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ON THE ONE-ENDEDNESS OF GRAPHS OF GROUPS

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We give a technical result that implies a straightforward necessary and sufficient condition for a graph of groups with virtually cyclic edge groups to be one-ended. For arbitrary graphs of groups, we show that if their fundamental group is not one-ended, then we can blow up vertex groups to graphs of groups with simpler vertex and edge groups. As an application, we generalize a theorem of Swarup to decompositions of virtually free groups.

1. Introduction

A finitely generated group $G = \langle S \rangle$ is said to be *one-ended* if the corresponding Cayley graph $\mathrm{Cay}(G, S)$ cannot be separated into two or more infinite components by removing a finite subset. Otherwise G is said to be *many-ended*. It is a classical result due to Stallings [1971] that a many-ended group decomposes as either an amalgamated free product or an HNN extension over a finite group.

Given the Bass–Serre correspondence between group actions on simplicial trees and their decompositions, or splittings, as (fundamental groups of) graphs of groups (see [Serre 1980]), a finitely generated group G is many-ended if and only if it acts minimally, without inversions, and cocompactly on a simplicial tree T in which for some edge e the stabilizer G_e is finite.

It is often the case that a graph of groups with many-ended vertex groups is itself one-ended. For example, the fundamental group of a closed surface is one-ended but it is an amalgamated free product of free groups, which are many-ended. Theorem 3.1, stated and proved in Section 3, essentially characterizes one-ended graphs of groups. This result is rather technical, but has many "nontechnical" corollaries which we now present.

We say that G is *one-ended relative to a collection $\mathcal{H}$ of subgroups* if for any minimal nontrivial G-tree T with finite edge stabilizers, there exists a subgroup $H \in \mathcal{H}$ that acts without a global fixed point. Otherwise G is said to be *many-ended relative to $\mathcal{H}$*. In this case, G admits a nontrivial splitting as a graph of groups relative to $\mathcal{H}$ (i.e., groups in $\mathcal{H}$ are conjugate into vertex groups) with finite edge groups.

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Corollary 1.1. *If G_1 is one-ended relative to a collection $\mathcal{H}_1 \cup \{C_1\}$, and G_2 is one-ended relative to $\mathcal{H}_2 \cup \{C_2\}$ with $C_1 \approx C_2$ virtually cyclic groups, then any free product with amalgamation of the form*

$$G_1 *_{C_1=C_2} G_2$$

is one-ended relative to $\mathcal{H}_1 \cup \mathcal{H}_2$.

In the case of graphs of free groups with cyclic edge groups, this corollary (actually its natural generalization, see Corollary 1.5) is proved in [Wilton 2012, Theorem 18] and implied by results in [Diao and Feighn 2005]. Corollary 1.1 is false if we do not require the amalgamating subgroups to be virtually cyclic or, synonymously, two-ended. Nonetheless, we can still understand the failure of one-endedness of general graphs of groups.

Definition 1.2. A G-equivariant map $S \to T$ of simplicial G-trees is called a *collapse* if T is obtained by identifying some edge orbits of S to points. In this case we also say that S is obtained from T by a *blow up*. We call the preimage $\check{T}_v \subset S$ of a vertex $v \in T$ its *blowup*.

Definition 1.3. We write $H \preccurlyeq G$ to signify that G splits essentially as a graph of groups with finite edge groups and H is a vertex group. A group G is *accessible* if it admits no infinite proper chains

$$G \succ G_1 \succ G_2 \succ \dots.$$

For example, if F is a free group and $H \preccurlyeq F$, then H is a free factor of F. This next theorem, a formal consequence of Theorem 3.1, states that if a graph of groups with finitely generated infinite edge group is not one-ended, then we can blow up some of its vertex groups.

Theorem 1.4. *Suppose that T is a G-tree (in which a collection of subgroups $\mathcal{H}$ act elliptically) with infinite edge groups, and that G is not one-ended relative to $\mathcal{H}$. Then there is a vertex $v \in$ Vertices(T) and an edge $e \in$ Edges(T) with $v \in e$ such that the orbit of v can be blown up with G_v acting minimally on the nontrivial blowups $\check{T}_v$ satisfying the following properties:*

- *$G_e \leq G_v$ is the stabilizer of a vertex in $\check{T}_v$.*
- *The edge groups of $\check{T}_v$ are conjugate in G_v to the vertex groups of an essential amalgamated free product or HNN decomposition of G_e with a finite edge group.*

In particular, in the tree S obtained by blowing up the orbit of v in T to $\check{T}_v$, each vertex or edge stabilizer of S is $\preccurlyeq$ a vertex or edge stabilizer of T, and at least one of these inclusions is strict. Furthermore the groups in $\mathcal{H}$ act elliptically on S.

