SHOWING DISTINCTNESS OF SURFACE LINKS
BY TAKING 2-DIMENSIONAL BRAIDS

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For an oriented surface link S, we can take a satellite construction called a 2-dimensional braid over S, which is a surface link in the form of a covering over S. We demonstrate that 2-dimensional braids over surface links are useful for showing the distinctness of surface links. We investigate nontrivial examples of surface links with free abelian groups of rank two, concluding that their link types are infinitely many.

1. Introduction

A *surface link* is the image of a smooth embedding of a closed surface into Euclidean space $\mathbb{R}^4$. Two surface links are *equivalent* if there is an orientation-preserving self-diffeomorphism of $\mathbb{R}^4$ carrying one to the other. In this paper, we assume that surface links are oriented. In [Nakamura 2014a], we investigated a satellite construction called a 2-dimensional braid over an oriented surface link, and introduced its graphical presentation called an *m*-chart on a surface diagram. A 2-dimensional braid over a surface link S is a surface link in the form of a covering over S, and can be regarded as an analog to a double of a classical link. One of the expected applications of the notion of a 2-dimensional braid is that it will provide us with a method for showing the distinctness of surface links. The aim of this paper is to demonstrate such use for 2-dimensional braids.

Our main theorem is as follows. For a positive integer k, let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the standard generators of the $(k + 1)$-braid group. Take $X_1 = \sigma_1^2$ and $X_k = \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_k$, and let Δ be a $(k + 1)$-braid with a positive half twist. Consider $S_k = \mathcal{S}_{k+1}(X_k, \Delta^2)$, a torus-covering T^2-link determined from the $(k + 1)$-braids X_k and Δ^2. We take the first component of S_k to be the one determined from the first strand of X_k, and likewise the second component from the second strand; see Section 2 for the construction. Here, a T^2-*link* is a surface link each of whose components is of genus one.

Theorem 1.1. *Abelian T^2-links of rank two*, S_k *and* S_l, *are not equivalent for distinct positive integers k and l. Thus, the link types of abelian T^2-links of rank two are infinitely many.*

An *abelian surface link* of rank n is a surface link whose link group is a free abelian group of rank n [Ito and Nakamura 2014]; note that n is the number of the components. We remark that our abelian T^2-links of rank two cannot be distinguished by using link groups, and that by a homological argument we cannot show that their link types are infinitely many, but only that there are two link types; see Section 2B. Our abelian T^2-link S_k of rank two is a sublink of the surface link given in [Ito and Nakamura 2014], where we gave examples of abelian T^2-links of rank four, and we showed that their link types are infinitely many by calculations of triple linking numbers (see also Remark 2.3).

Triple linking numbers are integer-valued invariants of surface links with at least three components, so we cannot use them directly for our case. In order to overcome this situation, we take a 2-dimensional braid over S_k such that each component of S_k is split into two components. This has four components, so we can calculate triple linking numbers. A 2-dimensional braid over a surface link is obtained from the "standard" 2-dimensional braid by the addition of braiding information. Unfortunately, for the standard 2-dimensional braid, the triple linking number is trivial (Proposition 5.1). However, addition of braiding information makes a 2-dimensional braid with nontrivial triple linking, and enables us to show that S_k and S_l are not equivalent for distinct positive integers k and l. For a similar result, we refer to Suciu's paper [1985] where it is shown that there are infinitely many ribbon 2-knots in S^4 with knot group the trefoil knot group.

The paper is organized as follows. In Section 2, we review torus-covering links and explain our example S_k, and we review triple linking numbers of torus-covering links. In Section 3, we discuss the notion of a 2-dimensional braid over a surface link. In Section 4, we observe that a 2-dimensional braid of degree m over a surface link is presented by a finite graph called an m-chart on a surface diagram, and that 2-dimensional braids of degree m are equivalent if their surface diagrams with m-charts are related by local moves called Roseman moves. In Section 5, we prove Proposition 5.1. In Section 6, we calculate triple linking numbers of a certain 2-dimensional braid over S_k and prove Theorem 1.1.

2. Abelian T^2-links of rank two

The example S_k given in Theorem 1.1 is a surface link called a torus-covering link. In this section, we review torus-covering T^2-links; see [Nakamura 2011] for details. We briefly observe that S_k is an abelian surface link of rank two, and that we cannot show that the link types of our examples are infinitely many by using a

homological argument. Further, we review a formula for the triple linking numbers of torus-covering links [Ito and Nakamura 2014].

2A. *Torus-covering links.* Let T be a standard torus in $\mathbb{R}^4$, the boundary of an unknotted (standardly embedded) solid torus in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$.

Definition 2.1. A *torus-covering T^2-link* S is a surface link in the form of a 2-dimensional braid over the standard torus T, i.e., S is a T^2-link in $\mathbb{R}^4$ such that S is contained in a tubular neighborhood $N(T)$ and $\pi|_S : S \to T$ is an unbranched covering map, where $\pi : N(T) \to T$ is the natural projection.

Let S be a torus-covering T^2-link. Fix a base point $x_0 = (x_0', x_0'')$ of $T = S^1 \times S^1$. Take two simple closed curves on T, $\boldsymbol{m} = \partial B^2 \times \{x_0''\}$ and $\boldsymbol{l} = \{x_0'\} \times S^1$. Recall that T is embedded as $T = \partial(B^2 \times S^1) \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Let us consider the intersections $S \cap \pi^{-1}(\boldsymbol{m}) \subset B^2 \times \boldsymbol{m}$ and $S \cap \pi^{-1}(\boldsymbol{l}) \subset B^2 \times \boldsymbol{l}$. They are regarded as closed m-braids in the 3-dimensional solid tori, where m is the degree of the covering map $\pi|_S : S \to T$. Cutting open the solid tori along the 2-disk $\pi^{-1}(x_0) = B^2 \times \{x_0\}$, we obtain two m-braids a and b. The assumption that $\pi|_S$ is an unbranched covering implies that a and b commute. We call the commutative braids (a, b) the *basis braids* of S. Conversely, starting from a pair of commutative m-braids (a, b), we can uniquely construct a torus-covering T^2-link with basis braids (a, b) [Nakamura 2011, Lemma 2.8]. For commutative m-braids a and b, we denote by $\mathscr{S}_m(a, b)$ the torus-covering T^2-link with basis braids (a, b).

