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**GAMMA FACTORS OF DISTINGUISHED
REPRESENTATIONS OF $GL_n(\mathbb{C})$**

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Let (π, V) be a $GL_n(\mathbb{R})$ -distinguished, irreducible, admissible representation of $GL_n(\mathbb{C})$, let π' be an irreducible, admissible, $GL_m(\mathbb{R})$ -distinguished representation of $GL_m(\mathbb{C})$, and let ψ be a nontrivial character of \mathbb{C} which is trivial on \mathbb{R} . We prove that the Rankin–Selberg gamma factor at $s = 1/2$ is $\gamma(1/2, \pi \times \pi'; \psi) = 1$. The result follows as a simple consequence from the characterization of $GL_n(\mathbb{R})$ -distinguished representations in terms of their Langlands data.

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1. Introduction

Let $G_n(\mathbb{C}) = GL_n(\mathbb{C})$, $G_n(\mathbb{R}) = GL_n(\mathbb{R})$. Let $B_n = B_n(\mathbb{C})$ be the Borel subgroup of upper triangular matrices in $G_n(\mathbb{C})$. Denote the complex conjugation by $x \rightarrow \bar{x}$. We identify $G_n(\mathbb{C})/G_n(\mathbb{R})$ with the space of matrices

$$X_n = \{x \in G_n(\mathbb{C}) : x \cdot \bar{x} = I_n\},$$

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via the isomorphism $gG_n(\mathbb{R}) \mapsto g \cdot \bar{g}^{-1}$. See [Serre 2002, Chapter 3, Section 1, Lemma 1] for the proof of the surjectivity of this map. Given a representation π of $G_n(\mathbb{C})$, the representation $\bar{\pi}$ is defined by the formula $\bar{\pi}(g) := \pi(\bar{g})$.

The group $G_n(\mathbb{C})$ acts on X_n by the twisted conjugation, where the action is induced by the natural action $l(g)g'G_n(\mathbb{R}) := gg'G_n(\mathbb{R})$. Namely, we have

$$g'G_n(\mathbb{R}) \leftrightarrow g' \cdot \bar{g}'^{-1} := x \quad \text{and} \quad l(g)(g'G_n(\mathbb{R})) := gg'G_n(\mathbb{R}) \leftrightarrow gg' \bar{g}'^{-1} \bar{g}^{-1}.$$

Hence, the action of $G_n(\mathbb{C})$ on X is given by $l(g)x := gx\bar{g}^{-1}$.

For a topological vector space V , we denote by V^* the topological dual of V , i.e., the space of all continuous maps from V to \mathbb{C} . In this paper we work with the category of the admissible smooth Fréchet representations of moderate growth (see [Wallach 1992, Section 11.5; Aizenbud et al. 2008, Section 2.1]).

A representation (π, V) of $G_n(\mathbb{C})$ is called $G_n(\mathbb{R})$ -distinguished if there exists a nonzero continuous linear map $L : V \rightarrow \mathbb{C}$ such that

$$L(\pi(h)v) = L(v) \quad \text{for all } v \in V, h \in G_n(\mathbb{R}).$$

We denote the space of all such linear maps by $(V^*)^{G_n(\mathbb{R})}$. We denote the set of equivalence classes of irreducible representations of $G_n(\mathbb{C})$ by $\text{Irr}(G_n(\mathbb{C}))$ and the set of equivalence classes of irreducible $G_n(\mathbb{R})$ -distinguished representations of $G_n(\mathbb{C})$ by $\text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$.

Let $\psi : \mathbb{C} \rightarrow \mathbb{C}^\times$ be a nontrivial unitary character which is trivial on \mathbb{R} , for example

$$\psi(x) = e^{\pi(x-\bar{x})}.$$

We let $U_n(\mathbb{C})$ be the group of upper triangular matrices with unit diagonal and we denote by $\theta_{\psi,n}$ the character $\theta_{\psi,n} : U_n(\mathbb{C}) \rightarrow \mathbb{C}^\times$ defined by

$$\theta_{\psi,n}(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

A ψ -form on V is a nonzero continuous linear form $\lambda : V \rightarrow \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \theta_{\psi,n}(u)\lambda(v),$$

for each $v \in V$ and each $u \in U_n(\mathbb{C})$. We say that π is a generic representation if there exists a ψ -form on V .

Theorem 1.1. *Let $\pi \in \text{Irr}_{G_t(\mathbb{R})}(G_t(\mathbb{C}))$ and let $\pi' \in \text{Irr}_{G_r(\mathbb{R})}(G_r(\mathbb{C}))$. If ψ is a nontrivial character of \mathbb{C} with a trivial restriction to \mathbb{R} then the value of the Rankin–Selberg gamma factor at $s = 1/2$ is*

$$\gamma\left(\frac{1}{2}, \pi \times \pi'; \psi\right) = 1.$$

A similar theorem is proved in [Offen 2011, Theorem 0.1] for the p -adic case (see also [Ok 1997]). See Section 5 for the definition of Rankin–Selberg integrals, Rankin–Selberg gamma factors and for the proof of Theorem 1.1.

We will deduce Theorem 1.1 from the characterization of irreducible $G_n(\mathbb{R})$ -distinguished representations of $G_n(\mathbb{C})$. Let χ be a character of B_n . We denote by $I(\chi)$ the normalized parabolic induction representation

$$I(\chi) := \text{Ind}_{B_n}^{G_n(\mathbb{C})}(\chi)$$

of the character $\chi = (\chi_1, \dots, \chi_n)$ from B_n to $G_n(\mathbb{C})$. We remind the reader that this space consists of smooth functions such that $f(bg) = (\chi \delta^{1/2})(b) f(g)$ for all $b \in B_n$ and all $g \in G_n(\mathbb{C})$. The group $G_n(\mathbb{C})$ acts on $I(\chi)$ by right translations, and the group of permutations on n elements, S_n , acts naturally on the characters of B_n . We will call a character $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ of B_n dominant if

$$|\chi(t)| = |t_1|^{\lambda_1} |t_2|^{\lambda_2} \cdots |t_n|^{\lambda_n} \quad \text{with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

In Section 4 we will prove the following.

Theorem 1.2. *Let π be an element of $\text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ and let $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ be a dominant character of B_n . Suppose π is the Langlands quotient of $I(\chi)$, that is, the unique irreducible quotient of $I(\chi)$. Then there exists an involution $w \in S_n$ such that $w\chi = (\chi^{-1})$. Moreover, we can choose this w such that for every fixed point i of w we have $\chi_i(-1) = 1$.*

Remark 1.3. Note that the conditions $w(i) = i$, $\chi_{w(i)} = \bar{\chi}_i^{-1}$ and $\chi_i(-1) = 1$ imply that χ_i is $GL_1(\mathbb{R})$ -distinguished. Indeed, $\chi_i = \bar{\chi}_i^{-1}$ implies that χ_i is \mathbb{R}_+ -invariant. Together with the condition $\chi_i(-1) = 1$ this means that χ_i is \mathbb{R}^\times -invariant (i.e., $GL_1(\mathbb{R})$ -invariant).

As a consequence of Theorem 1.2 we obtain the following analogue of [Aizenbud and Lapid 2012, Theorem B.1].

Theorem 1.4. *Let $\pi \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ and suppose π is a generic representation of $G_n(\mathbb{C})$. Then $\bar{\pi} \simeq \tilde{\pi}$, where $\tilde{\pi}$ is the contragredient representation of π .*

Let (π, V) be an irreducible representation of $G_n(\mathbb{C})$. The existence of $I(\chi)$ with the properties stated in Theorem 1.2 is a well-known fact (see [Wallach 1988, Theorem 5.4.1]). It is also well-known that the Langlands quotient of $I(\chi)$ is generic if and only if $I(\chi)$ is irreducible (see Appendix A). Therefore, $\bar{\pi} = I(\bar{\chi})$ and $\tilde{\pi} = I(\chi^{-1})$. Since $\bar{\pi}$ is irreducible, for every $w_0 \in S_n$ we have $I(w_0(\bar{\chi})) \simeq I(\bar{\chi})$. In particular, for $w \in S_n$ such that $w(\bar{\chi}) = \chi^{-1}$, we have

$$\bar{\pi} \simeq I(\bar{\chi}) \simeq I(w(\bar{\chi})) \simeq I(\chi^{-1}) \simeq \tilde{\pi}.$$

A similar result was proved by Marie-Noelle Panichi in her Ph.D. thesis [2001, Theorem 3.3.6].

The structure of the paper is as follows. In Section 3 we recall basic facts about the structure of $G_n(\mathbb{C})$. In Section 4 we prove Theorem 1.2 by analyzing the geometry of the action of B_n on the variety $G_n(\mathbb{C})/G_n(\mathbb{R})$. In Section 5, as an application of our classification we deduce Theorem 1.1. In Section 8 we prove a new type of integral identity for Whittaker functions on generic $G_n(\mathbb{R})$ -distinguished representations which in turn proves [Lapid and Mao 2014, Assumption 5.2]. A similar identity was proved in the p -adic case in [Offen 2011, Corollary 7.2]. Our proof is similar to the proof in the p -adic case, but in the archimedean case there are many analytical difficulties. We overcome them in Sections 5–7.

Finally, in Appendix B we prove a converse-type theorem. We prove that if $(\pi, V) = I(\chi)$ is an irreducible, generic, admissible unitary representation of $G_n(\mathbb{C})$ such that for every unitary character $\chi'(z) = (z/|z|)^{2m}$ with $m \in \mathbb{Z}$ we have

$$\gamma\left(\frac{1}{2}, \pi \times \chi', \psi\right) = 1,$$

then π is $G_n(\mathbb{R})$ -distinguished. The proof is done by a combinatorial argument combined with the Tadic–Vogan classification of the unitary dual of $G_n(\mathbb{C})$.

2. Notation and preliminaries

Let $M(a \times b, F)$ be the space of matrices with a rows and b columns with entries in F , where F is either \mathbb{R} or \mathbb{C} . Let $\eta_n = (0, 0, \dots, 1)$ be an element of $M(1 \times n, \mathbb{R})$, and let $P_n(\mathbb{R})$ be the subgroup of $G_n(\mathbb{R})$ consisting of all $n \times n$ matrices with the last row equal to η_n .

Let $U_n(F)$ be the group of all upper triangular matrices in $M(n \times n, F)$ with unit diagonal. Let

$$K_n = \{g \in G_n(\mathbb{C}) : g \cdot {}^t\bar{g} = I\}$$

be the standard maximal compact subgroup of $G_n(\mathbb{C})$.

For $g \in G_n(\mathbb{C})$, define

$$\|g\| := \sqrt{\sum_{i,j=1}^n |g_{ij}|^2} \quad \text{and} \quad \|g\|_H := \max(\|g\|, \|g\|^{-1}).$$

The value $\|g\|_H$ is called a norm on g (see [Wallach 1988, Section 2.A.2] for a general discussion on norms on a reductive group).

Let G be a group and H its subgroup. We say that a function $f : G \rightarrow \mathbb{C}$ is H -finite if the dimension of the space spanned by right H -translations of f is finite. In this work we will often consider K_n -finite functions on $G_n(\mathbb{C})$.

For V , a finite dimensional vector space over \mathbb{R} , we denote by $\mathcal{S}(V)$ the Schwartz space of all infinitely differentiable functions $f : V \rightarrow \mathbb{C}$ of rapid decay.

Let $\Phi \in \mathcal{S}(V)$, where $V = M(a \times b, \mathbb{C})$. We denote by $\hat{\Phi}$ the Fourier transform of Φ . It is a function on the same space, defined by

$$\hat{\Phi}(X) = \int \Phi(Y) \psi(-\text{Tr}({}^tXY)) dY.$$

For $\Phi \in \mathcal{S}(\mathbb{C}^n)$ and $g \in G_n(\mathbb{C})$ we denote by $(R(g)\Phi)(x) := \Phi(xg)$ the right translation of Φ by g .

For $z = x + iy \in \mathbb{C}$ we denote by $|z| = \sqrt{x^2 + y^2}$ the usual absolute value of z and by $|z|_{\mathbb{C}} = |z|^2 = x^2 + y^2$ the square of the usual absolute value. Note that $\mu(zA) = |z|_{\mathbb{C}} \mu(A)$, where $A \subset \mathbb{C}$ is an open set and μ is a Haar measure on \mathbb{C} .

Let W_n equal S_n and let $W_{n,2} = \{w \in W_n : w^2 = 1\}$ be the set of involutions in W_n . For $w \in W_{n,2}$ set

$$I_w = \{(i, j) : i > j, w(i) > w(j)\},$$

and define for any function $\kappa : I_w \rightarrow \mathbb{Z}_{\geq 0}$ a character α_{κ} of B_n by the formula

$$\alpha_{\kappa}(\text{diag}(t_1, \dots, t_n)) = \prod_{(i,j) \in I_w} \left[\frac{t_i}{t_j} \right]^{\kappa(i,j)}.$$

We will identify α_{κ} with the one-dimensional representation of B_n on the vector space \mathbb{C} with the action of α_{κ} . By abuse of notation we will denote both the function and the one-dimensional representation by the same symbol, α_{κ} .