We note that blowing up a G-tree is equivalent to *refining* a graph of groups. If G acts on a tree with accessible vertex and edge stabilizers then the order $\prec$ actually tells us that the vertex groups of the blowup given by Theorem 1.4 have lower complexity, in the sense that the process of successively blowing up vertex groups in this manner must terminate in finitely many steps.

Accessible groups, in turn, are abundant. Linnell [1983] showed that if there is a global bound on the order of finite order elements in a finitely generated group, then the group is accessible. Dunwoody [1985] showed that finitely presented groups are accessible. We now use Theorem 1.4 to give a proof of Corollary 1.1.

Proof of Corollary 1.1. We show the contrapositive. Let T be the Bass–Serre tree dual to the splitting $G = G_1 *_C G_2$, and suppose that G is not one-ended relative to $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Note that any decomposition of a virtually cyclic group as an HNN extension or an essential amalgamated free product must have finite edge groups. It follows that in all cases, by Theorem 1.4, some orbit of vertices Gv can be blown up to minimal gG_vg^{-1}-trees with finite edge groups. This implies that one of the vertex groups G_i fixing some vertex $v \in \text{Vertices}(T)$ acts minimally on $\check{T}_v$ with finite edge stabilizers, with

$$\mathcal{H}_i = \{H \in \mathcal{H} \mid H \cap G_i \neq \{1\}\}$$

and $C_i = G_e$ for some $v \in e \in \text{Edges}(T)$ acting elliptically. It follows that G_i is not one-ended relative to $\mathcal{H}_i \cup \{C\}$. □

This proof is easily adapted to give:

Corollary 1.5. *The fundamental group G of a graph of groups with two-ended edge groups is one-ended* (*relative to a collection $\mathcal{H}$ of subgroups*) *if and only if every vertex group G_v is one-ended relative to the incident edge groups* (*and the collection $\{H^g \cap G_v \mid g \in G, H \in \mathcal{H}\}$*).

Using the full strength of Theorem 3.1, we also generalize a result of Swarup [1986] on the decomposition of free groups to virtually free groups. This result was already partially generalized by Cashen [2012] to decompositions of virtually free groups with virtually cyclic edge groups.

Theorem 1.6. *Let G be finitely generated and virtually free.*

(1) *If G splits as an amalgamated free product $G = A *_C B$ with C finitely generated and infinite*, *then there is some $C_1 \preccurlyeq C$ such that $C_1 \preccurlyeq A$ or $C_1 \preccurlyeq B$.*

(2) *If G splits as an HNN extension $G = A*_{C,t}$ with C finitely generated and infinite*, *then there is an infinite subgroup $C_1 \preccurlyeq C$ and a splitting Δ of A as a graph of groups with finite edge groups relative to $\{C_1, t^{-1}C_1t\}$ such that either C_1 or $t^{-1}C_1t$ is a vertex group of Δ.*

Unlike in Swarup's proof, we do not use homological methods. Our proof is more along the lines of the geometric arguments found in [Wilton 2012; Louder 2008; Bestvina and Feighn 1994; Diao and Feighn 2005] using graphs of spaces X with $\pi_1(X) = G$. The presence of torsion, however, can make the attaching maps in the graphs of spaces difficult to describe. By using the more abstract G-cocompact core of the product of two G-trees [Guirardel 2005], we sidestep these difficulties. The core has been used before to study pairs of group splittings. In particular, Fujiwara and Papasoglu [2006] use it to show the existence of QH subgroups for one-ended groups that have hyperbolic-hyperbolic pairs of slender splittings; this is the main technicality in constructing group theoretical JSJ decompositions. Although it could be noted that the action of our group on the core gives rise to a G-orbihedron à la [Haefliger 1991], we will not need this machinery; in fact, modulo classical Bass–Serre theory and Guirardel's Core Theorem for simplicial trees (Theorem 2.3, of which we sketch a proof), our argument is self-contained.

2. Preliminaries

Group actions. All group actions will be from the left. Let X be a G-set. If $S \subset X$ is a subset, we denote by G_S the (setwise) stabilizer $\{g \in G \mid gS = S\}$. If $S = \{x\}$ is a singleton, then we write G_x instead of $G_{\{x\}}$. We call a subset $S \subset X$ *G-regular* if for any $x, y \in S$ in the same G-orbit, there is some $g \in G_S$ such that $gx = y$. The following lemma is immediate.

Lemma 2.1. *Let X be a G-set. If $S \subset X$ is G-regular, then we have an embedding*

$$G_S \backslash S \hookrightarrow G \backslash X.$$

In this paper, all trees will be simplicial. In particular we consider them to be topological spaces, equipped with a CW-structure, which also makes them into graphs. We further metrize these graphs by viewing edges as real intervals of length 1.

All G-trees T will be *without inversions*, meaning that for any edge $e \in \mathrm{Edges}(T)$, if $ge = e$ then g fixes e pointwise. Equivalently, if $u, v \in \mathrm{Vertices}(T)$ are the vertices at the ends of the edge e, then we have inclusions

$$G_u \geq G_e \leq G_v.$$

We call vertex stabilizers *vertex groups*, and edge stabilizers *edge groups*. We assume the reader is familiar with Bass–Serre theory and we switch freely between G-trees and splittings as graphs of groups, viewing the two as being equivalent.

