**K-THEORY AND HOMOTOPIES OF 2-COCYCLES
ON HIGHER-RANK GRAPHS**

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This paper continues our investigation into the question of when a homotopy of 2-cocycles on a locally compact Hausdorff groupoid gives rise to an isomorphism of the *K*-theory groups of the twisted groupoid *C**-algebras. Our main result, which builds on work by Kumjian, Pask, and Sims, shows that a homotopy of 2-cocycles on a row-finite higher-rank graph Λ gives rise to twisted groupoid *C**-algebras with isomorphic *K*-theory groups. (The groupoid in question is the path groupoid of Λ.) We also establish a technical result: any homotopy of 2-cocycles on a locally compact Hausdorff groupoid $\mathcal{G}$ gives rise to an upper semicontinuous bundle of *C**-algebras.

1. Introduction

Higher-rank graphs, or *k*-graphs, provide a *k*-dimensional analogue of directed graphs. They were introduced by Kumjian and Pask [2000] to provide a combinatorial model for the higher-rank Cuntz–Krieger algebras studied by Robertson and Steger [1999]. Much of the interest in the *C**-algebras $C^*(\Lambda)$ associated to *k*-graphs Λ stems from the multiple ways one can model $C^*(\Lambda)$ — the *k*-graph Λ reflects many of the properties of $C^*(\Lambda)$, but we can also describe $C^*(\Lambda)$ as a universal *C**-algebra for certain generators and relations, or as a groupoid *C**-algebra $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$.

The class of groupoids includes groups, group actions, equivalence relations, and group bundles. Renault [1980] initiated the study of groupoid *C**-algebras, and the theory and applications of groupoid *C**-algebras have since been developed by many researchers. Given a 2-cocycle $\omega \in Z^2(\mathcal{G}, \mathbb{T})$ on a groupoid $\mathcal{G}$, Renault also explained how to construct the twisted groupoid *C**-algebra $C^*(\mathcal{G}, \omega)$. These objects have received relatively little attention until quite recently, but it has now become clear that twisted groupoid *C**-algebras can help answer many questions about the structure of untwisted groupoid *C**-algebras (see [Muhly and Williams 1992; Muhly et al. 1996; Clark and an Huef 2012; an Huef et al. 2011; Brown

and an Huef 2014]), as well as classifying those C^*-algebras which admit diagonal subalgebras (also known as Cartan subalgebras) — see [Kumjian 1986]. In another direction, [Tu et al. 2004] establishes a connection between the K-theory of twisted groupoid C^*-algebras and the classification of D-brane charges in string theory.

Two recent papers have explored the effect of a homotopy $\{\omega_t\}_{t \in [0,1]}$ of 2-cocycles on the K-theory of the twisted groupoid C^*-algebras. Echterhoff, Lück, Phillips, and Walters showed in [Echterhoff et al. 2010, Theorem 1.9] that if G is a group that satisfies the Baum–Connes conjecture with respect to the coefficient algebras $\mathcal{K}$ and $C([0,1], \mathcal{K})$, and if $\{\omega_t\}_{t \in [0,1]}$ is a homotopy of 2-cocycles on G, then the K-theory groups of the reduced twisted group C^*-algebras are unperturbed by the homotopy

$$(1) \qquad\qquad K_*(C_r^*(G, \omega_0)) \cong K_*(C_r^*(G, \omega_1)).$$

In particular, taking $G = \mathbb{Z}^2$, we obtain another proof of the fact, established by Pimsner and Voiculescu [1980], that all of the rotation algebras $\{A_\theta\}_{\theta \in [0,1]}$ have isomorphic K-theory groups.

Kumjian, Pask, and Sims also studied the effect of a homotopy of 2-cocycles on K-theory in [Kumjian et al. 2013]. Theorem 5.4 of [Kumjian et al. 2013] establishes that if Λ is a row-finite, source-free k-graph and c is a 2-cocycle on Λ such that $c(\lambda, \mu) = e^{2\pi i \sigma(\lambda, \mu)}$ for some $\mathbb{R}$-valued 2-cocycle σ, then we have $K_*(C^*(\Lambda)) \cong K_*(C^*(\Lambda, c))$. Defining $c_t(\lambda, \mu) = e^{2\pi i t \sigma(\lambda, \mu)}$ for $t \in [0,1]$ gives us a homotopy of 2-cocycles linking c and the trivial 2-cocycle. Moreover, Corollary 7.8 of [Kumjian et al. 2015] tells us that $C^*(\Lambda, c)$ is isomorphic to a twisted groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda, \omega_c)$. Thus, we can view [Kumjian et al. 2013, Theorem 5.4] as a result about homotopic 2-cocycles on groupoids.

Inspired by the above-mentioned results, we have begun exploring the question of when a homotopy of 2-cocycles on a locally compact Hausdorff groupoid $\mathcal{G}$ induces an isomorphism of the K-theory groups of the (full or reduced) twisted groupoid C^*-algebras. In a previous article [Gillaspy 2015], we extended the above-mentioned Theorem 1.9 of [Echterhoff et al. 2010] to the case when $\mathcal{G} = G \ltimes X$ is a transformation group, where X is locally compact Hausdorff and G satisfies the Baum–Connes conjecture with coefficients.

We prove the following generalization of [Kumjian et al. 2013, Theorem 5.4].

Theorem 4.1. *Let Λ be a row-finite k-graph with no sources and let $\{c_t\}_{t \in [0,1]}$ be a homotopy of 2-cocycles in $\underline{Z}^2(\Lambda, \mathbb{T})$. Then $\{c_t\}_{t \in [0,1]}$ gives rise to a homotopy $\{\sigma_{c_t}\}_{t \in [0,1]}$ of 2-cocycles on $\mathcal{G}_\Lambda$ such that*

$$K_*(C^*(\mathcal{G}_\Lambda, \sigma_{c_0})) \cong K_*(C^*(\mathcal{G}_\Lambda, \sigma_{c_1})).$$

As of this writing, we are unaware of any examples of groupoids $\mathcal{G}$ and homotopies $\omega = \{\omega_t\}_{t \in [0,1]}$ of 2-cocycles on $\mathcal{G}$ where the homotopy does not induce an isomorphism of the K-theory groups of the twisted groupoid C^*-algebras.

Outline. This paper begins by recalling the definitions of a higher-rank graph and of a groupoid in Section 2, as well as the definition of a 2-cocycle in each category, and sketching the procedure by which we can construct a C^*-algebra from these objects. In Section 3 we define a homotopy of 2-cocycles on a k-graph and on a groupoid, and show that the definitions are compatible. We also prove a technical result (Theorem 3.3), namely, that a homotopy $\{\omega_t\}_{t \in [0,1]}$ of 2-cocycles on a groupoid $\mathcal{G}$ gives rise to a $C([0, 1])$-algebra with fiber algebra $C^*(\mathcal{G}, \omega_t)$ at $t \in [0, 1]$. We expect that this result will prove useful in future work, as we search for more classes of groupoids where a homotopy of 2-cocycles induces an isomorphism of the K-theory groups of the twisted groupoid C^*-algebras.

In Section 4 we begin the proof of Theorem 4.1. Our proof technique consists of proving a stronger version of Theorem 4.1 in a simple case, and then showing how to use this simple case to obtain our desired result for general k-graphs. To be precise, Proposition 4.2 shows that when the degree map d on Λ satisfies $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, the $C([0, 1])$-algebra associated to a homotopy of 2-cocycles on Λ is actually a trivial continuous field. We then show how to exploit the triviality of the continuous field in this special case to see that a homotopy of 2-cocycles on any row-finite, source-free k-graph Λ induces an isomorphism $K_*(C^*(\Lambda, c_0)) \cong K_*(C^*(\Lambda, c_1))$. The argument closely parallels Section 5 of [Kumjian et al. 2013].

2. Groupoids and *k*-graphs

Definition 2.1 [Kumjian and Pask 2000, Definition 1.1]. A *higher-rank graph* of degree k, or a k-graph, is a nonempty countable small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$ (the *degree map*) satisfying the following *factorization property*: Given a morphism $\lambda \in \Lambda$ with $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$.

The simplest example of a k-graph is $\mathbb{N}^k$, equipped with the identity morphism id $: \mathbb{N}^k \to \mathbb{N}^k$.

In this article we use the arrows-only picture of category theory, so that we think of the objects of a category Λ as identity morphisms. Hence, $\lambda \in \Lambda$ means that λ is a morphism in Λ. Given an element λ in a category Λ, write $s(\lambda)$ for the domain, or *source*, of the morphism λ, and write $r(\lambda)$ for its target, or *range*. We say a k-graph Λ is *row-finite* if, for any $v \in \mathrm{Obj}(\Lambda)$ and any $n \in \mathbb{N}^k$, the set

$$v\Lambda^n := \{\lambda \in \Lambda : r(\lambda) = v, \ d(\lambda) = n\}$$

is finite. We say Λ has *no sources* if $v\Lambda^n \neq \varnothing$ for every $v \in \mathrm{Obj}(\Lambda)$ and every $n \in \mathbb{N}^k$. We only consider k-graphs which are row-finite and have no sources,

since these are the k-graphs which we can study via the groupoid method that was introduced in [Kumjian and Pask 2000] and which we will explain in Section 2.