For the convenience of the reader we write here notation and formulations of some of the theorems that appear in [Aizenbud and Lapid 2012], in versions that are suitable for this work.

Let G be an arbitrary group.

- For any G -set X and a point $x \in X$, we denote by $G(x)$ the G -orbit of x and by G^x the stabilizer of x .
- For any representation of G on a vector space V and a character χ of G , we denote by $V^{G,\chi}$ the subspace of (G, χ) -equivariant vectors in V .
- Given manifolds $L \subseteq M$, we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M and by $CN_L^M := (N_L^M)^*$ the conormal bundle. For any point $y \in L$, we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space to L in M at the point y .
- The symmetric algebra of a vector space V is denoted by

$$\text{Sym}(V) = \bigoplus_{k \geq 0} \text{Sym}^k(V).$$

- The Fréchet space of Schwartz functions on a Nash manifold X is denoted by $\mathcal{S}(X)$ and the dual space of Schwartz distributions by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$.
- For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

See [Aizenbud and Lapid 2012, p. 309] for more details.

Suppose X is a smooth manifold with G acting on X . Recall that $X = \bigcup_{i=1}^l X_i$ is called a G -invariant stratification if all sets X_i are G -invariant and there is some reordering $X_{i_1}, X_{i_2}, \dots, X_{i_l}$ of X_1, \dots, X_l such that all the sets

$$X_{i_1}, X_{i_1} \cup X_{i_2}, \dots, X_{i_1} \cup X_{i_2} \cup \dots \cup X_{i_k}, \dots, X = X_{i_1} \cup \dots \cup X_{i_l}$$

are open in X .

Lemma 2.1. *Let a real algebraic Lie group G act on a real algebraic smooth manifold X . Let $X = \bigcup_{i=1}^l X_i$ be a G -invariant stratification, and let χ be a character of G . If*

$$\mathcal{S}^*(X)^{G, \chi} \neq 0,$$

then there exist an $1 \leq i \leq l$ and $k \geq 0$ such that

$$\mathcal{S}^*(X_i, \text{Sym}^k(\text{CN}_{X_i}^X))^{G, \chi} \neq 0.$$

This lemma is a special case of [Aizenbud and Lapid 2012, Proposition B.3].

Theorem 2.2 [Aizenbud and Lapid 2012, Theorem B.6]. *Let G be a real algebraic group acting transitively on a real algebraic smooth manifold Z and let $\varphi : X \rightarrow Z$ be a G -equivariant smooth map. Fix $z \in Z$ and let X_z be the fiber of z . Let χ be a tempered character of G [Aizenbud et al. 2008, Definition 5.1.1], and let δ_G and δ_{G_z} be the modulus characters of the groups G and G_z respectively. Then $\mathcal{S}^*(X)^{G, \chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z, \chi \delta_{G_z}^{-1} \delta_G}$.*

Moreover, for any G -equivariant bundle E on X , the space $\mathcal{S}^(X, E)^{G, \chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z, E|_{X_z})^{G_z, \chi \delta_{G_z}^{-1} \delta_G}$.*

3. Some matrix spaces decompositions

In this section we obtain some matrix space decompositions that will be used in this work. In the following lemma we analyze the structure of orbits of the action of the Borel subgroup B_n on X_n . Let $W_n = S_n$ be the Weyl group of $G_n(\mathbb{C})$.

Lemma 3.1. *There is a bijection between $B_n \backslash X_n = B_n \backslash G_n(\mathbb{C}) / G_n(\mathbb{R})$ and the space of involutions $W_{n,2} = \{w \in W_n : w^2 = 1\}$.*

Proof. Recall that $X_n = \{x \in G_n(\mathbb{C}) : x \cdot \bar{x} = I\}$ and let $x \in X_n$. Let

$$T = \{\text{diag}(d_1, \dots, d_n) : d_i \in \mathbb{C}^* \text{ for all } i\}$$

be a maximal torus in $G_n(\mathbb{C})$. From [Lapid and Rogawski 2003, Lemma 4.1.1] (see also [Springer 1984]), the B_n -orbit of x intersects the normalizer

$$N(T) := \{g \in G_n(\mathbb{C}) : gTg^{-1} = T\}.$$

It is a well-known fact that $N(T) = \{d \cdot w : d \in T, w \in W_n\}$. Thus, we may assume $x = dw$, where $w \in W_n$ and $d = \text{diag}(d_1, d_2, \dots, d_n)$. Note that w is uniquely determined by x . Since $x \cdot \bar{x} = I$, we have $dw = w^{-1}\bar{d}^{-1}$. We obtain $w = w^{-1}$ and therefore $w^2 = 1$, i.e., $w \in W_{n,2}$.

Therefore, we can assume that $w \in W_{n,2}$ in the decomposition $x = dw$. On the other hand, it is clear that different involutions $w, w' \in W_{n,2}$ belong to disjoint orbits of B_n . Indeed, $l(b)w := bw\bar{b}^{-1} \neq w'$ for all $b \in B_n$.

It remains to show that the B_n -orbit of $x = dw$ contains the point w , i.e., there is some $b \in B_n$ such that $l(b)x = w$. Since w is an involution it is enough to check the claim for 1×1 and 2×2 matrices. For the 1×1 matrix $x = (b)_{1 \times 1}$, the assumption $x\bar{x} = I$ gives $b\bar{b} = 1$, and we want to prove that $b = \mu\bar{\mu}^{-1}$. Clearly, there is such a μ .

For a 2×2 matrix of the form $b = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, the assumption

$$x = b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_1 \\ d_2 & 0 \end{pmatrix} \in X$$

gives the condition $d_1\bar{d}_2 = 1$ on the entries d_1, d_2 . We seek an invertible matrix $\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ such that

$$(3-1) \quad \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} 0 & d_1 \\ d_2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mu}_1^{-1} & 0 \\ 0 & \bar{\mu}_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Matrix multiplication gives the condition $d_1\mu_1\bar{\mu}_2^{-1} = 1$, and clearly there are such a μ_1 and μ_2 . □

In the next paragraph let us fix n and define $G = G_n(\mathbb{C})$, $H = G_n(\mathbb{R})$. Our goal is to obtain a generalized Cartan decomposition $G = KAH$, where K is a maximal compact subgroup of G consisting of all unitary matrices in G and A is a torus which we will now describe. Let $m = [n/2]$. Note that $H = G^\sigma$ and $K = G^\tau$, where $\sigma(g) = \bar{g}$ and $\tau(g) = g^* = {}^t\bar{g}$. Let \mathfrak{g} be the Lie algebra of G over the field \mathbb{C} . Following [Kobayashi 2007, Fact 2.1, p. 7], we take \mathfrak{a} to be a maximal abelian subspace in

$$\mathfrak{g}^{-\sigma, -\tau} = \{X \in \mathfrak{g} : \tau X = \sigma X = -X\}.$$

Following this recipe, let us define

$$\mathfrak{a} = \sum_{j=1}^m i\mathbb{R}(E_{2j+1,2j} - E_{2j,2j+1}).$$

Recall that

$$\exp\begin{pmatrix} 0 & it \\ -it & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{ch}(t) & i \operatorname{sh}(t) \\ -i \operatorname{sh}(t) & \operatorname{ch}(t) \end{pmatrix}$$

and let A be the Lie group corresponding to \mathfrak{a} . Denote by $a(t_1, t_2, \dots, t_m)$ the $n \times n$ matrix which consists of m 2×2 diagonal blocks of the form $\exp\begin{pmatrix} 0 & it_j \\ -it_j & 0 \end{pmatrix}$, where $j = 1, 2, \dots, m$ if $n = 2m$ is even, and which consists of these blocks and $a_{nn} = 1$ in the last diagonal place if $n = 2m + 1$ is odd. For example, if $n = 4$ then

$$a(t_1, t_2) = \begin{pmatrix} \cosh(t_1) & i \sinh(t_1) & 0 & 0 \\ -i \sinh(t_1) & \cosh(t_1) & 0 & 0 \\ 0 & 0 & \cosh(t_2) & i \sinh(t_2) \\ 0 & 0 & -i \sinh(t_2) & \cosh(t_2) \end{pmatrix}.$$

We have

$$A = \{a(t_1, t_2, \dots, t_m) : t_1, t_2, \dots, t_m \in \mathbb{R}\}.$$

Define

$$A^+ = \{a(t_1, t_2, \dots, t_m) : t_1 \geq t_2 \geq \dots \geq t_m \geq 0\}.$$

Theorem 3.2. *There is a decomposition $G = KA^+H$. That is, every element $g \in G$ can be written as*

$$(3-2) \quad g = kah, \quad \text{where } k \in K, a \in A^+, h \in H.$$

Moreover, the $a \in A^+$ in decomposition (3-2) is uniquely determined by g .

Remark 3.3. By taking the transpose of (3-2) we obtain a similar decomposition $G = HA^+K$. That is, every $g \in G$ can be written as

$$(3-3) \quad g = hak, \quad \text{where } h \in H, a \in A^+, k \in K,$$

and $a \in A^+$ in this decomposition is uniquely determined by g . Actually, after taking the transpose of (3-2), we obtain that $a^t \in A$ and in general $a^t \notin A^+$. But the permutation group S_n is naturally contained in both K and H and we can replace $a \in A$ with $a' = w_1 a w_2$ such that $a' \in A^+$.

Proof of Theorem 3.2. To prove the existence part we will show that $G = KAH$. Since permutation matrices are clearly in $H \cap K$, the equality $G = KA^+H$ will easily follow from the equality $G = KAH$. Let $g \in G$. Our goal is to achieve a decomposition $g = kah$ with $h \in H$, $a \in A$, and $k \in K$. Suppose that g is of such a form. Then, since $h^* = {}^t h$, $a^* = a$, and $k^* = k^{-1}$ we get

$$(3-4) \quad g^* g = {}^t h a^2 h, \quad h \in H, a \in A.$$

On the other hand, suppose that every matrix of the form $g^* g$ can be written as (3-4). Then write $g = ((g^*)^{-1} {}^t h a) a h$, and let us show that $k = (g^*)^{-1} {}^t h a$ is a unitary

matrix. Indeed,

$$k^*k = a^*({}^t h)^*((g^*)^{-1})^*(g^*)^{-1}{}^t h a = ah(g^*g)^{-1}{}^t h a = ah({}^t h a^2 h)^{-1}{}^t h a = I.$$

Therefore, to prove the existence part of the theorem, it is enough to prove that every matrix of the form g^*g can be written in the form (3-4). For this purpose write $g^*g = x + iy$, $x, y \in H$. Then $x = {}^t x$ is symmetric, and $y = -{}^t y$ is antisymmetric. Also, ${}^t v g^* g v > 0$ for every $0 \neq v \in \mathbb{R}^n$. Hence, x is a positive definite matrix; that is, ${}^t v x ({}^t x) v = {}^t v g^* g v > 0$ for every $0 \neq v \in \mathbb{R}^n$. Thus, there is a matrix $h \in H$ such that ${}^t h x h = I$. Then ${}^t h g^* g h = I + i({}^t h y h)$. The matrix $z := {}^t h y h$ is antisymmetric and it is a standard fact in linear algebra that it is diagonalizable by a real orthogonal matrix. Consequently, $h' z h'^{-1} = d$, with d consisting of $m = \lfloor n/2 \rfloor$ 2×2 blocks of the form

$$\begin{pmatrix} d_{2i-1,2i-1} & d_{2i-1,2i} \\ d_{2i,2i-1} & d_{2i,2i} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

in the case when n is even, and m such blocks and the last row zero in the case when n is odd. Note also that the numbers λ_i are uniquely determined up to a permutation by the matrix gg^* since they are eigenvalues of $hy{}^t h$. Clearly, every 2×2 block of the form $\begin{pmatrix} 1 & i\lambda \\ -i\lambda & 1 \end{pmatrix}$ can be transformed by a diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ to the form

$$\begin{pmatrix} \text{ch}(\mu) & i \text{sh}(\mu) \\ -i \text{sh}(\mu) & \text{ch}(\mu) \end{pmatrix} = \exp\left(\mu \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right).$$

Taking in every block a of the form $\exp\left(\frac{\mu}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)$ proves the existence of the decomposition $g^*g = {}^t h a^2 h$ and thus establishes the existence of the decomposition $g = kah$.

