Pacific Journal of Mathematics

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Volume 277 No. 1 September 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 277, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.277.149

ON THE FLAG CURVATURE OF A CLASS OF FINSLER METRICS PRODUCED BY THE NAVIGATION PROBLEM

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One of the important approaches in discussing Finsler geometry is the navigation problem. In this paper, we determine the flag curvature of a Finsler metric produced from *any* Finsler metric and *any* conformal field in terms of the navigation problem, and therefore we provide a unifying framework for the fundamental equations due to Bao, Robles, and Shen, Cheng and Shen, Foulon, and Mo and Huang.

1. Introduction

The navigation problem (or shortest time problem [Shen 2003]) was first studied by E. Zermelo [1931]. Bao, Robles, and Shen [Bao et al. 2004] classified Randers metrics of constant flag curvature via the navigation problem on Riemannian manifolds. Flag curvature is an important quantity in Finsler geometry because it takes the place of sectional curvature in the Riemannian case [Bao and Chern 1993]. The complete classification of Randers metrics of constant flag curvature, due to Bao, Robles, and Shen, is motivated by the following result [Bao et al. 2004; Chern and Shen 2005]:

Theorem. *A Randers metric F is of constant flag curvature $K = \lambda$ if and only if* (i) *h has constant sectional curvature $\mu = \lambda + c^2$ and* (ii) *V is a homothetic field of h with dilation c, where (h, V) is the navigation data of F.*

Condition (ii) is equivalent to F having constant S-curvature [Shen and Xing 2008; Xing 2005]. Recently, Cheng and Shen [2009] established a relationship between the flag curvature of F and h for a Randers metric F of isotropic S-curvature (see also [Chern and Shen 2005]), generalizing the flag curvature nonincreasing equation of [Bao et al. 2004]. More generally, they obtained a relationship between the Riemann curvature of F and h. Using this, they locally classified Randers metrics of scalar flag curvature with isotropic S-curvature [Cheng and Shen 2009;

Xiaohuan Mo is the corresponding author. Huang was supported by the National Natural Science Foundation of China 11301283. Mo was supported by the National Natural Science Foundation of China 11371032.
MSC2010: 58E20.
Keywords: Finsler metric, flag curvature, conformal field, navigation problem.

Chern and Shen 2005; Shen 2004]. Mo [2008] gave a global classification for these metrics on compact manifolds by using a formula of Cheng and Shen [2009]. It is worth mentioning the recent result by Xing [Shen and Xing 2008] that a Randers metric F is of isotropic S-curvature $S = (n+1)c(x)F$ if and only if V is conformal with respect to h. The theorems of Cheng and Shen [2003] and Mo generalize results previously only known in the case of locally projectively flat Randers metrics with isotropic S-curvature. Recall that all locally projectively flat Finsler metrics are of scalar curvature [Chern and Shen 2005, Proposition 6.1.3].

Recall that a vector field V on a Finsler manifold (M, F) is called *conformal with dilation* $c(x)$ if its flow Φ_t satisfies

$$F(\Phi_t(x), \Phi_{t*}(y)) = e^{2\sigma_t(x)} F(x, y), \quad \forall x \in M,\ y \in T_x M,$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$ [Shen and Xia 2012; Huang and Mo 2013]. In particular, V is called a *homothetic* field if c is constant, and V is called a *Killing* field if $c = 0$ [Huang and Mo 2011; Mo and Hang 2007].

At the 2004 International Conference on Riemann–Finsler Geometry at Nankai University, P. Foulon announced that if F is a Finsler metric and V is a Killing field, then F and $\widetilde{F}$ have the same flag curvature. Mo and Huang [2007] studied the navigation problem for any Finsler metric F and any homothetic field (for instance, the Funk metric on a strongly convex domain) in the spirit of the flag curvature nonincreasing equation of Bao, Robles, and Shen and the announcement of P. Foulon. They showed that for a homothetic field, the navigation representation satisfies the flag curvature nonincreasing equation. In particular, the navigation problem has the flag curvature preserving property for a Killing field. Applying this result, Hu and Deng [2012] established a principle to classify homogeneous Randers spaces with (almost) isotropic S-curvature and positive flag curvature, and then they gave a complete classification of these homogeneous Randers spaces.

In this paper, we provide a unifying framework for [Bao et al. 2004; Cheng and Shen 2009; Mo and Hang 2007]. We study the Finsler metric $\widetilde{F}$ produced from *any* Finsler metric F and *any* conformal field V in terms of the shortest time problem and give the relation between the flag curvatures of F and $\widetilde{F}$. Precisely we show the following:

Theorem 1.1. *Let $F = F(x, y)$ be a Finsler metric on a manifold M with Cartan torsion A and V be a vector field on M with $F(x, V_x) < 1$. Let $\widetilde{F} = \widetilde{F}(x, y)$ denote the Finsler metric on M defined in* (2-2). *Suppose that V is conformal with dilation $c(x)$. Then the flag curvatures of $\widetilde{F}$ and F are related by*

$$K_{\widetilde{F}}(y, y \wedge u) - \left[3\frac{y^i c_{x^i}}{\widetilde{F}(x, y)} - c^2 + 2V(c)\right] = K_F(\tilde{y}, \tilde{y} \wedge u) - 2\frac{A_{(x,[\tilde{y}])}(u, \nabla c, u)}{h_{(x,[\tilde{y}])}(u, u)},$$

where $\tilde{y} = y + F(x, \tilde{y})V$ and h is the angular metric of F.

For the definition of a conformal field V with dilation $c(x)$, see Section 2. In Theorem 1.1, we denote the partial derivative $\partial c/\partial x^i$ by c_{x^i}. The case where F is a Riemannian manifold implies a formula of Cheng and Shen [2009], whilst V is homothetic implies the curvature nonincreasing equation of Mo and Huang [2007]. In particular, if $\widetilde{F}$ has constant flag curvature and is of Randers type, our formula has been obtained by Bao, Robles, and Shen [2004].

Our approach to proving Theorem 1.1 is partially in the contact geometry [Blair 2002]. Recall that a Finsler metric is Riemannian if and only if its Cartan torsion vanishes [Chern and Shen 2005].

As an application of Theorem 1.1, we determine the flag curvature of a Finsler metric produced by a generalized Poincaré metric and its nonhomothetic conformal field via the navigation problem (see Section 5).

Finally, we should point out that very recently [Shen and Xia 2012; Xia 2013] established the relationship between the flag curvatures of $\widetilde{F}$ and F, where F is a Randers metric with some special curvature properties and $\widetilde{F}$ is produced from (F, V) via the navigation problem, where V is a conformal field.

2. Preliminaries

Let (M, F) be a Finsler manifold with Hilbert form ω. Let SM be the projective sphere bundle of M, obtained from TM by identifying nonzero vectors which differ from each other by a positive multiplicative factor. It is easy to verify that

$$\omega \wedge (d\omega)^{n-1} \neq 0, \quad n = \dim M.$$

That is, ω defines a contact structure on SM [Chern 1996]. Hence there is a unique vector field X on SM that satisfies $\omega(X) = 1$ and $X \lrcorner (d\omega) = 0$. This vector field X is known as the *Reeb vector field* [Blair 2002; Bryant 2002; Huang and Mo 2011].

