 (the geometric meaning of cyclic completion). From Example 2.9 recall that the completion of the truncated symmetric algebra $(\mathrm{Sym}\, L[1]^*)\widehat{\ }$ (we omit $\tau^{>0}$ for simplicity) can be interpreted as the structure sheaf of the graded linear manifold $M = L[1]$.

The odd cotangent bundle of the graded manifold M, denoted as $T^*[-1]M$, is defined to be the graded manifold $L[1] \oplus (L[1]^*[-1])$. As graded vector spaces,

$T^*[-1]M$ is the same as $L_\omega[1]$. Then, $\mathcal{O}_{T^*[-1]M}$ coincides with $(\operatorname{Sym} L_\omega[1]^*)\widehat{\ }$ as graded algebras. The L_∞ products $\bar{\mu}_k$ defines a derivation on $\mathcal{O}_{T^*[-1]M}$ and the cyclic pairing κ defines an odd two-form on $T^*[-1]M$. In fact, this process is functorial. Hence, passing to the cyclic completion of an L_∞ algebra is an algebraic counterpart for taking the odd cotangent bundle of a dg manifold.

The L_∞ (or A_∞) structure of the Yoneda algebra L has been studied for a long time in the representation theory of finite dimensional algebras. The following boundedness theorem turns out to be very important for the purpose of this paper.

Theorem 4.4. *The L_∞ products (higher brackets) μ_k on L vanish when $k > \chi(X)$.*

Proof. Let A be a finite dimensional algebra and $\{S_i\}$ be the collection of simple A-modules. It is well known that the Yoneda algebra $R = \operatorname{Ext}^\bullet_A(\bigoplus_i S_i, \bigoplus_i S_i)$ controls the deformation of A. If A is presented as a path algebra of a quiver with relations, then the A_∞ products m_k on R can be interpreted as relations of the path algebra; see [Keller 2006, Section 7.8].

Since in our situation the quiver is constructed from a strong exceptional collection of line bundles on X (recall the construction in Section 3), the elements in the path algebra A carry an extra grading given by the ordering on the strong exceptional collection. The A_∞ products preserve this extra grading. Therefore, the length of the strong exceptional collection, which is equal to the Euler characteristic $\chi(X)$, gives an upper bound for number of nonvanishing m_k. This is intuitively clear since, on a directed quiver generated by, say, a length 4 strong exceptional collection, there cannot be a relation involving length 5 paths.

Finally, we pass from an A_∞ algebra to an L_∞ algebra. Since L is the antisymmetrization of R, we get $\mu_k = 0$ when $m_k = 0$. □

5. Moduli spaces and Chern–Simons functions

We fix the ground field $\boldsymbol{k} = \mathbb{C}$. Let Γ be the Grothendieck group of D_ω. By derived Morita equivalence, Γ also equals the Grothendieck group of the derived category of nilpotent representations of $\mathcal{Q}_\omega$. It is a free abelian group of rank $n+1$ generated by the collection of simple modules $\iota_* S_0, \dots, \iota_* S_n$. If we fix these simple modules as a $\mathbb{Z}$-basis of Γ, every effective class can be written as a vector $\boldsymbol{d} = (d_0, \dots, d_n)$ with nonnegative entries. We call such a choice of $\boldsymbol{d}$ a *dimension vector*.

Theorem 5.1. *Let X be a toric Fano stack of dimension two and Y the total space of its canonical bundle. Pick a strong exceptional collection constructed in [Borisov and Hua 2009] and denote the corresponding quiver of Y by $\mathcal{Q}_\omega$. Let $\mathfrak{M}_\gamma$ be a bounded family of sheaves on Y support on X with class $\gamma \in \Gamma$. There exists a dimension vector $\boldsymbol{d}$ and an open immersion of Artin stacks from $\mathfrak{M}_\gamma$ to the quotient stack* $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$, *where* $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$ *is the space of representations of $\mathcal{Q}_\omega$ with* *dimension vector* $\boldsymbol{d}$ *and* $G_{\boldsymbol{d}}$ (*defined later in this section*) *is the gauge group acting by changing of basis.*

Theorem 5.2. *Given a class* $\gamma \in \Gamma$, *a bounded family of sheaves on* Y *supported on* X *with class* γ *is the critical set of an algebraic function* $f_{\boldsymbol{d}}$.

We call such a function a *Chern–Simons (CS) function*. The infinitesimal deformation of representations is controlled by the following L_∞ algebras. Fix a dimension vector $\boldsymbol{d}$, define

$$L_{\boldsymbol{d}} := \mathrm{Ext}^\bullet\left(\bigoplus_{i=0}^{n} S_i \otimes V_i,\ \bigoplus_{i=0}^{n} S_i \otimes V_i\right)$$

and

$$L_{\omega,\boldsymbol{d}} := \mathrm{Ext}^\bullet\left(\bigoplus_{i=0}^{n} \iota_* S_i \otimes V_i,\ \bigoplus_{i=0}^{n} \iota_* S_i \otimes V_i\right),$$

where each V_i is a vector space of dimension d_i. They are generalizations of the Yoneda algebras: if we take $\boldsymbol{d} = (1, \dots, 1)$ we obtain the Yoneda algebras. All the results in Section 4 clearly generalize to $L_{\boldsymbol{d}}$ and $L_{\omega,\boldsymbol{d}}$.

The space $L^1_{\boldsymbol{d}}$ can be identified with the space $\bigoplus_{a \in \mathcal{Q}_1} \mathrm{Hom}(V_i, V_j)$ of matrices, summing over all the arrows, and similarly for $L^1_{\omega,\boldsymbol{d}}$ with $a \in \mathcal{Q}_{\omega 1}$. It carries a natural bigrading by the source and target of each arrow. The space $L^0_{\boldsymbol{d}}$ can be identified with the space $\bigoplus_{i \in \mathcal{Q}_0} \mathrm{End}(V_i)$, which is the Lie algebra associated to the group $\prod_{i \in \mathcal{Q}_0} \mathrm{GL}(V_i)$. We denote this group by $G_{\boldsymbol{d}}$ for simplicity. It acts on $L_{\boldsymbol{d}}$ by conjugation. Analogously, the space $L^0_{\omega,\boldsymbol{d}}$ can be identified with the Lie algebra associated to the same group, which acts on $L_{\omega,\boldsymbol{d}}$.

The following lemma is well known in representation theory of quivers.

Lemma 5.3. *The elements of* $\mathrm{MC}(L_{\boldsymbol{d}})$ *are in one to one correspondence with the representations of* $\mathcal{Q}$ *of dimension vector* $\boldsymbol{d}$, *and analogously for the elements of* $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$ *and the representations of* $\mathcal{Q}_\omega$. *Two representations are isomorphic if and only if they belong to the same orbits of* $G_{\boldsymbol{d}}$.