We now prove the uniqueness part of the theorem. Note that H acts on the space of positive definite matrices of the form g^*g by $h \cdot x := h{}^t x h$. Let us take $h, b, c \in H$ and suppose $h \cdot (I + ib) = I + ic$. Then h is an orthogonal matrix, ${}^t h h = I$, and thus $c = h^{-1} b h$. In particular, the eigenvalues of b and c are equal. Now, to prove the uniqueness of $a \in A^+$ in the decomposition (3-2) let us write $a = \text{Re}(a) + i \text{Im}(a)$ and note that $H \cdot a = H \cdot (I + i \text{Im}(a))$. Since the eigenvalues of $i \text{Im}(a)$ are $\pm \sinh(\lambda_1), \dots, \pm \sinh(\lambda_n)$ we see that if $a_1, a_2 \in A^+$ and $a_1 \neq a_2$, then $H \cdot (I + i \text{Im}(a_1)) \neq H \cdot (I + i \text{Im}(a_2))$, and therefore $H \cdot a_1 \neq H \cdot a_2$. It follows that the $a^2 \in A^+$ part in $g^*g = {}^t h a^2 h$ is uniquely determined by g . As a result, $a \in A^+$ is uniquely determined by g . \square

4. Proof of Theorem 1.2

In this paragraph n is fixed and $G = G_n(\mathbb{C})$, $H = G_n(\mathbb{R})$, and $B = B_n(\mathbb{C})$. We denote by M the standard maximal torus in G and by $W_2 = W_{n,2}$ the set of involutions

in S_n . As a starting point of the proof, observe that

$$I(\chi)^* = \mathcal{S}^*(G)^{B, \chi \delta_0^{-\frac{1}{2}}},$$

where B acts on the space of tempered distributions $\mathcal{S}^*(G)$ from the left. We have

$$\mathrm{Hom}_H(I(\chi), \mathbb{C}) = \mathcal{S}^*(G/H)^{B, \chi \delta_0^{-\frac{1}{2}}}.$$

We will stratify $X := G/H$ by B -orbits. By [Lemma 3.1](#), we have $B \backslash X = W_2$. Suppose $\mathrm{Hom}_H(I(\chi), \mathbb{C}) \neq 0$. By [Lemma 2.1](#) there exists an involution $w \in W_2$ and a $k \geq 0$ such that

$$\mathcal{S}^*(B(w), \mathrm{Sym}^k(CN_{B(w)}^X))^{B, \chi \delta_0^{-\frac{1}{2}}} \neq 0.$$

Note that B acts on $B(w)$ transitively, the stabilizer of w under the action of B is B_w , and $\delta_0^{1/2}|_{B^w} = \delta_{B^w}$. Therefore, by Frobenius reciprocity ([Theorem 2.2](#)),

$$\begin{aligned} \mathcal{S}^*(B(w), \mathrm{Sym}^k(CN_{B(w)}^X))^{B, \chi \delta_0^{-\frac{1}{2}}} &= \mathcal{S}^*({w}, \mathrm{Sym}^k(CN_{B(w), w}^X))^{B^w, \chi \delta_0^{-\frac{1}{2}} \delta_{B^w}^{-1} \delta_0} \\ &= \mathcal{S}^*({w}, \mathrm{Sym}^k(CN_{B(w), w}^X))^{B^w, \chi} \\ &= (\mathrm{Sym}^k(N_{B(w), w}^X) \otimes_{\mathbb{R}} \mathbb{C})^{B^w, \chi}. \end{aligned}$$

Observe that $M^w \subset B^w$. Hence $(\mathrm{Sym}^k(N_{B(w), w}^X) \otimes_{\mathbb{R}} \mathbb{C})^{B^w, \chi} \neq 0$ implies

$$(\mathrm{Sym}^k(N_{B(w), w}^X) \otimes_{\mathbb{R}} \mathbb{C})^{M^w, \chi} \neq 0.$$

Note that

$$\begin{aligned} M^w &= \{t \in M : t^{-1} w \bar{t} = w\} = \{t \in M : t = w \bar{t} w\} \\ &= \{t = \mathrm{diag}(t_1, t_2, \dots, t_n) \in M : t_i = \overline{t_w(i)} \text{ for } 1 \leq i \leq n\}. \end{aligned}$$

It will be useful to obtain one more formula for M^w . It is easy to see, by examining the case of 1×1 and 2×2 matrices, that

$$(4-1) \quad M^w = \{t(w \bar{t} w) a : t \in M, a = \mathrm{diag}(a_1, a_2, \dots, a_n), \\ a_i = 1 \text{ if } w(i) \neq i, a_i = \pm 1 \text{ if } w(i) = i\}.$$

In the next lemma we perform a calculation of the normal space $N_{B(w), w}^X$. Note that this is a finite-dimensional vector space over \mathbb{R} . Since the group M^w preserves the tangent space $T_w^{B(w)}$ and clearly preserves the tangent space T_w^X , there is an action of M^w on the normal space $N_{B(w), w}^X$. By taking the scalar extension $N_{B(w), w}^X \otimes \mathbb{C}$, we get a complex representation of M^w . Since M^w is abelian, this representation decomposes into a direct sum of irreducible, one-dimensional representations.

Lemma 4.1. *We have*

$$N_{B(w),w}^X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{(i,j) \in I_w} \alpha_{\delta(i,j)}$$

as a representation of M^w .

Before proving this lemma, we give the following corollary.

Corollary 4.2. *We have*

$$\text{Sym}(N_{B(w),w}^X \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{\kappa: I_w \rightarrow \mathbb{Z}_{\geq 0}} \alpha_{\kappa}$$

as a representation of M^w .

Proof of Lemma 4.1. Let us denote by $e_{i,j}$ the elementary matrix with 1 at the (i, j) -th entry and zeros in all other entries. The tangent space of X at w is equal to

$$\begin{aligned} T_w^X &= \{A \in \text{Mat}_n(\mathbb{C}) : Aw + w\bar{A} = 0\} = \{A \in \text{Mat}_n(\mathbb{C}) : wAw = -\bar{A}\} \\ &= \text{Span}_{\mathbb{R}}\{-e_{i,j} + e_{w(i),w(j)}, \sqrt{-1}(e_{i,j} + e_{w(i),w(j)}) : 1 \leq i, j \leq n\}. \end{aligned}$$

On the other hand,

$$T_w^{B(w)} = \{-Aw + w\bar{A} : A \in \text{Mat}_n(\mathbb{C}), A \text{ is upper triangular}\}.$$

Since $e_{i,j}w = e_{i,w(j)}$ and $we_{i,j} = e_{w(i),j}$, we obtain that

$$\begin{aligned} T_w^{B(w)} &= \text{Span}_{\mathbb{R}}\{-e_{i,w(j)} + e_{w(i),j}, \sqrt{-1}(e_{i,w(j)} + e_{w(i),j}) : 1 \leq i \leq j \leq n\} \\ &= \text{Span}_{\mathbb{R}}\{-e_{i,j} + e_{w(i),w(j)}, \sqrt{-1}(e_{i,j} + e_{w(i),w(j)}) : i \leq w(j)\} \\ &= \text{Span}_{\mathbb{C}}\{e_{i,j}, e_{w(i),w(j)} : i \leq w(j)\} \cap T_w^X \\ &= \text{Span}_{\mathbb{C}}\{e_{i,j} : i \leq w(j) \text{ or } j \geq w(i)\} \cap T_w^X. \end{aligned}$$

Hence

$$\begin{aligned} (4-2) \quad N_{B(w),w}^X &\cong \text{Span}_{\mathbb{C}}\{e_{i,j} : i > w(j), w(i) > j\} \cap T_w^X \\ &= \text{Span}_{\mathbb{C}}\{e_{i,w(j)} : i > j, w(i) > w(j)\} \cap T_w^X \\ &= \text{Span}_{\mathbb{C}}\{e_{i,w(j)} : (i, j) \in I_w\} \cap T_w^X. \end{aligned}$$

Let us denote $V = \text{Span}_{\mathbb{C}}\{e_{i,w(j)} : (i, j) \in I_w\}$. Note that if $e_{i,w(j)} \in V$ then also $e_{w(i),j} \in V$, since w is an involution and for an involution

$$(i, j) \in I_w \iff (w(i), w(j)) \in I_w.$$

Let us use the lexicographic ordering on pairs (i, j) : write $(i, j) < (i', j')$ if $i < i'$ or if $i = i'$ and $j < j'$. Then we may rewrite (4-2) as

$$(4-3) \quad N_{B(w),w}^X \cong \text{Span}_{\mathbb{R}}\{\sqrt{-1}e_{i,w(j)} : (i, j) \in I_w, (i, j) = (w(i), w(j))\} \\ \oplus \text{Span}_{\mathbb{R}}\{e_{i,w(j)} - e_{w(i),j}, \sqrt{-1}(e_{i,w(j)} + e_{w(i),j}) : \\ (i, j) \in I_w, (i, j) < (w(i), w(j))\}.$$

For $t = \text{diag}(t_1, \dots, t_n) \in M$ we have

$$te_{i,j}\bar{t}^{-1} = (t_i/\bar{t}_j)e_{i,j},$$

and for $t \in M^w$ we also have

$$te_{i,w(j)}\bar{t}^{-1} = (t_i/\bar{t}_{w(j)})e_{i,w(j)} = (t_i/t_j)e_{i,w(j)}.$$

Therefore, the action of M^w on $e_{i,w(j)}$ is given by $\alpha_{\delta(i,j)}$. We obtain that, as a representation of M^w ,

$$N_{B(w),w}^X \otimes_{\mathbb{R}} \mathbb{C} \\ \cong \bigoplus_{\{(i,j) \in I_w : (i,j) = (w(i), w(j))\}} \alpha_{\delta(i,j)} \oplus \bigoplus_{\{(i,j) \in I_w : (i,j) < (w(i), w(j))\}} (\alpha_{\delta(i,j)} \oplus \alpha_{\delta(w(i), w(j))}) \\ = \bigoplus_{\{(i,j) \in I_w : (i,j) = (w(i), w(j))\}} \alpha_{\delta(i,j)} \oplus \bigoplus_{\{(i,j) \in I_w : (i,j) < (w(i), w(j))\}} \alpha_{\delta(i,j)} \oplus \bigoplus_{\{(i,j) \in I_w : (i,j) > (w(i), w(j))\}} \alpha_{\delta(i,j)} \\ = \bigoplus_{\{(i,j) \in I_w\}} \alpha_{\delta(i,j)}. \quad \square$$

Lemma 4.3. *If $(\text{Sym}^k(N_{B(w),w}^X \otimes_{\mathbb{R}} \mathbb{C}))^{M^w, \chi} \neq 0$ then $k = 0$, $w(\chi) = \bar{\chi}^{-1}$, and $\chi_i(-1) = 1$ for all $1 \leq i \leq n$ such that $w(i) = i$.*

Proof. Note that, for $t \in M$ and $w \in W_2$, the element wtw is also diagonal and its diagonal entries are the permutation of diagonal entries of t by w , i.e.,

$$(wtw)_{ii} = t_{w(i),w(i)}.$$

By (4-1), if $\alpha_{\kappa}|_{M^w} = \chi|_{M^w}$, then for every $t \in M$ we have

$$\alpha_{\kappa}(t(w\bar{t}w)) = \alpha_{\kappa}(t)\overline{\alpha_{\kappa}(t)} = \chi(t)\overline{\chi(w(t))} = \chi(t(w\bar{t}w)).$$

That is,

$$(4-4) \quad (\alpha_{\kappa}|_{M^w} = \chi|_{M^w}) \Rightarrow (\bar{\alpha}_{\kappa}w(\alpha_{\kappa}) = \bar{\chi}w(\chi)).$$

To obtain (4-4) just put $a = 1$ in (4-1). The set of κ s that satisfy

$$(4-5) \quad \bar{\alpha}_{\kappa}w(\alpha_{\kappa}) = \bar{\chi}w(\chi)$$

is $\{\kappa \equiv 0\}$ if $w(\chi) = \overline{\chi}^{-1}$ and is empty otherwise. Indeed, we take the absolute value on both sides of (4-5) to obtain

$$(4-6) \quad \prod_{(i,j) \in I_w} \left| \frac{t_i}{t_j} \right|^{\kappa(i,j) + \kappa(w(i), w(j))} = \prod_{i=1}^n |t_i|^{\lambda_i + \lambda_{w(i)}}.$$

First, we will deduce from the last equation that the right-hand side of this equation is 1. Note that from (4-6) it follows, by substituting $t_i = c$ for all i with a generic $c \in \mathbb{C}^*$, that

$$\lambda_1 + \dots + \lambda_n = 0.$$

Since no pair $(1, i)$ belongs to I_w , it follows that $\lambda_1 + \lambda_{w(1)} \leq 0$. Let i be the first index such that $\lambda_i + \lambda_{w(i)} > 0$. Then on the left-hand side of (4-6) the power of $|t_i|$ is positive, thus there is a j such that $(i, j) \in I_w$. Hence $i > j$, $w(i) > w(j)$ and from the assumption $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ we obtain $\lambda_i \leq \lambda_j$ and $\lambda_{w(i)} \leq \lambda_{w(j)}$. Thus

$$0 < \lambda_i + \lambda_{w(i)} \leq \lambda_j + \lambda_{w(j)} \leq 0,$$

a contradiction! Therefore, for every i , there is an inequality $\lambda_i + \lambda_{w(i)} \leq 0$. Since the sum of all λ s is equal to 0, we obtain $\lambda_i + \lambda_{w(i)} = 0$ for every i . Hence, $\lambda_1 \geq 0$ and $\lambda_n \leq 0$.

Now, we can deduce $\kappa \equiv 0$. Let j be the minimal index such that there exists a pair $(i, j) \in I_w$ with the property

$$\kappa(i, j) \neq 0 \quad \text{or} \quad \kappa(w(i), w(j)) \neq 0.$$

The power of $|t_j|$ on the right-hand side of (4-5) must equal 0, thus there is a pair $(j, k) \in I_w$ such that $\kappa(j, k) \neq 0$ or $\kappa(w(j), w(k)) \neq 0$. In both cases we obtain a contradiction to the minimality of j . As a conclusion, we obtain that (4-5) implies $\kappa \equiv 0$.