2B. *Our abelian T^2-links of rank two.* We can verify that $S_k = \mathscr{S}_{k+1}(X_k, \Delta^2)$ is an abelian surface link as follows. The link group of a torus-covering link $\mathscr{S}_m(a, b)$ is a quotient group of the classical link group of the closure of a such that the abelianization is a free abelian group [Nakamura 2011, Proposition 3.1]. Since the link group of the closure of X_k, a Hopf link, is a free abelian group of rank two, so is the link group of S_k.

We remark that a homological argument cannot show that our examples are infinitely many, but only that there are two link types. Let us consider the one-point compactification of $\mathbb{R}^4$, and regard S_k to be in the Euclidean 4-sphere S^4. Recall that we take the first and second components F_1 and F_2 of S_k to be determined from the first and second strands of X_k, respectively. By Alexander's duality, we see that $H_2(S^4 - F_1; \mathbb{Z}) \cong H_1(F_1; \mathbb{Z})$, whence $[F_2] = \mu + k\lambda \in H_2(S^4 - F_1; \mathbb{Z})$, where (μ, λ) is a preferred basis of $H_1(F_1; \mathbb{Z}) \cong H_2(S^4 - F_1; \mathbb{Z})$ represented by a meridian and a preferred longitude of F_1. Similarly, let us denote by F_1' and F_2' the first and second components of S_l. Then we can see that $[F_2'] = \mu' + l\lambda' \in H_2(S^4 - F_1'; \mathbb{Z})$, where (μ', λ') is a preferred basis of $H_1(F_1'; \mathbb{Z}) \cong H_2(S^4 - F_1'; \mathbb{Z})$ represented by a meridian and a preferred longitude of F_1'. Now, the standardly embedded tori F_1 and F_1' are related by an orientation-preserving self-diffeomorphism of S^4 if and

only if

$$\begin{pmatrix} \mu' \\ \lambda' \end{pmatrix} = A \begin{pmatrix} \mu \\ \lambda \end{pmatrix}$$

for

$$A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \in GL_+(2; \mathbb{Z})$$

such that $\alpha + \beta + \gamma + \delta \equiv 0 \pmod 2$ [Montesinos 1983], which implies that $[F_2] = [F_2'] \in H_2(S^4 - F_1; \mathbb{Z})$ if and only if $k \equiv l \pmod 2$ (see [Iwase 1988]).

Remark 2.2. The abelian surface link S_1, i.e., $\mathscr{S}_2(\sigma_1^2, \sigma_1^2)$, is the twisted Hopf 2-link we will mention in the proof of Proposition 5.1; see also [Carter et al. 2001].

Remark 2.3. It is known [Kawauchi 1996, Theorem 6.3.1–Exercise 6.3.3] that for classical links, the rank of an abelian link is at most two, and for abelian links of rank two, there are exactly two link types: a positive and a negative Hopf link.

Remark 2.4. Set $T_m = \mathscr{S}_{k+1}(X_k, X_k^m)$ for an integer m. It is known ([Boyle 1993], see also [Iwase 1988; Nakamura 2011]) that T_m and T_n are equivalent for $m \equiv n \pmod 2$. Fix the first component of T_m in the form of the standard torus. By a homological argument as above, we see that T_m cannot be taken to T_n for $n \neq m$ by an orientation-preserving self-diffeomorphism of $\mathbb{R}^4$ relative to the first component.

2C. *Triple linking numbers of torus-covering links.* The triple linking number of a surface link S is defined as follows [Carter et al. 2003, Definition 9.1]. For the i-th, j-th, and k-th components F_i, F_j, F_k of S with $i \neq j$ and $j \neq k$, the *triple linking number* $\mathrm{tlk}_{i,j,k}(S)$ of the i-th, j-th, and k-th components of S is the total number of positive triple points minus the total number of negative triple points of a surface diagram of S such that the top, middle, and bottom sheet are from F_i, F_j, and F_k, respectively. The triple linking number is a link bordism invariant [Carter et al. 2004; 2001; Sanderson 1987; 1993]; for other properties, see [Carter et al. 2003; 2004]. Triple linking numbers are useful for showing the distinctness of surface links with at least three components [Ito and Nakamura 2014; Nakamura 2012; 2014b].

From [Ito and Nakamura 2014], we have a formula for the triple linking numbers of a torus-covering T^2-link $\mathscr{S}_m(a, b)$. Let A_i be the components of the closure of a which are from the i-th component of $\mathscr{S}_m(a, b)$. Take one of the connected components of A_i and denote it by A_i^1. We define by the classical linking number

$$\mathrm{lk}_{i,j}^a = \mathrm{lk}(A_i^1, A_j),$$

where we regard A_i^1 and A_j as oriented links in $\mathbb{R}^3$. The notation $\mathrm{lk}_{i,j}^b$ for the other basis braid is defined similarly. Note that $\mathrm{lk}_{i,j}^a$ does not depend on the choice of

a connected component A_i^1 [Ito and Nakamura 2014, Remark 5.5], and note that $\mathrm{lk}_{i,j}^a$ is not always symmetric, i.e., $\mathrm{lk}_{i,j}^a$ is not always equal to $\mathrm{lk}_{j,i}^a$.

Now, for a torus-covering T^2-link, the triple linking number of the i-th, j-th, and k-th components is given by

$$(2\text{-}1) \qquad \mathrm{tlk}_{i,j,k}(\mathscr{S}_m(a,b)) = -\,\mathrm{lk}_{j,i}^a\,\mathrm{lk}_{j,k}^b + \mathrm{lk}_{j,k}^a\,\mathrm{lk}_{j,i}^b,$$

where $i \neq k$ and $j \neq k$ [Ito and Nakamura 2014, Theorem 5.4 and Remark 5.7].

3. Two-dimensional braids over a surface link

A 2-dimensional braid, also called a simple braided surface, over a 2-disk is analogous to a classical braid [Kamada 1992; 2002; Rudolph 1983]. We can modify this notion to a 2-dimensional braid over a closed surface [Nakamura 2011], and further to a 2-dimensional braid over a surface link [Carter et al. 2004, Section 2.4.2; Nakamura 2014a]. In this section, we review this notion of a 2-dimensional braid over a surface link.

3A. *Two-dimensional braids over a surface link.* We use 2-dimensional braids without branch points over a closed surface, so our definition here is restricted to such surfaces; see [Nakamura 2011; 2014a] for the general definition.

Let Σ be a closed surface, let B^2 be a 2-disk, and let m be a positive integer.

Definition 3.1. A closed surface $\widetilde{\Sigma}$ embedded in $B^2 \times \Sigma$ is called a *2-dimensional braid over Σ of degree m* if the restriction $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ is an unbranched covering map of degree m, where $\pi : B^2 \times \Sigma \to \Sigma$ is the natural projection.