Proof. See [Keller 2006, Section 7.8] or [Segal 2008, Proposition 3.8]. □

The L_∞ algebra L controls the infinitesimal deformation of representations in the following sense. Let M be an A-module with dimension vector $\boldsymbol{d}$. We denote its corresponding Maurer–Cartan element by x. The homology groups $H^i(L_{\boldsymbol{d}}, \delta^x)$ are isomorphic to $\mathrm{Ext}^i_A(M, M)$. In general, $L_{\boldsymbol{d}}$ is just the formal tangent space at the point $\bigoplus_i S_i \otimes V_i$. However, in our situation because of the boundedness of μ_k (Theorem 4.4), the Maurer–Cartan equation actually converges. An analogous argument holds for the L_∞ algebra $L_{\omega,\boldsymbol{d}}$, with M a B-module with dimension vector $\boldsymbol{d}$, in which case the homology groups $H^i(L_{\omega,\boldsymbol{d}}, \delta^x)$ are isomorphic to

$\mathrm{Ext}^i_B(M, M)$. Therefore the moduli space can be constructed globally as mentioned in the previous Lemma.

Proof of Theorem 5.1. Given Lemma 5.3, it suffices to show the existence of an open immersion of $\mathfrak{M}_\gamma$ into $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$.

First, we need to construct a monomorphism of stacks. Let's pick an ample line bundle L on X. If T is a tilting bundle on X then $T \otimes L^{-N}$ is again a tilting bundle for any integer N. Therefore, the functor $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), -)$ induces an equivalence from $\mathrm{D}^b(Y)$ to $\mathrm{D}^b(\text{mod-}B)$. Because T is direct sum of line bundles, we can choose $N \gg 0$ such that for any sheaf $\mathcal{E} \in \mathfrak{M}_\gamma$, $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), \mathcal{E})$ is concentrated in degree zero, i.e., is a module over B.[1] Let $\boldsymbol{d}$ be its dimension vector, which depends on both γ and N. Then we obtain a morphism between stacks. Because of the derived Morita equivalence, this is clearly an injection.

Next we need to argue this morphism is étale. Let $A' \to A \to \mathbb{C}$ be a small extension of pointed $\mathbb{C}$-algebras, and let $T = \mathrm{Spec}\, A$ and $T' = \mathrm{Spec}\, A'$. Consider the 2-commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & \mathfrak{M}_\gamma \arrow[d, hook, "\mathbf{R}\mathrm{Hom}(\pi^*(T\otimes L^{-N}),-)"] \\ T' \arrow[r] \arrow[ur, dashed] & {[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]} \end{tikzcd}$$

of solid arrows. We have to prove that the dotted arrow exists, uniquely, up to a unique 2-isomorphism. This follows from standard deformation theory. We need that $\mathbf{R}\mathrm{Hom}(\pi^*(T \otimes L^{-N}), -)$ induces a bijection on deformation spaces and an injection on obstruction spaces (associated to the above diagram). They follow immediately for the equivalence between $\mathrm{D}^b(Y)$ and $\mathrm{D}^b(\text{mod-}B)$. In fact, all the obstruction groups are isomorphic. $\square$

Proof of Theorem 5.2. As we have seen in Definition 2.5, there is always a formal function

$$f_{\boldsymbol{d}}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{(k+1)!}\,\kappa(\bar{\mu}_k(z, \dots, z), z)$$

associated to the cyclic L_∞ algebra $L_{\omega,\boldsymbol{d}}$, where $z \in L^1_{\omega,\boldsymbol{d}}$. The critical set of $f_{\boldsymbol{d}}$ coincides with $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$.

By the boundedness in Theorem 4.4, such a formal function is, in fact, a polynomial function of degree at most $\chi(X)$. Therefore, $\mathrm{MC}(L_{\omega,\boldsymbol{d}})$, as a subvariety of $L^1_{\omega,\boldsymbol{d}}$, is the critical scheme of $f_{\boldsymbol{d}}$. Since the $G_{\boldsymbol{d}}$ action is induced from the action of the Lie subalgebra $L^0_{\omega,\boldsymbol{d}}$, it is clear that $f_{\boldsymbol{d}}$ is invariant under this action.

[1]This is *not* true when T contains torsion.

By Theorem 5.1, $\mathfrak{M}_\gamma$ is an open substack of $[\mathrm{MC}(L_{\omega,\boldsymbol{d}})/G_{\boldsymbol{d}}]$ for an appropriate choice of $\boldsymbol{d}$. The theorem follows since we can restrict the function $f_{\boldsymbol{d}}$. □

Remark 5.4. Recall that by Theorem 4.2, $L^1_{\omega,\boldsymbol{d}}$ decomposes into $L^1_{\boldsymbol{d}} \oplus (L^2_{\boldsymbol{d}})^*$. The CS function $f_{\boldsymbol{d}}$ has a nice property coming from this decomposition:

If we write the cyclic pairing $\kappa(x, y)$ as $\mathrm{tr}(x \circ y)$, then the function $f_{\boldsymbol{d}}$ can be written as the trace of the cyclic invariant polynomial of matrices. Definition 4.1 tells us that the variables in $(L^2_{\boldsymbol{d}})^*$ appear exactly once (in degree one) in all the monomials. This means that we can always write $f_{\boldsymbol{d}}$ as an inner product of a polynomials of elements in $L^1_{\boldsymbol{d}}$ and elements of $(L^2_{\boldsymbol{d}})^*$. This property plays a central role in Section 8.

As a summary of Sections 4 and 5, we give an algorithm to compute CS functions on local toric Fano surfaces.

STEP 1 Choose a strong exceptional collection of line bundles on X. By results in Section 3, this completely determines the quiver $\mathcal{Q}$, together with its relations.

STEP 2 Compute the A_∞ structures on the Yoneda algebra R using the correspondence between m_k and the relations on $\mathcal{Q}$.

STEP 3 Apply Theorem 4.2 to compute $\overline{m}_k$ for R_ω.

STEP 4 Plug in specific dimension vector $\boldsymbol{d}$, antisymmetrize $R_{\omega,\boldsymbol{d}}$ to $L_{\omega,\boldsymbol{d}}$, and apply Definition 2.5 to compute $f_{\boldsymbol{d}}$.

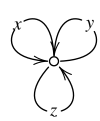
6. Examples of CS functions

In these section, we discuss some examples of CS functions.

6A. *Complex affine 3-space* $\mathbb{C}^3$. The easiest example of a Calabi–Yau 3-fold is the three dimensional affine space. Rigorously speaking, it is not a local surface but still the CS function can be computed using the same philosophy.

Let B be the polynomial algebra with three variables. The category $\mathrm{Coh}(\mathbb{C}^3)$ equals mod-B. Consider the quiver $\mathcal{Q}_\omega$:

(4)



with relations $xy - yx$, $yz - zy$, $zx - xz$. Its path algebra is equal to B.

Given a positive integer n, let $L_{\omega,n}$ be the Yoneda algebra $\mathrm{Ext}^\bullet_{\mathbb{C}^3}(\mathcal{O}_{\{0\}}, \mathcal{O}_{\{0\}}) \otimes \mathfrak{gl}_n$. Since the only nonvanishing product is $\bar{\mu}_2$, $L_{\omega,n}$ is a graded Lie algebra. Now, let A, B, C be $n \times n$ matrices associated to x, y, z. The CS function f_n is equal to $\mathrm{tr}((AB - BA)C)$.