Every vector $y \in T_xM\backslash\{0\}$ uniquely determines a covector $p \in T_x^*M\backslash\{0\}$ by

$$p(u) := \frac{1}{2}\frac{d}{dt}\big(F^2(x, y+tu)\big)\Big|_{t=0}, \quad u \in T_xM.$$

The resulting map $L_x^F : y \in T_xM \to p \in T_x^*M$ is called the *Legendre transformation* at x.

Define a nonnegative scalar function $H = H(x, p)$ by

$$H(x, p) := \max_{y \in T_xM\backslash\{0\}} \frac{p(y)}{F(x, y)}. \tag{2-1}$$

The function H is C^∞ on $T^*M\backslash\{0\}$ and $H_x := H|_{T_x^*M}$ is a Minkowski norm on T_x^*M for $x \in M$. Such a function is called a *Cartan metric* [Miron et al. 2001; Mo and Hang 2007] (or co-Finsler metric [Shen 2004; 2002]). The pair (M, H) is called a *Cartan manifold*.

Every covector $p \in T_x^*M\backslash\{0\}$ uniquely determines a vector $y \in T_xM\backslash\{0\}$ by

$$q(y) := \frac{1}{2}\frac{d}{dt}\big(H^2(x, p+tq)\big)\Big|_{t=0}, \quad q \in T_x^*M.$$

The resulting map $L_x^{F*} : p \in T_x^*M \to y \in T_xM$ is called the *inverse Legendre transformation* at x. Indeed L_x^F and L_x^{F*} are inverses of each other. Moreover, they preserve the Minkowski norms $H(x, p) = F(x, L_x^{F*}p)$.

Recently, one of the important approaches in discussing Finsler metrics is the (Zermelo) navigation problem [Bao et al. 2004; Hu and Deng 2012; Huang and Mo 2011; Shen 2003; Zermelo 1931; Xia 2013]. The main technique of the navigation problem is described as follows. Given a Finsler metric F and a vector field V with $F(x, V_x) < 1$, define a new Finsler metric $\widetilde{F}$ by

$$F\bigg(x, \frac{y}{\widetilde{F}(x,y)} + V_x\bigg) = 1, \quad \forall x \in M,\ y \in T_xM. \tag{2-2}$$

A *(local) flow* (or *local one-parameter group*) on a manifold M is a map $\Phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\Phi_t := \Phi(t, \cdot)$, satisfying

- $\Phi_0 = \mathrm{id} : M \to M$;
- $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow Φ_t on M is a flow $\hat{\Phi}_t$ on T^*M,

$$\hat{\Phi}_t(x, p) := \big(\Phi_t(x), (\Phi_t^*)^{-1}(p)\big). \tag{2-3}$$

By the relationship between vector fields and flows, (2-3) induces a natural way a lift of a vector field V on M to a vector field X_V^* on T^*M.

A vector field V on a Finsler manifold (M, F) is called *conformal with dilation* $c(x)$ if its flow Φ_t satisfies

$$F\big(\Phi_t(x), \Phi_{t*}(y)\big) = e^{2\sigma_t(x)}F(x, y), \quad \forall x \in M,\ y \in T_xM, \tag{2-4}$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$ [Shen and Xia 2012]. In particular, V is called a *homothetic* field if c is constant.

Similarly, a vector field V on a Cartan manifold (M, H) is called *conformal with dilation* $c(x)$ if its flow Φ_t is a conformal transformation on (M, H), i.e.,

$$H\big(\Phi_t(x), (\Phi_t^*)^{-1}(p)\big) = e^{-2\sigma_t(x)}H(x, p), \quad \forall x \in M,\ p \in T_x^*M, \tag{2-5}$$

where $c(x) = [d\sigma_t(x)/dt]_{t=0}$.

Lemma 2.1. *Let V be a conformal field on a Finsler manifold (M, F) with dilation $c(x)$ and H its Cartan metric defined by* (2-1). *Then V is a conformal field of H with dilation $c(x)$.*

Proof. By using (2-1) and (2-4) we have

$$\begin{aligned} H\big(\Phi_t(x),(\Phi_t^*)^{-1}(p)\big) &= \max_{\tilde{y}\in T_{\Phi_t(x)}M\setminus\{0\}} \frac{[(\Phi_t^*)^{-1}(p)](\tilde{y})}{F(\Phi_t(x),\tilde{y})} \\ &= \max_{\tilde{y}\in T_{\Phi_t(x)}M\setminus\{0\}} \frac{p((\Phi_{t*})^{-1}(\tilde{y}))}{F(\Phi_t(x),\tilde{y})} \\ &= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(\Phi_t(x),\Phi_{t*}(y))} \\ &= \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{e^{2\sigma_t(x)}F(x,y)} \\ &= e^{-2\sigma_t(x)} \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(x,y)} = e^{-2\sigma_t(x)}H(x,p), \end{aligned}$$

where $y:=(\Phi_{t*})^{-1}(\tilde{y})$. The lemma follows. □

The Hilbert form $\omega^\flat$ of the co-Finsler metric H is given by

$$\omega^\flat = \frac{p}{H} \tag{2-6}$$

[Mo and Hang 2007]. Let S^*M be the cosphere bundle of M and $\pi: S^*M \to M$ the natural projection. We call $\operatorname{Ker}\pi_*$ the *vertical distribution* of S^*M, denoted by VS^*M.

Lemma 2.2. *For an arbitrary function $f \in C^\infty(S^*M)$, there is a unique vector field X_f on S^*M satisfying*

$$\omega^\flat(X_f) = f, \quad X_f \lrcorner (d\omega^\flat) = -df + X^\flat(f)\omega^\flat. \tag{2-7}$$

This vector field X_f is called the Reeb field associated with f.

Proof. The Hilbert form $\omega^\flat$ defines a contact structure on S^*M. By using [Blair 2002, Theorem 4.4], there exists an almost contact metric structure $(\Phi, X^\flat, \omega^\flat, g)$ such that $g(X,\Phi Y) = d\omega^\flat(X,Y)$. A direct computation tells us that the second equation of (2-7) is equivalent to $\mathcal{L}_{X_f}\omega^\flat = X^\flat(f)\omega^\flat$. Together with [loc. cit., Theorem 5.7], we have $X_f = -\Phi Df + fX^\flat$, where $g(Df,Y) = Y(f)$. □

Remark. (i) It is easy to see that $X_1 = X^\flat$ is known as the Reeb vector field.

(ii) Let $\{e_\alpha, X^\flat, e_{\bar\alpha}\}$ be a locally orthonormal frame on S^*M such that $e_\alpha \in HS^*M$ (see (2-10) below) and $e_{\bar\alpha} \in VS^*M$. By using (2-7), we have

$$X_f = fX^\flat + \Sigma_\alpha e_{\bar\alpha}(f)e_\alpha - \Sigma_\alpha e_\alpha(f)e_{\bar\alpha}.$$

By the definition of VS^*M, we have $e_{\bar\alpha}(f) = 0$ for $f \in C^\infty(M)$. It follows that

$$Y_f := X_f - fX^\flat = -\Phi Df \in VS^*M \quad \text{for } f \in C^\infty(M). \tag{2-8}$$

(iii) Note that the $d\omega^\flat$ adopted here differs from that of D. E. Blair [2002], where $d\omega^\flat$ is defined by

$$d\omega^\flat(X,Y)=\tfrac{1}{2}\big(X(\omega^\flat(Y))-Y(\omega^\flat(X))-\omega^\flat([X,Y])\big).$$

In the same work, X_f is called an *infinitesimal contact transformation*.