A G-tree T is *essential* if every edge of T divides it into two infinite components. T is *minimal* if there are no proper subtrees $S \subset T$ with $G_S = G$. T is *cocompact* if $G \backslash T$ is compact. An element g or a subgroup H of G are said to *act elliptically on T* if the groups $\langle g \rangle$ or H fix some $v \in \mathrm{Vertices}(T)$.

Products of trees, cores, and leaf spaces. If T_1 and T_2 are G-trees, then we have a natural induced action $G \curvearrowright T_1 \times T_2$. Since the trees T_1, T_2 are 1-dimensional CW complexes, we may consider their product $T_1 \times T_2$ as a *square complex*, i.e., a 2-dimensional CW complex whose cells consist of vertices, edges, and squares. There are natural projections $p_i : T_1 \times T_2 \twoheadrightarrow T_i$. The following lemma is immediate.

Lemma 2.2. *If the actions $G \curvearrowright T_1$ and $G \curvearrowright T_2$ are without inversions, then so is the action $G \curvearrowright T_1 \times T_2$, i.e., if $\sigma \supset \epsilon$ is an inclusion of cells* (*e.g., a square containing an edge*), *then $G_\sigma \le G_\epsilon$.*

If $\mathcal{H}$ is a collection of subgroups acting elliptically on T_1 and T_2, then each subgroup in $\mathcal{H}$ fixes a vertex of $T_1 \times T_2$.

The action $G \curvearrowright T_1 \times T_2$ is not cocompact in general. It turns out, however, that we can extract a useful subset, namely Guirardel's cocompact core. We state the special case of his result applied to simplicial trees.

Theorem 2.3 (the Core Theorem, see [Guirardel 2005, Théorème principal and Corollaire 8.2]). *Let $G \curvearrowright T_1$, $G \curvearrowright T_2$ be two minimal actions of a finitely generated group G on simplicial trees T_1, T_2 with finitely generated edge stabilizers. Suppose furthermore that T_1, T_2 do not equivariantly collapse to a common nontrivial tree.*

Then there is a G-invariant subset $\mathcal{C} \subset T_1 \times T_2$, called the core of the action $G \curvearrowright T_1 \times T_2$, which is defined as the smallest connected G-invariant subset such that the restrictions of the projections $p_i|_{\mathcal{C}} : \mathcal{C} \twoheadrightarrow T_i$ have connected fibers. The quotient $\mathcal{S} = G \backslash \mathcal{C}$ is compact.

Suppose for the remainder of this section that T_1, T_2 satisfy the hypotheses of Theorem 2.3. The restrictions of the projections $p_i|_\sigma : \sigma \to T_i$ are well defined for each cell (i.e., a vertex, edge, square) $\sigma \subset T_1 \times T_2$. If σ is a square then the projection is onto an edge $p_i(\sigma) \in \mathrm{Edges}(T)_i$. If $\lambda_1, \lambda_2 \subset \sigma$ are two fibers of such a projection (see Figure 1), we can define a distance $d_i^\sigma(\lambda_1, \lambda_2)$ to be the distance

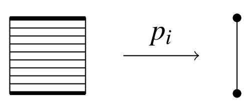

Figure 1. The projection of a square on an edge and some of its fibers.

in $p_i(\sigma)$ between the points $p_i(\lambda_1)$ $p_i(\lambda_2)$, thus putting a metric d_i^σ on the set of p_i-fibers in a cell σ. We now define the *i-leaf space* $\mathcal{L}_i$ of a subset $Z \subset T_1 \times T_2$ to be the set of connected unions of p_i-fibers of cells in Z, called *leaves*, so that we see Z as being *foliated* by the leaves in $\mathcal{L}_i$. $\mathcal{L}_i$ is a 1-complex with metrized edges; therefore we can endow $\mathcal{L}_i$ with the path metric d_i. As a consequence of the direct product structure we have the following.

Lemma 2.4. *If $Z \subset T_1 \times T_2$, then the leaf spaces $\mathcal{L}_i$ are forests* (see Figure 2).

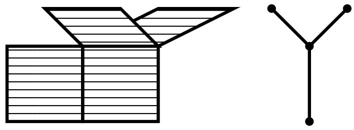

Figure 2. The i-leaves in a square complex and the resulting leaf space, which is a tree.

If $\mathscr{C} \subset T_1 \times T_2$ is a core then the leaf spaces $\mathscr{L}_i$ are homeomorphic to the trees T_i. Later, however, we will be performing operations that will alter the leaf spaces.