The Morita equivalence in this case is the classical Koszul duality between symmetric and exterior algebras

$$\mathrm{D}^b(\mathrm{Coh}(V)) = D^b\big(\text{mod-}{\textstyle\bigwedge}^\bullet(V)\big).$$

The quiver $\mathcal{Q}_\omega$ gives combinatorial description for both $\mathbb{C}^3$ and the cotangent bundle of the three dimensional torus. The first is clear since the path algebra of $\mathcal{Q}_\omega$ is the algebra of functions on $\mathbb{C}^3$. For the second, we can think of the quiver as the 1-skeleton of T^3 and the relations as the gluing conditions of two cells.

The stack $[\mathrm{crit}(f_{\boldsymbol{d}})/G_{\boldsymbol{d}}]$ is related to two interesting moduli spaces. The first one[2] is the moduli space of length n sheaves on $\mathbb{C}^3$ and the second one[3] is the moduli space of flat GL_n vector bundles on T^3. These two moduli spaces are related by homological mirror symmetry.

6B. ***The local projective plane $\boldsymbol{\omega_{\mathbb{P}^2}}$.*** Using the calculations done in Example 3.6, the CS function for the local projective plane is

$$\mathrm{tr}\big(C''(A'B - B'A) + A''(B'C - C'B) + B''(C'A - A'C)\big)$$

where A, B, C, A', B', C', A'', B'', C'' are matrices associated, respectively, to the arrows x, y, z, x', y', z', x'', y'', z''.

6C. ***The Calabi–Yau 3-folds $\boldsymbol{\omega_{\mathbb{P}(1:3:1)}}$ and $\boldsymbol{\omega_{\mathbb{P}(2:1:2)}}$.*** In this subsection, we will compute the CS functions of $\omega_{\mathbb{P}(1:3:1)}$ and $\omega_{\mathbb{P}(2:1:2)}$. These two Calabi–Yau 3-folds are K-equivalent; consequently, there is some interesting symmetry between these two CS functions.

For simplicity, we set $X_1 := \mathbb{P}(1:3:1)$ and $X_2 := \mathbb{P}(2:1:2)$. The stacky fan Σ_1 of X_1 has rays $(0,1)$, $(1,-1)$, $(-1,-2)$; the stacky fan Σ_2 of X_2 has rays $(0,2)$, $(1,0)$, $(-1,-1)$. Denote their canonical bundles by Y_1 and Y_2.

The Picard groups of X_1 and X_2 both equal $\mathbb{Z}$. We denote the positive generator by $\mathcal{O}(1)$. On X_1, $\mathcal{O}(1)$ can be written as $\mathcal{O}(D_2)$, with D_2 being the toric invariant divisor for $(1,-1)$. On X_2, $\mathcal{O}(1)$ can be written as $\mathcal{O}(D_1)$ with D_1 being the toric invariant divisor for $(0,2)$. For both $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$,

$$\mathcal{O},\quad \mathcal{O}(1),\quad \mathcal{O}(2),\quad \mathcal{O}(3),\quad \mathcal{O}(4)$$

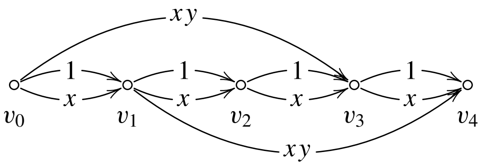
form a full strong exceptional collection. The quivers associated to these two collections are denoted by $\mathcal{Q}_1$ and $\mathcal{Q}_2$. The sets of vertices $\{0, 1, 2, 3, 4\}$ correspond

[2] One can modify the construction slightly to include the Hilbert scheme of points; see [Behrend et al. 2013].

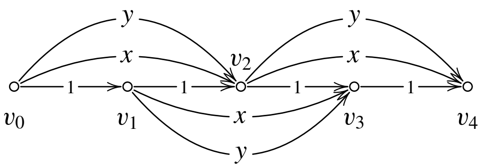
[3] One needs to include a stability condition to make it hold rigorously.

to $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4)$.

(5) $\mathcal{Q}_1$



(6) $\mathcal{Q}_2$



Notice that Σ_1 and Σ_2 are related by a shift of origin. This shift changes the stack completely. But surprisingly, the full strong exceptional collections on X_{Σ_1} and X_{Σ_2} are related [Borisov and Hua 2009]. We denote the arrows from the i-th node to the j-th node by A_{ij}, B_{ij} or C_{ij} and the relations from the i-th node to the j-th node by R_{ji}. Because the quivers are directed, i is strictly less than j.

Using the algorithm at the end of last section, the CS function for $\omega_{\mathbb{P}(1:3:1)}$ is

$$
\begin{aligned}
(7)\quad f = \operatorname{tr}\big(&R_{20}(B_{12}A_{01} - A_{12}B_{01}) + R_{31}(B_{23}A_{12} - A_{23}B_{12}) \\
&+ R_{42}(B_{34}A_{23} - A_{34}B_{23}) + R_{40}(A_{34}C_{03} - C_{14}A_{01}) \\
&+ S_{40}(B_{34}C_{03} - C_{14}B_{01})\big).
\end{aligned}
$$

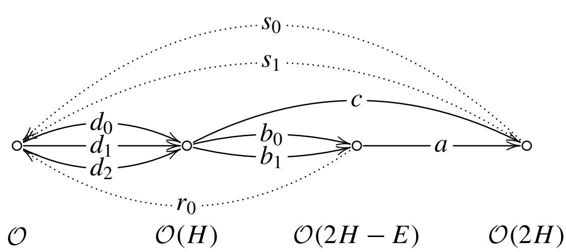
The CS function for $\omega_{\mathbb{P}(2:1:2)}$ is

$$
\begin{aligned}
(8)\quad f = \operatorname{tr}\big(&R_{30}(A_{23}B_{02} - B_{13}A_{01}) + R_{41}(B_{24}A_{12} - A_{34}B_{13}) \\
&+ S_{30}(A_{23}C_{02} - C_{13}A_{01}) + S_{41}(C_{24}A_{12} - A_{34}C_{13}) \\
&+ R_{40}(B_{24}C_{02} - C_{24}B_{02})\big).
\end{aligned}
$$

6D. *Blow-up of the projective plane $\mathbb{P}^2$ at one point.* The first example which involves μ_k terms with $k > 2$ is the local DelPezzo surface of degree one. It was first computed in [Aspinwall and Fidkowski 2006].