Let F be a Finsler metric and $\widetilde{F}$ denote the Finsler metric defined in (2-2). With the help of the inverse Legendre transformation at x, we obtain co-Finsler metrics $H(x,p)$ and $\widetilde{H}(x,p)$ respectively. Then H and $\widetilde{H}$ are related by

$$\widetilde{H}(x,p)=H(x,p)-p(V) \tag{2-9}$$

[Mo and Hang 2007]. Furthermore the Hilbert form $\tilde{\omega}^\flat$ of the co-Finsler metric $\widetilde{H}$ satisfies $\tilde{\omega}^\flat=p/\widetilde{H}$. Taking this together with (2-6), we obtain $\operatorname{Ker}\omega^\flat=\operatorname{Ker}\tilde{\omega}^\flat$. The vertical endomorphism $\mathcal{V}^\flat$ is characterized by

$$\mathcal{V}^\flat(v)=0,\quad \mathcal{V}^\flat(X^\flat)=0,\quad \mathcal{V}^\flat[X^\flat,v]=-v,\qquad \forall\, v\in VS^*M.$$

The horizontal endomorphism $\mathcal{H}^\flat$ is given by

$$\mathcal{H}^\flat(v)=-[X^\flat,v]-\tfrac{1}{2}\mathcal{V}^\flat[X^\flat,[X^\flat,v]],\quad \mathcal{H}^\flat(X^\flat)=0,\quad \mathcal{H}^\flat(\mathcal{H}^\flat(v))=0$$

for $v\in VS^*M$. The horizontal distribution of S^*M is defined by

$$HS^*M=\mathcal{H}^\flat(VS^*M). \tag{2-10}$$

It is easy to see that

$$TS^*M=HS^*M\oplus VS^*M\oplus\operatorname{Span}\{X^\flat\}=\operatorname{Ker}\omega^\flat\oplus\operatorname{Span}\{X^\flat\}.$$

We denote the projection to VS^*M (resp. HS^*M) by $P^\flat_{\mathcal{V}}:=\mathcal{V}^\flat\circ\mathcal{H}^\flat$ (resp. $P^\flat_{\mathcal{H}}:=\mathcal{H}^\flat\circ\mathcal{V}^\flat$). Define the Riemann tensor of $\mathcal{R}^\flat$ by

$$\mathcal{R}^\flat(v)=\mathcal{V}^\flat\circ\mathcal{H}^\flat[X^\flat,\mathcal{H}^\flat(v)],\quad v\in VS^*M. \tag{2-11}$$

Then the flag curvature $K^\flat$ is given by

$$K^\flat(v)=\frac{h^\flat(\mathcal{R}^\flat(v),v)}{h^\flat(v,v)},\quad v\in VS^*M\backslash\{0\}, \tag{2-12}$$

where $h^\flat$ is the angular metric on VS^*M which satisfies

$$h^\flat(v,v)=d\omega^\flat([X^\flat,u],v)=d\omega^\flat(u,\mathcal{H}^\flat(v)).$$

The Cartan torsion $A^\flat$ is characterized by

$$2A^\flat(u,v,w)=u(d\omega^\flat([X^\flat,v],w)+d\omega^\flat([u,[X^\flat,v]],w)+d\omega^\flat([u,[X^\flat,w]],v))$$

for $u, v, w \in VS^*M$. We require the following result in Lemma 3.5, the proof of which is omitted.

Lemma 2.3. *There is a unique affine connection $\nabla : VS^*M \times VS^*M \to VS^*M$ satisfying*

$$\nabla_u v = \mathcal{V}^\flat[u, \mathcal{H}^\flat(v)], \quad \nabla_u v - \nabla_v u = [u, v], \quad (\nabla_u h^\flat)(v, w) = 2A^\flat(u, v, w)$$

for $u, v, w \in VS^*M$.

The following lemma will be used in Section 4.

Lemma 2.4 [Mo and Hang 2007]. *Assume that Cartan metrics H and $\widetilde{H}$ are related by* (2-9). *Then vertical endomorphisms $\mathcal{V}^\flat$ and $\widetilde{\mathcal{V}}^\flat$ are related by $\mathcal{V}^\flat = \widetilde{\mathcal{V}}^\flat - \widetilde{\mathcal{V}}^\flat(X_V^*) \otimes \omega^\flat$, where X_V^* is the left of V on T^*M.*

3. Conformal transformations

In this section, we establish some properties for a conformal transformation on a Cartan manifold required in next section. For the definition of conformal transformation, see (2-5) above.

Lemma 3.1. *Let φ be a conformal transformation on a Cartan manifold (M, H), i.e., $\hat{\varphi}^* H = e^{-2\sigma(x)} H$, where $\hat{\varphi}(x, p) = (\varphi(x), (\varphi^*)^{-1}(p))$. Then*

$$\hat{\varphi}_* X^\flat = X_{e^{2\sigma(x)}},$$

where $X^\flat$ denotes the Reeb field of H.

Proof. By (2-5) and (2-6), we have

$$\hat{\varphi}^* \omega^\flat = e^{2\sigma(x)} \omega^\flat. \tag{3-1}$$

Hence $\hat{\varphi} : S^*M \to S^*M$ is a contact transformation [Blair 2002]. It follows that

$$\omega^\flat(\hat{\varphi}_* X^\flat) = (\hat{\varphi}^* \omega^\flat) X^\flat = e^{2\sigma(x)} \omega^\flat(X^\flat) = e^{2\sigma(x)}$$

and

$$\begin{aligned}
\hat{\varphi}_* X^\flat \lrcorner (d\omega^\flat) &= X^\flat \lrcorner (\hat{\varphi}^* d\omega^\flat) \\
&= X^\flat \lrcorner [d(\hat{\varphi}^* \omega^\flat)] \\
&= X^\flat \lrcorner [d(e^{2\sigma(x)} \omega^\flat)] \\
&= X^\flat \lrcorner [de^{2\sigma(x)} \wedge \omega^\flat + e^{2\sigma(x)} d\omega^\flat] \\
&= de^{2\sigma(x)}(X^\flat)\omega^\flat - \omega^\flat(X^\flat) de^{2\sigma(x)} + e^{2\sigma(x)} X^\flat \lrcorner (d\omega^\flat) \\
&= -de^{2\sigma(x)} + X^\flat(e^{2\sigma(x)})\omega^\flat.
\end{aligned}$$

The lemma follows from the uniqueness of the Reeb field associated with $e^{2\sigma(x)}$. □

Proposition 3.2. *Let φ be a conformal transformation on a Cartan manifold (M, H), i.e., $\hat{\varphi}^* H = e^{-2\sigma(x)} H$. Then $\hat{\varphi}_* X^\flat = e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)})$, where $Y_{\sigma(x)}$ is defined in* (2-8).