Take a base point x_0 of Σ. Two 2-dimensional braids over Σ of degree m are *equivalent* if there is a fiber-preserving ambient isotopy of $B^2 \times \Sigma$ rel $\pi^{-1}(x_0)$ which carries one to the other.

A surface link is said to be *of type Σ* when it is the image of an embedding of Σ. Let S be a surface link of type Σ, and let $N(S)$ be a tubular neighborhood of S in $\mathbb{R}^4$.

Definition 3.2. A *2-dimensional braid $\widetilde{S}$ over S* is the image of a 2-dimensional braid over Σ in $B^2 \times \Sigma$ by an embedding $B^2 \times \Sigma \to \mathbb{R}^4$ which identifies $N(S)$ with $B^2 \times \Sigma$ as a B^2-bundle over a surface. We define the *degree* of $\widetilde{S}$ as that of S.

Two 2-dimensional braids $\widetilde{S}$ and $\widetilde{S}'$ over surface links S and S' are *equivalent* if there is an ambient isotopy of $\mathbb{R}^4$ carrying $\widetilde{S}$ to $\widetilde{S}'$ and $N(S) = B^2 \times S$ to $N(S') = B^2 \times S'$ as a B^2-bundle over a surface.

Equivalent 2-dimensional braids over surface links are also equivalent as surface links. A 2-dimensional braid $\widetilde{S}$ over S is a specific satellite with companion S; see [Carter et al. 2004, Section 2.4.2] as well as [Lickorish 1997, Chapter 1].

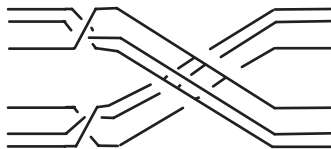a double point curve   a triple point   a branch point
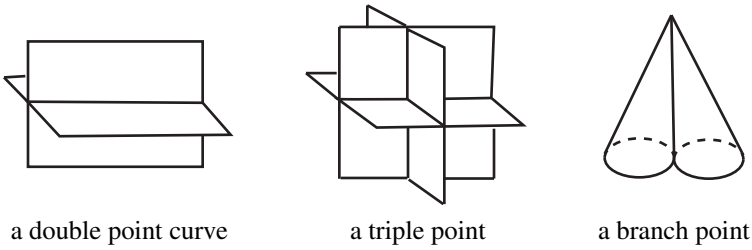
Figure 1. The singularity of a surface diagram.

3B. *Standard 2-dimensional braids.* In this section, we define the standard 2-dimensional braid over a surface link S. Using this notion, we will explain in the next section that a 2-dimensional braid is presented by a finite graph called an m-chart on a surface diagram D of S. The standard 2-dimensional braid over S is the 2-dimensional braid presented by an empty m-chart on D [Nakamura 2014a].

We first review surface diagrams of a surface link S; see [Carter et al. 2004]. For a projection $p : \mathbb{R}^4 \to \mathbb{R}^3$, the closure of the self-intersection set of $p(S)$ is called the singularity set. Let p be a generic projection, meaning that the singularity set of the image $p(S)$ consists of double points, isolated triple points, and isolated branch points; see Figure 1. The closure of the singularity set forms a union of immersed arcs and loops, called double point curves. Triple points form the intersection points of the double point curves, and branch points form the end points. A *surface diagram* of S is the image $p(S)$ equipped with over/under information along each double point curve with respect to the projection direction.

We define the $2m$-braid $\tilde{\sigma}_1$ obtained from a 2-braid σ_1 as follows. More generally, we construct an mn-braid $\tilde{b}$ from an n-braid b, needed for the proof of Theorem 1.1. Let Q_m be m interior points of B^2. For a standard generator σ_i of an n-braid, let $\tilde{\sigma}_i$ be the mn-braid obtained from σ_i in the form of a Q_m-bundle over σ_i by splitting each strand into a bundle of m parallel strands with a negative half twist at the initial points of each bundle; see Figure 2. The map taking σ_i to $\tilde{\sigma}_i$ determines a homomorphism from the n-braid group to the mn-braid group. For an n-braid b, let $\tilde{b}$ denote the image of b by this homomorphism.

Definition 3.3. Let S be a surface link. A surface diagram D of S consists of the following local parts: around (1) a regular point, i.e., a nonsingular point,

Figure 2. The $2m$-braid $\tilde{\sigma}_1$.

(2) a double point curve, (3) a triple point, and (4) a branch point. The diagram around a regular point (1) consists of an embedded 2-disk B^2 with no singularities, and the diagram around a double-point curve (2) can be expressed as the product of a 2-braid σ_1 and an interval I.

We define the *standard 2-dimensional braid over S* locally for such local parts of D as follows: for (1), it is m parallel copies of B^2, and for (2), it is the product of the $2m$-braid $\widetilde{\sigma}_1$ and I. Then, for the other cases (3) and (4), the standard 2-dimensional braid is naturally defined [Nakamura 2014a, Definition 5.1 and Proposition 5.2].

4. Chart presentation of 2-dimensional braids and Roseman moves

In this section, we recall that a 2-dimensional braid of degree m over a surface link S is presented by a finite graph called an m-chart on a surface diagram D of S. For two 2-dimensional braids of degree m, they are equivalent if their surface diagrams with m-charts are related by a finite sequence of local moves called Roseman moves. See [Nakamura 2014a].

4A. *Chart presentation of 2-dimensional braids over a surface link.* The graphical method called an m-chart on a 2-disk was introduced to present a simple surface braid which is a 2-dimensional braid over a 2-disk with trivial boundary condition [Kamada 1992; 2002]. By regarding an m-chart on a 2-disk as drawn on a 2-sphere S^2, it presents a 2-dimensional braid over S^2 [Kamada 1992; 2002; Nakamura 2011]. This notion can be modified to an m-chart on a closed surface, and further to an m-chart on a surface diagram D of a surface link S [Nakamura 2011; 2014a]. A 2-dimensional braid over S is presented by an m-chart on D [Nakamura 2014a].

In this paper, we treat 2-charts with vertices of degree 2. We now review the graphical form of an m-chart of a 2-dimensional braid over a surface link. See [Nakamura 2014a] for details.