Suppose now that $w(\chi) = \overline{\chi}^{-1}$ and thus $\kappa \equiv 0$. Then $\alpha_\kappa = 1$, the identity character. We want to prove that $\chi_i(-1) = 1$ for all i such that $w(i) = i$. This follows from $\chi(a) = \alpha_\kappa(a) = 1$ for $a = \text{diag}(a_1, \dots, a_n)$, where $a_i = \pm 1$ whenever $w(i) = i$ and $a_i = 1$ otherwise. \square

5. Calculation of Rankin–Selberg gamma factors

In this section we recall the notion of Rankin–Selberg integrals and apply the results of previous sections to calculate special values of Rankin–Selberg gamma factors. The exposition and notation follows [Jacquet 2009]. Let $\chi : B_n \rightarrow \mathbb{C}^\times$ be a multiplicative character and let $\lambda : I(\chi) \rightarrow \mathbb{C}$ be a ψ -form on $I(\chi)$. Recall that such a λ always exists and it is unique up to a scalar multiple. If $f \in V$, $g \in G_n$, we set

$$W_f(g) = \lambda(R(g)f).$$

Let $\mathcal{W}(I(\chi), \psi)$ be the space spanned by the functions of the form W_f .

For every n , we denote by w_n the $n \times n$ permutation matrix whose antidiagonal entries are 1. If $n > n'$, we define

$$w_{n,n'} = \begin{pmatrix} 1_{n'} & 0 \\ 0 & w_{n-n'} \end{pmatrix}.$$

If $f \in I(\chi)$, then the function \tilde{f} is defined by

$$\tilde{f}(g) := f(w_n {}^t g^{-1}).$$

Let π be an irreducible representation of $G_n(\mathbb{C})$ and let π' be an irreducible representation of $G_{n'}(\mathbb{C})$. Suppose π is the Langlands quotient of $I(\chi)$ and π' is the Langlands quotient of $I(\chi')$. We choose a ψ -form λ on $I(\chi)$ and a $\bar{\psi}$ -form λ' on $I(\chi')$. Rankin–Selberg integrals are defined as follows. For $f \in I(\chi)$, $f' \in I(\chi')$, set

$$W = W_f, \quad W' = W_{f'}.$$

For $W = W_f$, set

$$\tilde{W}_f := W_f(w_n {}^t g^{-1}).$$

Note that $\tilde{W}_f(g) = W_{\tilde{f}}(g)$ and $W_{\tilde{f}}(g) \in \mathcal{W}(I(\chi^{-1}), \bar{\psi})$.

If $n > n'$, we set

$$(5-1) \quad \Psi(s, W, W') = \int W \begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} W'(g) |\det g|_{\mathbb{C}}^{s - \frac{n-n'}{2}} dg.$$

In addition, for $0 \leq j \leq n - n' - 1$, we set

$$(5-2) \quad \Psi_j(s, W, W') = \int W \begin{pmatrix} g & 0 & 0 \\ X & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix} W'(g) |\det g|_{\mathbb{C}}^{s - \frac{n-n'}{2}} dX dg.$$

Here X is integrated over the space $M(m \times j, \mathbb{C})$ of matrices with m rows and j columns. In each integral, g is integrated over the quotient $U_n(\mathbb{C}) \backslash G_n(\mathbb{C})$.

If $n = n'$, we let Φ be a Schwartz function on \mathbb{C}^n and we set

$$(5-3) \quad \Psi(s, W, W', \Phi) = \int W(g) W'(g) \Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{C}}^s dg.$$

The Rankin–Selberg gamma factor $\gamma(s, \pi \times \pi', \psi)$ is a proportionality factor appearing in functional equations on certain Rankin–Selberg integrals. We quote here [Jacquet 2009, Theorem 2.1].

- Theorem 5.1.** (1) *The integrals (5-1), (5-2), and (5-3) converge for $\text{Re}(s) \gg 0$.*
 (2) *Each integral extends to a meromorphic function of s which is a holomorphic multiple of $L(s, \pi \times \pi')$ bounded at infinity in vertical strips. See [Jacquet 2009] for the definition of $L(s, \pi \times \pi')$.*
 (3) *The following functional equations are satisfied. If $n > n'$,*

$$\Psi(1-s, \widetilde{W}, \widetilde{W}') = \omega_{I(\chi)}(-1)^{n-1} \omega_{I(\chi')}(-1) \gamma(s, I(\chi) \times I(\chi'), \psi) \Psi(s, W, W').$$

If $n > n' + 1$ and $\beta = n - n' - 1 - j$,

$$\Psi_j(1-s, \pi(w_{n,n'}) \widetilde{W}, \widetilde{W}') = \omega_{I(\chi)}(-1)^n \omega_{I(\chi')}(-1) \gamma(s, \pi \times \pi', \psi) \Psi_\beta(s, W, W').$$

If $n = n'$,

$$\Psi(1-s, \widetilde{W}, \widetilde{W}', \hat{\Phi}) = \omega_{I(\chi)}(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) \Psi(s, W, W', \Phi).$$

We will calculate the special values of the Rankin–Selberg gamma factor of $G_n(\mathbb{R})$ -distinguished representations. The main tool will be the classification of such representations obtained in Theorem 1.2 and basic properties of Rankin–Selberg gamma factors from [Jacquet 2009, Lemma 16.3].

Let us recall some facts about one-dimensional Tate gamma factors. Let χ be a one-dimensional character $\chi : \mathbb{C}^* \rightarrow \mathbb{C}^*$. We have the following functional equation for Tate gamma factors:

$$(5-4) \quad \gamma(s, \chi, \psi) \gamma(1-s, \overline{\chi}^{-1}, \overline{\psi}^{-1}) = 1.$$

Since we assume ψ is trivial on \mathbb{R} , we obtain $\overline{\psi}^{-1} = \psi$, and for $s = 1/2$ we get

$$(5-5) \quad \gamma\left(\frac{1}{2}, \chi, \psi\right) \gamma\left(\frac{1}{2}, \overline{\chi}^{-1}, \psi\right) = 1.$$

For a real character χ , that is, for χ satisfying $\chi^2 = 1$, we obtain $\gamma(1/2, \chi, \psi)^2 = 1$, and thus $\gamma(1/2, \chi, \psi) \in \{1, -1\}$. The value of $\gamma(1/2, \chi, \psi)$ depends on $\chi(-1)$. Whenever $\chi(-1) = 1$ we obtain

$$(5-6) \quad \gamma\left(\frac{1}{2}, \chi, \psi\right) = 1.$$

Proof of Theorem 1.1. Recall that if π is the Langlands quotient of $I(\chi)$ and π' is the Langlands quotient of $I(\chi')$, then

$$\gamma(s, \pi \times \pi', \psi) = \gamma(s, I(\chi) \times I(\chi'), \psi).$$

It is well-known that $\chi = (\chi_1, \dots, \chi_t)$, where the χ_i 's are one-dimensional characters of \mathbb{C} . Similarly, $\chi' = (\chi'_1, \dots, \chi'_r)$, where the χ'_i 's are one-dimensional

characters of \mathbb{C} . Thus,

$$(5-7) \quad \gamma(s, I(\chi) \times I(\chi'), \psi) = \prod_{i=1}^t \gamma(s, \chi_i \times I(\chi'), \psi) = \prod_{i=1}^t \prod_{j=1}^r \gamma(s, \chi_i \chi'_j, \psi).$$

Using [Theorem 1.2](#), there exist involutions $w \in S_t$ and $w' \in S_r$ such that $w(\chi) = \bar{\chi}^{-1}$ and $w'(\chi') = \bar{\chi}'^{-1}$ and for every fixed point i of w , and j of w' , we have $\chi_i(-1) = 1$ and $\chi'_j(-1) = 1$. The formula in (5-7) may be rewritten as

$$\gamma(s, I(\chi) \times I(\chi'), \psi) = I_1 I_2,$$

where

$$I_1 = \prod_{\{(i,j):(w(i),w'(j))=(i,j)\}} \gamma\left(\frac{1}{2}, \chi_i \chi'_j, \psi\right),$$

$$I_2 = \prod_{\{(i,j):i < w(i) \text{ or } (i=w(i) \text{ and } w'(j) < j)\}} \gamma\left(\frac{1}{2}, \chi_i \chi'_j, \psi\right) \gamma\left(\frac{1}{2}, \chi_{w(i)} \chi'_{w'(j)}, \psi\right).$$

Let us prove that every term appearing in the product I_1 is 1. Indeed, by [Theorem 1.2](#) the character $\chi_i \chi'_j$ appearing as the argument in the gamma factor in I_1 is a real character satisfying $\chi_i \chi'_j(-1) = 1$, and therefore, by (5-6), we get

$$\gamma\left(\frac{1}{2}, \chi_i \chi'_j, \psi\right) = 1.$$

Each term in the product I_2 also equals 1, since $\chi_{w(i)} \chi'_{w'(j)} = \overline{(\chi_i \chi'_j)}^{-1}$ and by applying (5-5). Finally, $I_1 = I_2 = 1$ and we obtain

$$\gamma\left(\frac{1}{2}, \pi \times \pi', \psi\right) = 1. \quad \square$$

We will need the following technical result about Rankin–Selberg integrals in [Section 8](#).

Lemma 5.2. *Let (π, V) , (π', V') be generic representations of $G_n(\mathbb{C})$ and let*

$$\mathcal{W}(\pi, \psi), \quad \mathcal{W}(\pi', \psi^{-1})$$

be their Whittaker models. Suppose (π, V) is unitarizable and (π', V') is tempered. Let $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in \mathcal{S}(\mathbb{C}^n)$. Then the Rankin–Selberg integral

$$\int_{U_n(\mathbb{C}) \backslash G_n(\mathbb{C})} W(g) W'(g) \Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{C}}^s dg$$

converges absolutely at $s = 1/2$.

Proof. Define

$$T_n = \{\text{diag}(t_1, \dots, t_n) : t_i \in \mathbb{R} \text{ and } t_1 \geq t_2 \geq \dots \geq t_n > 0\}$$

and let K_n be a maximal compact subgroup of $G_n(\mathbb{C})$ consisting of all unitary matrices in $G_n(\mathbb{C})$. Let δ be the modular character of $B_n(\mathbb{C})$. By [Lapid and Mao 2014, Lemma 2.1], we know that there exist a $\lambda > -1/2$, a $d > 0$ and a continuous seminorm μ on $\mathcal{W}(\pi, \psi)$ such that

$$|W(tk)| \leq \delta^{\frac{1}{2}}(t) |\det t|_{\mathbb{C}}^{\lambda} |t_n|_{\mathbb{C}}^{-n\lambda} (1 + \|\log t\|)^d \mu(W)$$

for all $t \in T_n$, $k \in K_n$ and every $W \in \mathcal{W}(\pi, \psi)$. Similarly, there exist an $\epsilon < 0$ such that $\lambda + \epsilon > -1/2$, a $d' > 0$ and a continuous seminorm μ' on $\mathcal{W}(\pi', \psi^{-1})$ such that

$$|W'(tk)| \leq \delta^{\frac{1}{2}}(t) |\det t|_{\mathbb{C}}^{\epsilon} |t_n|_{\mathbb{C}}^{-n\epsilon} (1 + \|\log t\|)^{d'} \mu'(W')$$

for all $t \in T_n$, $k \in K_n$ and every $W' \in \mathcal{W}(\pi', \psi^{-1})$. Let us define $\lambda := \lambda + \epsilon$, $d := d + d'$. All that matters for the estimates is that $\lambda > 1/2$ and $d > 0$. There is a $\phi \in \mathcal{S}(\mathbb{R})$ such that

$$\begin{aligned} |W(tk)W'(tk)\Phi((0, 0, \dots, 0, 1)g)| \\ \leq \delta(t) |\det t|_{\mathbb{C}}^{\lambda} |t_n|_{\mathbb{C}}^{-n\lambda} (1 + \|\log t\|)^d \phi(t_n) \mu(W) \mu'(W') \end{aligned}$$

for all $t \in T_n$, $k \in K_n$, every $W \in \mathcal{W}(\pi, \psi)$ and every $W' \in \mathcal{W}(\pi', \psi^{-1})$. For fixed functions W, W' the numbers $\mu(W), \mu'(W')$ are constant and we can move them to the function ϕ . Thus we can rewrite the last estimate as

$$|W(tk)W'(tk)\Phi((0, 0, \dots, 0, 1)g)| \leq \delta(t) |\det t|_{\mathbb{C}}^{\lambda} |t_n|_{\mathbb{C}}^{-n\lambda} (1 + \|\log t\|)^d \phi(t_n)$$

for all $t \in T_n$, $k \in K_n$. Let us rewrite the expression

$$\int_{U_n(\mathbb{C}) \backslash G_n(\mathbb{C})} |W(g)W'(g)\Phi((0, 0, \dots, 0, 1)g)| |\det g|_{\mathbb{C}}^{\text{Re}(s)} dg$$

using the Iwasawa decomposition to obtain

$$(5-8) \quad \int_{K_n} \int_{T_n} |W(tk)W'(tk)\Phi((0, 0, \dots, 0, 1)g)| |\det t|_{\mathbb{C}}^{\text{Re}(s)} \delta^{-1}(t) dt dk.$$