Proof. By virtue of (2-8), we conclude that

$$Y_{e^{2\sigma(x)}} = -\Phi D e^{2\sigma(x)} = 2e^{2\sigma(x)}(-\Phi D\sigma(x)) = 2e^{2\sigma(x)} Y_{\sigma(x)}.$$

It follows that

$$\begin{aligned}\hat{\varphi}_* X^\flat &= X_{e^{2\sigma(x)}} \\ &= Y_{e^{2\sigma(x)}} + e^{2\sigma(x)} X^\flat \\ &= 2e^{2\sigma(x)} Y_{\sigma(x)} + e^{2\sigma(x)} X^\flat = e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}).\end{aligned}$$ □

Lemma 3.3. *For a conformal transformation φ on a Cartan manifold (M, H), we have*

$$\hat{\varphi}_* \circ \mathcal{V}^\flat = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*.$$

Proof. For $v \in VS^*M$ and $\hat{\varphi}_* v \in VS^*M$, it follows that

$$\hat{\varphi}_* \circ \mathcal{V}^\flat(v) = 0 = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(v).$$

Similarly, from (i) we have $\hat{\varphi}_* \circ \mathcal{V}^\flat(X^\flat) = e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(X^\flat)$. For $u \in HS^*M$, we write $u = \mathcal{H}^\flat(v)$, where $v \in VS^*M$. Then

$$\begin{aligned}\hat{\varphi}_* \circ \mathcal{V}^\flat(u) &= \hat{\varphi}_* \circ \mathcal{V}^\flat(-[X^\flat, v] - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat, [X^\flat, v]]) \\ &= -\hat{\varphi}_* \circ \mathcal{V}^\flat[X^\flat, v] - \tfrac{1}{2}\hat{\varphi}_* \circ \mathcal{V}^\flat \circ \mathcal{V}^\flat[X^\flat, [X^\flat, v]] = \hat{\varphi}_* v,\end{aligned}$$

and

$$\begin{aligned}e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(u) &= e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(\mathcal{H}^\flat(v)) \\ &= e^{-2\sigma(x)} \mathcal{V}^\flat \circ \hat{\varphi}_*(-[X^\flat, v] - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat, [X^\flat, v]]) \\ &= -e^{-2\sigma(x)} \mathcal{V}^\flat[\hat{\varphi}_* X^\flat, \hat{\varphi}_* v] \\ &= -e^{-2\sigma(x)} \mathcal{V}^\flat[e^{2\sigma(x)} X^\flat, \hat{\varphi}_* v] \\ &= -e^{-2\sigma(x)} e^{2\sigma(x)} \mathcal{V}^\flat[X^\flat, \hat{\varphi}_* v] = \hat{\varphi}_* v.\end{aligned}$$ □

Lemma 3.4. *Write $X^\flat(f) = \dot{f}$ for an arbitrary function $f \in C^\infty(M)$. Then*

$$[X^\flat, X_f] = X_{\dot{f}}.$$

Proof. Simple calculations give

$$\omega^\flat([X^\flat, X_f]) = \dot{f}, \quad [X^\flat, X_f] \lrcorner (d\omega^\flat) = -d\dot{f} + \ddot{f}\omega^\flat.$$

The lemma now follows from the uniqueness of the Reeb field associated with $\dot{f}$. □

Lemma 3.5. *If $f \in C^\infty(M)$ and $v \in VS^*M$, then*

$$\mathcal{V}^\flat[X_{\dot f}, v] = -2A^\flat(Y_f, v), \tag{3-2}$$

where $h^\flat(A^\flat(Y_f, v), u) := A^\flat(v, Y_f, u)$.

Proof. By (2-8) and Lemma 3.4, we have

$$\begin{aligned}\mathcal{V}^\flat X_{\dot f} &= \mathcal{V}^\flat[X^\flat, X_f]\\ &= \mathcal{V}^\flat[X^\flat, Y_f + fX^\flat]\\ &= \mathcal{V}^\flat[X^\flat, Y_f] + \mathcal{V}^\flat[X^\flat(f)X^\flat]\\ &= -Y_f + X^\flat(f)\mathcal{V}^\flat(X^\flat) = -Y_f.\end{aligned} \tag{3-3}$$

Note that $[P^\flat_{\mathcal{V}} X_{\dot f}, v] \in VS^*M$. It follows that

$$\mathcal{V}^\flat[P_{\mathcal{V}} X_{\dot f}, v] = 0. \tag{3-4}$$

Together with (2-7) and (3-3), we obtain

$$\begin{aligned}\mathcal{V}^\flat[X_{\dot f}, v] &= \mathcal{V}^\flat[\dot f X^\flat + P^\flat_{\mathcal{H}} X_{\dot f} + P^\flat_{\mathcal{V}} X_{\dot f}, v]\\ &= \mathcal{V}^\flat[\dot f X^\flat + \mathcal{H}^\flat \circ \mathcal{V}^\flat X_{\dot f}, v]\\ &= \mathcal{V}^\flat[\dot f X^\flat - \mathcal{H}^\flat Y_f, v] = \mathcal{V}^\flat[\dot f X^\flat, v] - \mathcal{V}^\flat[\mathcal{H}^\flat Y_f, v].\end{aligned} \tag{3-5}$$

On the other hand,

$$\mathcal{V}^\flat[\dot f X^\flat, v] = -v(\dot f)\mathcal{V}^\flat(X^\flat) + \dot f \mathcal{V}^\flat[X^\flat, v] = -\dot f v.$$

Plugging this into (3-5) yields $\mathcal{V}^\flat[X_{\dot f}, v] = -\dot f v + \mathcal{V}^\flat[v, \mathcal{H}^\flat Y_f]$. It follows that

$$\begin{aligned}h^\flat(\mathcal{V}^\flat[X_{\dot f}, v], u) &= -\dot f h^\flat(v, u) + h^\flat(\mathcal{V}^\flat[v, \mathcal{H}^\flat Y_f], u)\\ &= -\dot f h^\flat(v, u) + h^\flat(\nabla_v Y_f, u).\end{aligned} \tag{3-6}$$

By Lemma 2.3, we have

$$\begin{aligned}h^\flat(\nabla_v Y_f, u) &= -(\nabla_v h^\flat)(Y_f, u) - h^\flat(Y_f, \nabla_v u) + v(h^\flat(Y_f, u))\\ &= -2A^\flat(v, Y_f, u) - h^\flat(Y_f, \nabla_v u) + v(h^\flat(Y_f, u)).\end{aligned} \tag{3-7}$$

By a straightforward computation, one obtains

$$h^\flat(Y_f, v) = -\mathcal{H}^\flat(v)(f) = -df(\mathcal{H}^\flat(v)), \quad v \in VS^*M.$$

It follows that

$$h^\flat(Y_f, \nabla_v u) = -(P^\flat_{\mathcal{H}}[v, \mathcal{H}^\flat(u)])(f) \tag{3-8}$$

and

$$v(h^\flat(Y_f, u)) = -(P^\flat_{\mathcal{H}}[v, \mathcal{H}^\flat(u)])(f) + \dot f h^\flat(u, v), \tag{3-9}$$

where we have used $h^\flat(u,v) = -\omega^\flat[u, \mathcal{H}^\flat(v)]$. Substituting (3-8) and (3-9) into (3-7) and then combining it with (3-6), we have (3-2). □

Proposition 3.6. *For a conformal transformation φ on a Cartan manifold (M, H), we have*

$$\hat{\varphi}_*\mathcal{H}^\flat(v) = e^{2\sigma(x)}[\mathcal{H}^\flat(\hat{\varphi}_* v) + 2\dot{\sigma}\hat{\varphi}_* v - 2A^\flat(Y_\sigma, \hat{\varphi}_* v)]. \tag{3-10}$$