Definition 2.2 [Renault 1980, Definition I.1.12; Kumjian et al. 2015, Section 3]. For a category Λ, let $\Lambda^{*2} = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda : s(\lambda_1) = r(\lambda_2)\}$. A function $c : \Lambda^{*2} \to \mathbb{T}$ is called a *2-cocycle* on Λ if

$$(2) \qquad\qquad c(\lambda, \mu\nu)c(\mu, \nu) = c(\lambda\mu, \nu)c(\lambda, \mu)$$

whenever $(\lambda, \mu), (\mu, \nu) \in \Lambda^{*2}$ and $c(\lambda, s(\lambda)) = c(r(\lambda), \lambda) = 1$ for all $\lambda \in \Lambda$. We write $\underline{Z}^2(\Lambda, \mathbb{T})$ for the set of 2-cocycles on Λ.

If $c, \tilde{c}$ are two 2-cocycles on Λ, we say that $c, \tilde{c}$ are *cohomologous* if there exists a function $b : \Lambda \to \mathbb{T}$ such that

$$\tilde{c}(\mu, \nu) := b(\mu)b(\nu)b(\mu\nu)^{-1}c(\mu, \nu) = \delta b(\mu, \nu)c(\mu, \nu) \quad \text{for all } (\mu, \nu) \in \Lambda^{*2}.$$

We note that cohomologous 2-cocycles give rise to isomorphic twisted C^*-algebras (see [Kumjian et al. 2015, Proposition 5.6; Renault 1980, Proposition II.1.2]).

The only cocycles we consider in this paper are 2-cocycles, so we will occasionally drop the "2" and refer to them simply as *cocycles*.

Definition 2.3 [Kumjian et al. 2015, Definition 5.2]. The *twisted higher-rank-graph algebra* $C^*(\Lambda, c)$ associated to a k-graph Λ and a 2-cocycle c on Λ is the universal C^*-algebra generated by a collection $\{s_\lambda\}_{\lambda \in \Lambda}$ of partial isometries satisfying the following *twisted Cuntz–Krieger relations*:

(CK1) $\{s_v\}_{v \in \mathrm{Obj}(\Lambda)}$ is a collection of mutually orthogonal projections;

(CK2) $s_\mu s_\nu = c(\mu, \nu)s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(CK3) $s_\mu^* s_\mu = s_{s(\mu)}$ for all $\mu \in \Lambda$;

(CK4) $s_v = \sum_{\mu \in v\Lambda^n} s_\mu s_\mu^*$ for all $v \in \mathrm{Obj}(\Lambda)$ and all $n \in \mathbb{N}^k$.

Note that every k-graph Λ admits at least one 2-cocycle: the trivial cocycle, obtained by setting $c(\lambda, \mu) = 1$ for all $(\lambda, \mu) \in \Lambda^{*2}$. In this case, the definition above of $C^*(\Lambda, c)$ agrees with that of $C^*(\Lambda)$ given in [Kumjian and Pask 2000, Definition 1.5]. For example, if Λ is $\mathbb{N}^k$ and c is the trivial cocycle, then we have $C^*(\Lambda, c) \cong C(\mathbb{T}^k)$. More generally, if Λ is $\mathbb{N}^2$, let $c_\theta : \Lambda^{*2} \to \mathbb{T}$ be given by $c_\theta((m, n), (j, k)) = e^{2\pi i \theta n j}$. Then c_θ is a 2-cocycle on Λ and $C^*(\Lambda, c_\theta)$ is isomorphic to the rotation algebra A_θ.

Groupoids. In this section, we review the construction of a twisted groupoid C^*-algebra set forth in [Renault 1980], as well as the procedure given in the seminal article [Kumjian and Pask 2000] for associating a groupoid to a k-graph. Theorem 3.3 applies to arbitrary locally compact Hausdorff groupoids, so we

present in full generality all the definitions necessary for the construction of a twisted groupoid C^*-algebra.

A *groupoid* $\mathcal{G}$ is a small category with inverses. We use the notation of [Renault 1980] to denote groupoid elements and operations; for example, $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ denotes the set of composable pairs and $\mathcal{G}^{(0)}$ denotes the unit space. If $u \in \mathcal{G}^{(0)}$, we write

$$\mathcal{G}_u = \{x \in \mathcal{G} : s(x) = u\}, \quad \mathcal{G}^u = \{x \in \mathcal{G} : r(x) = u\}.$$

In this article, we restrict our attention to groupoids which admit a locally compact Hausdorff topology in which the operations of composition (or multiplication) and inversion are continuous.

In addition to the groupoids associated to k-graphs, examples of groupoids include groups, vector bundles, and transformation groups. For details and examples, see [Goehle 2009].

Given a row-finite, source-free k-graph Λ, Section 2 of [Kumjian and Pask 2000] describes how to form the associated path groupoid $\mathcal{G}_\Lambda$:

Definition 2.4 [Kumjian and Pask 2000, Example 1.7(ii)]. Define the k-graph Ω_k to be the category with $\mathrm{Obj}(\Omega_k) = \mathbb{N}^k$ and morphisms $\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : n \geq m\}$. We have $r(m, n) = m$, $s(m, n) = n$, $d(m, n) = n - m$. Composition in Ω_k is given by $(m, n)(n, \ell) = (m, \ell)$.

For a k-graph Λ, let Λ^∞ denote the set of degree-preserving functors $x : \Omega_k \to \Lambda$. When $k = 1$, the elements $x \in \Lambda^\infty$ are the infinite paths in Λ.

Given $p \in \mathbb{N}^k$, define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$. When Λ is row-finite and source-free, Proposition 2.3 in [Kumjian and Pask 2000] shows that if $\lambda \in \Lambda$, $x \in \Lambda^\infty$ satisfy $s(\lambda) = x(0)$, there is a unique $y \in \Lambda^\infty$ such that $\sigma^{d(\lambda)}(y) = x$; we often write $y = \lambda x$.

Definition 2.5 [Kumjian and Pask 2000, Definition 2.1]. Given a row-finite, source-free k-graph Λ, the associated path groupoid $\mathcal{G}_\Lambda$ is the groupoid associated to the equivalence relation on Λ^∞ of "shift equivalence with lag". In other words,

$$\mathcal{G}_\Lambda := \{(x, n - m, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : n, m \in \mathbb{N}^k, \ \sigma^n(x) = \sigma^m(y)\}$$

and $\mathcal{G}_\Lambda^{(0)} = \Lambda^\infty$, with $r(x, \ell, y) = x$, $s(x, \ell, y) = y$ and multiplication and inversion in $\mathcal{G}_\Lambda$ given by $(x, \ell, y)(y, m, z) = (x, \ell + m, z)$, $(x, \ell, y)^{-1} = (y, -\ell, x)$.

When Λ is a row-finite, source-free k-graph, Proposition 2.8 in [Kumjian and Pask 2000] tells us that the sets

$$Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) : x(0) = s(\mu) = s(\nu)\}$$

form a basis of compact open sets for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$ (in fact, with this topology $\mathcal{G}_\Lambda$ is an ample étale groupoid).

To build a C^*-algebra out of a groupoid $\mathcal{G}$ we start by putting a $*$-algebra structure on $C_c(\mathcal{G})$, and to do this we need to integrate over the groupoid $\mathcal{G}$. A Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ (the groupoid analogue of Haar measure for groups; see [Renault 1980, Definition I.2.2]) will allow us to do this. Unlike in the group case, one cannot make existence or uniqueness statements about Haar systems for groupoids, so one usually starts by hypothesizing the existence of a fixed Haar system. For example, we obtain a Haar system $\{\lambda^x\}_{x \in \Lambda^\infty}$ on $\mathcal{G}_\Lambda$ by setting

$$\lambda^x(E) = \#\{e \in E : e = (x, n, y) \text{ for some } n \in \mathbb{Z}^k, \ y \in \Lambda^\infty\}.$$

We will always use this Haar system on $\mathcal{G}_\Lambda$ in this paper.