Let X be the blow-up of $\mathbb{P}^2$ at one point. Denote the pull back of a hyperplane by H and the exceptional divisor by E. The derived category $\mathrm{D}^b(X)$ has a strong exceptional collection, consisting of $\mathcal{O}$, $\mathcal{O}(H)$, $\mathcal{O}(2H - E)$, $\mathcal{O}(2H)$, and the corresponding quiver is

(9) $\mathcal{Q}_3$



The graded piece L^2 of the Yoneda algebra has dimension three. We denote the basis by r_0, s_0, s_1. If we denote the matrices associated to each arrow by uppercase letters, then the CS function is

$$f = \mathrm{tr}\big(R_0(B_0D_1 - B_1D_0) + S_0(AB_0D_2 - CD_0) + S_1(AB_1D_2 - CD_1)\big).$$

7. Integrality of generalized DT invariants

In this section, we give the first geometric application of CS functions. The main result is Theorem 7.4, where we show that the L_∞ products vanish at semistable points of the moduli space of sheaves of local surfaces. As a consequence, the generalized Donaldson–Thomas invariants defined by Joyce and Song [2012] are integral on local surfaces.

We only consider sheaves on Y that belong to the category D_ω, i.e., set theoretically supported on X. Furthermore, we assume they are supported in dimension bigger than zero. The integrality of the zero dimensional sheaves has been proved in [Joyce and Song 2012, Section 6.3].

Let L be an ample line bundle on X. The Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on Y is defined to be $\chi(\mathcal{E} \otimes \pi^* L^k)$ for $k \gg 0$. The slope of $\mathcal{E}$, denoted by $\mu(\mathcal{E})$ is defined to be the quotient of the second nonzero coefficient of its Hilbert polynomial by the first. We will adopt the notation of Joyce and Song [loc. cit.]. A sheaf $\mathcal{E}$ is called *τ-stable* if for any proper subsheaf $\mathcal{F}$, the slopes satisfy $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Similarly, $\mathcal{E}$ is called *τ-semistable* if $\mu(\mathcal{F}) \le \mu(\mathcal{E})$. The moduli space of τ semistable sheaves on Y with class $\gamma \in \Gamma$ is denoted by $\mathfrak{M}^\tau(Y, \gamma)$.

Lemma 7.1. *Assume X is a Gorenstein toric Fano stack of dimension two. If $\mathcal{E}$ is a τ-stable sheaf on Y, then $\mathcal{E}$ is supported on X scheme theoretically.*

Proof. Let Z be the scheme theoretical support of a τ-stable sheaf $\mathcal{E}$. There is a trace map $\mathrm{tr}_\mathcal{E} : \underline{\mathrm{Hom}}^\bullet(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_Z$ and a map $i_\mathcal{E} : \mathcal{O}_Z \to \underline{\mathrm{Hom}}^0(\mathcal{E}, \mathcal{E})$ such that $\mathrm{tr}_\mathcal{E} \circ i_\mathcal{E} = \mathrm{rk}_Z(\mathcal{E})$. (We refer the reader to [Huybrechts and Lehn 1997, §10.1] for the precise definitions of these maps.) Since the rank of $\mathcal{E}$ (over Z) is positive, $i_\mathcal{E}$ must be an injection. By local-to-global spectral sequence, there is an injection $H^0(Z, \mathcal{O}_Z) \to \mathrm{Ext}^0_Z(\mathcal{E}, \mathcal{E})$.

By stability, $\mathcal{E}$ must be pure. We first assume $\mathcal{E}$ is supported in dimension two. Then Z is an order n thickening of X in the normal direction. The cohomology group $H^0(Z, \mathcal{O}_Z)$ is equal to $\bigoplus_{i=0}^n H^0(X, \omega_X^{-i})$. The dimension of $H^0(X, \omega_X^{-1})$ can be identified with number of lattice points in $\Delta^\vee$ in $M_\mathbb{R} := \mathrm{Hom}(M, \mathbb{R})$, where M is the dual lattice of N. Recall that the polytope supporting the fan Σ lives in $N_\mathbb{R}$. In general, $\Delta^\vee$ is only a rational polytope. However, since the origin is always in the interior of $\Delta^\vee$, the dimension of $H^0(X, \omega_X^{-1})$ is at least one. Therefore, the dimension of $H^0(Z, \mathcal{O}_Z)$ is strictly bigger than one. We get a contradiction since a stable sheaf can only have one dimensional endomorphisms.

Now, let Z be a thickening of a divisor C in X. Similarly, it suffices to show that $H^0(C, \omega_X^{-1})$ is nonzero. There is a morphism $H^0(X, \omega_X^{-1}) \to H^0(C, \omega_X^{-1})$. Let us denote the toric divisors of X by E_i. Because C is an effective divisor, it can be written as a linear combination $\sum_i a_i E_i$ where all a_i are nonnegative integers and at least one of them is positive. Consider the short exact sequence

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

The cohomology group $H^0(C, \omega_X^{-1})$ vanishes only if the morphism

$$H^0(X, I_C \otimes \omega_X^{-1}) \to H^0(X, \omega_X^{-1})$$

is a bijection. The first group can be written as $H^0\big(X, \mathcal{O}\big(\sum_i (1-a_i)E_i\big)\big)$. The Gorenstein condition implies that Δ and $\Delta^\vee$ are both lattice polytopes. The dimension of $H^0\big(X, \mathcal{O}\big(\sum_i (1-a_i)E_i\big)\big)$ is equal to the number of lattice points inside the polytope that is obtained from $\Delta^\vee$ by translating its faces towards origin. Because at least one a_i is positive and $\Delta^\vee$ is a lattice polytope to begin with, the number of lattice points will decrease when one face is pushed. As a consequence, $H^0(C, \omega_X^{-1})$ is nonzero. □

Lemma 7.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be τ-semistable sheaves on X such that $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$. Then*, $\operatorname{Ext}^2(\mathcal{E}_1, \mathcal{E}_2) = 0$.

Proof. By Serre duality, $\operatorname{Ext}^2_X(\mathcal{E}_1, \mathcal{E}_2) = \operatorname{Hom}_X(\mathcal{E}_2, \mathcal{E}_1 \otimes \omega_X)^*$. Because ω_X^{-1} is ample and $\mathcal{E}_1$, $\mathcal{E}_2$ have dimension bigger than zero, $\mu(\mathcal{E}_1 \otimes \omega_X) < \mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$. Hence, $\operatorname{Ext}^2(\mathcal{E}_1, \mathcal{E}_2)$ vanishes by stability. □

Lemma 7.1 doesn't hold for semistable sheaves. For example, if we take a proper but nonreduced curve in Y, then its structure sheaf can be semistable but not stable.

Lemma 7.3. *Let $\mathcal{E}$ be a τ-semistable sheaf on Y. Then the restriction $\mathcal{E}|_X$ is a semistable sheaf on X.*

Proof. Because $\mathcal{E}$ is set theoretically supported on X, it can be written as consequent extensions of stable sheaves on X with the same slope (the Jordan–Holder filtration). Furthermore, the natural morphism $\mathcal{E} \to \mathcal{E}|_X$ is always a surjection of sheaves. Since $\mu(\mathcal{E}|_X) = \mu(\mathcal{E})$, any quotient sheaf that destabilizes $\mathcal{E}|_X$ will destabilize $\mathcal{E}$ as well. □

From now on, we will assume X is Gorenstein.