Induced splittings. Let $v \in \text{Vertices}(T_i)$, $e \in \text{Edges}(T_i)$ and let m_e be the midpoint of e. Let $\tau_v = p_i^{-1}(\{v\}) \cap \mathscr{C}$ and $\tau_e = p_i^{-1}(\{m_e\}) \cap \mathscr{C}$. By Theorem 2.3 the preimages τ_v, τ_e are connected and are therefore leaves in $\mathscr{L}_i$.

Since we have an action $G \curvearrowright \mathscr{C}$, since τ_v, τ_e are defined as T_i-point preimages via a G-equivariant map, and since G_v, G_e are exactly the stabilizers of these points v, m_e, the subsets τ_v, $\tau_e \leq \mathscr{C}$ are G-regular. So, by Lemma 2.1 we have embeddings

$$G_v \backslash \tau_v \hookrightarrow G \backslash \mathscr{C} \hookleftarrow G_e \backslash \tau_e.$$

By Theorem 2.3, $G \backslash \mathscr{C}$ is compact so the quotients $G_v \backslash \tau_v$, $G_v \backslash \tau_v$ must be as well. Moreover, because τ_v, τ_e are contained in p_i-fibers, for $j \neq i$ the restrictions

$$p_j|_{\tau_v} : \tau_v \to T_j, \quad p_j|_{\tau_e} : \tau_e \to T_j$$

are injective. Finally, the projection $p_j|_{\mathscr{C}} : \mathscr{C} \twoheadrightarrow T_j$ is G-equivariant. We have shown the following.

Lemma 2.5. *If $v \in \text{Vertices}(T_i)$, $e \in \text{Edges}(T_i)$, $j \neq i$, then the fibers τ_v, τ_e are mapped injectively via p_j to subtrees that are G_v, G_e-invariant, respectively. Viewed as subsets of the core $\mathscr{C} \subset T_1 \times T_2$, τ_v and τ_e coincide with their j-leaf spaces.*

The actions $G_e \curvearrowright \tau_e$, $G_v \curvearrowright \tau_v$ are cocompact. Moreover τ_v, τ_e are infinite if and only if the actions of the subgroups $G_v \curvearrowright T_j$, $G_e \curvearrowright T_j$ are without global fixed points.

The G_v, G_e-trees τ_v, τ_e give splittings induced by the action on T_j. The blowups of Theorem 3.1 will be obtained by modifying the trees τ_v. For aficionados of CAT(0) cube complexes, it is worth remarking that the core $\mathscr{C}$ is a CAT(0) square complex, in fact a $\mathscr{V}\mathscr{H}$-complex, and that the set of fibers τ_e, $e \in \text{Edges}(T_i)$ is the set of hyperplanes.

Spurs, free faces, and cleavings. In the previous subsection we obtained cocompact G_v, G_e-trees τ_v, τ_e. We say a tree has a *spur* if it has a vertex of degree 1. An edge adjacent to a spur is called a *hair*. We now give a shaving process.

Lemma 2.6. *Let T be a cocompact G-tree. T is minimal if T doesn't have any spurs. If T is not minimal, then we can obtain the minimal subtree $T(G)$ as the final term of a finite sequence*

$$T = T_0, \ldots, T_k = T(G),$$

where T_{i+1} is obtained from T_i by G-equivariantly contracting one G-orbit of hairs to points.

Proof. Let $v \in \text{Vertices}(T)$ be a spur adjacent to an edge $e \in \text{Edges}(T)$ and let $u \in \text{Vertices}(T)$ be the other endpoint of e. The map $T \to T$ obtained by G-equivariantly collapsing ge onto gu for $g \in G$ is a deformation retraction onto a proper G-invariant subtree, so T is not minimal.

Suppose now that T is not minimal. Then there is some proper G-invariant subtree $S \subset T$. Let K be the closure of some connected component of $T \setminus S$. Then $K \cap S = \{v\}$ for some $v \in \text{Vertices}(S)$. Since S is G-invariant and connected, we must have $G_K \leq G_v$. It follows that for any $w \in \text{Vertices}(K)$ and any $g \in G_K$ the distance $d_T(w, v) = d_T(gw, v)$, i.e., the action of G_K on K is the action on a rooted tree with root v. Since K is G-regular, we have an embedding $G_K \backslash K \hookrightarrow G \backslash T$ which is compact; thus K must have finite radius since G_K preserves distances from the root.

Since K is a rooted tree with finite diameter it must have a nonroot vertex of valence 1. By the argument at the beginning of the proof, we can G_K-equivariantly collapse hairs, and since $G_K \curvearrowright K$ is cocompact, after finitely many collapses we will have collapsed K to v. Again since $G \curvearrowright T$ is cocompact, there are only finitely many orbits of connected components of $T \setminus S$, so the result follows. □

If σ is a square in some $Z \subset T_1 \times T_2$, then we say an edge $\epsilon \subset \sigma$ is a *free face* if it is only contained in one square. The following terminology is due to Wise [2004].