Definition 2.6. Let $\mathcal{G}$ be a locally compact Hausdorff groupoid equipped with a Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ and a continuous 2-cocycle ω. We define a $*$-algebra structure on $C_c(\mathcal{G})$ as follows: for $f, g \in C_c(\mathcal{G})$ let

$$f *_\omega g(a) = \int_{\mathcal{G}_{s(a)}} f(ab)g(b^{-1})\omega(ab, b^{-1}) \, d\lambda^{s(a)}(b),$$

$$f^*(a) = \overline{f(a^{-1})\omega(a, a^{-1})}.$$

From [Renault 1980, Proposition II.1.1] we know that the multiplication is well-defined (that is, that $f *_\omega g \in C_c(\mathcal{G})$ as claimed) and associative, and that $(f^*)^* = f$ so that the involution is involutive. The proof of associativity relies on the cocycle condition (2).

Given the fundamental role that the cocycle ω plays in the multiplication and involution on $C_c(\mathcal{G})$, we will often write $C_c(\mathcal{G}, \omega)$ to denote the set $C_c(\mathcal{G})$ equipped with the $*$-algebra structure of Definition 2.6. We define $C^*(\mathcal{G}, \omega)$ to be the completion of $C_c(\mathcal{G}, \omega)$ in the maximal C^*-norm, as described in Chapter II of [Renault 1980].

Definition 2.7. When $\mathcal{G} = \mathcal{G}_\Lambda$ is the groupoid associated to a row-finite k-graph Λ with no sources, Lemma 6.3 of [Kumjian et al. 2015] explains how, given a cocycle $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, we can construct a cocycle $\sigma_c \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$. Then Corollary 7.8 of [Kumjian et al. 2015] shows that $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$. The construction of σ_c is rather technical, but since we will need the details later, we present it here.

Lemma 6.6 of [Kumjian et al. 2015] establishes the existence of a subset

$$\mathcal{P} \subseteq \{Z(\mu, \nu) : s(\mu) = s(\nu)\}$$

that partitions $\mathcal{G}_\Lambda$. In other words, every $a \in \mathcal{G}_\Lambda$ has exactly one representation of the form $a = (\mu_a x, d(\mu_a) - d(\nu_a), \nu_a x)$ with $Z(\mu_a, \nu_a) \in \mathcal{P}$. Note that if $(a, b) \in \mathcal{G}_\Lambda^{(2)}$, we need not have $\mu_a = \mu_{ab}$ or $\nu_b = \nu_{ab}$. However, given $(a, b) \in \mathcal{G}_\Lambda^{(2)}$, Lemma 6.3(i) of [Kumjian et al. 2015] shows that we can always find $y \in \Lambda^\infty$ and $\alpha, \beta, \gamma \in \Lambda$

such that

$$a = (\mu_a \alpha y, d(\mu_a) - d(\nu_a), \nu_a \alpha y),$$
$$b = (\mu_b \beta y, d(\mu_b) - d(\nu_b), \nu_b \beta y),$$
$$ab = (\mu_{ab} \gamma y, d(\mu_{ab}) - d(\nu_{ab}), \nu_{ab} \gamma y).$$

Then, given a 2-cocycle c on Λ, we define a 2-cocycle σ_c on $\mathcal{G}_\Lambda$ by

$$\sigma_c(a, b) = c(\mu_a, \alpha)c(\mu_b, \beta)c(\nu_{ab}, \gamma)\overline{c(\nu_a, \alpha)c(\nu_b, \beta)c(\mu_{ab}, \gamma)}.$$

Since c satisfies the cocycle condition (2), it's straightforward to check that σ_c does also. Lemma 6.3 of [Kumjian et al. 2015] checks that σ_c is well-defined and continuous, so we can construct the groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda, \sigma_c)$ as outlined above. Corollary 7.8 of [Kumjian et al. 2015] tells us that $C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$.

Theorem 6.5 of [Kumjian et al. 2015] establishes that different choices of partitions $\mathcal{P}$ give rise to cohomologous groupoid cocycles, and hence to isomorphic twisted groupoid C^*-algebras.

3. Homotopies of cocycles

In order to define a homotopy of groupoid 2-cocycles, we begin by observing that, given any locally compact Hausdorff groupoid $\mathcal{G}$, we can make $\mathcal{G} \times [0, 1]$ into a locally compact Hausdorff groupoid by equipping it with the product topology and setting $(\mathcal{G} \times [0, 1])^{(2)} := \mathcal{G}^{(2)} \times [0, 1]$. In other words, $(\mathcal{G} \times [0, 1])^{(0)} = \mathcal{G}^{(0)} \times [0, 1]$ and

$$r(\gamma, t) = (r(\gamma), t), \quad s(\gamma, t) = (s(\gamma), t).$$

Moreover, if $\mathcal{G}$ has a Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$, then setting $\lambda^{u,t} := \lambda^u$ for every $t \in [0, 1]$ gives rise to a Haar system on $\mathcal{G} \times [0, 1]$. We will always use this Haar system on $\mathcal{G} \times [0, 1]$ in this paper.

Definition 3.1 [Gillaspy 2015, Definition 2.12]. A *homotopy of* (2-)*cocycles* on a locally compact Hausdorff groupoid $\mathcal{G}$ is a 2-cocycle $\omega \in Z^2(\mathcal{G} \times [0, 1], \mathbb{T})$. We say that two cocycles $\omega_0, \omega_1 \in Z^2(\mathcal{G}, \mathbb{T})$ are *homotopic* if there exists a homotopy $\omega \in Z^2(\mathcal{G} \times [0, 1], \mathbb{T})$ such that $\omega_i = \omega|_{\mathcal{G} \times \{i\}}$ for $i = 0, 1$.

If ω is a homotopy of cocycles on $\mathcal{G}$ linking ω_0, ω_1, Theorem 3.3 below tells us that $C^*(\mathcal{G}, \omega_0)$ and $C^*(\mathcal{G}, \omega_1)$ are quotients of $C^*(\mathcal{G} \times [0, 1], \omega)$. This is fundamental to the proof of our main result, Theorem 4.1.

Definition 3.2 [Williams 2007, Definition C.1]. Let X be a locally compact Hausdorff space. A C^*-algebra A is a $C_0(X)$-*algebra* if we have a $*$-homomorphism $\Phi : C_0(X) \to ZM(A)$ such that

$$A = \overline{\text{span}}\{\Phi(f)a : f \in C_0(X), \, a \in A\}.$$

We usually write $f \cdot a$ for $\Phi(f)a$.

If A is a $C_0(X)$-algebra, then, for any $x \in X$, $\overline{\mathrm{span}}C_0(X\backslash x) \cdot A$ is an ideal I_x. We call $A_x := A/I_x$ the *fiber* of A at $x \in X$.

Theorem 3.3. *Let ω be a homotopy of cocycles on a locally compact Hausdorff groupoid $\mathcal{G}$ with Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$. Then $C^*(\mathcal{G} \times [0, 1], \omega)$ is a $C([0, 1])$-algebra, with fiber $C^*(\mathcal{G}, \omega_t)$ at $t \in [0, 1]$.*

Proof. We begin by checking that $C^*(\mathcal{G} \times [0, 1], \omega)$ is a $C([0, 1])$-algebra. For $f \in C([0, 1])$, $\phi \in C_c(\mathcal{G} \times [0, 1], \omega)$, define

$$f \cdot \phi(a, t) = f(t)\phi(a, t).$$

It's not difficult to check that this action extends to a $*$-homomorphism

$$\Phi : C([0, 1]) \to ZM(C^*(\mathcal{G} \times [0, 1], \omega))$$

such that $\|\Phi(f)\phi\| \leq \|f\|_\infty \|\phi\|$, or to check that

$$\Phi(C([0, 1])) \cdot C_c(\mathcal{G} \times [0, 1], \omega) = C_c(\mathcal{G} \times [0, 1], \omega)$$

is dense in $C^*(\mathcal{G} \times [0, 1], \omega)$. In other words, Φ makes $C^*(\mathcal{G} \times [0, 1], \omega)$ into a $C([0, 1])$-algebra as claimed.

Fix $t \in [0, 1]$ and denote by $q_t : C_c(G \times [0, 1], \omega) \to C_c(G, \omega_t)$ the evaluation map. Then q_t is bounded by the I-norm (see [Renault 1980, Section II.1]), and hence extends to a surjective $*$-homomorphism $q_t : C^*(\mathcal{G} \times [0, 1], \omega) \to C^*(\mathcal{G}, \omega_t)$. In other words, $C^*(\mathcal{G}, \omega_t)$ is a quotient of $C^*(\mathcal{G} \times [0, 1], \omega)$. To see that $C^*(\mathcal{G}, \omega_t) \cong C^*(\mathcal{G} \times [0, 1], \omega)_t$, we need to check that $\ker q_t = I_t$. A standard approximation argument will show that $\ker q_t \supseteq I_t$; thus, we will only detail the proof that $\ker q_t \subseteq I_t$.