For $f : G_n(\mathbb{C}) \rightarrow \mathbb{C}$ such that the following integrals are absolutely convergent, we have

$$\begin{aligned} \int_{G_n(\mathbb{C})} f(g) dg &= \int_{U_n(\mathbb{C})} \int_{T_n} \int_{K_n} f(tuk) du dt dk \\ &= \int_{U_n(\mathbb{C})} \int_{T_n} \int_{K_n} f((tut^{-1})tk) du dt dk \\ &= \int_{U_n(\mathbb{C})} \int_{T_n} \int_{K_n} f(utk)\delta^{-1}(t) du dt dk. \end{aligned}$$

Let us define $\alpha_i(t) = t_i/t_{i+1}$ for $t \in T_n$. Note that

$$\det t = \prod_{i=1}^{n-1} \alpha_i(t)^i t_n^n.$$

The integrand in (5-8) is bounded by

$$\begin{aligned} |W(tk)W'(tk)\Phi(\eta_n g)| |\det t|_{\mathbb{C}}^{\operatorname{Re}(s)} \delta^{-1}(t) \\ \leq (1 + \|\log t\|)^d \phi(t_n) \left(\prod_{i=1}^{n-1} \alpha_i(t)^{2i(\operatorname{Re}(s)+\lambda)} \right) t_n^{2n \operatorname{Re}(s)}. \end{aligned}$$

There exists an $e > 0$ such that

$$(1 + \|\log t\|)^d \leq \left(\prod_{j=1}^{n-1} (1 + \log \alpha_j(t))^e \right) (1 + \log |t_n|)^e$$

for all $t \in T_n$. We have the estimate

$$\begin{aligned} \int_{T_n} (1 + \|\log t\|)^e \phi(t_n) |\det t|_{\mathbb{C}}^{\operatorname{Re}(s)+\lambda} |t_n|_{\mathbb{C}}^{-n\lambda} dt \\ \leq \prod_{j=1}^{n-1} \int_1^\infty (1 + \log t_j)^e t_j^{2j(\operatorname{Re}(s)+\lambda)} d^\times t_j \times \int_0^\infty (1 + |\log t_n|)^e \phi(t_n) t_n^{2n \operatorname{Re}(s)} d^\times t_n. \end{aligned}$$

It follows that the integral absolutely converges for s satisfying $\operatorname{Re}(s) > -\lambda$ and $\operatorname{Re}(s) > 0$. As $\lambda > -1/2$, we obtain the absolute convergence of the Rankin–Selberg integral at $s = 1/2$. \square

6. Integral representation of Whittaker functions

Let $n \geq 2$ be fixed and let $K = U_n(\mathbb{C})$ be a maximal compact subgroup of $G_n(\mathbb{C})$. The next lemma gives a convenient formula for the $G_n(\mathbb{R})$ -period of a unitary and generic representation $(\pi, V) \in \operatorname{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$. It is proved in [Lapid and Mao

2014, Lemma 1.2]. We state it and, for the convenience of the reader, provide a full proof here.

Lemma 6.1 [Lapid and Mao 2014, Lemma 1.2]. *Let $(\pi, V) \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ be unitarizable and generic and let $\mathcal{W}(\pi, \psi)$ be its Whittaker model. The functional*

$$\mu : W \mapsto \int_{U_{n-1}(\mathbb{R}) \backslash G_{n-1}(\mathbb{R})} W(h) dh$$

defines a $P_n(\mathbb{R})$ -invariant functional on $\mathcal{W}(\pi, \psi)$. Moreover, there exists an $N > 0$ and a seminorm ν on $\mathcal{W}(\pi, \psi)$ such that for all $g \in G_n(\mathbb{C})$ and $W \in \mathcal{W}(\pi, \psi)$ we have the inequality

$$|\mu(\pi(g)W)| \leq \|g\|_H^N \nu(W).$$

Proof. By [Wallach 1992, Theorem 15.2.5], there exist a continuous seminorm ν' on $\mathcal{W}(\pi, \psi)$, a $\lambda > -1/2$ and a $d > 0$ such that

$$|W(tk)| \leq \delta_0^{\frac{1}{2}}(t) |\det t|_{\mathbb{C}}^{\lambda} |t_n|_{\mathbb{C}}^{-n\lambda} (1 + \|\log t\|)^d \nu'(W)$$

for every $W \in \mathcal{W}(\pi, \psi)$, $t \in T_n$, and $k \in K_n$. For $g \in G_{n-1}(\mathbb{R})$ we have $t_n = 1$. Let us denote

$$T_{n-1} = \{t \in T_n : t_n = 1\}.$$

Let $\delta_0(t)$ be the modular character of $B_n(\mathbb{R})$. For $t \in B_n(\mathbb{R})$ we have that $\delta^{1/2}(t) = \delta_0(t)$. Multiplying ν' by a scalar we have the estimate

$$(6-1) \quad \int_{U_{n-1}(\mathbb{R}) \backslash G_{n-1}(\mathbb{R})} |W(h)| dh \leq \int_{K_{n-1}(\mathbb{R})} \int_{T_{n-1}} |W(tk)| \delta_0^{-1}(t) dt dk \leq \nu'(W) \int_{T_{n-1}} |\det t|_{\mathbb{C}}^{\lambda} (1 + \|\log t\|)^d dt$$

for every $W \in \mathcal{W}(\pi, \psi)$. By the estimates of the previous lemma we obtain that the last integral converges absolutely for $\lambda > -1/2$. It follows that $|\mu(W)| \leq \nu''(W)$ for a continuous seminorm ν'' on $\mathcal{W}(\pi, \psi)$. Since π is of moderate growth, there exist another continuous seminorm ν on $\mathcal{W}(\pi, \psi)$ and an $N > 0$ such that

$$|\mu(\pi(g)W)| \leq \nu''(\pi(g)W) \leq \|g\|_H^N \nu(W)$$

for every $W \in \mathcal{W}(\pi, \psi)$ and every $g \in G$. □

We will identify the functional μ on $\mathcal{W}(\pi, \psi)$ with the corresponding linear functional on V , which we will, by abuse of notation, also denote by μ . By the uniqueness of the Whittaker model, this identification defines $\mu \in V^*$ in a unique way, up to a scalar multiple. Since $\mu \in (V^*)^{P_n(\mathbb{R})}$ and $(V^*)^{P_n(\mathbb{R})} = (V^*)^{G_n(\mathbb{R})}$ (see [Kemarsky 2015, Theorem 1.1]), we obtain that $\mu \in (V^*)^{G_n(\mathbb{R})}$. Clearly, $\mu \neq 0$.

The functional μ defines an embedding of V into the space of functions on $G_n(\mathbb{R}) \backslash G_n(\mathbb{C})$ via

$$V \ni v \mapsto (g \mapsto \mu(\pi(g)v)).$$

By abuse of notation we denote this embedding again by μ . Denote the image of the embedding μ by $\mathcal{C}_{G_n(\mathbb{R})}(\pi)$. In the other direction, we can define a map

$$\theta : \mathcal{C}_{G_n(\mathbb{R})}(\pi) \rightarrow \mathcal{W}(\pi, \psi)$$

by

$$(6-2) \quad \theta : f \mapsto \left(g \mapsto \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f(ug) \psi^{-1}(u) du \right).$$

In this section we will prove that for every n there exists an irreducible representation (π, V) of $G_n(\mathbb{C})$ that is $G_n(\mathbb{R})$ -distinguished and such that the integral (6-2) is absolutely convergent for every K -finite vector in (π, V) .

Suppose that we have a generic representation $(\pi, V) \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ and that the integral (6-2) is absolutely convergent for a K -finite function $f \in \mathcal{C}_{G_n(\mathbb{R})}(\pi)$. Then, from [Lapid and Mao 2014], the composition of maps $\theta(\mu(f))$ is equal to cf for some constant $0 \neq c \in \mathbb{C}$.

Lemma 6.2. *Let $(\pi, V) \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ be a generic representation. Suppose the integral*

$$\int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} W(u) \psi^{-1}(u) du$$

absolutely converges for every K_n -finite function $W \in \mathcal{W}(\pi, \psi)$. Then for every $W \in \mathcal{W}(\pi, \psi)$ there exists an $f \in \mathcal{C}_{G_n(\mathbb{R})}(\pi)$ such that

$$W(g) = \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f(ug) \psi^{-1}(u) du.$$

Recall the decomposition (3-3):

$$G_n(\mathbb{C}) = G_n(\mathbb{R})A^+K.$$

The involution $g \rightarrow {}^t g^{-1}$ preserves this decomposition. Let $\check{W}(g) = W({}^t g^{-1})$. The Whittaker model $\mathcal{W}(\tilde{\pi}, \psi^{-1})$ of the contragredient representation of (π, V) is given by

$$\mathcal{W}(\tilde{\pi}, \psi^{-1}) = \{\check{W} : W \in \mathcal{W}(\pi, \psi)\}.$$

If the conditions of Lemma 6.2 are satisfied for $\mathcal{W}(\pi, \psi)$ then they are also satisfied for the contragredient representation $\mathcal{W}(\tilde{\pi}, \psi^{-1})$. Explicitly, if $W \in \mathcal{W}(\pi, \psi)$

is equal to

$$W(g) = \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f(ug)\psi^{-1}(u) du,$$

then we have

$$\check{W}(g) = \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} \check{f}(ug)\psi(u) du,$$

where $\check{f}(g) := f({}^t g^{-1})$.

Lemma 6.3. *Let $N > 0$. Then there exists a $(\pi, V) \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ such that for every K -finite function $f \in V$ there is a constant $C > 0$, depending only on f , that satisfies, for every $k \in K$, $a \in A$, $h \in G_n(\mathbb{R})$, the inequality*

$$(6-3) \quad |f(hak)| \leq C(f) \|a\|_H^{-N}.$$

Proof. By [Flensted-Jensen 1980], there exists a relatively discrete series $\mathcal{H} := L^2_{d_s}(G_n(\mathbb{R}) \backslash G_n(\mathbb{C}))$. Moreover, every irreducible representation in \mathcal{H} is isomorphic to some $I(\chi)$, where

$$\chi(z) = ((z/|z|)^{i_1}, (z/|z|)^{i_2}, \dots, (z/|z|)^{i_n})$$

and $i_1, \dots, i_n \in \mathbb{Z}$. If $C > 0$ is big enough and if $|i_k - i_j| > C > 0$ for all $i \neq j$, then the $G_n(\mathbb{R}) \backslash G_n(\mathbb{C})$ model of the space $I(\chi)$ lies in \mathcal{H} , and the (\mathfrak{g}, K) -module generated by a K -finite function $0 \neq f_\lambda \in I(\chi)$ satisfies the properties of the lemma. Indeed, by [Flensted-Jensen 1980, p. 254] (see also [Kassel and Kobayashi 2013, Proposition 5.1]), if $C > 0$ is big enough and if $|i_k - i_j| > C > 0$ for all $j \neq k$, then $f_\lambda(hak) \leq C' \|a\|^{-N}$ for all $h \in H$, $a \in A^+$, and $k \in K$.

Clearly, f_λ and right translations of f_λ by K satisfy the properties of our lemma. We should prove that the derivatives of f_λ also satisfy similar growth properties. This is achieved by a classical idea, which is attributed to Harish-Chandra (see also an expository article by [Cowling et al. 1988]). The function f_λ is K -finite, hence there exists a smooth function e_α of compact support such that $f_\lambda * e_\alpha = f_\lambda$. Thus, for $X \in \mathfrak{g}$ we have $dX(f_\lambda) = f_\lambda * dX(e_\alpha)$. It follows that the derivatives of f_λ have the same decay properties that f_λ has.

Finally, the (\mathfrak{g}, K) -module generated by f_λ is of finite length. Consequently, it contains an irreducible admissible (\mathfrak{g}, K) -submodule satisfying the decay property (6-3). \square

If every K -finite function in (π, V) satisfies (6-3) we say that the representation (π, V) decays faster than N . Note that if (π, V) decays faster than N , then its contragredient $(\tilde{\pi}, \tilde{V})$ also decays faster than N . Indeed, we can realize $(\tilde{\pi}, \tilde{V})$ as $\tilde{V} = \{\check{f} : f \in V\}$, where $\check{f}(g) := f({}^t g^{-1})$. If $g = hak$, then ${}^t g^{-1} = {}^t h^{-1} a {}^t k^{-1}$. Hence, the property of fast decay is true for \check{f} if and only if it is true for f .

To obtain estimates of convergence of integrals over the unipotent matrices we need the next elementary result. Define Ω_n as the subset of all upper triangular unipotent matrices in $G_n(\mathbb{C})$ with u_{ij} purely imaginary for $j > i$. Note that Ω_n is a fundamental domain for $U_n(\mathbb{R}) \backslash U_n(\mathbb{C})$.

Lemma 6.4. *There exist a $C > 0$ and $d > 0$, which depend only on n , such that for every $u \in \Omega_n$ we have*

$$\|u\| \leq C \|u\bar{u}^{-1}\|^d.$$

Proof. The proof is by induction on n . For $n = 2$ it follows by direct computation: if $u = \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix}$, then $u\bar{u}^{-1} = \begin{pmatrix} 1 & 2ix \\ 0 & 1 \end{pmatrix}$ and the claim is satisfied.