Definition 2.7. Let $e \in \text{Edges}(T_i)$ and let $\tau_e \subset \mathscr{C}$ be the fiber of e as in Lemma 2.5. The *hypercarrier* $\mathscr{H}_{\mathscr{C}}(\tau_e)$ is the union of squares of $\mathscr{C}$ intersecting τ_e nontrivially.

We note that for $e \in T_i$, a hypercarrier is mapped to an edge of T_i and that $\mathscr{H}_{\mathscr{C}}(\tau_e)$ is homeomorphic to $\tau_e \times [-1, 1]$.

Definition 2.8. We say an edge ϵ in some $Z \subset T_1 \times T_2$ is i-transverse if it coincides with its i-leaf space, or equivalently if it is mapped monomorphically via $p_i|_\epsilon$, or equivalently if it is contained in a j-leaf.

An immediate consequence of Lemma 2.6 and Figure 3 is the following.

Lemma 2.9. *Let $e \in \text{Edges}(T_i)$. If $G_e \curvearrowright \tau_e$ is not minimal then $\mathscr{H}_{\mathscr{C}}(\tau_e)$ has a square σ containing an i-transverse free face ϵ.*

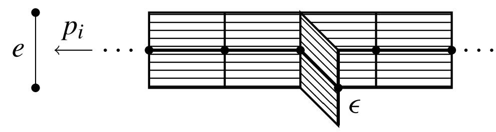

Figure 3. A spur of τ_e and the corresponding free face ϵ in the hypercarrier $\mathcal{H}_{\mathcal{C}}(\tau_e)$.

We now borrow some terminology from [Diao and Feighn 2005].

Definition 2.10. A simplicial map $S \to T$ between two trees that is obtained by identifying edges sharing a common vertex is called a *folding*. If T is obtained from S by a folding, then we say S is obtained from T by a *cleaving*.

The next lemma is now immediate (see Figure 4).

Lemma 2.11. *Let $\epsilon \subset Z \subset T_1 \times T_2$ be an i-transverse free face in a square σ. If we collapse σ onto the face opposite to ϵ, then the leaf space $\mathcal{L}_i$ is unchanged and the leaf space $\mathcal{L}_j$ gets cleaved.*

In fact this lemma can be used backwards to give a proof of Theorem 2.3. We will sketch it, leaving the details to an interested reader familiar with folding sequences [Bestvina and Feighn 1991; Stallings 1991; Dunwoody 1998; Kapovich et al. 2005].

Sketch of the proof of Theorem 2.3. Pick some vertex $v \in T_1 \times T_2$ and consider its G-orbit. We can add finitely many connected G-orbits of edges to get a connected G-complex $Gv \subset \mathcal{C}_1 \subset T_1 \times T_2$. $\mathcal{C}_1$ has leaf spaces $\mathcal{L}_1, \mathcal{L}_2$ which project onto T_1, T_2. The disconnectedness of the fibers of the projections $p_i|_{\mathcal{C}_1} : \mathcal{C}_1 \twoheadrightarrow T_i$ coincides with

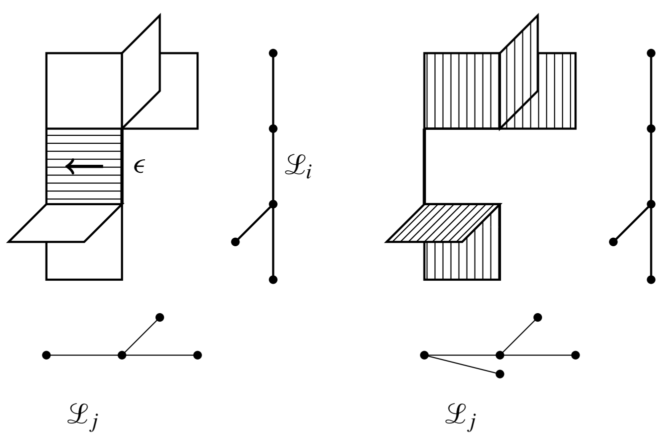

Figure 4. The effects of collapsing an i-transverse free face ϵ: the leaf space $\mathcal{L}_j$ gets cleaved, $\mathcal{L}_i$ remains unchanged. On the right the j-leaves are drawn.

the failure of injectivity of the projections $\mathcal{L}_i \twoheadrightarrow T_i$. By Lemma 2.11 (backwards) adding a square can give a folding of one of the leaf spaces. Since the edge groups of T_1, T_2 are finitely generated, and because adding all the squares of $T_1 \times T_2$ folds $\mathcal{L}_i$ to T_i, it follows that the leaf spaces $\mathcal{L}_i$ can be made to coincide with T_i after adding finitely many G-orbits of squares. □

3. The statement and proof of the main theorem

For this section we fix a collection $\mathcal{H}$ of subgroups of G. We let T_∞ and $T_{\mathcal{F}}$ be cocompact, minimal G-trees in which the subgroups in $\mathcal{H}$ act elliptically. We further require that edge groups of T_∞ are infinite and finitely generated and that edge groups of $T_{\mathcal{F}}$ are finite. Note that any nontrivial tree obtained by a collapse of T_∞ has infinite edge groups whereas any collapse of $T_{\mathcal{F}}$ has finite edge groups. It follows that T_∞ and $T_{\mathcal{F}}$, having no nontrivial common collapses, satisfy the hypotheses of Theorem 2.3.