Note that the fiber algebra $C^*(\mathcal{G} \times [0, 1], \omega)_t \cong C^*(\mathcal{G} \times [0, 1], \omega)/I_t$ can be calculated as a completion $\overline{C_c(\mathcal{G} \times [0, 1], \omega)}$ with respect to the norm given by

$$\|f\|_t := \sup\{\|L(f)\| : L(I_t) = 0, \ L \text{ is an } I\text{-norm-bounded representation}\}.$$

Thus, to show that $\ker q_t \subseteq I_t$, we will show that each such representation L factors through q_t.

Given such a representation $L : C_c(\mathcal{G} \times [0, 1], \omega) \to B(\mathcal{H})$, define $L' : C_c(\mathcal{G}, \omega_t) \to B(\mathcal{H})$ by $L'(q_t(f)) := L(f)$. We claim that L' is an I-norm-bounded representation of $C_c(\mathcal{G}, \omega_t)$. To see this, it suffices to check that L' is well-defined and bounded.

Lemma 3.4. *If $f, g \in C_c(\mathcal{G} \times [0, 1], \omega)$ satisfy $q_t(f) = q_t(g)$, then the function $h = f - g \in C_c(\mathcal{G} \times [0, 1], \omega)$ lies in I_t. Consequently, $L(f) = L(g)$ and L' is well-defined on $C_c(\mathcal{G}, \omega_t)$.*

Proof. Let $\{f_i\}_{i \in I}$ be an approximate unit for $C_0([0, 1]\backslash t)$ such that $f_i(s) \nearrow 1$ for every $s \neq t$; moreover, suppose that for each i there exists a $\delta_i > 0$ such that $f_i(s) = 1$ if $|s - t| \geq \delta_i$. We will show that the I-norm $\|h - f_i h\|_I$ tends to 0. Consequently, $h = \lim_i f_i h$ in $C^*(\mathcal{G} \times [0, 1], \omega)$, so $h \in I_t$.

For any $k \in C_c(\mathcal{G} \times [0, 1], \omega)$, the axioms of a Haar system tell us that the function $(u, t) \mapsto \int |k(a, t)| d\lambda^{u,t}(a)$ is in $C_0(\mathcal{G}^{(0)} \times [0, 1])$. In particular, if we take k to be a function that equals 1 where h is nonzero and vanishes rapidly off $\mathrm{supp}\, h$, this shows us that $\phi(u, t) := \lambda^{u,t}(\mathrm{supp}\, h)$ is a pointwise limit of functions in $C_0(\mathcal{G}^{(0)} \times [0, 1])$, and hence is bounded. Let $K = \max \phi$.

Let $\epsilon > 0$ be given. Since h is compactly supported and since $h(a, t) = 0$ for all $a \in \mathcal{G}$, we can choose a $\delta > 0$ such that $|h(a, s)| < \epsilon/K$ for all $a \in \mathcal{G}$ whenever $|s - t| < \delta$, and we can choose a j such that $i \geq j$ means $\delta_i < \delta$. Then, if $|s - t| < \delta$,

$$\int_{\mathcal{G}^u} |h(a, s) - f_i(s)h(a, s)| \, d\lambda^u(a) = (1 - f_i(s)) \int_{\mathcal{G}^u} |h(a, s)| \, d\lambda^u(a) < 1 \cdot \epsilon.$$

On the other hand, if $|s - t| \geq \delta > \delta_i$, then $f_i(s) = 1$ and

$$\int_{\mathcal{G}^u} |h(a, s) - f_i(s)h(a, s)| \, d\lambda^u(a) = 0$$

for any $u \in \mathcal{G}^{(0)}$. In either case, given any $\epsilon > 0$ we can always choose j such that $i \geq j$ implies

$$\|h - f_i h\|_I = \max\left\{ \sup_{s \in [0,1]} \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |h(a, s) - f_i(s)h(a, s)| \, d\lambda^u(a), \right.$$
$$\left. \sup_{s \in [0,1]} \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |h(a^{-1}, s) - f_i(s)h(a^{-1}, s)| \, d\lambda^u(a) \right\} < \epsilon.$$

Since $\|h - f_i h\| \leq \|h - f_i h\|_I$ and I_t is closed, it follows that $h \in I_t$ as desired, and so $L(h) = 0$. □

Having seen that L' is well-defined, we now proceed to show that it is bounded.

Lemma 3.5. *For any fixed $f \in C_c(\mathcal{G} \times [0, 1], \omega)$, the map $s \mapsto \|q_s(f)\|_I$ is continuous.*

Proof. Fix $f \in C_c(\mathcal{G} \times [0, 1], \omega)$ and fix $t \in [0, 1]$. As in the proof of Lemma 3.4, let K denote the supremum of the function $(u, s) \mapsto \lambda^{u,s}(\mathrm{supp}\, f)$. Since f has compact support, given $\epsilon > 0$ we can choose a δ such that

$$|s - t| < \delta \Longrightarrow |f(a, t) - f(a, s)| < \frac{\epsilon}{2K} \quad \text{for all } a \in \mathcal{G}.$$

Now, by definition of the I-norm, there exists a $u \in \mathcal{G}^{(0)}$ such that either

$$\|q_s(f)\|_I < \int_{\mathcal{G}^u} |f(a, s)| \, d\lambda^u(a) + \frac{\epsilon}{2}$$

or

$$\|q_s(f)\|_I < \int_{\mathcal{G}^u} |f(a^{-1}, s)| \, d\lambda^u(a) + \frac{\epsilon}{2}.$$

It follows that either

$$\|q_s(f)\|_I < \int_{\mathcal{G}^u} |f(a,t)| + \frac{\epsilon}{2K} \, d\lambda^u(a) + \frac{\epsilon}{2} \leq \int_{\mathcal{G}^u} |f(a,t)| \, d\lambda^u(a) + \epsilon$$

or

$$\|q_s(f)\|_I < \int_{\mathcal{G}^u} |f(a^{-1},t)| + \frac{\epsilon}{2K} \, d\lambda^u(a) + \frac{\epsilon}{2} \leq \int_{\mathcal{G}^u} |f(a^{-1},t)| \, d\lambda^u(a) + \epsilon.$$

Thus,

$$\|q_s(f)\|_I < \max\left\{ \int_{\mathcal{G}^u} |f(a,t)| \, d\lambda^u(a), \int_{\mathcal{G}^u} |f(a^{-1},t)| \, d\lambda^u(a) \right\} + \epsilon$$

$$\leq \max\left\{ \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |f(a,t)| \, d\lambda^u(a), \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |f(a^{-1},t)| \, d\lambda^u(a) \right\} + \epsilon$$

$$= \|q_t(f)\|_I + \epsilon$$

if $|s-t| < \delta$. Reversing the roles of s and t in the above argument tells us that

$$|s-t| < \delta \Rightarrow \left| \|q_s(f)\|_I - \|q_t(f)\|_I \right| < \epsilon. \qquad \square$$

Now we can finish the proof of Theorem 3.3. Set $S_t = \{\psi \in C([0,1]) : \psi(t) = 1\}$. For any $\psi \in S_t$ and any $f \in C_c(\mathcal{G} \times [0,1], \omega)$, we have

$$\|L(\psi \cdot f)\| = \|L'(q_t(\psi \cdot f))\| = \|L'(q_t(f))\|.$$

Consequently,

$$\|L'(q_t(f))\| = \inf_{\psi \in S} \|L(\psi \cdot f)\| \leq \inf_\psi \|\psi \cdot f\|_I$$

$$= \inf_\psi \max\left\{ \sup_{s \in [0,1]} \sup_{u \in \mathcal{G}^{(0)}} \int |\psi(s) f(a,s)| \, d\lambda^u(a), \right.$$

$$\left. \sup_{s \in [0,1]} \sup_{u \in \mathcal{G}^{(0)}} \int |\psi(s) f(a^{-1},s)| \, d\lambda^u(a) \right\}$$

$$= \inf_\psi \sup_{s \in [0,1]} \|q_s(\psi \cdot f)\|_I.$$

Let $\epsilon > 0$ be given. Choose a δ such that $|s-t| < \delta \Rightarrow \left| \|q_s(f)\|_I - \|q_t(f)\|_I \right| < \epsilon$; choose $\psi_\epsilon \in C([0,1])$ such that $\psi_\epsilon(t) = 1$ and $|s-t| \geq \delta \Rightarrow \psi_\epsilon(s) = 0$. Then, since $\psi_\epsilon \in S_t$, we have

$$(3) \qquad \|q_s(\psi_\epsilon \cdot f)\|_I = \psi_\epsilon(s) \|q_s(f)\|_I < \psi_\epsilon(s) (\|q_t(f)\|_I + \epsilon) \leq \|q_t(f)\|_I + \epsilon$$

if $|s-t| < \delta$; otherwise we have $\|q_s(\psi_\epsilon \cdot f)\|_I = 0$, and (3) still holds.