Theorem 7.4. *The L_∞ products $\bar{\mu}_k$ of L_ω vanish at semistable points.*

Proof. Let $\mathcal{E}$ be a τ-semistable sheaf on Y. It follows from Theorem 5.1 that we can define a cyclic L_∞ algebra L_ω such that $\mathcal{E}$ is mapped to a Maurer–Cartan element $\bar{x}$.

Moreover, $\mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})$ coincides with $H^i(L_\omega,\delta^{\bar{x}})$. The L_∞ products $\bar{\mu}_k$ uniquely defines L_∞ products on $H^\bullet(L_\omega,\delta^{\bar{x}})$ up to L_∞ isomorphisms. We say that $\bar{\mu}_k$ vanish at x if they vanish after passing to $H^\bullet(L_\omega,\delta^{\bar{x}})$.

An MC element $\bar{x}$ of L_ω decomposes into (x,ϵ), with respect to the decomposition $L^1_\omega = L^1 \oplus (L^2)^*$. It follows from Theorem 4.2 that x is an MC element of L. The cohomology $H^\bullet(L_\omega,\delta^{\bar{x}})$ can be computed as the cohomology of the total complex of

$$\begin{tikzcd}
(L^2)^* \arrow[r] & (L^1)^* \arrow[r] & (L^0)^* \\
L^0 \arrow[r] \arrow[u] & L^1 \arrow[r] \arrow[u] & L^2 \arrow[u]
\end{tikzcd}$$

where the horizontal differential is δ^x and the vertical differential is induced by $[\epsilon,-]$.

If $\bar{x}$ is the image of a sheaf of the form $\iota_*\mathcal{E}$ for some sheaf $\mathcal{E}$ on X then $\bar{x}=(x,0)$. In that case, the associated spectral sequence will degenerate at E_1 page.

If $\epsilon \neq 0$, we need to pass to the E_2 page of

$$\begin{tikzcd}
H^2(L,\delta^x)^* \arrow[r, "0"] & H^1(L,\delta^x)^* \arrow[r] & H^0(L,\delta^x)^* \\
H^0(L,\delta^x) \arrow[r] \arrow[u, "{[\epsilon,-]}"] & H^1(L,\delta^x) \arrow[r] \arrow[u, "{[\epsilon,-]}"] & H^2(L,\delta^x) \arrow[u, "{[\epsilon,-]}"]
\end{tikzcd}$$

The MC element $(x,0)$ is exactly the one corresponding to $\mathcal{E}|_X$. So $H^i(L,\delta^x) = \mathrm{Ext}^i_X(\mathcal{E}|_X,\mathcal{E}|_X)$. Now by Lemmas 7.3 and 7.2, $H^2(L,\delta^x)$ vanishes. By the previous commutative diagram, $H^1(L_\omega,\delta^{\bar{x}})$ and $H^2(L_\omega,\delta^{\bar{x}})$ are equal to the kernel and cokernel of the map

$$H^1(L,\delta^x) \xrightarrow{[\epsilon,-]} H^1(L,\delta^x)^*.$$

The L_∞ structure $\bar{\mu}_k$ on $H^\bullet(L_\omega,\delta^{\bar{x}})$ is obtained from μ_k by transferring. The vanishing of $H^2(L,\delta^x)$ and $H^2(L,\delta^x)^*$ together with Theorem 4.2 implies $\mu_k=0$. Therefore $\bar{\mu}_k$ must vanish after transferring to cohomology with respect to $[\epsilon,-]$. □

Remark 7.5. A corollary of Theorem 7.4 is that the moduli space of τ-semistable sheaves on Y is smooth as an Artin stack since the images of $\bar{\mu}_k$ are nothing but obstructions to smoothness of moduli space.

We are not going to define Joyce's generalized DT invariants and state the general form of the integrality conjecture since it requires too much work. The interested reader can refer to [Joyce and Song 2012] for the full story.

Corollary 7.6. *The generalized Donaldson–Thomas invariants $\widehat{\mathrm{DT}}(\tau)$ for τ-semistable sheaves are integers on local surfaces.*

Proof. The integrality has been proved for the DT invariants of a quiver without relations. The proof can be found in [Joyce and Song 2012] or [Reineke 2011]. By [Joyce and Song 2012, Proposition 7.28], the formal neighborhood of a point of the moduli space of sheaves is isomorphic as formal schemes to a formal neighborhood of the origin of the moduli space of representation of the Ext-quiver (see the proposition for the definition). By Theorems 7.4 and 4.4 the relations of the Ext-quiver vanish when the point is taken to be semistable. Jiang [2010] proved that Behrend function only depends on the formal neighborhood of a moduli space. Therefore, the integrality of the moduli space of semistable sheaves is equivalent to the integrality of the moduli space of representations of quivers without relations. □

8. A dimension reduction formula for virtual motives

In this section, we give the second application of CS functions. We prove a decomposition theorem for virtual motives of $f_{\boldsymbol{d}}$, which partially generalizes [Behrend et al. 2013, Section 3]. If we could identify geometric stability with the appropriate quiver stability condition, then we would obtain a decomposition theorem of virtual motives of Hilbert schemes, which generalizes the most interesting part of [loc. cit.]. However, so far we have no idea how to deal with geometric stability.

Let $\mathbb{L}$ be the motive of the affine line. Given a scheme X, we will denote its motive by $[X]$.

Consider a smooth scheme M with an action of a special algebraic group G, together with a G-invariant regular function $f : M \to \mathbb{C}$. Denef and Loeser [2001] defined the motivic vanishing cycle $[\phi_f]$ in a suitable augmented Grothendieck ring of varieties (called the ring of motivic weights). Since our result is not going to involve the precise definition of this ring, we refer to [Behrend et al. 2013, Section 1] for the precise definition of the ring of motivic weights.

Definition 8.1. [Behrend et al. 2013] In the appropriate ring of motivic weights, we define the *virtual motive of a degeneracy locus* by

$$[\mathrm{crit}^G(f)]_{\mathrm{vir}} := -\mathbb{L}^{-\frac{\dim M-\dim G}{2}} \cdot \frac{[\phi_f]}{[G]}.$$

We will try to get some property of the virtual motive of the CS function $f_{\boldsymbol{d}}$. The following lemma guarantees that the main technical result [Behrend et al. 2013, Proposition 1.11] applies.