Proof. By Lemma 3.3, we have

$$\begin{aligned}\hat{\varphi}_*\mathcal{H}^\flat(v) &= -\hat{\varphi}_*[X^\flat, v] - \tfrac{1}{2}\hat{\varphi}_* \circ \mathcal{V}^\flat[X^\flat, [X^\flat, v]] \\ &= -[\hat{\varphi}_* X^\flat, \hat{\varphi}_* v] - \tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat \circ \hat{\varphi}_*[X^\flat, [X^\flat, v]] \\ &= -[e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}), \hat{\varphi}_* v] + (I),\end{aligned} \tag{3-11}$$

where

$$\begin{aligned}(I) &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat[\hat{\varphi}_* X^\flat, \hat{\varphi}_*[X^\flat, v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat[e^{2\sigma(x)}(X^\flat + 2Y_{\sigma(x)}), \hat{\varphi}_*[X^\flat, v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\mathcal{V}^\flat\big(-\hat{\varphi}_*[X^\flat, v](e^{2\sigma(x)})(X^\flat + 2Y_{\sigma(x)})\big) \\ &\qquad\qquad - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, [\hat{\varphi}_* X^\flat, \hat{\varphi}_* v]] \\ &= -\tfrac{1}{2}e^{-2\sigma(x)}\big(-\hat{\varphi}_*[X^\flat, v](e^{2\sigma(x)})\mathcal{V}^\flat(X^\flat + 2Y_{\sigma(x)})\big) \\ &\qquad - \tfrac{1}{2}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, e^{2\sigma(x)}[X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v]] \\ &= -\tfrac{1}{2}\big((II) + e^{2\sigma(x)}\mathcal{V}^\flat[X^\flat + 2Y_{\sigma(x)}, [X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v]]\big),\end{aligned} \tag{3-12}$$

and

$$\begin{aligned}(II) &= \mathcal{V}^\flat(X^\flat + 2Y_{\sigma(x)})(e^{2\sigma(x)})[X^\flat + 2Y_{\sigma(x)}, \hat{\varphi}_* v] \\ &= X^\flat(e^{2\sigma(x)})\big(\mathcal{V}^\flat[X^\flat, \hat{\varphi}_* v] + 2\mathcal{V}^\flat[Y_{\sigma(x)}, \hat{\varphi}_* v]\big) = -X^\flat(e^{2\sigma(x)})\hat{\varphi}_* v.\end{aligned}$$

Plugging this into (3-12) and combining with (3-11), we obtain

$$\hat{\varphi}_*\mathcal{H}^\flat(v) = e^{2\sigma(x)}\big(\mathcal{H}^\flat(\hat{\varphi}_* v) - [Y_{\sigma(x)}, \hat{\varphi}_* v] + X^\flat(v)\hat{\varphi}_* v - \mathcal{V}^\flat[Y_{\sigma(x)}, [X^\flat, \hat{\varphi}_* v]]\big). \tag{3-13}$$

By using the Jacobi identity and Lemma 3.4, we have

$$\begin{aligned}-\mathcal{V}^\flat[Y_{\sigma(x)}, [X^\flat, \hat{\varphi}_* v]] &= \mathcal{V}^\flat[X^\flat, [\hat{\varphi}_* v, Y_{\sigma(x)}]] - \mathcal{V}^\flat[\hat{\varphi}_* v, [X^\flat, Y_{\sigma(x)}]] \\ &= -[\hat{\varphi}_* v, Y_{\sigma(x)}] - \mathcal{V}^\flat[\hat{\varphi}_* v, X_{\dot{\sigma}}].\end{aligned}$$

Plugging this into (3-13) and using Lemma 3.5, we get (3-10). □

4. Conformal navigation problems

We call the navigation problem (2-2) *conformal* if V is a conformal field. In this section, we explore some properties of conformal navigation problems and prove Theorem 1.1.

Lemma 4.1. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in* (2-9). *Then for $v \in VS^*M$*

$$\mathcal{H}^\flat(v) = \widetilde{\mathcal{H}}^\flat(v) - cv, \tag{4-1}$$

where $\mathcal{H}^\flat$ (resp. $\widetilde{\mathcal{H}}^\flat$) is the horizontal endomorphism of H (resp. $\widetilde{H}$).

Proof. By [Mo and Hang 2007, Lemma 4.10], we have

$$[X^\flat, v] \in \operatorname{Ker} \omega^\flat = HS^*M \oplus VS^*M, \quad [X^\flat, [X^\flat, v]] \in \operatorname{Ker} \omega^\flat. \tag{4-2}$$

Together with Lemma 2.4 we get

$$-\mathcal{H}^\flat(v) = [X^\flat, v] + \tfrac{1}{2}\widetilde{\mathcal{V}}^\flat[X^\flat, [X^\flat, v]]. \tag{4-3}$$

According to [loc. cit., Lemma 6.2], the Reeb fields of $X^\flat$ and $\widetilde{X}^\flat$ satisfy

$$X^\flat = \widetilde{X}^\flat + X_V^*, \tag{4-4}$$

where

$$X_V^* = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial p_i},$$

with $V = v^i (\partial / \partial x^i)$. It follows that

$$\begin{aligned} \widetilde{\mathcal{V}}^\flat[X^\flat, [X^\flat, v]] &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X^\flat, v]] + \widetilde{\mathcal{V}}^\flat[X_V^*, [X^\flat, v]] \\ &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [\widetilde{X}^\flat, v]] + \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X_V^*, v]] + \widetilde{\mathcal{V}}^\flat[X_V^*, [X^\flat, v]]. \end{aligned} \tag{4-5}$$

Let $\hat{\varphi}_t$ be flow of X_V^*. Then $(\hat{\varphi}_t)_* v$ is vertical for $v \in VS^*M$. Hence,

$$[X_V^*, v] := \lim_{t \to 0} \frac{v - (\hat{\varphi}_t)_* v}{t} \tag{4-6}$$

is also vertical. It follows that

$$\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat, [X_V^*, v]] = -[X_V^*, v]. \tag{4-7}$$

By using the Jacobi identity, we have

$$[X_V^*, [X^\flat, v]] = -[X^\flat, [v, X_V^*]] - [v, [X_V^*, X^\flat]]. \tag{4-8}$$

Now we assume that V is a conformal field of Cartan metric H with dilation $c(x)$; that is, the flow φ_t of V satisfies

$$\hat{\varphi}_t^* H = e^{-2\sigma_t(x)} H, \quad c(x) = \left[\frac{d\sigma_t(x)}{dt} \right]_{t=0}. \tag{4-9}$$

Differentiating the first of these equations with respect to t at $t = 0$ yields

$$\begin{aligned}-2c(x)H &= \frac{\partial}{\partial t}(e^{-2\sigma_t(x)}H)\big|_{t=0}\\ &= \frac{\partial}{\partial t}(\hat{\varphi}_t^* H)\big|_{t=0}\\ &= \frac{\partial}{\partial t}(H\circ\hat{\varphi}_t)\big|_{t=0}\\ &= \frac{\partial\hat{\varphi}_t}{\partial t}\Big|_{t=0}H = X_V^*(H).\end{aligned}$$

Recall that $VS^*M = \operatorname{Ker}\pi_* = \{v\in TSM \mid v(f)=0,\ \forall f\in C^\infty(M)\subset C^\infty(S^*M)\}$. Together with (4-2), we have

$$[v, 2cX^\flat] = v(2c)X^\flat - 2c[X^\flat, v] = -2c[X^\flat, v]\in \operatorname{Ker}\omega^\flat.$$

Note that the vertical distribution is involutive. We obtain

$$\widetilde{\mathcal{V}}^\flat\left[v, \frac{\partial c}{\partial x^i}H\frac{\partial}{\partial p_i}\right] = 0.$$