Theorem 3.1 (Main Theorem). *Let $\mathcal{H}$ be a collection of subgroups of G and let T_∞ and $T_{\mathcal{F}}$ be cocompact, minimal G-trees in which the subgroups in $\mathcal{H}$ act elliptically. Suppose furthermore that the edge groups of $T_{\mathcal{F}}$ are finite and that the edge groups of T_∞ are infinite. Then there exists a vertex $v \in$ Vertices(T_∞) and a nontrivial, cocompact, minimal G_v-tree $\check{T}_v$ such that*

(i) *for every $f \in$ Edges(T_∞) incident to v the subgroups $G_f \leq G_v$ act elliptically on $\check{T}_v$, and*

(ii) *for every $H \in \mathcal{H}$ and $g \in G$ the subgroup $H^g \cap G_v \leq G_v$ acts elliptically on $\check{T}_v$.*

Moreover, either

(1) *every edge group of $\check{T}_v$ is finite, or*

(2) *there is some edge $e \in$ Edges(T_∞), incident to v, that not only satisfies (i), but also satisfies the following:*
   - (a) *G_e splits essentially as an amalgamated free product or an HNN extension with finite edge group.*
   - (b) *$G_e = G_{v_e}$ for some vertex $v_e \in$ Vertices$\big(\check{T}_v\big)$.*
   - (c) *The edge stabilizers of $\check{T}_v$ are conjugate in G_v to the vertex group(s) of the splitting of G_e found in (a); in particular, the edge groups of $\check{T}_v$ are $\prec G_e$.*
   - (d) *The vertex groups of $\check{T}_v$ that are not conjugate in G_v to G_e are also vertex groups of a one-edge splitting of G_v with a finite edge group; in particular these vertex groups of $\check{T}_v$ are $\prec G_v$.*

An example of what happens in situation (2) is shown in Figure 7.

Proof. Let $\mathcal{C}$ be the core of $T_\infty \times T_{\mathcal{F}}$. The ∞-leaf space $\mathcal{L}_\infty$ is the tree T_∞, and we can see $\mathcal{C}$ as a tree of spaces (see [Scott and Wall 1979] for details) which is a union of vertex spaces τ_v for $v \in \text{Vertices}(T_\infty)$ and edge spaces $\mathcal{H}_{\mathcal{C}}(\tau_e) = \tau_e \times [-1, 1]$ for $e \in \text{Edges}(T_\infty)$ attached to the τ_v along the subspaces $\tau_e \times \{\pm 1\}$.

It may be that for some $e \in \text{Edges}(T_\infty)$, the G_e-trees τ_e are not minimal. By Lemmas 2.9, 2.6, and 2.11, we can repeatedly G-equivariantly collapse ∞-transverse free faces, so that after finitely many steps we obtain a *shaved core* $\mathcal{C}'_s$ such that the $\tau_e \cap \mathcal{C}'_s$ are minimal G_e-trees. Although the $\mathcal{F}$-leaf space was cleaved repeatedly in the shaving process given by Lemma 2.6, the ∞-leaf space is unchanged. We still write $\mathcal{L}_\infty = T_\infty$.

We now construct a complex $\mathcal{C}_s \subset \mathcal{C}'_s \subset \mathcal{C}$, called the *$\infty$-minimal core*. Its principal feature is that for every $v \in \text{Vertices}(T_\infty)$ and $e \in \text{Edges}(T_\infty)$, the trees $\tau_v \cap \mathcal{C}_s$ and $\tau_e \cap \mathcal{C}_s$ are minimal G_v- and G_e-trees, respectively. Define $\mathcal{H}_{\mathcal{C}'_s}(\tau_e) = \mathcal{H}_{\mathcal{C}}(\tau_e) \cap \mathcal{C}'_s$. We call $\mathcal{H}_{\mathcal{C}'_s}(\tau_e)$ the *$\mathcal{C}'_s$-hypercarrier* attached to a vertex space τ_v in $\mathcal{C}'_s$. Note that $\tau_e \cap \mathcal{C}'_s$ naturally projects injectively into τ_v as a minimal G_e-invariant subtree where $G_e \le G_v$. If T is a G-tree and $H \le G$, denoting by $T(S)$ the minimal S-invariant subtree, we have $T(H) \subset T(G)$. It therefore follows that all the $\mathcal{C}'_s$-hypercarriers attached to τ_v are actually attached to the minimal G_v-invariant subtree of τ_v. By Lemma 2.6, after finitely many equivariant spur collapses we can make the vertex spaces τ_v into minimal G_v-trees. None of these collapses will affect the attached $\mathcal{C}'_s$-hypercarriers $\mathcal{H}_{\mathcal{C}'_s}(\tau_e)$, and the leaf space $\mathcal{L}_\infty = T_\infty$ is preserved. We have therefore constructed $\mathcal{C}_s$, the ∞-minimal core. Denote $\mathcal{H}_{\mathcal{C}_s}(\tau_e) = \mathcal{H}_{\mathcal{C}'_s}(\tau_e) \cap \mathcal{C}_s$. By what was written above, $\mathcal{H}_{\mathcal{C}_s}(\tau_e) = \mathcal{H}_{\mathcal{C}'_s}(\tau_e)$, and we now call $\mathcal{H}_{\mathcal{C}_s}(\tau_e)$ a *$\mathcal{C}_s$-hypercarrier.*