For a general n , let us define A_j to be the set of the entries in the j -th upper diagonal in g :

$$A_0 = \{g_{11}, g_{22}, \dots, g_{nn}\}, A_1 = \{g_{12}, g_{23}, \dots, g_{(n-1)n}\}, \dots, A_{n-1} = \{g_{1n}\}.$$

Define $B_j := \bigcup_{0 \leq i < j} A_i$. The crucial observation is that entry (i, j) of \bar{u}^{-1} with indices satisfying $j - i = k$ equals

$$\bar{u}_{ij}^{-1} = u_{ij} + P_{ij}(u),$$

where $P_{ij} \in \mathbb{C}[B_k]$ is a fixed polynomial which depends only on the entries u_{lm} with indices $l - m < k$. Similarly,

$$(u\bar{u}^{-1})_{ij} = 2u_{ij} + Q_{ij}(u),$$

where $Q_{ij} \in \mathbb{C}[B_k]$ is a fixed polynomial which depends only on the entries u_{lm} with indices $l - m < k$. For example, let $n = 3$. Then

$$u = \begin{pmatrix} 1 & ix & iy \\ 0 & 1 & iz \\ 0 & 0 & 1 \end{pmatrix}, \bar{u}^{-1} = \begin{pmatrix} 1 & ix & iy - xz \\ 0 & 1 & iz \\ 0 & 0 & 1 \end{pmatrix}, u\bar{u}^{-1} = \begin{pmatrix} 1 & ix & 2iy - 2xz \\ 0 & 1 & iz \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $P_{12} = P_{23} = 0$, $P_{13} = -xz$, $Q_{12} = Q_{23} = 0$, $Q_{13} = -2xz$. Define $v = u\bar{u}^{-1}$ and define “partial seminorms” of u by

$$\|u\|_k = \sqrt{\sum_{(i,j):j-i \leq k} |u_{ij}|^2}.$$

We will prove by induction on k , with base $k = 1$, that for every k , there exist $C_k, d_k > 0$ such that $\|u\|_k^2 \leq C_k \|v\|_k^{d_k}$. As $\|u\|_n = \|u\|$, the result follows.

Let $k = 1$. For $C_1 = 1$, $d_1 = 1$ we obtain the desired inequality. Suppose the claim is true for $k - 1$; that is, suppose

$$\|u\|_{k-1}^2 \leq C_{k-1} \|v\|_{k-1}^{d_{k-1}}.$$

We want to show a similar inequality for k . There exist $C, d > 0$ such that for every $1 \leq i \leq n - k$ we have $|v_{i,i+k}| \geq |u_{i,i+k}| - C\|u\|_{k-1}^d$. For example, one can choose

$$d = \max\{\deg(P_{ij}) : i - j = k\}$$

and a big enough constant C . Let u be a given upper triangular unipotent matrix with purely imaginary entries above the diagonal. There exist constants C', C'' such that if for all i we have $|u_{i,i+k}| \leq 2C\|u\|_{k-1}^d$, then

$$\|u\|_k^2 = \|u\|_{k-1}^2 + \sum_i |u_{i,i+k}|^2 \leq C'\|u\|_{k-1}^{2d} \leq C''\|v\|_{k-1}^{2dd_{k-1}} \leq C'''\|v\|_k^{2dd_{k-1}}.$$

On the other hand, if for some i we have $|u_{i,i+k}| > 2C\|u\|_{k-1}^d$, then we have the inequality $|v_{i,i+k}| > |u_{i,i+k}|/2$ and there exists a constant C'''' such that

$$\sum_{i=1}^{n-k} |u_{i,i+k}|^2 \leq C'''' \sum_{i=1}^{n-k} |v_{i,i+k}|^2 \leq C''''\|v\|_k \leq C'''''\|v\|_k^{2dd_{k-1}}.$$

Therefore, in both cases there are constants C_k, d_k such that

$$\|u\|_k^2 \leq C_k\|v\|_k^{d_k}. \quad \square$$

Corollary 6.5. *There exist a $C > 0$ and $d > 0$, which depend only on n , such that for every $u \in \Omega_n$ we have*

$$\|u\|_H \leq C\|u\bar{u}^{-1}\|_H^d.$$

Proof. From Lemma 6.4 we know that there exist $C_1, d_1 > 0$ such that for every $u \in \Omega_n$ we have $\|u\| < C_1\|u\bar{u}^{-1}\|^{d_1}$. Similarly, one proves that there exist $C_2, d_2 > 0$ such that for every $u \in \Omega_n$ we have $\|u\| < C_2\|\bar{u}u^{-1}\|^{d_2}$. Define $C = \max\{C_1, C_2\}$, $d = \max\{d_1, d_2\}$. Then $\|u\|_H \leq C\|u\bar{u}^{-1}\|_H^d$ for every $u \in \Omega_n$. \square

Lemma 6.6. *Let $N > 0$ be sufficiently large. Then, for every irreducible, $G_n(\mathbb{R})$ -distinguished representation (π, V) of $G_n(\mathbb{C})$ with decay faster than N , the integral*

$$\int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f(ug) du$$

absolutely converges for every $g \in G_n(\mathbb{C})$ and every K -finite function $f \in V$.

Proof. Let (π, V) be a $G_n(\mathbb{R})$ -distinguished representation of $G_n(\mathbb{C})$ such that for every K -finite function $f \in V$ there exists a $C > 0$ depending only on f such that

$$|f(hak)| < C\|a\|_H^{-N}$$

for every $h \in G_n(\mathbb{R})$, $a \in A^+$, $k \in K$. Let $ug = hak$. Then $(\bar{u}g)^{-1}ug = \bar{g}^{-1}(\bar{u}^{-1}u)g$. Since g is fixed, there exists a $C_1 > 0$ such that for every matrix $u \in G_n$ we have

$$C_1^{-1}\|\bar{u}^{-1}u\| < \|(\bar{u}g)^{-1}ug\| < C_1\|\bar{u}^{-1}u\|.$$

By [Lemma 6.4](#), for $u \in \Omega$ we have

$$\|u\| < C_2 \|\bar{u}^{-1}u\|^d.$$

On the other hand,

$$(\bar{u}g)^{-1}ug = \bar{k}^{-1}(a\bar{a}^{-1})k = \bar{k}^{-1}a^2k.$$

Note that $k \in K$ is a unitary matrix, and therefore

$$\|\bar{k}^{-1}a^2k\| = \|a^2k\| = \|a^2\|.$$

Combining these inequalities we get

$$\|a^2\| = \|(\bar{u}g)^{-1}ug\| > C_3 \|u\bar{u}^{-1}\| > C_4 \|u\|^{1/d}.$$

Finally, we obtain that there exist constants C, d' such that for $ug = hak$, where u is in Ω_n and $g \in G_n$ is fixed, we have

$$\|a\| > C \|u\|^{1/d'}.$$

Therefore,

$$(6-4) \quad \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} |f(ug)| du \leq \int_{\Omega} C \|u\|_H^{-N/d'} du.$$

The integral in (6-4) converges for N big enough, thus the lemma is proved. \square

Corollary 6.7. *Let $N > 0$ be sufficiently large. Then, for every irreducible $G_n(\mathbb{R})$ -distinguished representation (π, V) of $G_n(\mathbb{C})$ with decay faster than N , the integral (6-2) is absolutely convergent.*

7. Archimedean Asai integrals

Nonarchimedean Asai integrals were introduced by Flicker [1988] who then used them to analyze the local and global Asai L and ϵ -factors [1993]. In this section we introduce an archimedean analog of Asai integrals and prove that they are of moderate growth. We also state a functional equation analogous to [Offen 2011, Lemma 4.2] that is satisfied by the integrals.

Let (π, V) be a generic irreducible unitarizable representation of $G_n(\mathbb{C})$ and let $\mathcal{W}(\pi, \psi)$ be its Whittaker model. For $W \in \mathcal{W}(\pi, \psi)$, we define an archimedean Asai integral to be

$$(7-1) \quad Z(s, W, \Phi) = \int_{U_n(\mathbb{R}) \backslash G_n(\mathbb{R})} W(g)\Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{R}}^s dg.$$

Lemma 7.1. *Let $\Phi \in \mathcal{S}(\mathbb{C}^n)$ and $\operatorname{Re}(s) \geq 1$. Then $W \mapsto Z(s, W, \Phi)$ defines a continuous functional on $\mathcal{W}(\pi, \psi)$ for $\operatorname{Re}(s) \geq 1$. That is, there exist a continuous seminorm μ on $\mathcal{W}(\pi, \psi)$ and a continuous seminorm ν on $\mathcal{S}(\mathbb{C}^n)$ such that*

$$|Z(s, W, \Phi)| \leq \mu(W)\nu(\Phi)$$

for every $\Phi \in \mathcal{S}(\mathbb{C}^n)$ and every $W \in \mathcal{W}(\pi, \psi)$. As a consequence, there exist an $N > 0$, a continuous seminorm μ' on $\mathcal{W}(\pi, \psi)$ and a continuous seminorm ν on $\mathcal{S}(\mathbb{C}^n)$ such that

$$\int_{U_n(\mathbb{R}) \backslash G_n(\mathbb{R})} |W'(hg)\Phi((0, 0, \dots, 0, 1)hg)| |\det h|_{\mathbb{R}} dh \leq \mu'(W)\nu(\Phi) \|g\|_H^N$$

for every $g \in G$ and every $W' \in \mathcal{W}(\pi, \psi)$.

Proof. Let δ_0 be the modulus function of $B_n(\mathbb{R})$. Using the Iwasawa decomposition we obtain

(7-2)

$$|Z(s, W, \Phi)| \leq \int_{K_n(\mathbb{R})} \int_{T_n} |W(tk)\Phi((0, 0, \dots, 0, 1)tk)| |\det(t)|_{\mathbb{R}}^{\operatorname{Re}(s)} \delta_0^{-1}(t) dt dk.$$

By [Lapid and Mao 2014, Corollary 2.2] there exist a $\lambda > -1/2$, a $d > 0$ and a continuous seminorm μ on $\mathcal{W}(\pi, \psi)$ such that

$$|W(tk)| \leq \delta^{\frac{1}{2}}(t) |\det t|_{\mathbb{R}}^{2\lambda} |t_n|_{\mathbb{R}}^{-2n\lambda} (1 + \|\log t\|)^d \mu(W)$$

for every $t \in T_n$, every $k \in K_n$ and every $W \in \mathcal{W}(\pi, \psi)$. Note that

$$\delta^{\frac{1}{2}}(t)\delta_0(t)^{-1} = 1$$

for all $t \in T_n$. The expression $\Phi((0, 0, \dots, 0, 1)tk) = \Phi((0, 0, \dots, 0, t_n)k)$ does not depend on t_1, t_2, \dots, t_{n-1} . As in the proof of Lemma 5.2, we obtain for some $e > 0$

$$\begin{aligned} |Z(s, W, \Phi)| &\leq \mu(W) \prod_{j=1}^{n-1} \int_1^{\infty} t_j^{j(2\lambda + \operatorname{Re}(s))} (1 + \log t)_j^e d^\times t_j \\ &\times \int_{K_n(\mathbb{R})} \int_0^{\infty} t_n^{n\operatorname{Re}(s)} (1 + |\log t_n|)^e \Phi((0, 0, \dots, 0, t_n)k) d^\times t_n dk. \end{aligned}$$

Thus, $Z(s, W, \Phi)$ converges absolutely for $\operatorname{Re}(s) > \max(-2\lambda, 0)$. Since $\lambda > -1/2$ for π unitary and generic (see [Lapid and Mao 2014, p. 8]), the integral converges absolutely for $\operatorname{Re}(s) \geq 1$. For such s , there is a continuous seminorm ν on $\mathcal{S}(\mathbb{R}^n)$

such that

$$\left| \int_{K_n(\mathbb{R})} \int_0^\infty t_n^{n \operatorname{Re}(s)} (1 + |\log t_n|)^e \Phi((0, 0, \dots, 0, t_n)k) d^\times t_n dk \right| \leq \nu(\Phi)$$

for every $\Phi \in \mathcal{S}(\mathbb{R}^n)$. As a consequence, there exist a continuous seminorm μ on $\mathcal{S}(\mathbb{R}^n)$ and a continuous seminorm ν on $\mathcal{W}(\pi, \psi)$ such that

$$|Z(s, W, \Phi)| \leq \mu(\Phi)\nu(W)$$

for every $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and every $W \in \mathcal{W}(\pi, \psi)$. Thus there exist $M_1, M_2 > 0$ and continuous seminorms μ' on $\mathcal{S}(\mathbb{R}^n)$ and ν' on $\mathcal{W}(\pi, \psi)$ such that

$$|Z(s, \pi(g)W, R(g)\Phi)| \leq \mu(R(g)\Phi)\nu(\pi(g)W) \leq \|g\|_H^{M_1} \|g\|_H^{M_2} \mu'(\Phi)\nu'(W)$$

for every $g \in G$ and every $W \in \mathcal{W}(\pi, \psi)$. \square

The next lemma provides a functional equation for archimedean Asai integrals.