A direct calculation (see [Huang and Mo 2011, Lemma 3.2]) gives the formula

$$[X^\flat, X_V^*] = 2cX^\flat - 2\frac{\partial c}{\partial x^i}H\frac{\partial}{\partial p_i}.$$

By Lemma 2.4, we obtain

$$\widetilde{\mathcal{V}}^\flat[v,[X_V^*, X^\flat]] = 2\widetilde{\mathcal{V}}^\flat\left[v, \frac{\partial c}{\partial x^i}H\frac{\partial}{\partial p_i}\right] - \widetilde{\mathcal{V}}^\flat[v, 2cX^\flat] = -2cv.$$

Together with (4-6) and (4-8), we have

$$\begin{aligned}\widetilde{\mathcal{V}}^\flat[X_V^*,[X^\flat, v]] &= -\widetilde{\mathcal{V}}^\flat[X^\flat,[v, X_V^*]] - \widetilde{\mathcal{V}}^\flat[v,[X_V^*, X^\flat]]\\ &= -\mathcal{V}^\flat[X^\flat,[v, X_V^*]] + 2cv = [v, X_V^*] + 2cv.\end{aligned}$$

Plugging this and (4-7) into (4-5) yields

$$\begin{aligned}\widetilde{\mathcal{V}}^\flat[X^\flat,[X^\flat, v]] &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat,[\widetilde{X}^\flat, v]] - [X_V^*, v] + [v, X_V^*] + 2cv\\ &= \widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat,[\widetilde{X}^\flat, v]] - 2[X_V^*, v] + 2cv.\end{aligned}$$

Substituting this into (4-3) and using (4-4), we deduce that

$$\begin{aligned}-\mathcal{H}^\flat(v) &= [X^\flat, v] + \tfrac{1}{2}\big(\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat,[\widetilde{X}^\flat, v]] - 2[X_V^*, v] + 2cv\big)\\ &= [\widetilde{X}^\flat, v] + \tfrac{1}{2}\widetilde{\mathcal{V}}^\flat[\widetilde{X}^\flat,[\widetilde{X}^\flat, v]] + cv = -\widetilde{\mathcal{H}}^\flat(v) + cv.\end{aligned}$$

This gives (4-1). □

Lemma 4.2. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in* (2-9). *Then on $HS^*M \oplus VS^*M$, we have $P^\flat_{\widetilde{\mathcal{V}}} = P^\flat_{\mathcal{V}} - c\widetilde{\mathcal{V}}^\flat = P^\flat_{\mathcal{V}} - c\mathcal{V}^\flat$, where $P^\flat_{\mathcal{V}}$ (resp. $P^\flat_{\widetilde{\mathcal{V}}}$) is the projection of H (resp. $\widetilde{H}$).*

Proof. The second equality follows from Lemma 2.4. For $v \in VS^*M$,

$$\widetilde{\mathcal{V}}^\flat(v) = 0, \quad P^\flat_{\widetilde{\mathcal{V}}}(v) = P^\flat_{\mathcal{V}}.$$

It follows that

$$P^\flat_{\widetilde{\mathcal{V}}}(v) = (P^\flat_{\mathcal{V}} - c\widetilde{\mathcal{V}}^\flat)(v), \quad \forall\, v \in VS^*M.$$

For $u \in HS^*M$, we write $u \in \mathcal{H}^\flat(v)$, where $v \in VS^*M$. By the definition of $\widetilde{\mathcal{H}}^\flat$ and Lemma 4.1, we obtain

$$\begin{aligned} P^\flat_{\widetilde{\mathcal{V}}}(u) + c\widetilde{\mathcal{V}}^\flat(u) &= \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(\mathcal{H}^\flat(v)) + c\widetilde{\mathcal{V}}^\flat(\mathcal{H}^\flat(v)) \\ &= \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(\widetilde{\mathcal{H}}^\flat(v) - cv) + c\widetilde{\mathcal{V}}^\flat(\widetilde{\mathcal{H}}^\flat(v) - cv) \\ &= -c\widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(v) + \widetilde{\mathcal{V}}^\flat \circ \widetilde{\mathcal{H}}^\flat(v) - c^2\widetilde{\mathcal{V}}^\flat(v) \\ &= 0 = P^\flat_{\mathcal{V}}(u). \end{aligned}$$

□

Proposition 4.3. *Let V be a conformal field of H with dilation $c(x)$. Then*

$$[X^*_V, \mathcal{H}^\flat(v)] = -2c\mathcal{H}^\flat(v) + \mathcal{H}^\flat[X^*_V, v] - 2\dot{c}v + 2A^\flat(Y_c, v). \tag{4-10}$$

Proof. By using Proposition 3.6, we have

$$\begin{aligned} [X^*_V, \mathcal{H}^\flat(v)] &= -\frac{d}{dt}\Big|_{t=0} \hat{\varphi}_{t*}\mathcal{H}^\flat(v) \\ &= -\frac{d}{dt}\Big|_{t=0} \big(e^{2\sigma_t(x)}\big[\mathcal{H}^\flat(\hat{\varphi}_{t*}v) + 2\dot{\sigma}_t\hat{\varphi}_{t*}v - 2A^\flat(Y_{\sigma_t}, \hat{\varphi}_{t*}v)\big]\big), \end{aligned} \tag{4-11}$$

where φ_t is the flow of V. By direct calculations, we have

$$-\frac{d}{dt}\Big|_{t=0} \mathcal{H}^\flat(\hat{\varphi}_{t*}v) = \mathcal{H}^\flat[X^*_V, v], \quad -\frac{d}{dt}\Big|_{t=0} A^\flat(Y_{\sigma_t}, \hat{\varphi}_{t*}v) = A^\flat(Y_c, v),$$
$$-\frac{d}{dt}\Big|_{t=0} (\dot{\sigma}_t\hat{\varphi}_{t*}v) = \frac{d\dot{\sigma}_t}{dt}\Big|_{t=0} \hat{\varphi}_{0*}v + \dot{\sigma}_t\Big|_{t=0} \frac{d}{dt}\hat{\varphi}_{t*}v = X^\flat(c)v = \dot{c}v.$$

Plugging them into (4-11), we have (4-10). □

Proposition 4.4. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in* (2-9). *Then*

$$\tilde{\mathcal{R}}^\flat(v) = \mathcal{R}^\flat(v) + [3\widetilde{X}^\flat(c) - c^2 + 2X^*_V(c)]v - 2A^\flat(Y_c, v), \tag{4-12}$$

where $\mathcal{R}^\flat$ (resp. $\tilde{\mathcal{R}}^\flat$) is the Riemann tensor of H (resp. $\widetilde{H}$).