Lemma 8.2. *Let $f_{\boldsymbol{d}} : L^1_{\omega,\boldsymbol{d}} \to \mathbb{C}$ be the CS function constructed in Section 5. There is a $\mathbb{C}^*$ action on $L^1_{\omega,\boldsymbol{d}}$ such that*:

(1) *For $\lambda \in \mathbb{C}^*$, $f_{\boldsymbol{d}}(\lambda \cdot z) = \lambda f_{\boldsymbol{d}}(z)$.*

(2) *The limit $\lim_{\lambda\to 0} \lambda \cdot z$ exists in $L^1_{\omega,\boldsymbol{d}}$.*

Proof. Let us choose coordinate $z = (y_1, \dots, y_j, \dots, w_1^*, \dots, w_i^*, \dots)$ on $L^1_{\omega,\boldsymbol{d}}$ with respect to the decomposition $L^1_{\omega,\boldsymbol{d}} = L^1_{\boldsymbol{d}} \oplus (L^2_{\boldsymbol{d}})^*$. As mentioned in Remark 5.4,

$$f_{\boldsymbol{d}} = \sum_{i=1}^{\dim L^2_{\boldsymbol{d}}} a_i(\dots, y_j, \dots)\, w_i^*$$

where the a_i are polynomials in y_j. We define the $\mathbb{C}^*$ action by scaling w_i^*. The limit of the orbits of this one parameter subgroup is $L^1_{\boldsymbol{d}}$. □

Theorem 8.3. *Take X, Y and $L_{\boldsymbol{d}}$, $L_{\omega,\boldsymbol{d}}$ as before. We have the dimension reduction formula*

$$[\phi_{f_{\boldsymbol{d}}}] = -[(L^2_{\boldsymbol{d}})^*] \cdot [\mathrm{MC}(L_{\boldsymbol{d}})].$$

Proof. The existence of the $\mathbb{C}^*$ action in Lemma 8.2 implies that the Milnor fibration given by $f_{\boldsymbol{d}}$ is Zariski trivial outside the central fiber. Hence

$$[f_{\boldsymbol{d}}^{-1}(1)] = \frac{[L^1_{\omega,\boldsymbol{d}}] - [f_{\boldsymbol{d}}^{-1}(0)]}{(\mathbb{L}-1)}.$$

Furthermore, Lemma 8.2 together with [Behrend et al. 2013, Proposition 1.11] implies that

$$[\phi_{f_{\boldsymbol{d}}}] = [f_{\boldsymbol{d}}^{-1}(1)] - [f_{\boldsymbol{d}}^{-1}(0)].$$

Recall that

$$f_{\boldsymbol{d}} = \sum_{i=1}^{r} a_i(y_1, \dots, y_j, \dots) \cdot w_i^*,$$

where $r = \dim L^2_{\boldsymbol{d}}$. We can stratify $L^1_{\boldsymbol{d}}$ by the union of $\{a_i = 0 \mid i = 1, \dots, r\}$ and its complement. The first subscheme is nothing but $\mathrm{MC}(L_{\boldsymbol{d}})$. Using this stratification, we obtain

$$\begin{aligned}[f_{\boldsymbol{d}}^{-1}(0)] &= [(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + ([L^1_{\boldsymbol{d}}] - [\mathrm{MC}(L_{\boldsymbol{d}})])[(L^2_{\boldsymbol{d}})^*]\mathbb{L}^{-1} \\ &= (1-\mathbb{L}^{-1})[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + \mathbb{L}^{-1}[L^1_{\omega,\boldsymbol{d}}].\end{aligned}$$

Then we obtain the formula for $[\phi_{f_{\boldsymbol{d}}}]$:

$$\begin{aligned}(10)\qquad [\phi_{f_{\boldsymbol{d}}}] &= [f_{\boldsymbol{d}}^{-1}(1)] - [f_{\boldsymbol{d}}^{-1}(0)] = -[f^{-1}(0)]\frac{\mathbb{L}}{\mathbb{L}-1} + \frac{[L^1_{\omega,\boldsymbol{d}}]}{\mathbb{L}-1} \\ &= -\left(\frac{\mathbb{L}-1}{\mathbb{L}}[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})] + \frac{L^1_{\omega,\boldsymbol{d}}}{\mathbb{L}}\right)\frac{\mathbb{L}}{\mathbb{L}-1} + \frac{[L^1_{\omega,\boldsymbol{d}}]}{\mathbb{L}-1} \\ &= -[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})].\end{aligned}$$

□

9. Virtual motives of the moduli space of framed representations

In this section, we will compute the virtual motive of the moduli space of framed representations, which is a noncommutative analogue of Hilbert schemes. The main result is the formula

$$Z(t) = \frac{C(\mathbb{L}^{\frac{1}{2}}t)}{C(\mathbb{L}^{-\frac{1}{2}}t)}.$$

where $C(\mathbb{L}^{\frac{1}{2}}t)$ is a generating series defined in (16).

Using the Chern–Simons function we obtained, this is a straightforward generalization of the work in [Behrend et al. 2013] in the case of $\mathbb{C}^3$. The same calculation is also obtained independently by Morrison [2012].

We fix the following notations for motives:

$$[d]_{\mathbb{L}}! := (\mathbb{L}^d - 1)(\mathbb{L}^{d-1} - 1)\cdots(\mathbb{L} - 1), \qquad [\boldsymbol{d}]_{\mathbb{L}}! := \prod_{i=0}^{n} [d_i]_{\mathbb{L}}!,$$

$$\begin{bmatrix} d \\ d' \end{bmatrix} := \frac{[d]_{\mathbb{L}}!}{[d-d']_{\mathbb{L}}!\,[d']_{\mathbb{L}}!}, \qquad \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} := \prod_{i=0}^{n} \begin{bmatrix} d_i \\ d_i' \end{bmatrix}.$$

Let $\mathrm{GL}_{\boldsymbol{d}} = \prod_{i=0}^{n} \mathrm{GL}_{d_i}$ and $\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}} = \prod_{i=0}^{n} \mathrm{Gr}(d_i', d_i)$. It is easy to show that

$$[\mathrm{GL}_{\boldsymbol{d}}] = \mathbb{L}^{\sum_{i=0}^{n}\binom{d_i}{2}}[\boldsymbol{d}]_{\mathbb{L}}! \quad \text{and} \quad [\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}}] = \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}}.$$

Definition 9.1. Consider the quiver $\mathcal{Q}_\omega$ defined in the previous section. Given a dimension vector $\boldsymbol{d}$, let $V_0, \dots, V_n$ be the sequence of vector spaces of dimensions $d_0, \dots, d_n$ over the nodes. A *framed representation* V of $\mathcal{Q}_\omega$ with dimension vector $\boldsymbol{d}$ is a representation of $\mathcal{Q}_\omega$ together with a vector $v = (v_0, \dots, v_n)$ such that $v_i \in V_i$. A framed representation V is called *cyclic* if $v_0, \dots, v_n$ generate V.

Denote the submodule generated by v by M_v, and let

$$Y_{\boldsymbol{d}} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \dots \times V_n \mid f_{\boldsymbol{d}} = 0\},$$
$$Z_{\boldsymbol{d}} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \dots \times V_n \mid f_{\boldsymbol{d}} = 1\}.$$

Then $Y_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Y_{\boldsymbol{d}}^{\boldsymbol{d}'}$ and $Z_{\boldsymbol{d}} = \bigsqcup_{\boldsymbol{d}'<\boldsymbol{d}} Z_{\boldsymbol{d}}^{\boldsymbol{d}'}$, where

$$Y_{\boldsymbol{d}}^{\boldsymbol{d}'} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \dots \times V_n \mid f_{\boldsymbol{d}} = 0, \mathrm{cl}(M_v) = \boldsymbol{d}'\},$$
$$Z_{\boldsymbol{d}}^{\boldsymbol{d}'} = \{(A, v) \in L^1_{\omega,\boldsymbol{d}} \times V_0 \times \dots \times V_n \mid f_{\boldsymbol{d}} = 1, \mathrm{cl}(M_v) = \boldsymbol{d}'\}.$$

Now, write $w_{\boldsymbol{d}} = [Y_{\boldsymbol{d}}] - [Z_{\boldsymbol{d}}]$ and $w_{\boldsymbol{d}}^{\boldsymbol{d}'} = [Y_{\boldsymbol{d}}^{\boldsymbol{d}'}] - [Z_{\boldsymbol{d}}^{\boldsymbol{d}'}]$.