Since we can find such a ψ_ϵ for any $\epsilon > 0$, it follows that

$$\|L'(q_t(f))\| \leq \inf_{\psi \in S_t} \sup_{s \in [0,1]} \|q_s(\psi \cdot f)\|_I \leq \inf_\epsilon \sup_s \|q_s(\psi_\epsilon \cdot f)\|_I$$

$$\leq \inf_\epsilon \|q_t(f)\|_I + \epsilon = \|q_t(f)\|_I.$$

The fact that q_t is onto now tells us that L' is a bounded representation of $C_c(\mathcal{G}, \omega_t)$ as claimed. In other words, every representation L of $C_c(\mathcal{G} \times [0, 1], \omega)$ that kills I_t also factors through q_t, so $\ker q_t \subseteq I_t$. This completes the proof that the fiber algebra $C^*(\mathcal{G} \times [0, 1], \omega)/I_t$ of the $C([0, 1])$-algebra $C^*(\mathcal{G} \times [0, 1], \omega)$ is simply $C^*(\mathcal{G}, \omega_t)$. $\qquad\square$

In order to apply Theorem 3.3 to a homotopy of cocycles on a k-graph, we first need to define such a homotopy. Unlike for groupoids, there is no obvious way to make $\Lambda \times [0, 1]$ into a higher-rank graph, so our definition of a homotopy of k-graph cocycles will look rather different than Definition 3.1 above. However, Proposition 3.8 below shows that the two definitions are compatible.

Definition 3.6. Let Λ be a k-graph. A family $\{c_t\}_{t \in [0,1]}$ of 2-cocycles in $\underline{Z}^2(\Lambda, \mathbb{T})$ is a *homotopy of* (*2-*)*cocycles* on Λ if for each pair $(\lambda, \mu) \in \Lambda^{*2}$ the function $t \mapsto c_t(\lambda, \mu) \in \mathbb{T}$ is continuous.

Definition 3.7. Let $\{c_t\}_{t \in [0,1]}$ be a homotopy of cocycles on a k-graph Λ. Define $\omega \in Z^2(\mathcal{G}_\Lambda \times [0, 1], \mathbb{T})$ by

$$\omega((a, t), (b, t)) = \sigma_{c_t}(a, b),$$

where σ_{c_t} is the cocycle on $\mathcal{G}_\Lambda$ associated to c_t as in Definition 2.7.

A moment's thought will reveal that ω satisfies the cocycle condition (2), since each σ_{c_t} is a cocycle. Thus, in order to see that ω is a homotopy of cocycles on $\mathcal{G}_\Lambda$, we merely need to check that $\omega : (\mathcal{G}_\Lambda \times [0, 1])^{(2)} \to \mathbb{T}$ is continuous.

Proposition 3.8. *The cocycle ω described in Definition 3.7 is continuous, and hence is a homotopy of groupoid cocycles on $\mathcal{G}_\Lambda$.*

Proof. We will show that if $\{(a_i, b_i, t_i)\}_{i \in I} \subseteq \mathcal{G}_\Lambda^{(2)} \times [0, 1]$ is a net which converges to (a, b, t), then

(4) $$\omega((a_i, t_i), (b_i, t_i)) := \sigma_{c_{t_i}}(a_i, b_i) = \sigma_{c_{t_i}}(a, b)$$

for large enough i. Recall from Definition 2.7 that $\sigma_{c_{t_i}}(a, b)$ is a finite product of terms of the form $c_{t_i}(\mu, \nu)$ and their inverses, where the elements μ, ν depend only on the elements a, b and on the choice of partition $\mathcal{P}$ of $\mathcal{G}_\Lambda$ — but not on the 2-cocycle c_{t_i}. Thus, (4) and the continuity of the maps $t \mapsto c_t(\mu, \nu)$ will imply that $\omega((a_i, t_i), (b_i, t_i)) \to \sigma_{c_t}(a, b) = \omega((a, t), (b, t))$.

In what follows, we use the notation of Definition 2.7. If $(a_i, b_i, t_i) \to (a, b, t)$, then, for large enough i, we have $a_i \in Z(\mu_a, \nu_a)$, $b_i \in Z(\mu_b, \nu_b)$, and in addition $a_i b_i \in Z(\mu_{ab}, \nu_{ab})$. In other words, we can write

$$a = (\mu_a \alpha y, d(\mu_a) - d(\nu_a), \nu_a \alpha y), \quad a_i = (\mu_a \alpha_i y_i, d(\mu_a) - d(\nu_a), \nu_a \alpha_i y_i),$$
$$b = (\mu_b \beta y, d(\mu_b) - d(\nu_b), \nu_b \beta y), \quad b_i = (\mu_b \beta_i y_i, d(\mu_b) - d(\nu_b), \nu_b \beta_i y_i),$$
$$ab = (\mu_{ab} \gamma y, d(\mu_{ab}) - d(\nu_{ab}), \nu_{ab} \gamma y),$$
$$a_i b_i = (\mu_{ab} \gamma_i y_i, d(\mu_{ab}) - d(\nu_{ab}), \nu_{ab} \gamma_i y_i)$$

for some $\alpha, \beta, \gamma, \alpha_i, \beta_i, \gamma_i \in \Lambda$ and $y, y_i \in \Lambda^\infty$.

Since $a_i \to a$ we must also have $\alpha_i y_i \to \alpha y$ in Λ^∞. Thus, for large enough i, $\alpha_i y_i \in Z(\alpha) := \{\alpha y : y \in \Lambda^\infty, \ y(0) = s(\alpha)\}$ (see Proposition 2.8 of [Kumjian and Pask 2000]). It follows that

$$a_i = (\mu_a \alpha y_i', d(\mu_a) - d(\nu_a), \nu_a \alpha y_i'), \quad b_i = (\mu_b \beta z_i', d(\mu_b) - d(\nu_b), \nu_b \beta z_i')$$
$$a_i b_i = (\mu_{ab} \gamma w_i', d(\mu_{ab}) - d(\nu_{ab}), \nu_{ab} \gamma w_i'),$$

where (since each pair (a_i, b_i) is in $\mathcal{G}_\Lambda^{(2)}$ by hypothesis)

$$\nu_a \alpha y_i' = \mu_b \beta z_i', \quad \mu_a \alpha y_i' = \mu_{ab} \gamma w_i', \quad \nu_b \beta z_i' = \nu_{ab} \gamma w_i'.$$

Now, $\nu_a \alpha = \mu_b \beta$ by [Kumjian et al. 2015, Lemma 6.3], and thus $y_i' = z_i'$. A similar argument gives $z_i' = w_i'$ as well, so $y_i' = z_i' = w_i'$. In other words, for large enough i,

$$\sigma_{c_{t_i}}(a_i, b_i) = c_{t_i}(\mu_a, \alpha) c_{t_i}(\mu_b, \beta) c_{t_i}(\nu_{ab}, \gamma) \overline{c_{t_i}(\nu_a, \alpha) c_{t_i}(\nu_b, \beta) c_{t_i}(\mu_{ab}, \gamma)}$$
$$= \sigma_{c_{t_i}}(a, b),$$

as claimed. As observed in the first paragraph of the proof, it now follows that ω is a homotopy of cocycles on $\mathcal{G}_\Lambda$ as desired. $\square$

Corollary 3.9. *Let $\{c_t\}$ be a homotopy of cocycles on a k-graph Λ, and define a cocycle ω on $\mathcal{G}_\Lambda \times [0, 1]$ as in Definition 3.7. Then $C^*(\mathcal{G}_\Lambda \times [0, 1], \omega)$ is a $C([0, 1])$-algebra with fiber algebra $C^*(\mathcal{G}_\Lambda, \sigma_{c_t}) \cong C^*(\Lambda, c_t)$ at $t \in [0, 1]$.*

Proof. Proposition 3.8 tells us that ω is a homotopy of cocycles on $\mathcal{G}_\Lambda$. Theorem 3.3 tells us that the fiber over $t \in [0, 1]$ of the $C([0, 1])$-algebra $C^*(\mathcal{G}_\Lambda \times [0, 1], \omega)$ is $C^*(\mathcal{G}_\Lambda, \sigma_{c_t})$. The final isomorphism is provided by Corollary 7.8 of [Kumjian et al. 2015]. $\square$

4. The main theorem

Theorem 4.1. *Let Λ be a row-finite k-graph with no sources and let $\{c_t\}_{t \in [0,1]}$ be a homotopy of cocycles on Λ. Then*

$$K_*(C^*(\Lambda, c_0)) \cong K_*(C^*(\Lambda, c_1)).$$

Moreover, this isomorphism preserves the K-theory class of the vertex projection s_v for each $v \in \mathrm{Obj}(\Lambda)$.