Let $\widetilde{S}$ be a 2-dimensional braid over a surface link S. Let D be a surface diagram of S by a projection $p : \mathbb{R}^4 \to \mathbb{R}^3$ which is generic with respect to both S and $\widetilde{S}$. We can assume that the singularity set of the surface diagram $p(\widetilde{S})$ is the union of the singularity set of the diagram of the standard 2-dimensional braid over S and some finite graph Γ [Nakamura 2014a, Theorem 5.5]. Project Γ to D by the projection $p(N(S)) = B^2 \times D \to D$. Thus we obtain a finite graph on the surface diagram D. An m-chart on a surface diagram D is such a finite graph equipped with certain additional information of orientations and labels assigned to the edges, where m is the degree of the 2-dimensional braid. Owing to the additional information, we can regain the original 2-dimensional braid from the m-chart on D [Nakamura 2014a] (see also [Kamada 2002]).

We can define an m-chart on D in graphical terms, where the labels of edges are from 1 to $m-1$; see [Nakamura 2014a, Definitions 5.3 and 5.4]. Around a double
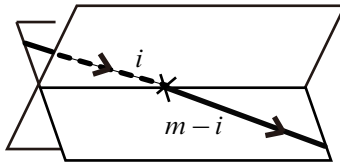
Figure 3. An m-chart around a double point curve. Here i is one
of $1, \ldots, m-1$. For simplicity, we omit the over/under information
of each sheet.

point curve, an m-chart is as in Figure 3, with a vertex of degree 2. A 2-dimensional
braid over S is presented by an m-chart on D [Nakamura 2014a, Theorem 5.5].

4B. *Roseman moves.* Roseman moves are local moves of surface diagrams, as
illustrated in Figure 4. It is known [Roseman 1998] that two surface links are
equivalent if and only if their surface diagrams are related by a finite sequence
of Roseman moves and ambient isotopies of the diagrams in $\mathbb{R}^3$. In [Nakamura
2014a], we introduced the notion of Roseman moves for surface diagrams with
m-charts.

An m-chart is said to be *empty* if it is an empty graph.

Definition 4.1. We define *Roseman moves for surface diagrams with m-charts* by
the local moves as shown in Figures 4 and 5, where we regard the diagrams in
Figure 4 as equipped with empty m-charts.
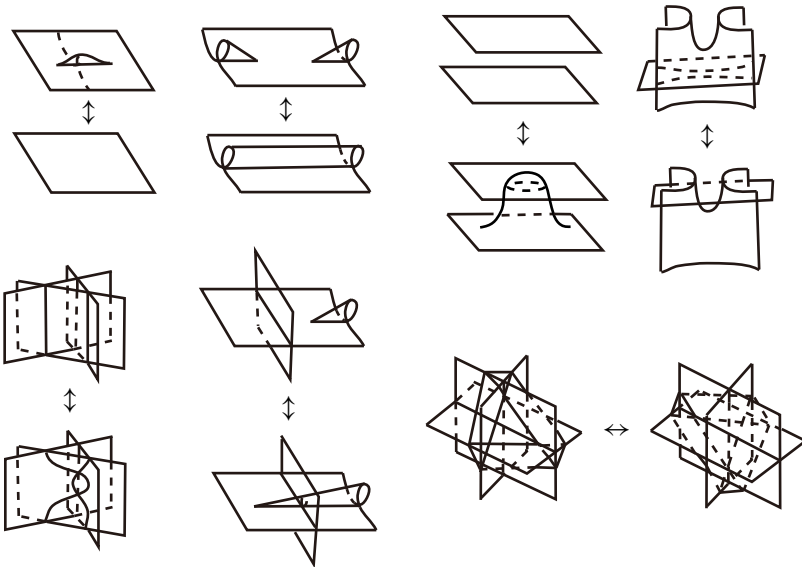
Figure 4. Roseman moves. We omit the over/under information
of each sheet.
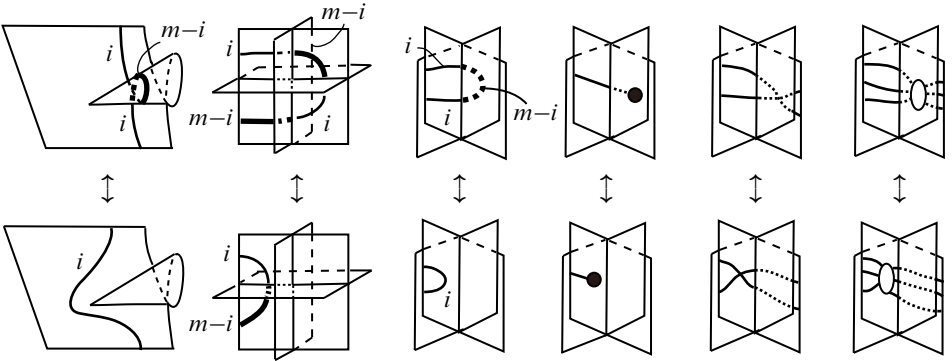
Figure 5. Roseman moves for surface diagrams with m-charts, where $i \in \{1, \ldots, m-1\}$. We omit the over/under information of each sheet, and orientations and labels of edges of m-charts.

Roseman moves for surface diagrams with m-charts, as illustrated in Figures 4 and 5, are well-defined. That is, for each pair of Roseman moves, the m-charts on the indicated diagrams present equivalent 2-dimensional braids [Nakamura 2014a, Theorem 6.2].

5. Triple linking numbers of standard 2-dimensional braids

Recall the triple linking numbers from Section 2C. We will say that a surface link S has *trivial* triple linking if every triple linking number of S is zero or S consists of less than three components.

Proposition 5.1. *For the standard 2-dimensional braid $\widetilde{S}$ over a surface link S, if S has trivial triple linking, then so does $\widetilde{S}$.*

Proof. Assume that S has trivial triple linking. Recall from [Carter et al. 2001] that the link bordism class of a surface link is determined from triple linking numbers and double linking numbers (another kind of link bordism invariant), and that a surface link with trivial triple linking is link bordant to a split union of a finite number of trivial spheres and surface links called twisted Hopf 2-links, which has a surface diagram with no triple points (see also Remark 2.2). Hence, S is link bordant to a surface link S' whose surface diagram has no triple points. By the well-definedness of Roseman moves, $\widetilde{S}$ is link bordant to the standard 2-dimensional braid $\widetilde{S}'$ over S'. Since the surface diagram of a standard 2-dimensional braid has triple points only around triple points of the companion surface [Nakamura 2014a], the surface diagram of $\widetilde{S}'$ has no triple points. Thus $\widetilde{S}$ is link bordant to a surface link with no triple points, which implies that $\widetilde{S}$ has trivial triple linking.   □

6. Proof of Theorem 1.1

In this section, we consider a 2-dimensional braid $\widetilde{S}$ over a surface link S presented by a 2-chart consisting of a finite number of loops on a surface diagram of S. Here, a *loop* is a union of edges connected by vertices of degree 2 as in Figure 3. For our 2-charts, the edges are labeled by 1 and the orientations are coherent around a vertex of degree 2, so we can ignore the label information, and we regard the 2-chart on a surface diagram of S as oriented loops. Further, we consider that the loops are on S itself. By the well-definedness of Roseman moves, a 2-dimensional braid presented by a 2-chart Γ on S is equivalent to the 2-dimensional braid presented by a 2-chart $f(\Gamma)$ on $f(S)$ for an orientation-preserving self-diffeomorphism f of $\mathbb{R}^4$.