For every $k \in \text{Edges}(T_{\mathcal{F}})$, G_k is finite, therefore a minimal G_k-tree is a point; thus, by cocompactness and regularity, the trees $\tau_k \in \mathcal{C}$ have finite diameter and the same must be true of every connected component of $\tau_k \cap \mathcal{C}_s$. So, every connected component of $\tau_k \cap \mathcal{C}_s$ has a spur. It therefore follows that $\mathcal{C}_s$ must have an $\mathcal{F}$-transverse free face ϵ containing a spur of some connected component of $\tau_k \cap \mathcal{C}_s$ for some $k \in \text{Edges}(T_{\mathcal{F}})$. Furthermore, the stabilizer $G_\epsilon \le G_{p_{\mathcal{F}}(\epsilon)}$ is an edge stabilizer of $T_{\mathcal{F}}$, and therefore finite. This $\mathcal{F}$-transverse free face ϵ must be contained in some $\tau_v \cap \mathcal{C}_s$ for $v \in \text{Vertices}(T_\infty)$. Suppose first that ϵ was not contained in any $\mathcal{C}_s$-hypercarrier attached to $\tau_v \cap \mathcal{C}_s$. Then for every $e \ni v$ in $\text{Edges}(T_\infty)$, G_e fixes some $\mathcal{C}_s$-hypercarrier $\mathcal{H}_{\mathcal{C}_s}(\tau_e)$ such that $\mathcal{H}_{\mathcal{C}_s}(\tau_e) \cap \tau_v = \tau_e^+$ is contained in the complement $(\tau_v \cap \mathcal{C}_s) \setminus G_v\epsilon$.

Definition 3.2. Let T be a minimal G-tree and $e \in \text{Edges}(T)$. We denote by $C(T, e)$ the *non-e-collapse of T*, the tree whose edges are the orbit $Ge \subset T$ and whose vertices are the closures of the connected components of $T \setminus Ge$, with a vertex v adjacent to an edge e' in $C(T, e)$ if and only if, viewed as subsets of T, $e' \cap v \neq \varnothing$.

It therefore follows that $\check{T}_v = C(\tau_v \cap \mathcal{C}_s, \epsilon)$ is a tree with finite edge groups, in which each $G_e \le G_v$ ($e \in \text{Edges}(T_\infty)$) acts elliptically, and also conjugates of groups in $\mathcal{H}$ intersecting G_v act elliptically. Thus (i), (ii) and (1) are satisfied.

Otherwise, the free face $\epsilon \subset \tau_v \cap \mathscr{C}_s$ is, by definition of a free face, contained in *exactly one* $\mathscr{C}_s$-hypercarrier $\mathscr{H}_{\mathscr{C}_s}(\tau_e)$. We now construct the G_v-tree $\breve{T}_v$ satisfying (2). This construction is illustrated in Figure 5. We first take the subset

$$Z = \Big(\tau_v \cup \bigcup_{e \ni v} \mathscr{H}_{\mathscr{C}_s}(\tau_e)\Big) \cap \mathscr{C}_s,$$

i.e., $\tau_v \cap \mathscr{C}_s$ to which we attach all adjacent $\mathscr{C}_s$-hypercarriers. Now the G_v-translates of ϵ are contained in the $\mathscr{C}_s$-hypercarriers $\mathscr{H}_{\mathscr{C}_s}(\tau_{ge})$ for $g \in G_v$. For each such $\mathscr{C}_s$-hypercarrier we denote by τ_{ge}^- the connected component of $\tau_e \times \{\pm 1\} \subset \mathscr{H}_{\mathscr{C}_s}(\tau_{ge})$ not contained in $\tau_v \cap \mathscr{C}_s$ (see the top of Figure 5).