We begin by proving a stronger version of Theorem 4.1 in the simpler case when the degree functor d satisfies $d(\lambda) = \delta b(\lambda) := b(s(\lambda)) - b(r(\lambda))$ for some function $b : \mathrm{Obj}(\Lambda) \to \mathbb{Z}^k$; this is Proposition 4.2 below. We then combine Proposition 4.2 with techniques from [Kumjian et al. 2013] to prove Theorem 4.1 in full generality.

The AF case. If (Λ, d) is a k-graph such that $d = \delta b$, then Lemma 8.4 of [Kumjian et al. 2015] tells us that $C^*(\Lambda, c)$ and $C^*(\Lambda)$ are both AF-algebras, with the same approximating subalgebras and multiplicities of partial inclusions. Consequently, $C^*(\Lambda, c) \cong C^*(\Lambda)$. In order to fix notation for what follows, we describe this isomorphism in some detail.

Lemma 3.1 of [Kumjian and Pask 2000] shows that if Λ is a row-finite, source-free k-graph, then $\{s_\lambda s_\mu^* : s(\lambda) = s(\mu)\}$ spans a dense $*$-subalgebra of $C^*(\Lambda)$. Moreover, when $d = \delta b$, Lemma 5.4 of [Kumjian and Pask 2000] tells us that $\{s_\lambda s_\mu^* : b(s(\lambda)) = b(s(\mu)) = n\}$ forms a collection of matrix units for the subalgebra

$$A_n = \overline{\mathrm{span}}\{s_\lambda s_\mu^* : b(s(\lambda)) = b(s(\mu)) = n\} \cong \bigoplus_{b(v)=n} \mathcal{K}\big(\ell^2(s^{-1}(v))\big).$$

Observe that we can think of A_n as a subalgebra of $C^*(\Lambda)$ or of $C^*(\Lambda, c)$. In fact, these subalgebras allow us to exhibit $C^*(\Lambda, c)$ and $C^*(\Lambda)$ as AF-algebras:

$$C^*(\Lambda, c) = \varinjlim(A_n, \phi_{m,n}^c) \quad \text{and} \quad C^*(\Lambda) = \varinjlim(A_n, \phi_{m,n}),$$

where the connecting maps $\phi_{m,n}, \phi_{m,n}^c : A_n \to A_m$ are given by

$$\phi_{m,n}^c(s_\lambda s_\mu^*) = \sum_{\substack{r(\alpha)=s(\lambda) \\ b(s(\alpha))=m}} c(\lambda, \alpha)\overline{c(\mu, \alpha)} s_{\lambda\alpha} s_{\mu\alpha}^*,$$

$$\phi_{m,n}(s_\lambda s_\mu^*) = \sum_{\substack{r(\alpha)=s(\lambda) \\ b(s(\alpha))=m}} s_{\lambda\alpha} s_{\mu\alpha}^*.$$

We can now describe explicitly the isomorphism $C^*(\Lambda, c) \cong C^*(\Lambda)$. As in Theorem 4.2 of [Kumjian et al. 2013], write $\mathbb{1}$ for $(1, \ldots, 1) \in \mathbb{N}^k$, and define $\kappa : \Lambda \to \mathbb{T}$ by

$$\kappa(\lambda) = \begin{cases} 1 & d(\lambda) \not\geq \mathbb{1}, \\ \kappa(\mu)c(\mu, \alpha) & d(\alpha) = \mathbb{1} \text{ and } \lambda = \mu\alpha. \end{cases}$$

For $n \in \mathbb{Z}^k$, let $U_n = \sum_{b(s(\lambda))=n} \kappa(\lambda) s_\lambda s_\lambda^* \in U(M(A_n))$. A quick computation will show that, for any λ, μ with $s_\lambda s_\mu^* \in A_n$,

$$(5) \qquad\qquad \mathrm{Ad}\, U_n(s_\lambda s_\mu^*) = \kappa(\lambda)\overline{\kappa(\mu)} s_\lambda s_\mu^*.$$

Moreover, the factorization property tells us that, for any $h \in \mathbb{Z}$,

$$\phi^c_{(h+1)\mathbb{1},h\mathbb{1}} \circ \operatorname{Ad} U_{h\mathbb{1}} = \operatorname{Ad} U_{(h+1)\mathbb{1}} \circ \phi_{(h+1)\mathbb{1},h\mathbb{1}}.$$

In other words, $\operatorname{Ad} U_*$ intertwines the connecting maps $\phi^c_{m,n}$, $\phi_{m,n}$, and hence implements the isomorphism $C^*(\Lambda) \to C^*(\Lambda, c)$.

We can now use this isomorphism to prove that a homotopy of cocycles on Λ gives rise to a trivial continuous field when $d = \delta b$.

Proposition 4.2. *Let (Λ, d) be a row-finite, source-free k-graph such that $d = \delta b$ for some function $b : \operatorname{Obj}(\Lambda) \to \mathbb{Z}^k$, let $\{c_t\}_{t \in [0,1]}$ be a homotopy of cocycles on Λ, and let ω be the cocycle on $\mathcal{G}_\Lambda \times [0, 1]$ associated to $\{c_t\}_{t \in [0,1]}$ as in Definition 3.7. We have an isomorphism of $C([0, 1])$-algebras*

$$C^*(\mathcal{G}_\Lambda \times [0, 1], \omega) \cong C^*(\mathcal{G}_\Lambda \times [0, 1]) \cong C([0, 1]) \otimes C^*(\Lambda).$$

Proof. Recall that

$$C^*(\mathcal{G}_\Lambda \times [0, 1])_t \cong C^*(\mathcal{G}_\Lambda, \sigma_{c_t}) \cong C^*(\Lambda, c_t) \cong C^*(\Lambda)$$

if $d = \delta b$. Thus, the $C([0, 1])$-algebras $C^*(\mathcal{G}_\Lambda \times [0, 1], \omega)$ and $C([0, 1]) \otimes C^*(\Lambda)$ have isomorphic fibers over each point $t \in [0, 1]$.

In order to prove the proposition, we need to show that these isomorphisms $C^*(\mathcal{G}_\Lambda, \sigma_{c_t}) \cong C^*(\Lambda)$ vary continuously in t, so that they patch together to give us an isomorphism of $C([0, 1])$-algebras $C^*(\mathcal{G}_\Lambda \times [0, 1], \omega) \cong C([0, 1]) \otimes C^*(\Lambda)$.

For each $t \in [0, 1]$, let $\pi^t : C^*(\Lambda, c_t) \to C^*(\mathcal{G}_\Lambda, \sigma_{c_t})$ denote the isomorphism described in Theorem 6.7 of [Kumjian et al. 2015]. Let $\pi : C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda)$ denote the equivalent isomorphism for the case of a trivial cocycle c. For each $n \in \mathbb{Z}^k$, write U_n^t for the unitary $U_n^t : A_n \to A_n$ associated to the cocycle c_t as above. Setting

$$\Psi_t := \pi^t \circ \operatorname{Ad} U_*^t \circ \pi^{-1}$$

consequently gives an isomorphism of C^*-algebras $\Psi_t : C^*(\mathcal{G}_\Lambda) \to C^*(\mathcal{G}_\Lambda, \sigma_{c_t})$.

We claim that $\Psi := \{\Psi_t\}_{t \in [0,1]}$ defines an isomorphism of $C([0, 1])$-algebras

$$\Psi : C^*(\mathcal{G}_\Lambda \times [0, 1]) \to C^*(\mathcal{G}_\Lambda \times [0, 1], \omega).$$

In order to prove this assertion, we begin by writing down an explicit formula for Ψ_t on the characteristic functions $1_{Z(\mu,\nu)} \in C_c(\mathcal{G}_\Lambda)$ where $Z(\mu, \nu) \in \mathcal{P}$ and where $\mathcal{P}$ is the partition of $\mathcal{G}_\Lambda$ described in Lemma 6.6 of [Kumjian et al. 2015].