Proof. From [Mo and Hang 2007, Lemma 4.9], we have

$$P^{\flat}_{\mathcal{V}}[X^{\flat}, v] = \mathcal{V}^{\flat}[X^{\flat}, \mathcal{H}^{\flat}(v)], \quad v \in VS^*M. \tag{4-13}$$

By (2-11), (4-1) and (4-4),

$$\begin{aligned}
\tilde{\mathcal{R}}^{\flat}(v) &= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \tilde{\mathcal{H}}^{\flat}(v)] \\
&= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \mathcal{H}^{\flat}(v) + cv] \\
&= P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, \mathcal{H}^{\flat}(v)] + P^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, cv] \\
&= P^{\flat}_{\tilde{\mathcal{V}}}[X^{\flat} - X^*_V, \mathcal{H}^{\flat}(v)] + P^{\flat}_{\tilde{\mathcal{V}}}\big(\widetilde{X}^{\flat}(c)v + c[\widetilde{X}^{\flat}, v]\big) \\
&= (I) - P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] + \widetilde{X}^{\flat}(c)v,
\end{aligned}$$

where

$$\begin{aligned}
(I) := {}& P^{\flat}_{\tilde{\mathcal{V}}}[X^{\flat}, \mathcal{H}^{\flat}(v)] + cP^{\flat}_{\tilde{\mathcal{V}}}[\widetilde{X}^{\flat}, v] \\
={}& (P^{\flat}_{\mathcal{V}} - c\mathcal{V}^{\flat})[X^{\flat}, \mathcal{H}^{\flat}(v)] + c(P^{\flat}_{\mathcal{V}} - c\mathcal{V}^{\flat})[X^{\flat} - X^*_V, v] \\
={}& P^{\flat}_{\mathcal{V}}[X^{\flat}, \mathcal{H}^{\flat}(v)] - c\mathcal{V}^{\flat}[X^{\flat}, \mathcal{H}^{\flat}(v)] + cP^{\flat}_{\mathcal{V}}[X^{\flat}, v] \\
&\qquad - c^2\mathcal{V}^{\flat}[X^{\flat}, v] - cP^{\flat}_{\mathcal{V}}[X^*_V, v] + c^2\mathcal{V}^{\flat}[X^*_V, v] \\
={}& \mathcal{R}^{\flat}(v) + c^2 v - c[X^*_V, v],
\end{aligned}$$

where we have used (4-13). It follows that

$$\tilde{\mathcal{R}}^{\flat}(v) = \mathcal{R}^{\flat}(v) - P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] - c[X^*_V, v] + [\widetilde{X}^{\flat}(c) + c^2]v. \tag{4-14}$$

From (4-4), we have

$$\widetilde{X}^{\flat}(c) = \dot{c} - X^*_V(c). \tag{4-15}$$

By using (4-11) and Lemma 4.2, we obtain

$$\begin{aligned}
P^{\flat}_{\tilde{\mathcal{V}}}[X^*_V, \mathcal{H}^{\flat}(v)] &= 2c^2 P^{\flat}_{\mathcal{V}} - cP^{\flat}_{\mathcal{V}}[X^*_V, v] - 2\dot{c}P^{\flat}_{\mathcal{V}}v \\
&\qquad + 2c\dot{c}\mathcal{V}^{\flat}(v) + 2P^{\flat}_{\mathcal{V}}A^{\flat}(Y_c, v) - 2c\mathcal{V}^{\flat}A^{\flat}(Y_c, v) \\
&= 2c^2 v - c[X^*_V, v] - 2\dot{c}v + 2A^{\flat}(Y_c, v).
\end{aligned}$$

Plugging this and (4-15) into (4-14) yields (4-12). □

Proposition 4.5. *Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\widetilde{H}$ be the Cartan metric given in* (2-9)*. Then*

$$\widetilde{K}^{\flat}(v) - [3\widetilde{X}^{\flat}(c) - c^2 + 2V(c)] = K^{\flat}(v) - 2\frac{A^{\flat}(v, Y_c, v)}{h^{\flat}(v, v)}, \tag{4-16}$$

where $K^{\flat}$ (*resp. $\widetilde{K}^{\flat}$*) *is the flag curvature of H* (*resp. $\widetilde{H}$*).

Proof. By [Mo and Hang 2007, Lemma 6.2], we have $h^\flat(v_1, v_2) = (\widetilde{H}/H)\tilde{h}^\flat(v_1, v_2)$. Together with (4-12) and (2-12), this yields

$$\widetilde{K}^\flat(v) = K^\flat(v) + 3\widetilde{X}^\flat(c) - c^2 + 2X_V^*(c) - 2\frac{A^\flat(v, Y_c, v)}{h^\flat(v, v)}. \tag{4-17}$$

On the other hand,

$$X_V^*(c) = \left(v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial p_i}\right) c(x) = v^i \frac{\partial c}{\partial x^i} = V(c).$$

Together with (4-17), we have (4-16). □

Proof of Theorem 1.1. Let F be a Finsler metric with flag curvature K, Cartan torsion A and angular metric h. Let V be a conformal field on M with $F(x, V_x) < 1$. Let $\widetilde{F}$ be the Finsler metric given in (2-2) with flag curvature $\widetilde{K}$. Then their Cartan metrics are related by (2-9). From Lemma 2.1, we obtain that V is a conformal field of H with dilation $c(x)$. Hence K and $\widetilde{K}$ satisfy (4-16). By (2-8), we have $A^\flat(v, Y_c, v) = -A^\flat(v, \Phi Dc, v)$. Plugging this into (4-16) yields

$$[\widetilde{K}^\flat(v)]_{(x,[p])} - [3\widetilde{X}^\flat(c) - c^2 + 2V(c)]_{(x,[p])} = [K^\flat(v)]_{(x,[p])} + 2\frac{A^\flat(v, \Phi Dc, v)_{(x,[p])}}{h^\flat(v, v)_{(x,[p])}}.$$

Pulling back to the sphere bundle, we have

$$[\widetilde{K}(u)]_{(x,[y])} - \left[3\frac{y^i c_{x^i}}{\widetilde{F}} - c^2 + 2V(c)\right] = [K(u)]_{(x,[\tilde{y}])} - 2\frac{A(u, \nabla c, u)_{(x,[\tilde{y}])}}{h(u, u)_{(x,[\tilde{y}])}},$$

where $u := (L_x^{F*})_* v$, $\nabla c := (L_x^{F*})_* \Phi Dc$ and where we have used $\partial\widetilde{H}/\partial p_i = y^i/\widetilde{F}$. By [Mo and Hang 2007, Lemma 3.9], we get the desired result. □

Remark. (i) The reader should note that the navigation problem adopted here differs from that of [Shen and Xia 2012; Shen 2003], where the navigation problem is defined by $F\big(x, y/\widetilde{F}(x, y) - V\big) = 1$; i.e., the $\widetilde{F}$ that we define with (F, V) is precisely the $\widetilde{F}$ that Shen defines with $(F, -V)$.

(ii) We have two special cases of Theorem 1.1:

(1) If V is homothetic, i.e., its dilation $c(x)$ is constant, then $\nabla c = 0$ and our formula is reduced to that of Mo and Huang [2007].

(2) If F is Riemannian and has sectional curvature $K = K(x)$, then our formula is reduced to that of Cheng and Shen [2009] (see also [Chern and Shen 2005]).

5. An example

In this section, we determine the flag curvature of a nontrivial example using Theorem 1.1.

Consider the case $\dim M = 2$; so $x = (x^1, x^2)$ and $y = (y^1, y^2)$. In order to avoid the excessive use of parentheses, we shall abbreviate x^1, x^2 as s, t and y^1, y^2 as p, q. Let

$$M := \{(s,t) \in \mathbb{R}^2 \mid t > 1\}.$$

Define $F : TM \to \mathbb{R}$ by

$$F(s,t;\ p,q) := \frac{1}{t}\Phi(p,q), \tag{5-1}$$

where

$$\Phi(p,q) := (p^4 + 2\epsilon p^2 q^2 + q^4)^{1/4}, \quad \epsilon \in (0,3), \tag{5-2}$$

is a Minkowski norm on $\mathbb{R}^2$ (see [Shen 2001, Example 1.1.3]) and F is a Finsler metric on M.