Lemma 7.2. *Let π be an irreducible, unitary, nondegenerate, $G_n(\mathbb{R})$ -distinguished representation of $G_n(\mathbb{C})$. For every $\Phi \in \mathcal{S}(\mathbb{C}^n)$ and $W \in \mathcal{W}(\pi, \psi)$ we have*

$$Z(1, \tilde{W}, \hat{\Phi}|_{\mathbb{R}^n}) = c(\pi)Z(1, W, \Phi|_{\mathbb{R}^n}).$$

Proof. For the proof see [Offen 2011, Lemma 4.2]. \square

We will use the following technical result in the next section.

Lemma 7.3. *Let (π', V') be nondegenerate unitary representation of $G_n(\mathbb{C})$ and let $\mathcal{W}(\pi', \psi^{-1})$ be its Whittaker model. Then there exists an $N > 0$ such that for every irreducible, $G_n(\mathbb{R})$ -distinguished representation (π, V) of $G_n(\mathbb{C})$ with decay faster than N and every function $f \in \mathcal{C}_{G_n(\mathbb{R})}(\pi)$, the following integral is absolutely convergent:*

$$\int_{G_n(\mathbb{R}) \backslash G_n(\mathbb{C})} |f(g)| |\det g|_{\mathbb{C}}^{\frac{1}{2}} \left(\int_{U_n(\mathbb{R}) \backslash G_n(\mathbb{R})} |W'(hg)\Phi((0, 0, \dots, 0, 1)hg)| |\det h|_{\mathbb{R}} dh \right) dg.$$

Proof. This is an immediate consequence of Lemma 7.1. \square

8. Equality of two functionals

Let $(\pi, V) \in \operatorname{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ be generic and unitarizable and let $\mathcal{W}(\pi, \psi)$ be its Whittaker model. Define linear functionals $\mu, \tilde{\mu} \in V^*$ on $\mathcal{W}(\pi, \psi)$ by

$$\mu : W \mapsto \int_{U_{n-1}(\mathbb{R}) \backslash G_{n-1}(\mathbb{R})} W(g) dg \quad \text{and} \quad \tilde{\mu} : W \mapsto \int_{U_{n-1}(\mathbb{R}) \backslash G_{n-1}(\mathbb{R})} W\left(\begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) dg.$$

Since $\mu, \tilde{\mu} \in (V^*)^{P_n(\mathbb{R})}$ and $(V^*)^{P_n(\mathbb{R})} = (V^*)^{G_n(\mathbb{R})}$ (see [Kemarsky 2015, Theorem 1.1]), we obtain that $\mu, \tilde{\mu} \in (V^*)^{G_n(\mathbb{R})}$. Clearly, the functionals $\mu, \tilde{\mu}$ are nonzero. The space of $G_n(\mathbb{R})$ -invariant continuous functionals on V is one-dimensional (see [Aizenbud and Gourevitch 2009, Theorem 8.2.5]), thus there exists a proportionality constant $c(\pi) \neq 0$ such that $\tilde{\mu} = c(\pi)\mu$.

The goal of this section is to calculate the proportionality factor $c(\pi)$ by proving the following theorem.

Theorem 8.1. *Let $(\pi, V) \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$. Then $c(\pi) = 1$.*

We now state an archimedean analogue of [Offen 2011, Lemma 6.1].

Lemma 8.2. *Let $\pi' \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ be generic and unitarizable. Then there exists a generic and unitarizable $\pi \in \text{Irr}_{G_n(\mathbb{R})}(G_n(\mathbb{C}))$ such that*

$$\gamma\left(\frac{1}{2}, \pi \times \pi'; \psi\right) = c(\pi').$$

Note that for π, π' as in Lemma 8.2 we already know that $\gamma(1/2, \pi \times \pi', \psi) = 1$. As a result, the equality $c(\pi') = 1$ follows.

The proof of Lemma 8.2 is similar to the proof of [Offen 2011, Lemma 6.1]. However, in the archimedean case, there are convergence issues that we need to check.

Proof of Lemma 8.2. Let $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\hat{\Phi} \in \mathcal{S}(\mathbb{C}^n)$. The idea is to prove an equality of Rankin–Selberg integrals of the type

$$(8-1) \quad \Psi\left(\frac{1}{2}, \tilde{W}, \tilde{W}'; \hat{\Phi}\right) = c(\pi')\Psi\left(\frac{1}{2}, W, W'; \hat{\Phi}\right).$$

Actually, it is enough to prove such an equality for a pair of functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi)$ such that at least one of the integrals $\Psi(1/2, \tilde{W}, \tilde{W}'; \hat{\Phi})$, $\Psi(1/2, W, W'; \hat{\Phi})$ is nonzero (and thus both integrals are nonzero).

We will obtain the necessary convergence estimates for every K_n -finite function $W \in \mathcal{W}(\pi, \psi)$ and every function $W' \in \mathcal{W}(\pi', \psi)$. By our classification of $G_n(\mathbb{R})$ -distinguished representations of $G_n(\mathbb{C})$, the central character ω_π of the $G_n(\mathbb{R})$ -distinguished representation satisfies $\omega_\pi(-1) = 1$. Thus, by Theorem 5.1, we have the equality

$$\Psi(1-s, \tilde{W}, \tilde{W}'; \hat{\Phi}) = \gamma(s, \pi \times \pi', \psi)\Psi(s, W, W'; \hat{\Phi}).$$

Let $f \in \mathcal{C}_{G_n(\mathbb{R})}(\pi)$ be such that

$$W(g) = \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f(ug)\psi^{-1}(u) du.$$

We will prove the absolute convergence of the following integrals at $s = 1/2$:

$$\begin{aligned}
 (8-2) \quad & \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} |W(g)W'(g)\Phi((0, 0, \dots, 0, 1)g)| |\det g|_{\mathbb{C}}^s dg \\
 & \leq \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} \left(\int_{U_n(\mathbb{R}) \setminus U_n(\mathbb{C})} |f(ug)| du \right) \\
 & \quad \times |W'(g)\Phi((0, 0, \dots, 0, 1)g)| |\det g|_{\mathbb{C}}^s dg \\
 & = \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} |f(g)W'(g)\Phi((0, 0, \dots, 0, 1)g)| |\det g|_{\mathbb{C}}^s dg \\
 & = \int_{G_n(\mathbb{R}) \setminus G_n(\mathbb{C})} |f(g)| |\det g|_{\mathbb{C}}^s \\
 & \quad \times \left(\int_{U_n(\mathbb{R}) \setminus G_n(\mathbb{R})} |W'(hg)\Phi((0, 0, \dots, 0, 1)hg)| |\det h|_{\mathbb{R}}^{2s} dh \right) dg.
 \end{aligned}$$

The left-hand side of (8-2) is absolutely convergent by Lemma 5.2 and the integrals on the right-hand side of (8-2) are absolutely convergent by Lemmas 7.1 and 7.3. Using absolute convergence for $s = 1/2$ of the integrals appearing in (8-2) we obtain the equality (8-1) by the following argument:

$$\begin{aligned}
 & \Psi\left(\frac{1}{2}, W, W'; \Phi\right) \\
 & = \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} W(g)W'(g)\Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} dg \\
 & = \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} \left(\int_{U_n(\mathbb{R}) \setminus U_n(\mathbb{C})} f(ug)\psi_n^{-1}(u) du \right) W'(g)\Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} dg \\
 & = \int_{U_n(\mathbb{C}) \setminus G_n(\mathbb{C})} f(g)W'(g)\Phi((0, 0, \dots, 0, 1)g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} dg \\
 & = \int_{G_n(\mathbb{R}) \setminus G_n(\mathbb{C})} f(g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} \left(\int_{U_n(\mathbb{R}) \setminus G_n(\mathbb{R})} W'(hg)\Phi((0, 0, \dots, 0, 1)hg) |\det h|_{\mathbb{R}} dh \right) dg \\
 & = \int_{G_n(\mathbb{R}) \setminus G_n(\mathbb{C})} f(g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} Z(1, \pi'(g)W', \Phi(\cdot g)|_{\mathbb{R}^n}) dg.
 \end{aligned}$$

Define $f^*(g) := f(g^{-1})$. Then, clearly, $f^* \in \mathcal{C}_{G_n(\mathbb{R})}(\tilde{\pi})$. Applying the change of variables $u \rightarrow w_n {}^t u^{-1} w_n^{-1}$ and the fact that $f(w_n g) = f(g)$, it follows from the

definitions that

$$\tilde{W}(g) = \int_{U_n(\mathbb{R}) \backslash U_n(\mathbb{C})} f^*(ug)\psi(u) du.$$

The same computation applied to $\tilde{\pi}$ and $\tilde{\pi}'$ yields

$$\Psi\left(\frac{1}{2}, \tilde{W}, \tilde{W}'; \hat{\Phi}\right) = \int_{G_n(\mathbb{R}) \backslash G_n(\mathbb{C})} f^*(g) |\det g|_{\mathbb{C}}^{\frac{1}{2}} Z(1, \tilde{\pi}'(g)\tilde{W}', \hat{\Phi}(\cdot g)|_{\mathbb{R}^n}) dg.$$

By [Lemma 7.2](#),

$$Z(1, \tilde{\pi}'(g^{-1})\tilde{W}', \hat{\Phi}(\cdot g^{-1})|_{\mathbb{R}^n}) = c(\pi') |\det g|_{\mathbb{C}} Z(1, \pi'(g)W', \Phi(\cdot g)|_{\mathbb{R}^n}).$$

Finally, we obtain

$$\Psi\left(\frac{1}{2}, \tilde{W}, \tilde{W}'; \hat{\Phi}\right) = c(\pi') \Psi\left(\frac{1}{2}, W, W'; \Phi\right)$$

for every K_n -finite function $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and every $\Phi \in \mathcal{S}(\mathbb{C}^n)$. It is well-known that there exist K_n -finite $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ such that $\Psi(1/2, W, W'; \Phi) \neq 0$. It follows that $c(\pi') = \gamma(1/2, \pi \times \pi'; \psi)$. \square

Appendix A: Generic Langlands quotient

In this section we sketch a proof of the well-known fact that the Langlands quotient of $I(\chi)$ is generic if and only if $I(\chi)$ is irreducible. This fact follows from the papers of Kostant [\[1978\]](#) and Vogan [\[1978\]](#). For the convenience of the reader we rewrite it here. Similar results for $GL_n(\mathbb{R})$ were obtained by Casselman and Zuckerman.

Let $\mathfrak{g} = M_n(\mathbb{C})$ be the Lie algebra of $G_n(\mathbb{C})$ and let K be the standard maximal compact subgroup of $G_n(\mathbb{C})$.

Definition. An irreducible (\mathfrak{g}, K) -module X is called large if its annihilator in the universal enveloping algebra $U(\mathfrak{g})$ is a minimal primitive ideal. We will say that a smooth irreducible representation (π, V) of $G_n(\mathbb{C})$ is large if the corresponding (\mathfrak{g}, K) -module consisting of K -finite vectors in V is large.

Let $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ be a character of $B_n(\mathbb{C})$ and suppose $|\chi_j(t)| = |t|^{\lambda_j}$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. By [\[Vogan 1978, Theorem 6.2\]](#), if (σ, W) is an irreducible subrepresentation of $I(\chi)$ then (σ, W) is large. Suppose (π, V) is the Langlands quotient of $I(\chi)$ and suppose (π, V) is generic. Then by Kostant's theorem (π, V) is large. On the other hand, [\[Vogan 1978, Corollary 6.7\]](#) states that there is a unique large composition factor in the composition series for $I(\chi)$. We obtain $(\pi, V) \simeq (\sigma, W)$ and thus $(\sigma, W) = I(\chi)$; that is, $I(\chi)$ is an irreducible representation.

Appendix B: Gamma factors: converse direction

Fix a smooth, irreducible, generic and admissible representation (π, V) of $G_n(\mathbb{C})$. Suppose we know that

$$(B-1) \quad \gamma\left(\frac{1}{2}, \pi \times \pi', \psi\right) = 1$$

for every $m \leq k$ and every smooth irreducible $G_m(\mathbb{R})$ -distinguished representation (π', V') of $G_m(\mathbb{C})$. What is the minimal k such that (B-1) implies that (π, V) is $G_n(\mathbb{R})$ -distinguished? In this section we give an answer to this question in the case when (π, V) is a unitary representation.

In the following two theorems we prove that $k = 1$ is enough. [Theorem B.1](#) is a particular case of [Theorem B.2](#). Nevertheless we state and prove it since the proof of [Theorem B.1](#) is simpler than and may aid in the understanding of the proof of [Theorem B.2](#).

Theorem B.1. *Let $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ be a unitary character of B_n and suppose that $\chi_j(z) = |z|_{\mathbb{C}}^{s_j} (z/|z|)^{k_j}$ with s_j purely imaginary and $k_j \in \mathbb{Z}$ for every $1 \leq j \leq n$. Suppose $(\pi, V) = I(\chi)$ is a smooth, generic and irreducible representation of $G_n(\mathbb{C})$. Finally, suppose*

$$\gamma\left(\frac{1}{2}, \pi \times \chi', \psi\right) = 1$$

for every \mathbb{R}^\times -distinguished unitary character $\chi' : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$. Then there exists an involution $w \in S_n$ such that $w\chi = \overline{(\chi^{-1})}$. Moreover, one can find an involution $w \in S_n$ such that $w\chi = \overline{(\chi^{-1})}$ and such that, for every fixed point $w(i) = i$, the integer k_i is even.