Let $|\boldsymbol{d}| = \sum_{i=0}^{n} d_i$. By Theorem 8.3, we have

$$w_{\boldsymbol{d}} = \mathbb{L}^{|\boldsymbol{d}|}[(L^2_{\boldsymbol{d}})^*][\mathrm{MC}(L_{\boldsymbol{d}})]. \tag{11}$$

There is a projection from $Y^{\boldsymbol{d}'}_{\boldsymbol{d}}$ to the Grassmannian $\mathrm{Gr}_{\boldsymbol{d}',\boldsymbol{d}}$, whose fiber is the set

$$\left\{\left(\left(\begin{smallmatrix} A^0 & A' \\ 0 & A^1 \end{smallmatrix}\right), v\right) \;\middle|\; f_{\boldsymbol{d}} = 0\right\},$$

where A^0 are matrices of size $\boldsymbol{d}' \times \boldsymbol{d}'$ (depending on the source and target vertices), A^1 are matrices of size $(\boldsymbol{d}-\boldsymbol{d}') \times (\boldsymbol{d}-\boldsymbol{d}')$ and A' are matrices of size $\boldsymbol{d}' \times (\boldsymbol{d}-\boldsymbol{d}')$. There is an embedding of $L^1_{\omega,\boldsymbol{d}'} \times L^1_{\omega,\boldsymbol{d}-\boldsymbol{d}'}$ into $L^1_{\omega,\boldsymbol{d}}$ by mapping to block diagonal matrices.

The CS function $f_{\boldsymbol{d}}$ satisfies

$$f_{\boldsymbol{d}}\left(\left(\begin{smallmatrix} A^0 & A' \\ 0 & A^1 \end{smallmatrix}\right), v\right) = f_{\boldsymbol{d}'}(A^0, v) + f_{\boldsymbol{d}-\boldsymbol{d}'}(A^1, v).$$

Denote the subgroup of $\mathrm{GL}_{\boldsymbol{d}}$ that preserves these Borel matrices by $B_{\boldsymbol{d},\boldsymbol{d}'}$ and the Euler form of $\mathcal{Q}_\omega$ by χ.

$$\begin{aligned}[Y^{\boldsymbol{d}'}_{\boldsymbol{d}}] = \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} &([Y^{\boldsymbol{d}'}_{\boldsymbol{d}'}] \cdot [Y_{\boldsymbol{d}-\boldsymbol{d}'}] \\ &+ (\mathbb{L}-1) \cdot [Z^{\boldsymbol{d}'}_{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}]) \cdot \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|}.\end{aligned}$$

A similar analysis yields

$$\begin{aligned}[Z^{\boldsymbol{d}'}_{\boldsymbol{d}}] = \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} &([Y^{\boldsymbol{d}'}_{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}] \\ &+ (\mathbb{L}-2) \cdot [Z^{\boldsymbol{d}'}_{\boldsymbol{d}'}] \cdot [Z_{\boldsymbol{d}-\boldsymbol{d}'}] + [Z^{\boldsymbol{d}'}_{\boldsymbol{d}'}] \cdot [Y_{\boldsymbol{d}-\boldsymbol{d}'}]) \cdot \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|}.\end{aligned}$$

The above formulas, combined with (11), yield

$$\begin{aligned} w^{\boldsymbol{d}'}_{\boldsymbol{d}} &= \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \mathbb{L}^{-|\boldsymbol{d}-\boldsymbol{d}'|} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} (w^{\boldsymbol{d}'}_{\boldsymbol{d}'} \cdot w_{\boldsymbol{d}-\boldsymbol{d}'}) \\ &= \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}][(L^2_{\boldsymbol{d}-\boldsymbol{d}'})^*]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} [\mathrm{MC}(L_{\boldsymbol{d}-\boldsymbol{d}'})] \cdot w^{\boldsymbol{d}'}_{\boldsymbol{d}'}. \end{aligned} \tag{12}$$

Because $Y_{\boldsymbol{d}} = \bigsqcup\limits_{\boldsymbol{d}'<\boldsymbol{d}} Y^{\boldsymbol{d}'}_{\boldsymbol{d}}$ and $Z_{\boldsymbol{d}} = \bigsqcup\limits_{\boldsymbol{d}'<\boldsymbol{d}} Z^{\boldsymbol{d}'}_{\boldsymbol{d}}$, we get

$$w^{\boldsymbol{d}}_{\boldsymbol{d}} = w_{\boldsymbol{d}} - \sum_{\boldsymbol{d}'<\boldsymbol{d}} w^{\boldsymbol{d}'}_{\boldsymbol{d}}.$$

Let $\tilde{c}_{\boldsymbol{d}} = [\mathrm{MC}(L_{\boldsymbol{d}})]/[\mathrm{GL}_{\boldsymbol{d}}]$. Applying (11) and (12), we obtain the recursion formula

$$
\begin{aligned}
(13)\quad w_{\boldsymbol{d}}^{\boldsymbol{d}} &= \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*][\mathrm{MC}(L_{\boldsymbol{d}})] \\
&\quad - \sum_{\boldsymbol{d}'<\boldsymbol{d}} \frac{[B_{\boldsymbol{d},\boldsymbol{d}'}][(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]}{[\mathrm{GL}_{\boldsymbol{d}'}][\mathrm{GL}_{\boldsymbol{d}-\boldsymbol{d}'}]} \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d}' \end{bmatrix}_{\mathbb{L}} [\mathrm{MC}(L_{\boldsymbol{d}-\boldsymbol{d}'})] \cdot w_{\boldsymbol{d}'}^{\boldsymbol{d}'} - \frac{[\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}}]} \\
&= \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} + \sum_{\boldsymbol{d}'<\boldsymbol{d}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*] \cdot \mathbb{L}^{-\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')} \tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot \frac{[\phi_{f_{\boldsymbol{d}'}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}'}]}
\end{aligned}
$$

Here $f_{\boldsymbol{d}}^{\mathrm{ss}}$ is the restriction of $f_{\boldsymbol{d}}$ to the semistable loci.