For the Finsler surface (M, F), its Gaussian curvature K takes the place of the flag curvature in general case. A direct calculation shows that the Gaussian curvature of F is given by

$$K_F(s,t;\ p,q) = \frac{[\Phi(p,q)]^2 Q(p,q)}{[\Delta(p,q)]^4}, \tag{5-3}$$

where

(5-4)

$$\begin{aligned} Q(p,q) := {} & \epsilon(2\epsilon^2-3)p^{14}+(17\epsilon^4-42\epsilon^3+18)p^{12}q^2+\epsilon(8\epsilon^4-50\epsilon^2+21)p^{10}q^4 \\ & +(9\epsilon^6-89\epsilon^4+81\epsilon^2-36)p^8q^6-5\epsilon(5\epsilon^4-4\epsilon^2+6)p^6q^8 \\ & +\epsilon^2(5\epsilon^4-5\epsilon^2-21)p^4q^{10}+\epsilon^3(5\epsilon^2-12)p^2q^{12}-\epsilon^4q^{14} \end{aligned}$$

and

$$\Delta(p,q) := \epsilon p^4 + (3-\epsilon^2)p^2q^2 + \epsilon q^4. \tag{5-5}$$

We denote the determinant of the fundamental tensor by g. Then

$$g = \frac{\Delta(p,q)}{t^4[\Phi(p,q)]^4}, \tag{5-6}$$

where we have used (5-1), (5-2) and (5-5). The Cartan form η is given by

$$\eta = \left(F\frac{\partial}{\partial y^j}\log\sqrt{g}\right)dx^j. \tag{5-7}$$

Then the main scalar I of F is given by

$$(5\text{-}8)\qquad \begin{aligned} I(x,y) &= \eta(e_1) \\ &= \frac{-1}{\sqrt{g}}\left(\left(\frac{\partial}{\partial p}\log\sqrt{g}\right)\left(\frac{F^2}{2}\right)_q - \left(\frac{\partial}{\partial q}\log\sqrt{g}\right)\left(\frac{F^2}{2}\right)_p\right) \\ &= \frac{3(1-\epsilon^2)pq}{[\Delta(p,q)]^{3/2}}(p^4-q^4), \end{aligned}$$

where $\{e_1, e_2\}$ is the Berwald frame with $\omega(e_1) = 0$. Let V denote a vector field on M defined by

$$(5\text{-}9)\qquad V := \frac{\partial}{\partial t}.$$

By using the isomorphism $T_x M \simeq \mathbb{R}^2$, we have $F(x, V_x) < 1$ on M. Denote the lift of V by X_V. Then

$$X_V = V + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i} = V$$

[Huang and Mo 2011]. It follows that

$$X_V(F) = \frac{\partial F}{\partial t} = -\frac{1}{t}F,$$

where we have made use of (5-1). Thus V is conformal with dilation $c = -1/(2t)$ (see [Huang and Mo 2013, Lemma 3.1]). In particular, V is not homothetic.

Now we calculate the following scalar function on SM.

$$(5\text{-}10)\qquad \xi(x,y) := \frac{A_{(x,[y])}(u, \nabla c, u)}{h_{(x,[y])}(u,u)},$$

where $u \wedge y \neq 0$. Taking $u = e_1$ we obtain

$$(5\text{-}11)\qquad h_{(x,[y])}(e_1,e_1) = 1, \quad A_{(x,[y])}(e_1,e_1,e_1) = I(x,y), \quad A_{(x,[y])}(e_1,e_2,e_1) = 0.$$

Define ∇c by

$$(5\text{-}12)\qquad \nabla c = \lambda e_1 + \mu e_2,$$

where $\{e_1, e_2\}$ is the Berwald frame on M. Then

$$(5\text{-}13)\qquad \begin{aligned} \lambda(x,y) &= g_{(x,[y])}(\nabla c, e_1) \\ &= \frac{\partial c}{\partial s}\left(-\frac{F_q}{\sqrt{g}}\right) + \frac{\partial c}{\partial t}\frac{F_p}{\sqrt{g}} = \frac{p(p^2+\epsilon q^2)}{2F\sqrt{\Delta(p,q)}t^2}, \end{aligned}$$

where $\boldsymbol{g}$ denotes the fundamental tensor. From (5-10), (5-11) and (5-12), it follows that

$$\begin{aligned}\xi(x,y) &= \frac{A_{(x,[y])}(e_1,\lambda e_1+\mu e_2,e_1)}{h_{(x,[y])}(e_1,e_1)}\\ &= \lambda(x,y)A_{(x,[y])}(e_1,e_1,e_1)=\lambda(x,y)I(x,y),\end{aligned}$$

where λ and I are given in (5-13) and (5-8) respectively.

Now we consider the navigation data (F, V), where F and V are defined in (5-1) and (5-9) respectively. (F, V) produces a new Finsler metric $\widetilde{F}$ by

$$F\left(x, \frac{y}{\widetilde{F}(x,y)}+V_x\right)=1, \quad \forall x\in M,\ y\in T_xM. \tag{5-14}$$

By (5-1), (5-2) and (5-9), (5-14) holds if and only if

$$p^4+2\epsilon p^2(q+\widetilde{F})^2+(q+\widetilde{F})^4=t^4\widetilde{F}^4, \tag{5-15}$$

that is, $\widetilde{F}$ is the unique nonnegative solution of (5-15). By direct calculation we have

$$\frac{y^ic_{x^i}}{\widetilde{F}(x,y)}=\frac{q}{2t^2\widetilde{F}(x,y)}, \quad -c^2+2V(c)=\frac{3}{4t^2}.$$

For the Finsler surface (M, F), F is of scalar flag curvature. Using Theorem 1.1, we obtain that the Gaussian curvature $K_{\widetilde{F}}$ is given by

$$\begin{aligned}K_{\widetilde{F}}(x,y) &= K_F(x,\tilde{y})+\left[3\frac{y^ic_{x^i}}{\widetilde{F}(x,y)}-c^2+2V(c)\right]-2\frac{A_{(x,[\tilde{y}])}(u,\nabla c,u)}{h_{(x,[\tilde{y}])}(u,u)}\\ &= K_F(x,\tilde{y})+\frac{3q}{2t^2\widetilde{F}(x,y)}+\frac{3}{4t^2}-2\lambda(x,\tilde{y})I(x,\tilde{y}),\end{aligned}$$

where

$$\tilde{y}=y+F(x,y)V=\left(p,q+\frac{(p^4+2\epsilon p^2q^2+q^4)^{1/4}}{t}\right)$$

and K_F, λ, I are given in (5-3), (5-13) and (5-8) respectively.

Let us take a look at the special case when $\epsilon=1$,

$$F(s,t;\ p,q):=\frac{(p^2+q^2)^{1/2}}{t}.$$

F is the famous Poincaré metric of constant sectional curvature $K_F=-1$. In this case, $\widetilde{F}$ is of Randers type and its Gaussian curvature is given by

$$K_{\widetilde{F}}(x,y)=\frac{3}{4t^2}\left(\frac{2q}{\widetilde{F}(x,y)}+1\right)-1.$$

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Received December 15, 2013.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 277 No. 1 September 2015

0030-8730(201509)277:1;1-V