Recall that the value of $\sigma_{c_t}(a, b)$ depends only on the sets $Z(\mu, \nu) \in \mathcal{P}$ containing the points a, b, and ab in $\mathcal{G}_\Lambda$. Moreover, the proof of [Kumjian et al. 2015, Theorem 6.7] establishes that, if $1_{Z(\mu,\nu)}$ denotes the characteristic function

on $Z(\mu, \nu) \subseteq \mathcal{G}_\Lambda$, and if we write $a \in Z(\mu, \nu)$ as $a = bd$ where $b \in Z(\mu, s(\mu))$, $d \in Z(s(\nu), \nu)$, then

$$\pi^t(s_\mu s_\nu^*)(a) = 1_{Z(\mu, \nu)}(a)\sigma_{c_t}(b, d)\overline{\sigma_{c_t}(d^{-1}, d)} = 1_{Z(\mu, \nu)}(a)\overline{\sigma_{c_t}(bd, d^{-1})}.$$

In addition, we have $Z(\mu, s(\mu)) \in \mathcal{P}$ for all $\mu \in \Lambda$ by Lemma 6.6 of [Kumjian et al. 2015]. If we also have $Z(\mu, \nu) \in \mathcal{P}$, then the elements α, β, γ in the formula for $\sigma_{c_t}(bd, d^{-1})$ given in Definition 2.7 are all units, so, for any t, we have $\sigma_{c_t}(bd, d^{-1}) = 1$ by our hypothesis that any cocycle c satisfies the equality $c(\lambda, s(\lambda)) = c(r(\lambda), \lambda) = 1$. Thus,

$$Z(\mu, \nu) \in \mathcal{P} \Rightarrow \pi^t(s_\mu s_\nu^*) = 1_{Z(\mu, \nu)} \Rightarrow \Psi_t(1_{Z(\mu, \nu)}) = \kappa_t(\mu)\overline{\kappa_t(\nu)}1_{Z(\mu, \nu)}.$$

Now observe that each $f \in C_c(\mathcal{G}_\Lambda \times [0, 1])$ can be written as a finite sum $f(a, t) = \sum_{i \in N} f_i(a, t)$, where, for all i, we have $f_i \in C(Z(\mu_i, \nu_i) \times [0, 1])$ and $Z(\mu_i, \nu_i) \in \mathcal{P}$. Consequently, on $C_c(\mathcal{G}_\Lambda \times [0, 1])$, our map Ψ becomes

$$(6) \qquad \Psi\left(\sum_{i \in N} f_i\right)(a, t) = \sum_{i \in N} \Psi_t(f_i(\cdot, t))(a) = \sum_{i \in N} \kappa_t(\mu_i)\overline{\kappa_t(\nu_i)} f_i(a, t);$$

the fact that all the sums are finite implies that Ψ takes $C_c(\mathcal{G}_\Lambda \times [0, 1])$ onto $C_c(\mathcal{G}_\Lambda \times [0, 1])$.

Since Ψ is evidently $C([0, 1])$-linear and is a $*$-isomorphism in each fiber, Proposition C.10 of [Williams 2007] tells us that Ψ is norm-preserving. Moreover, Ψ is a $*$-homomorphism since the operations in $C_c(\mathcal{G}_\Lambda \times [0, 1])$ preserve the fiber over $t \in [0, 1]$, and each Ψ_t is a $*$-homomorphism.

In other words, Ψ extends to an isomorphism of $C([0, 1])$-algebras

$$\Psi : C^*(\mathcal{G}_\Lambda \times [0, 1]) \cong C^*(\mathcal{G}_\Lambda \times [0, 1], \omega).$$

A straightforward check will establish that the identity map on $C_c(\mathcal{G}_\Lambda \times [0, 1])$ induces an isomorphism id : $C^*(\mathcal{G}_\Lambda \times [0, 1]) \to C([0, 1], C^*(\mathcal{G}_\Lambda))$ of $C([0, 1])$-algebras; the isomorphism $C^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$ of [Kumjian and Pask 2000, Corollary 3.5(i)] now finishes the proof. $\qquad\square$

Remark 4.3. Note that Ψ induces an isomorphism $\Phi : C([0, 1]) \otimes C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda \times [0, 1], \omega)$ as follows. If $Z(\mu, \nu) \in \mathcal{P}$ and $f \in C([0, 1])$, then

$$(7) \qquad\qquad \Phi(f \otimes s_\mu s_\nu^*)(x, t) = f(t)1_{Z(\mu, \nu)}(x)\kappa_t(\mu)\overline{\kappa_t(\nu)}.$$

Remark 4.4. Since evaluation at $t \in [0, 1]$ induces a homotopy equivalence between $C([0, 1], C^*(\Lambda))$ and $C^*(\Lambda)$, the isomorphism established in the previous proposition implies that evaluation at t also induces a homotopy equivalence between $C^*(\mathcal{G}_\Lambda \times [0, 1], \omega)$ and its fiber algebra $C^*(\mathcal{G}_\Lambda, \sigma_{c_t})$ when $d = \delta b$.

To leverage Proposition 4.2 into the proof of Theorem 4.1, we will use the skew-product k-graphs $\Lambda \times_d \mathbb{Z}^k$.

Definition 4.5 [Kumjian and Pask 2000, Definition 5.1]. Given a k-graph (Λ, d), the *skew-product k-graph* $\Lambda \times_d \mathbb{Z}^k$ is the set $\Lambda \times \mathbb{Z}^k$, with the structure maps

$$r(\lambda, n) = (r(\lambda), n), \quad s(\lambda, n) = (s(\lambda), n + d(\lambda)), \quad d(\lambda, n) = d(\lambda),$$

and with multiplication given by $(\lambda, n)(\mu, n + d(\lambda)) = (\lambda\mu, n)$ for $(\lambda, \mu) \in \Lambda^{*2}$.

Observe that the function $b : (\Lambda \times_d \mathbb{Z}^k)^{(0)} = \Lambda^{(0)} \times \mathbb{Z}^k \to \mathbb{Z}^k$ given by $b(v, n) = n$ satisfies $\delta b = d$ on $\Lambda \times_d \mathbb{Z}^k$. Moreover, if Λ is row-finite and source-free, then so is $\Lambda \times_d \mathbb{Z}^k$.

We can now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\phi : \Lambda \times_d \mathbb{Z}^k \to \Lambda$ be the projection onto the first coordinate: $\phi(\lambda, n) = \lambda$. A cocycle c on Λ induces a cocycle $c \circ \phi$ on the skew-product k-graph $\Lambda \times_d \mathbb{Z}^k$:

$$c \circ \phi\big((\lambda, n), (\mu, n + d(\lambda))\big) := c(\lambda, \mu)$$

whenever $(\lambda, \mu) \in \Lambda^{*2}$. Note that if $\{c_t\}_{t \in [0,1]}$ is a homotopy of cocycles on Λ then $\{c_t \circ \phi\}_t$ is also a homotopy of cocycles on $\Lambda \times_d \mathbb{Z}^k$.

If ω is the homotopy of cocycles on $\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}$ associated to the homotopy $\{c_t\}_{t \in [0,1]}$ of cocycles on Λ, then Proposition 4.2 tells us that

$$C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \cong C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k).$$

Now we define an action of $\mathbb{Z}^k$ on $C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k)$ by setting

$$(8) \qquad\qquad\qquad f \otimes s_{\lambda, n} \cdot m := f \otimes s_{\lambda, n+m}.$$

To see that this formula gives a well-defined action of $\mathbb{Z}^k$ on $C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k)$, one checks first that, for each $m \in \mathbb{Z}^k$, the set $\{s_{\lambda, m+n} : \lambda \in \Lambda, \ n \in \mathbb{Z}^k\}$ is a collection of partial isometries satisfying the defining axioms (CK1)–(CK4) for $C^*(\Lambda \times_d \mathbb{Z}^k)$. Consequently, the universal property of $C^*(\Lambda \times_d \mathbb{Z}^k)$ implies that, for each fixed $m \in \mathbb{Z}^k$, the map $s_{\lambda, n} \mapsto s_{\lambda, n+m}$ determines a $*$-homomorphism

$$\alpha_m : C^*(\Lambda \times_d \mathbb{Z}^k) \to C^*(\Lambda \times_d \mathbb{Z}^k).$$

Each α_m is invertible with inverse α_{-m}; it follows that $m \mapsto \alpha_m$ defines a group action of $\mathbb{Z}^k$ on $C^*(\Lambda \times_d \mathbb{Z}^k)$. Thus, (8) describes a well-defined action $\mathrm{id} \otimes \alpha$ of $\mathbb{Z}^k$ on $C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k)$, given by $m \mapsto \mathrm{id} \otimes \alpha_m$. The fact that the degree map on $\Lambda \times_d \mathbb{Z}^k$ is a coboundary now allows us to combine the action $\mathrm{id} \otimes \alpha$ with the isomorphism $\Phi : C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k) \to C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega)$ of Remark 4.3 to obtain an action β of $\mathbb{Z}^k$ on $C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega)$:

$$\beta_n(\Phi(f \otimes s_{\mu, m} s^*_{v, m+d(\mu)-d(v)})) := \Phi(\mathrm{id} \otimes \alpha_n(f \otimes s_{\mu, m} s^*_{v, m+d(\mu)-d(v)})).$$

Moreover, since both $\mathrm{id} \otimes \alpha$ and Φ (and hence β) fix $C([0, 1])$ by construction, Lemma 5.3 of [Kumjian et al. 2013] tells us that the crossed product

$$C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k \cong (C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k)) \rtimes_{\mathrm{id} \otimes \alpha} \mathbb{Z}^k$$

is a $C([0, 1])$-algebra with fiber $C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t \circ \phi}) \rtimes_{\beta_t} \mathbb{Z}^k$, where

$$
\begin{aligned}
(\beta_t)_n(\Phi_t(s_{\mu,m} s^*_{v,m+d(\mu)-d(v)})) &= \Phi_t(\alpha_n(s_{(\mu,m)} s^*_{(v,m+d(\mu)-d(v))})) \\
&= \kappa_t(\mu)\overline{\kappa_t(v)} 1_{Z((\mu,m+n),(v,m+n+d(\mu)-d(v)))}
\end{aligned}
$$

whenever $Z\big((\mu, m+n), (v, m+n+d(\mu)-d(v))\big)$ is in the partition $\mathcal{P}$ of $\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}$ that we used in the proof of Proposition 4.2.