Define the virtual motive of the noncommutative Hilbert scheme $\mathrm{Hilb}^{\boldsymbol{d}}$ by

$$
(14)\qquad [\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}} := -\mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})-|\boldsymbol{d}|}{2}} \frac{[\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}]}{[\mathrm{GL}_{\boldsymbol{d}}]}.
$$

After replacing $\phi_{f_{\boldsymbol{d}}^{\mathrm{ss}}}$ by $[\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}}$, subject to the above formula, we obtain

$$
\begin{aligned}
(15)\quad \mathbb{L}^{|\boldsymbol{d}|}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{-\frac{2\chi(\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')+\chi(\boldsymbol{d}',\boldsymbol{d}')}{2}} \mathbb{L}^{\frac{|\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}} \\
\mathbb{L}^{\frac{|\boldsymbol{d}|}{2}}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{-\frac{\chi(\boldsymbol{d},\boldsymbol{d})-\chi(\boldsymbol{d}-\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')}{2}} \mathbb{L}^{-\frac{|\boldsymbol{d}-\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}} \\
\mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})}{2}}\mathbb{L}^{\frac{|\boldsymbol{d}|}{2}}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} &= \sum_{\boldsymbol{d}'\le\boldsymbol{d}} \mathbb{L}^{\frac{\chi(\boldsymbol{d}-\boldsymbol{d}',\boldsymbol{d}-\boldsymbol{d}')}{2}} \mathbb{L}^{-\frac{|\boldsymbol{d}-\boldsymbol{d}'|}{2}} [(L_{\boldsymbol{d}-\boldsymbol{d}'}^2)^*]\tilde{c}_{\boldsymbol{d}-\boldsymbol{d}'} \cdot [\mathrm{Hilb}^{\boldsymbol{d}'}]_{\mathrm{vir}}
\end{aligned}
$$

Define the generating series for $\tilde{c}_{\boldsymbol{d}}$ by

$$
(16)\qquad C(t) = \sum_{\boldsymbol{d}\in\mathbb{Z}_{\ge 0}^{n+1}} \mathbb{L}^{\frac{\chi(\boldsymbol{d},\boldsymbol{d})}{2}}[(L_{\boldsymbol{d}}^2)^*]\tilde{c}_{\boldsymbol{d}} \cdot t^{\boldsymbol{d}}
$$

and the generating series of noncommutative Hilbert schemes by

$$
Z(t) = \sum_{\boldsymbol{d}\in\mathbb{Z}_{\ge 0}^{n+1}} [\mathrm{Hilb}^{\boldsymbol{d}}]_{\mathrm{vir}} \cdot t^{\boldsymbol{d}}.
$$

Then the generating series of Hilbert schemes can be written as

$$
(17)\qquad Z(t) = \frac{C(\mathbb{L}^{\frac{1}{2}}t)}{C(\mathbb{L}^{-\frac{1}{2}}t)}.
$$

Finally, notice that $\mathbb{L}^{\chi(\boldsymbol{d},\boldsymbol{d})/2}[(L_{\boldsymbol{d}}^2)^*]$ is nothing but $\mathbb{L}^{\chi_{\mathcal{Q}}(\boldsymbol{d},\boldsymbol{d})/2}$ for the Euler form of the quiver $\mathcal{Q}$. So $C(t)$ is the generating series of the moduli space of representations of $\mathcal{Q}$ (without stability).

Acknowledgments

I would like to thank Jim Bryan, Kai Behrend and Kentaro Nagao for their interest in my work, comments, conversations, and helpful correspondence. The computations in Section 9 were also obtained independently by Andrew Morrison [2012] — thanks to Andrew Morrison for kindly sharing his preprint. Section 7 was finished during my visit to Imperial college London. It is motivated by a discussion with Martijn Kool. I would like to thank Martijn Kool and Richard Thomas for their very helpful comments. I would also like to thank Dominic Joyce for pointing out a mistake in the earlier draft. This work was partially funded by RGC Early Career Grant No. 27300214.

References

[Aspinwall and Fidkowski 2006] P. S. Aspinwall and L. M. Fidkowski, "Superpotentials for quiver gauge theories", *J. High Energy Phys.* 10 (2006), 047. MR 2007g:81091

[Behrend 2009] K. Behrend, "Donaldson–Thomas type invariants via microlocal geometry", *Ann. of Math.* (2) **170**:3 (2009), 1307–1338. MR 2011d:14098 Zbl 1191.14050

[Behrend et al. 2013] K. Behrend, J. Bryan, and B. Szendrői, "Motivic degree zero Donaldson–Thomas invariants", *Invent. Math.* **192**:1 (2013), 111–160. MR 3032328 Zbl 1267.14008

[Bondal 1990] A. I. Bondal, "Helices, representations of quivers and Koszul algebras", pp. 75–95 in *Helices and vector bundles*, edited by A. N. Rudakov, London Math. Soc. Lecture Note Ser. **148**, Cambridge Univ. Press, 1990. MR 92i:14016 Zbl 0742.14010

[Borisov and Hua 2009] L. Borisov and Z. Hua, "On the conjecture of King for smooth toric Deligne–Mumford stacks", *Adv. Math.* **221**:1 (2009), 277–301. MR 2010m:14064 Zbl 1210.14006

[Bridgeland 2005] T. Bridgeland, "T-structures on some local Calabi–Yau varieties", *J. Algebra* **289**:2 (2005), 453–483. MR 2006a:14067 Zbl 1069.14044

[Denef and Loeser 2001] J. Denef and F. Loeser, "Geometry on arc spaces of algebraic varieties", pp. 327–348 in *European Congress of Mathematics, I* (Barcelona, 2000), edited by C. Casacuberta et al., Progr. Math. **201**, Birkhäuser, Basel, 2001. MR 2004c:14037 Zbl 1079.14003

[Huybrechts and Lehn 1997] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics **E31**, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR 98g:14012 Zbl 0872.14002

[Jiang 2010] Y. Jiang, "Motivic Milnor fiber of cyclic L-infinity algebras", preprint, 2010. arXiv 0909.2858

[Joyce and Song 2012] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, vol. 217, Mem. Amer. Math. Soc. **1020**, Amer. Math. Soc., Providence, RI, 2012. MR 2951762 Zbl 1259.14054

[Keller 2006] B. Keller, "A_∞-algebras, modules and functor categories", preprint, 2006. arXiv math/0510508

[Kontsevich and Soibelman 2009] M. Kontsevich and Y. Soibelman, "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry", pp. 153–219 in *Homological mirror symmetry*, edited by A. Kapustin et al., Lecture Notes in Phys. **757**, Springer, Berlin, 2009. MR 2011f:53183 Zbl 1202.81120

[Morrison 2012] A. Morrison, "Motivic invariants of quivers via dimensional reduction", *Selecta Math.* (*N.S.*) **18**:4 (2012), 779–797. MR 3000467 Zbl 1256.14058

[Reineke 2011] M. Reineke, "Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants", *Compos. Math.* **147**:3 (2011), 943–964. MR 2012i:16031 Zbl 1266.16013

[Segal 2008] E. Segal, "The A_∞ deformation theory of a point and the derived categories of local Calabi–Yaus", *J. Algebra* **320**:8 (2008), 3232–3268. MR 2009k:16016 Zbl 1168.18005

Received May 23, 2014. Revised January 27, 2015.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 277 No. 1 September 2015

0030-8730(201509)277:1;1-V