For a component F of a torus-covering T^2-link, we take a preferred basis of $H_1(F; \mathbb{Z})$ represented by a pair of simple closed curves (μ, λ) such that μ is a connected component of $F \cap \pi^{-1}(\boldsymbol{m})$, and λ of $F \cap \pi^{-1}(\boldsymbol{l})$. Recall that $\pi : N(T) \to T$ is the natural projection for a standard torus T, and $\boldsymbol{m}$ and $\boldsymbol{l}$ are simple closed curves on T given in Section 2A. We will use the same notation (μ, λ) for the preferred basis, and we call simple closed curves in the homology classes μ and λ *meridians* and *preferred longitudes* of F, respectively. For a 2-chart Γ on F consisting of loops, we can assume that the intersections of the chart loops of Γ with a meridian μ and a preferred longitude λ of F are transverse. We assign each intersection point the sign $+1$ or -1 according to whether it presents a positive or negative crossing, and we denote by $I(\mu, \Gamma)$ and $I(\lambda, \Gamma)$ the sum of the signs of the intersection points of Γ with μ and λ. Note that we can assume that the chart loops are parallel by using local moves of charts called CI-moves of type (1) [Kamada 2002], and $I(\mu, \Gamma)$ and $I(\lambda, \Gamma)$ are well-defined for the homology classes μ and λ.

For the torus-covering T^2-link S, we take the first and second components of S as those determined from the first and second strands of each basis braid of S, respectively. Similarly, for the 2-dimensional braid $\widetilde{S}$, we take the i-th component of $\widetilde{S}$ as the one determined from the i-th strand of each basis braid of $\widetilde{S}$ for $i = 1, 2, 3, 4$.

We first calculate the triple linking numbers of a 2-dimensional braid of degree 2 over S_k as in Theorem 1.1.

Lemma 6.1. *For the torus-covering T^2-link S_k for a positive integer k, let us consider a 2-dimensional braid $\widetilde{S}_k$ of degree 2 over S_k, which is presented by a 2-chart Γ consisting of loops on S_k so that it consists of 4 components. Then* $\mathrm{tlk}_{i,j,3}(\widetilde{S}_k) = \mathrm{tlk}_{i,j,4}(\widetilde{S}_k)$ *for* $(i, j) = (1, 2)$ *or* $(2, 1)$, *and* $\mathrm{tlk}_{i,j,1}(\widetilde{S}_k) = \mathrm{tlk}_{i,j,2}(\widetilde{S}_k)$ *for* $(i, j) = (3, 4)$ *or* $(4, 3)$.

Proof. The 2-dimensional braid $\widetilde{S}_k$ is also a torus-covering T^2-link. We denote by (a, b) the basis braids presenting $\widetilde{S}_k$. Since $\mathrm{lk}^c_{j,3} = \mathrm{lk}^c_{j,4}$ for $j = 2, 1$, and $\mathrm{lk}^c_{j,1} = \mathrm{lk}^c_{j,2}$ for $j = 4, 3$ $(c = a, b)$, by (2-1) we have the result. $\square$

Lemma 6.2. *For the torus-covering T^2-link S_k, denote by F_1 and F_2 the first and second components of S_k. Let (μ_i, λ_i) be a preferred basis of $H_1(F_i; \mathbb{Z})$ for $i = 1, 2$. Consider a 2-dimensional braid $\widetilde{S}_k$, as in Lemma 6.1, such that $I(\mu_i, \Gamma) = 2p_i$ and $I(\lambda_i, \Gamma) = 2q_i$ for integers p_i and q_i for $i = 1, 2$. Then we have $\mathrm{tlk}_{1,2,3}(\widetilde{S}_k) = -kp_1 + q_1$ and $\mathrm{tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 + q_2$.*

Note that $\widetilde{S}_k$ consists of 4 components if and only if $I(\mu_i, \Gamma)$ and $I(\lambda_i, \Gamma)$ are even for $i = 1, 2$, since these conditions are equivalent to the condition that $\widetilde{S}_k \cap \pi_i^{-1}(\mu)$ and $\widetilde{S}_k \cap \pi_i^{-1}(\lambda)$ are closed pure braids, where $\pi_i : N(F_i) \to F_i$ is the natural projection.

Proof. The 2-dimensional braid $\widetilde{S}_k$ is also a torus-covering T^2-link. We denote by (a, b) the basis braids presenting $\widetilde{S}_k$. We use the notation given in Section 3B, taking $m = 2$ and $n = k + 1$. Then, $\mathrm{lk}_{2,1}^a$ is determined from the linking number coming from the linking consisting of $I(\mu_1, \Gamma)$ crossings and $\widetilde{X}_k$. That is,

$$\mathrm{lk}_{2,1}^a = p_1 + \mathrm{lk}_{2,1}^{\widetilde{X}_k},$$

and similarly,

$$\mathrm{lk}_{2,1}^b = q_1 + \mathrm{lk}_{2,1}^{\widetilde{\Delta}^2}.$$

By definition, for a braid c, the braid $\tilde{c}$ has a negative (respectively positive) half twist at the place which is a fiber of a point of each arc forming a positive (respectively negative) crossing of c. Hence,

$$\mathrm{lk}_{2,1}^{\widetilde{X}_k} = -\mathrm{lk}_{1,2}^{X_k}$$
$$\mathrm{lk}_{2,1}^{\widetilde{\Delta}^2} = -\mathrm{lk}_{1,2}^{\Delta^2},$$

from which it follows that

$$\mathrm{lk}_{2,1}^a = p_1 - \mathrm{lk}_{1,2}^{X_k}$$
$$\mathrm{lk}_{2,1}^b = q_1 - \mathrm{lk}_{1,2}^{\Delta^2}.$$

Further, $\mathrm{lk}_{2,3}^a = \mathrm{lk}_{1,2}^{X_k}$ and $\mathrm{lk}_{2,3}^b = \mathrm{lk}_{1,2}^{\Delta^2}$. Thus $\mathrm{tlk}_{1,2,3}(\widetilde{S}_k) = -p_1 \mathrm{lk}_{1,2}^{\Delta^2} + q_1 \mathrm{lk}_{1,2}^{X_k}$ by (2-1). Since $\mathrm{lk}_{1,2}^{X_k}$ is the linking number of the closure of X_k, $\mathrm{lk}_{1,2}^{X_k} = 1$. Since F_1 and F_2 are constructed by one strand and k strands of Δ^2, respectively, we have $\mathrm{lk}_{1,2}^{\Delta^2} = k$. Thus $\mathrm{tlk}_{1,2,3}(\widetilde{S}_k) = -kp_1 + q_1$.