Recall that we have a homotopy equivalence $q_t : C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \to C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t})$. A computation will show that q_t is equivariant with respect to the actions β, β_t of $\mathbb{Z}^k$; thus, Theorem 5.1 of [Kumjian et al. 2013] tells us that

$$(9) \qquad K_*(C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k) \cong K_*(C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t \circ \phi}) \rtimes_{\beta_t} \mathbb{Z}^k).$$

Thanks to Lemma 5.2 of [Kumjian et al. 2013], we know that

$$C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \rtimes_{lt} \mathbb{Z}^k \sim_{ME} C^*(\Lambda, c_t),$$

where $lt_m(s_{\lambda,n}) = s_{\lambda,n+m}$. To make use of this result, we need to show that β_t induces the action lt on $C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi)$.

Recall from the proof of Proposition 4.2 that $\pi^t(s_{\lambda,m}) = 1_{Z((\lambda,m),(s(\lambda),m+s(\lambda)))}$, since $Z\big((\lambda, m), (s(\lambda), m + s(\lambda))\big) \in \mathcal{P}$ always. Observe that

$$(10) \qquad C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t \circ \phi}) \rtimes_{\beta_t} \mathbb{Z}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \rtimes_{\gamma_t} \mathbb{Z}^k,$$

where

$$
\begin{aligned}
(\gamma_t)_n(s_{\lambda,m}) &:= (\pi^t)^{-1}\big((\beta_t)_n(\pi^t(s_{\lambda,m}))\big) = (\pi^t)^{-1}(\beta_t)_n(1_{Z((\lambda,m),(s(\lambda),m))}) \\
&= (\pi^t)^{-1}(\beta_t)_n\big(\Phi_t(\overline{\kappa_t(\lambda)} s_{(\lambda,m)})\big) = (\pi^t)^{-1}\big(\Phi_t(\alpha_n(\overline{\kappa_t(\lambda)} s_{\lambda,m}))\big) \\
&= (\pi^t)^{-1}\big(\Phi_t(\overline{\kappa_t(\lambda)} s_{\lambda,m+n})\big) = (\pi^t)^{-1}(1_{Z((\lambda,m+n),(s(\lambda),m+n))}) \\
&= s_{\lambda,m+n}.
\end{aligned}
$$

It follows that the action (γ_t) induced by β_t agrees with lt, as desired. Now, the Morita equivalence of Lemma 5.2 of [Kumjian et al. 2013] and (10) tell us that

$$(11) \qquad\qquad C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t \circ \phi}) \rtimes_{\beta_t} \mathbb{Z}^k \sim_{ME} C^*(\Lambda, c_t).$$

Combining (9) and (11) now yields

$$K_*(C^*(\Lambda, c_t)) \cong K_*(C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k)$$

for any $t \in [0, 1]$. It follows that, if $\{c_t\}_{t \in [0,1]}$ is a homotopy of cocycles on a row-finite k-graph Λ with no sources, then, for any $s, t \in [0, 1]$,

$$K_*(C^*(\Lambda, c_t)) \cong K_*(C^*(\Lambda, c_s)).$$

It remains to show that this isomorphism preserves the K-theory class of each vertex projection s_v. Essentially, this follows because the cocycles c_t, and thus the functions κ_t, are all trivial on any $v \in \mathrm{Obj}(\Lambda)$.

To be precise, let $v \in \mathrm{Obj}(\Lambda)$ and define $f_v \in C_c(\mathbb{Z}^k, C_c(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1])) \subseteq C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k$ by

$$f_v(n)(a, t) = \begin{cases} 1 & a \in Z_{(v,0),(v,0)} \text{ and } n = 0, \\ 0 & \text{else.} \end{cases}$$

Then the projection $q_t \rtimes \mathrm{id}(f_v)$ of f_v onto the fiber algebra $C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \omega_t) \rtimes_{\beta_t} \mathbb{Z}^k$ is independent of the choice of $t \in [0, 1]$:

$$q_t \rtimes \mathrm{id}(f_v)(n)(a) = \begin{cases} 1 & a \in Z_{(v,0),(v,0)} \text{ and } n = 0, \\ 0 & \text{else,} \end{cases}$$

for any $t \in [0, 1]$. Moreover, the isomorphism $\Phi_t : C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \to C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \sigma_{c_t \circ \phi})$ of Remark 4.3 satisfies

(12) $$\Phi_t \rtimes \mathrm{id}(j(s_{(v,0)})) = q_t \rtimes \mathrm{id}(f_v),$$

where $j : C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \to C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \rtimes_{lt} \mathbb{Z}^k$ is the canonical embedding of $C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi)$ into the crossed product.

The fact that the Morita equivalence $C^*(\Lambda, c_t) \sim_{ME} C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \rtimes_{lt} \mathbb{Z}^k$ takes $s_v \in C^*(\Lambda, c_t)$ to $j(s_{(v,0)})$ (see Lemma 5.2 in [Kumjian et al. 2013]) thus implies that our K-theoretic isomorphism $K_*(C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k) \to K_*(C^*(\Lambda, c_t))$, which is given by the composition of the Morita equivalence (11) with the $*$-homomorphism

$$q_t \rtimes \mathrm{id} : C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \times [0, 1], \omega) \rtimes_\beta \mathbb{Z}^k \to C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}, \omega_t) \rtimes_{\beta_t} \mathbb{Z}^k$$
$$\cong C^*(\Lambda \times_d \mathbb{Z}^k, c_t \circ \phi) \rtimes_{\gamma_t} \mathbb{Z}^k,$$

takes $[f_v]$ to $[s_v]$ for any $v \in \mathrm{Obj}(\Lambda)$ and any $t \in [0, 1]$. Consequently, the isomorphism $K_*(C^*(\Lambda, c_t)) \cong K_*(C^*(\Lambda, c_s))$ preserves the class of s_v, as claimed. $\square$

Remark 4.6. It's tempting to think that, since $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \cong C^*(\Lambda \times_d \mathbb{Z}^k)$ and $C^*(\Lambda, c) \sim_{ME} C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \rtimes_{lt} \mathbb{Z}^k$ for any cocycle c on Λ, any two twisted k-graph C^*-algebras should be Morita equivalent. This statement is false, however (the rotation algebras provide a counterexample). The flaw lies in the fact that the isomorphism $\mathrm{Ad}\, U_* : C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \to C^*(\Lambda \times_d \mathbb{Z}^k)$ is not equivariant

with respect to the left-translation action of $\mathbb{Z}^k$, so the isomorphism

$$C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \cong C^*(\Lambda \times_d \mathbb{Z}^k)$$

does not pass to an isomorphism $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \rtimes_{lt} \mathbb{Z}^k \to C^*(\Lambda \times_d \mathbb{Z}^k) \rtimes_{lt} \mathbb{Z}^k$. In other words, a *K*-theoretic equivalence of twisted *k*-graph *C**-algebras is the best result we can hope for in general.

5. Future work

The standing hypotheses of this paper, that our *k*-graphs be row-finite and source-free, are slightly more restrictive than the current standard for *k*-graphs. Thus, we would like to extend Theorem 4.1 to apply to all finitely aligned *k*-graphs. Finitely aligned *k*-graphs were introduced in [Raeburn and Sims 2005; Raeburn et al. 2004], and it seems that they constitute the largest class of *k*-graphs to which one can profitably associate a *C**-algebra. However, the Kumjian–Pask construction of a groupoid $\mathcal{G}_\Lambda$ associated to a *k*-graph Λ, which we described in Section 2 and which we use throughout the proof of Theorem 4.1, only works when Λ is row-finite and source-free. Farthing, Muhly, and Yeend [Farthing et al. 2005] provide an alternate construction of a groupoid $\mathcal{G}$ which can be associated to an arbitrary finitely aligned *k*-graph, and we hope that this approach will allow us to apply groupoid results such as Theorem 3.3 to study the effect on *K*-theory of homotopies of cocycles for finitely aligned *k*-graphs.

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