Proof. Observe that every \mathbb{R}^\times -distinguished unitary character $\chi' : \mathbb{C} \rightarrow \mathbb{R}^\times$ is of the form $\chi(z) = (z/|z|)^{2m}$ for $m \in \mathbb{Z}$. By [[Jacquet 2009](#), Lemma 16.3] we have

$$\gamma\left(\frac{1}{2}, \text{Ind}(\chi) \times \chi', \psi\right) = \prod_{i=1}^n \gamma\left(\frac{1}{2}, \chi_i \chi', \psi\right),$$

where $\gamma(1/2, \chi_i \chi', \psi)$ is the one-dimensional Tate gamma factor. Following Tate [[Cassels and Fröhlich 1967](#)], denote $c_m(z) = (z/|z|)^m$ and recall that the Tate gamma factor is given by

$$\gamma(s, c_m, \psi) = \epsilon_m \frac{(2\pi)^{1-s} \Gamma\left(s + \frac{|m|}{2}\right)}{(2\pi)^s \Gamma\left((1-s) + \frac{|m|}{2}\right)},$$

where

$$\epsilon_m = \begin{cases} 1 & \text{if } m \text{ is even or } m > 0, \\ -1 & \text{if } m \text{ is odd and } m < 0. \end{cases}$$

Let us rewrite the equality $\gamma(1/2, \text{Ind}(\chi) \times \chi', \psi) = 1$ as

$$(B-2) \quad \prod_{i=1}^n \epsilon_{2m+k_i} \frac{(2\pi)^{\frac{1}{2}-s_i} \Gamma\left(\frac{1}{2} + s_i + \frac{|k_i+2m|}{2}\right)}{(2\pi)^{\frac{1}{2}+s_i} \Gamma\left(\frac{1}{2} - s_i + \frac{|k_i+2m|}{2}\right)} = 1$$

for every $m \in \mathbb{Z}$. The product in (B-2) breaks into three products:

$$\begin{aligned} p_{m,1} &= \prod_{i=1}^n \epsilon_{2m+k_i}, \\ p_{m,2} &= \prod_{i=1}^n \frac{(2\pi)^{\frac{1}{2}-s_i}}{(2\pi)^{\frac{1}{2}+s_i}} = (2\pi)^{-2s_1-2s_2-\dots-2s_n}, \\ p_{m,3} &= \prod_{i=1}^n \frac{\Gamma\left(\frac{1}{2} + s_i + \frac{|k_i+2m|}{2}\right)}{\Gamma\left(\frac{1}{2} - s_i + \frac{|k_i+2m|}{2}\right)}. \end{aligned}$$

Note that the term $p_{m,2}$ is constant (does not depend on m) and the term $p_{m,1}$ stabilizes (that is, $p_{m,1} = p_{m+1,1}$ for large enough and for small enough m). Also, we have $|k_i + m| = k_i + m$ for m large enough. Let us take m large enough and look at the expression

$$\frac{p_{m+1,1} p_{m+1,2} p_{m+1,3}}{p_{m,1} p_{m,2} p_{m,3}}.$$

By our assumption this fraction equals 1 for every m . For m large enough we have $p_{m+1,1} p_{m+1,2} = p_{m,1} p_{m,2}$, so $p_{m+1,3}/p_{m,3} = 1$. By the functional equation $\Gamma(z+1) = z\Gamma(z)$ we obtain

$$1 = \frac{p_{m+1,3}}{p_{m,3}} = \prod_{i=1}^n \frac{\left(\frac{1}{2} + s_i + \frac{k_i+2m}{2}\right)}{\left(\frac{1}{2} - s_i + \frac{k_i+2m}{2}\right)}.$$

Thus,

$$\prod_{i=1}^n \left(\frac{1}{2} + s_i + \frac{k_i + 2m}{2}\right) = \prod_{i=1}^n \left(\frac{1}{2} - s_i + \frac{k_i + 2m}{2}\right)$$

for large enough $m \in \mathbb{Z}$. Since both sides are polynomials in m , the polynomials are equal. As a consequence, the zeros of these two polynomials coincide; that is, for every $1 \leq i \leq n$ there exists a $1 \leq j \leq n$ such that

$$\frac{1}{2} - s_i + \frac{k_i}{2} = \frac{1}{2} + s_j + \frac{k_j}{2}.$$

By our assumption the s_i s are purely imaginary and the k_i s are integers. Thus, $-s_i = s_j$ and $k_i = k_j$. Note that $\bar{s}_i = -s_i$ and this means exactly that we can define $w(i) = j$, $w(j) = i$ and $\chi_j = \bar{\chi}_i^{-1} = \chi_{w(i)}$. Therefore, there exists an involution $w \in S_n$ such that $w(\chi) = \bar{\chi}^{-1}$.

From the proof of the existence of an involution w it follows that $\sum_{j=1}^n s_j = 0$ and that the products $p_{m,2} = 1$ and $p_{m,3} = 1$ for every $m \in \mathbb{Z}$. This establishes the existence of an involution $w \in S_n$ such that $w(\chi) = \bar{\chi}^{-1}$. It remains to establish the second property: existence of an involution such that, in addition, for every fixed point $w(j) = j$ of the involution the corresponding integer k_j is even. Note that if i is a fixed point of w then $s_i = 0$. Without loss of generality assume that if $w(i) = i$ and $w(j) = j$, then $k_i \neq k_j$. Otherwise we can define an involution w' by $w'(i) = j$, $w'(j) = i$ and $w'(l) = w(l)$ for $l \neq i, j$ and the new involution w' also satisfies $w'(\chi) = \bar{\chi}^{-1}$.

Assume on the contrary that $w(i) = i$ but that k_i is odd. Then take two consecutive products $p_{m,1}$ and $p_{m+1,1}$ for $m = (-k_i - 1)/2$. Observe that $\epsilon_{2m+k_i} = -\epsilon_{2(m+1)+k_i}$ and that the other terms appearing in the products $p_{m,1}$ and $p_{m+1,1}$ equal each other respectively. As a consequence, $p_{m+1,1} = -p_{m,1}$. But from the preceding paragraph we have $p_{m,2} = p_{m+1,2} = 1$ and also $p_{m,3} = p_{m+1,3} = 1$ and thus $p_{m+1,1} = p_{m,1} = 1$. Contradiction!

Therefore, if $w(i) = i$ then the integer k_i is even; that is, $\chi_i(-1) = 1$. □

A small modification of this proof gives a stronger theorem.

Theorem B.2. *Let $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ be a character of B_n and suppose that $\chi_j(z) = |z|_{\mathbb{C}}^{s_j} (z/|z|)^{k_j}$ with $-1/2 < \text{Re}(s_j) < 1/2$ and $k_j \in \mathbb{Z}$ for every $1 \leq j \leq n$. Suppose $(\pi, V) = I(\chi)$ is a smooth, generic, irreducible representation of $G_n(\mathbb{C})$. Finally, suppose*

$$\gamma\left(\frac{1}{2}, \pi \times \chi', \psi\right) = 1$$

for every \mathbb{R}^\times -distinguished unitary character $\chi' : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$. Then there exists an involution $w \in S_n$ such that $w\chi = (\chi^{-1})$. Moreover, one can find an involution $w \in S_n$ such that $w\chi = (\overline{\chi^{-1}})$ and such that, for every fixed point $w(i) = i$, the integer k_i is even.

Proof. By the same argument as in the previous theorem we obtain that for every $1 \leq i \leq n$ there exists a $1 \leq j \leq n$ such that

$$\frac{1}{2} - s_i + \frac{k_i}{2} = \frac{1}{2} + s_j + \frac{k_j}{2}.$$

By subtracting $1/2$ from both sides of this equality and taking real parts we can replace s_j by $\text{Re}(s_j)$. Thus we can assume that for every $1 \leq i \leq n$ we have $-1/2 < s_i < 1/2$ and also for every $1 \leq i \leq n$ there exists a $1 \leq j \leq n$ such that $-s_i + k_i/2 = s_j + k_j/2$. Multiply both sides of this equation by 2 and replace s_j by $2s_j$. Then, we can assume that for every $1 \leq i \leq n$ we have $-1 < s_i < 1$ and also for every $1 \leq i \leq n$ there exists a $1 \leq j \leq n$ such that

$$-s_i + k_i = s_j + k_j.$$

Let us call this condition the “antisymmetry condition”. The claim is that the “antisymmetry condition” implies that there exists an involution $w \in S_n$ such that $w(\chi) = \bar{\chi}^{-1}$; that is, if $w(i) = j$ then $s_i = -s_j$ and $k_i = k_j$. The proof of the existence of an involution w is by induction on n . Clearly, for $n = 1$ the condition $-s_1 + k_1 = s_1 + k_1$ gives us $s_1 = 0$ and thus the identity involution $w(1) = 1$ works. For a general n it is enough that the “antisymmetry condition” implies that there is a pair i, j such that $s_i = -s_j$ and $k_i = k_j$. Note that if $i = j$ then $s_i = -s_i$ implies $s_i = 0$.

Suppose on the contrary that there are $\{s_i\}_{i=1}^n \subset (-1, 1)$ and $\{k_i\}_{i=1}^n \subset \mathbb{Z}$ that satisfy the “antisymmetry condition”, but there is no pair of indices $1 \leq i, j \leq n$ that satisfy $s_i = -s_j$ and $k_i = k_j$. In particular, there is some $1 \leq i \leq n$ such that $-s_1 + k_1 = s_i + k_i$. By our assumption we have $i > 1$, so without loss of generality assume $i = 2$. Let us assume $s_1 > 0$; the proof in the case $s_1 < 0$ is similar and $s_1 = 0$ is not possible by our assumption. We obtain $k_1 - k_2 = s_1 + s_2$. The left-hand side is an integer and we have $-1 < s_1 + s_2 < 2$. Thus $s_1 + s_2 = 0$ or $s_1 + s_2 = 1$. The case $s_1 + s_2 = 0$ is not possible by our assumption, thus $s_1 + s_2 = 1$ and as a corollary $s_2 > 0$ and $k_2 = k_1 - 1$. Similarly, there is some $1 \leq i \leq n$ such that $-s_2 + k_2 = s_i + k_i$. By the same argument we obtain $s_i > 0$ and $k_i = k_2 - 1$. Thus $i \neq 1, 2$ and without loss of generality we can assume $i = 3$. Continuing in this manner we obtain an infinite sequence of integers k_j such that $k_j = k_1 + (j - 1)$. Contradiction!

Thus there is a pair of indices $1 \leq i, j \leq n$ such that $s_i = -s_j$ and $k_i = k_j$. Removing them from our sequence of length n we obtain a shorter sequence which satisfies the “antisymmetry condition”.

Thus, we have proved that there is an involution $w \in S_n$ such that $w(\chi) = \bar{\chi}^{-1}$. The rest of the argument, that is, the proof of the existence of an involution w such that for every fixed point j of the involution the corresponding integer k_j is even, is the same as in the proof of the previous theorem. \square

As a corollary, using the Tadic–Vogan classification of the unitary dual of $G_n(\mathbb{C})$, we obtain the following theorem.

Theorem B.3. *Let $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ be a character of B_n and suppose that $(\pi, V) = \text{Ind}(\chi)$ is a smooth, generic, irreducible, and unitary representation of $G_n(\mathbb{C})$. Suppose*

$$\gamma\left(\frac{1}{2}, \pi \times \chi', \psi\right) = 1$$

for every \mathbb{R}^\times -distinguished unitary character $\chi' : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$. Then there exists an involution $w \in S_n$ such that $w\chi = \overline{(\chi^{-1})}$. Moreover, one can find an involution $w \in S_n$ such that $w\chi = \overline{(\chi^{-1})}$ and such that, for every fixed point $w(i) = i$, the integer k_i is even.

Proof. Let us define $\chi_j(z) = |z|_{\mathbb{C}}^{s_j} (z/|z|)^{k_j}$, where $s_j \in \mathbb{C}$ and $k_j \in \mathbb{Z}$. The theorem follows from [Theorem B.2](#) and the fact that the unitaricity of $\text{Ind}(\chi)$ implies $-1/2 < \text{Re}(s_j) < 1/2$ for every $1 \leq j \leq n$ (see [\[Tadić 1985, Theorem A\]](#)). \square

Finally, by [\[Panichi 2001, Theorem 3.3.6\]](#) we know that an irreducible tempered representation (π, V) of $G_n(\mathbb{C})$ is $G_n(\mathbb{R})$ -distinguished if and only if there exists an involution $w \in S_n$ such that $w\chi = (\chi^{-1})$ and such that, for every fixed point $w(i) = i$, the integer k_i is even. Therefore, an irreducible tempered representation (π, V) of $G_n(\mathbb{C})$ is $G_n(\mathbb{R})$ -distinguished if and only if

$$\gamma\left(\frac{1}{2}, \pi \times \chi', \psi\right) = 1$$

for every \mathbb{R}^\times -distinguished unitary character $\chi' : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$.

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