By the same argument, we have $\mathrm{tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 \mathrm{lk}_{2,1}^{\Delta^2} + q_2 \mathrm{lk}_{2,1}^{X_k}$ by (2-1), and $\mathrm{lk}_{2,1}^{X_k} = 1$. Since Δ^2 is a pure braid, we see that $\mathrm{lk}_{2,1}^{\Delta^2} = 1$. Thus $\mathrm{tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 + q_2$. $\qquad\square$

Proof of Theorem 1.1. Let k and l be positive integers. We denote by F_1 and F_2 the first and second components of S_k, and by F_1' and F_2' the first and second components of S_l.

First we show that for $k \neq l$, there does not exist an orientation-preserving self-diffeomorphism of $\mathbb{R}^4$ carrying F_1 to F_1' and F_2 to F_2'. Assume that there is such a diffeomorphism f. Let us consider a 2-dimensional braid over S_k, denoted by $\widetilde{S}_k^1$, which is presented by a 2-chart Γ on S_k such that $\Gamma \cap F_1$ consists of loops with $I(\mu_1, \Gamma) = 2p$ and $I(\lambda_1, \Gamma) = 2q$, where (μ_1, λ_1) is a preferred basis of $H_1(F_1; \mathbb{Z})$, and $\Gamma \cap F_2 = \varnothing$. Note that $\widetilde{S}_k^1$ consists of 4 components.

Since f is an orientation-preserving diffeomorphism which carries F_1 to F_1', $f|_{F_1}$ is an orientation-preserving diffeomorphism from a torus F_1 to a torus F_1'. Let

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{Z})$$

be a matrix determined by

$$(6\text{-}1) \qquad \begin{pmatrix} \mu_1' \\ \lambda_1' \end{pmatrix} = A \begin{pmatrix} f_*(\mu_1) \\ f_*(\lambda_1) \end{pmatrix},$$

where (μ_1', λ_1') is a preferred basis of $H_1(F_1'; \mathbb{Z})$.

Put $\Gamma' = f(\Gamma)$. By f, $\widetilde{S}_k^1$ is taken to a 2-dimensional braid $\widetilde{S}_l^1$ over S_l, presented by a 2-chart Γ' on S_l such that $\Gamma' \cap F_1'$ consists of loops and $\Gamma' \cap F_2' = \varnothing$. We see that $I(f_*(\mu_1), \Gamma') = I(\mu_1, \Gamma) = 2p$, and $I(f_*(\lambda_1), \Gamma') = I(\lambda_1, \Gamma) = 2q$. Set $p' = I(\mu_1', \Gamma')/2$ and $q' = I(\lambda_1', \Gamma')/2$; note that p' and q' are integers, since $\widetilde{S}_l^1$ consists of 4 components. It follows from (6-1) that

$$(6\text{-}2) \qquad \begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}.$$

Since the triple linking numbers $\mathrm{tlk}_{1,2,3}$ for $\widetilde{S}_k^1$ and $\widetilde{S}_l^1$ are the same, Lemma 6.2 implies that

$$(6\text{-}3) \qquad -kp + q = -lp' + q'.$$

Hence, it follows from (6-2) that $kp - q = (\alpha l - \gamma)p + (\beta l - \delta)q$. Since this equation holds true for any integers p and q,

$$(6\text{-}4) \qquad \begin{pmatrix} k \\ -1 \end{pmatrix} = A^T \begin{pmatrix} l \\ -1 \end{pmatrix},$$

where A^T is the transposed matrix of A.

Next we consider another 2-dimensional braid over S_k, denoted by $\widetilde{S}_k^2$, presented by a 2-chart $\widetilde{\Gamma}$ on S_k such that $\widetilde{\Gamma} \cap F_1 = \varnothing$ and $\widetilde{\Gamma} \cap F_2$ consists of loops on F_2, and

moreover that $\tilde{\Gamma} \cap F_2$ is the preimage by the projection $N(T) \to T$ of a 2-chart Γ on the standard torus T consisting of loops with $I(m, \Gamma) = 2p$ and $I(l, \Gamma) = 2q$, where (m, l) is a preferred basis of T. Note that $I(\mu_2, \tilde{\Gamma}) = 2kp$ and $I(\lambda_2, \tilde{\Gamma}) = 2q$, where (μ_2, λ_2) is a preferred basis of $H_1(F_2; \mathbb{Z})$.

Let g be an orientation-preserving diffeomorphism of $\mathbb{R}^4$ which carries F_2 sufficiently close to F_1 and $(g|_{F_i})_* = \text{id}_* : H_1(F_i; \mathbb{Z}) \to g_*(H_1(F_i); \mathbb{Z})$ for $i = 1, 2$. Further, we assume that T is sufficiently close to F_1. Then

$$\binom{m'}{l'} = A \binom{(f \circ g)_*(m)}{(f \circ g)_*(l)},$$

where (m', l') is a preferred basis of $T' = (f \circ g)(T)$. Put $\Gamma' = (f \circ g)(\Gamma)$. Then we have

(6-5)
$$\binom{I(m', \Gamma')}{I(l', \Gamma')} = A \binom{I(m, \Gamma)}{I(l, \Gamma)}.$$

Put $S' = (f \circ g)(S_k)$. The surface link S' is in the form of a 2-dimensional braid over T' of degree $k + 1$. For the natural projection $\pi' : N(T') = (f \circ g)(N(T)) \to T'$, a meridian m', and a preferred longitude l' of T', let us consider $S' \cap \pi'^{-1}(m')$ and $S' \cap \pi'^{-1}(l')$, which are closed $(k + 1)$-braids in the 3-dimensional solid tori. In the same way as obtaining basis braids, we obtain $(k + 1)$-braids from the closed braids by cutting open the solid tori along the 2-disk $\pi'^{-1}(x_0')$, where x_0' is the intersection point of m' and l'. We denote the braids by a and b. Note that here T' is a standard torus, and hence (a, b) are basis braids, but we can apply the same argument if T' is not a standard torus. Since S' consists of two components, a and b satisfy one of the following three cases.

Case 1: The closure of a is a link consisting of two components, and b is a pure braid.

Case 2: Each of the closures of a and b is a link consisting of two components.

Case 3: The braid a is a pure braid, and the closure of b is a link consisting of two components.

Set $\tilde{\Gamma}' = (f \circ g)(\tilde{\Gamma})$. By $f \circ g$, $\tilde{S}_k^2$ is taken to a 2-dimensional braid $\tilde{S}'$ presented by a 2-chart $\tilde{\Gamma}'$ on S'. We denote by F' the component $(f \circ g)(F_2)$ of S', and we denote by (μ', λ') a preferred basis of $H_1(F'; \mathbb{Z})$. Since $\tilde{\Gamma} \cap F_2$ is in the form of the preimage by $N(T) \to T$ of the 2-chart Γ on T, $\tilde{\Gamma}' \cap F'$ is in the form of the preimage by $N(T') \to T'$ of the 2-chart Γ' on T'. Hence, $I(\mu', \tilde{\Gamma}') = i \cdot I(m', \Gamma')$

and $I(\lambda', \widetilde{\Gamma}') = j \cdot I(\boldsymbol{l}', \Gamma')$ for $(i, j) = (k, 1)$ for Case 1, (k, k) for Case 2, and $(1, k)$ for Case 3. Thus

$$(6\text{-}6) \qquad \begin{pmatrix} I(\mu', \widetilde{\Gamma}') \\ I(\lambda', \widetilde{\Gamma}') \end{pmatrix} = B \begin{pmatrix} I(\boldsymbol{m}', \Gamma') \\ I(\boldsymbol{l}', \Gamma') \end{pmatrix},$$

where B is a diagonal matrix $\mathrm{diag}(i, j)$ such that $(i, j) = (k, 1)$ for Case 1, (k, k) for Case 2, and $(1, k)$ for Case 3.

Put $h = f \circ (f \circ g)^{-1}$. Then h is an orientation-preserving self-diffeomorphism of $\mathbb{R}^4$ which carries S' to S_l. In particular, h carries F' to the second component F_2' of S_l. Let

$$C = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{Z})$$

be a matrix determined by

$$\begin{pmatrix} \mu_2' \\ \lambda_2' \end{pmatrix} = C \begin{pmatrix} h_*(\mu') \\ h_*(\lambda') \end{pmatrix},$$

where (μ_2', λ_2') is a preferred basis of $H_1(F_2'; \mathbb{Z})$. Put $\Gamma'' = h(\widetilde{\Gamma}')$. Then

$$(6\text{-}7) \qquad \begin{pmatrix} I(\mu_2', \Gamma'') \\ I(\lambda_2', \Gamma'') \end{pmatrix} = C \begin{pmatrix} I(\mu', \widetilde{\Gamma}') \\ I(\lambda', \widetilde{\Gamma}') \end{pmatrix}.$$

Set $p'' = I(\mu_2', \Gamma'')/2$ and $q'' = I(\lambda_2', \Gamma'')/2$, which are both integers. Since $I(\boldsymbol{m}, \Gamma) = 2p$ and $I(\boldsymbol{l}, \Gamma) = 2q$, together with (6-5)–(6-7), we have

$$(6\text{-}8) \qquad \begin{pmatrix} p'' \\ q'' \end{pmatrix} = (CBA) \begin{pmatrix} p \\ q \end{pmatrix}.$$

By the composite diffeomorphism $h \circ f \circ g = f$, $\widetilde{S}_k^2$ is taken to a 2-dimensional braid over S_l, which will be denoted by $\widetilde{S}_l^2$. Since $\mathrm{tlk}_{2,3,4}$ is the same for $\widetilde{S}_k^2$ and $\widetilde{S}_l^2$, together with $I(\mu_2, \widetilde{\Gamma}) = 2kp$ and $I(\lambda_2, \widetilde{\Gamma}) = 2q$, Lemma 6.2 implies that

$$(6\text{-}9) \qquad -kp + q = -p'' + q''.$$

Since this equation holds true for any integers p and q, it follows from (6-8) that

$$\begin{pmatrix} k \\ -1 \end{pmatrix} = (CBA)^T \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, together with (6-4),

$$B^T C^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} l \\ -1 \end{pmatrix},$$

whence $i(\alpha' - \gamma') = l$ and $j(\beta' - \delta') = -1$. Let us assume $k > l > 0$. For Cases 1 and 2, $k(\alpha' - \gamma') = l$ from the first equation. This contradicts the assumption that $k > l > 0$. For Case 3, the second equation implies that $k(\delta' - \beta') = 1$, which contradicts the assumption that $k > 1$. Thus, for $k \neq l$, there does not exist an orientation-preserving self-diffeomorphism of R^4 which carries F_1 to F_1' and F_2 to F_2'.

Next we show that for $k \neq l$, there does not exist an orientation-preserving self-diffeomorphism of R^4 which carries F_1 to F_2' and F_2 to F_1'. We discuss a similar argument as in the former case of a diffeomorphism which carries F_1 to F_1' and F_2 to F_2', using the same notation except where noted.

Assume that there is such a diffeomorphism f, and consider Γ as before. Then, since $\mathrm{tlk}_{1,2,3}$ for $\widetilde{S}_k^1$ and $\mathrm{tlk}_{3,4,1} = \mathrm{tlk}_{4,3,2}$ (see Lemma 6.1) for $\widetilde{S}_l^1$ are the same, and since $\mathrm{tlk}_{4,3,2} = -\mathrm{tlk}_{2,3,4}$ [Carter et al. 2003], Lemma 6.2 implies that instead of (6-3) we have

$$(6\text{-}10) \qquad\qquad -kp + q = p' - q',$$

where $p' = I(\mu_2', \Gamma')/2$ and $q' = I(\lambda_2', \Gamma')/2$. Hence instead of (6-4) we have

$$(6\text{-}11) \qquad\qquad \begin{pmatrix} k \\ -1 \end{pmatrix} = A^T \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Next we will consider another 2-dimensional braid $\widetilde{S}_k^2$ over S_k, presented by the 2-chart $\widetilde{\Gamma}$ as in the former case. Then, by the same argument, we have (6-8), where $p'' = I(\mu_1', \Gamma'')/2$ and $q'' = I(\lambda_1', \Gamma'')/2$.

By the composite diffeomorphism $h \circ f \circ g$, $\widetilde{S}_k^2$ is carried to a 2-dimensional braid over S_l, which will be denoted by $\widetilde{S}_l^2$. Since $\mathrm{tlk}_{2,3,4}$ for $\widetilde{S}_k^2$ and $\mathrm{tlk}_{3,1,2} = \mathrm{tlk}_{3,2,1}$ (see Lemma 6.1) for $\widetilde{S}_l^2$ are the same, and since $\mathrm{tlk}_{3,2,1} = -\mathrm{tlk}_{1,2,3}$ [Carter et al. 2003], together with $I(\mu_2, \widetilde{\Gamma}) = 2kp$ and $I(\lambda_2, \widetilde{\Gamma}) = 2q$, Lemma 6.2 implies that

$$(6\text{-}12) \qquad\qquad -kp + q = lp'' - q''.$$

Since this equation holds true for any integers p and q, it follows from (6-8) that

$$\begin{pmatrix} k \\ -1 \end{pmatrix} = (CBA)^T \begin{pmatrix} -l \\ 1 \end{pmatrix}.$$

Thus, together with (6-11),

$$B^T C^T \begin{pmatrix} -l \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

whence $i(-l\alpha' + \gamma') = -1$ and $j(-l\beta' + \delta') = 1$. Let us assume $k > l > 0$. Since at least one of i and j is k for Cases 1, 2, and 3, these equations contradict the assumption that $k > 1$. Thus, for $k \neq l$, there does not exist an orientation-preserving

self-diffeomorphism of R^4 carrying F_1 to F_2' and F_2 to F_1'. Thus S_k and S_l are not equivalent for positive integers $k \neq l$. ∎

Acknowledgments

References

[Boyle 1993] J. Boyle, "The turned torus knot in S^4", *J. Knot Theory Ramifications* **2**:3 (1993), 239–249. MR 94i:57037 Zbl 0995.57502

[Carter et al. 2001] J. S. Carter, S. Kamada, M. Saito, and S. Satoh, "A theorem of Sanderson on link bordisms in dimension 4", *Algebr. Geom. Topol.* **1** (2001), 299–310. MR 2002d:57018 Zbl 0973.57010

[Carter et al. 2003] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, "Quandle cohomology and state-sum invariants of knotted curves and surfaces", *Trans. Amer. Math. Soc.* **355**:10 (2003), 3947–3989. MR 2005b:57048 Zbl 1028.57003

[Carter et al. 2004] S. Carter, S. Kamada, and M. Saito, *Surfaces in 4-space*, Encyclopaedia of Mathematical Sciences **142**, Springer, Berlin, 2004. MR 2005e:57065 Zbl 1078.57001

[Ito and Nakamura 2014] T. Ito and I. Nakamura, "On surface links whose link groups are abelian", *Math. Proc. Cambridge Philos. Soc.* **157**:1 (2014), 63–77. MR 3211808 Zbl 1298.57018

[Iwase 1988] Z. Iwase, "Dehn-surgery along a torus T^2-knot", *Pacific J. Math.* **133**:2 (1988), 289–299. MR 89m:57020 Zbl 0629.57009

[Kamada 1992] S. Kamada, "Surfaces in $\mathbf{R}^4$ of braid index three are ribbon", *J. Knot Theory Ramifications* **1**:2 (1992), 137–160. MR 93h:57039 Zbl 0763.57013

[Kamada 2002] S. Kamada, *Braid and knot theory in dimension four*, Mathematical Surveys and Monographs **95**, American Mathematical Society, Providence, RI, 2002. MR 2003d:57050 Zbl 0993.57012

[Kawauchi 1996] A. Kawauchi, *A survey of knot theory*, Birkhäuser, Basel, 1996. MR 97k:57011 Zbl 0861.57001

[Lickorish 1997] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics **175**, Springer, New York, 1997. MR 98f:57015 Zbl 0886.57001

[Montesinos 1983] J. M. Montesinos, "On twins in the four-sphere. I", *Quart. J. Math. Oxford Ser.* (2) **34**:134 (1983), 171–199. MR 86i:57025a Zbl 0522.57019

[Nakamura 2011] I. Nakamura, "Surface links which are coverings over the standard torus", *Algebr. Geom. Topol.* **11**:3 (2011), 1497–1540. MR 2012g:57048 Zbl 1230.57022

[Nakamura 2012] I. Nakamura, "Triple linking numbers and triple point numbers of certain T^2-links", *Topology Appl.* **159**:5 (2012), 1439–1447. MR 2879374 Zbl 1236.57030

[Nakamura 2014a] I. Nakamura, "Satellites of an oriented surface link and their local moves", *Topology Appl.* **164** (2014), 113–124. MR 3160466 Zbl 1295.57028

[Nakamura 2014b] I. Nakamura, "Surface links with free abelian groups", *J. Math. Soc. Japan* **66**:1 (2014), 247–256. MR 3161400 Zbl 1297.57056

[Roseman 1998] D. Roseman, "Reidemeister-type moves for surfaces in four-dimensional space", pp. 347–380 in *Knot theory* (Warsaw, 1995), edited by V. F. R. Jones et al., Banach Center Publ. **42**, Polish Acad. Sci., Warsaw, 1998. MR 99f:57029 Zbl 0906.57010

[Rudolph 1983] L. Rudolph, "Braided surfaces and Seifert ribbons for closed braids", *Comment. Math. Helv.* **58**:1 (1983), 1–37. MR 84j:57006 Zbl 0522.57017

[Sanderson 1987] B. J. Sanderson, "Bordism of links in codimension 2", *J. London Math. Soc.* (2) **35**:2 (1987), 367–376. MR 88d:57023 Zbl 0622.57026

[Sanderson 1993] B. J. Sanderson, "Triple links in codimension 2", pp. 457–471 in *Topology. Theory and applications, II* (Pécs, 1989), edited by Á. Császár, Colloq. Math. Soc. János Bolyai **55**, North-Holland, Amsterdam, 1993. MR 96a:57056 Zbl 0815.57016

[Suciu 1985] A. I. Suciu, "Infinitely many ribbon knots with the same fundamental group", *Math. Proc. Cambridge Philos. Soc.* **98**:3 (1985), 481–492. MR 87a:57025 Zbl 0596.57013

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