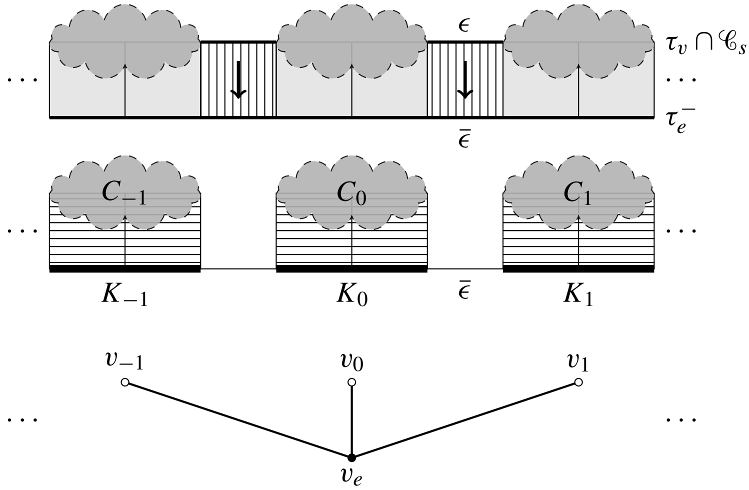

Figure 5. Constructing $\breve{T}_v$. The top shows a portion of Z, the middle shows the result of equivariantly collapsing the free face ϵ, and the bottom shows the corresponding ∞-leaf space.

We now G_v-equivariantly collapse the square $\sigma \supset \epsilon$ onto the opposite side $\bar{\epsilon}$, obtaining a connected G_v-subset $Z_c \subset Z$ (see the middle of Figure 5). The resulting intersection $\tau_v \cap Z_c$ consists of a collection of connected components $\{C_i \mid i \in I\}$. Similarly, the G_e-translates of $\bar{\epsilon}$ give connected components $\{K_i \mid i \in I\}$ of $\tau_e \setminus G_e\bar{\epsilon}$. Because G_e acts on $C(\tau_e^-, \bar{\epsilon})$, and by minimality of $\tau_e \cap \mathscr{C}_s$, this action is also minimal with one edge orbit. This gives us (a).

For every edge $f \in \mathrm{Edges}(T_\infty)$ incident to v that is not in the G_v-orbit of e, the orbit $G_v\epsilon$ does not intersect $\mathscr{H}_{\mathscr{C}_s}(\tau_f) \cap \tau_v$. It follows that each such $G_f \leq G_v$ stabilizes some component C_i. We now detach from Z_c all $\mathscr{C}_s$-hypercarriers not stabilized by a G_v-conjugate of G_e, producing a G_v-complex $Z_c' \subset Z_c$, specifically

$$Z_c' = Z_c \cap \Big(\tau_v \cup \bigcup_{g \in G_v} \mathscr{H}_{\mathscr{C}_s}(\tau_{ge})\Big).$$

Next, to get the G_v-tree T_v, we collapse each G_v-translate of τ_e^- to a vertex v_e, collapse each component C_i to a vertex v_i, and collapse each connected component of G_v-translates of $\mathcal{H}_{\mathscr{C}_S}(\tau_e) \cap Z_c'$ onto an edge connecting v_e and the corresponding vertex v_i. This is illustrated at the bottom of Figure 5.

Equivalently, if we consider the ∞-leaf space corresponding to the union of the $\mathscr{C}_s$-hypercarriers $g\mathcal{H}_{\mathscr{C}_s}(\tau_e)$ attached to $\tau_v \cap \mathscr{C}_s$ for $g \in G_v$, then we have a tree of radius 1, which is G_v-isomorphic to $\{v\} \cup \left(\bigcup_{g \in G_v} ge\right) \subset T_\infty$. After equivariantly collapsing the free face ϵ, Lemma 2.11 gives us a cleaving of this radius 1 subtree to the infinite tree $\breve{T}_v$ constructed above. See Figure 6. We note that if we took the ∞-leaf space of Z_c, i.e., had we not detached the other hypercarriers, the resulting leaf space would be a tree with many spurs. The tree $\breve{T}_v$ we obtain is a minimal G_v-tree that satisfies (b) and (i).

Moreover, we note that by construction, every subgroup $H^g \cap G_v$, for $g \in G$ and $H \in \mathcal{H}$, acts elliptically on $\breve{T}_v$. So (ii) is satisfied as well.

Since τ_e^- is G_v-regular, the vertex stabilizers of $C(\tau_e^-, \bar{\epsilon})$ coincide with the component stabilizers $(G_e)_{K_i} = (G_v)_{K_i}$. We also have $(G_v)_{C_i} \cap (G_v)_{\tau_e^-} = (G_v)_{K_i}$ (again referring to the middle of Figure 5). It now follows that the edge stabilizers of $\breve{T}_v$ satisfy (c).

Finally note that the vertex groups of $\breve{T}_v$ that are not stabilized by G_v-conjugates of G_e are also the vertex groups of $C(\tau_v, \epsilon)$ (see the top of Figure 5). Finally, since G_ϵ is finite, (d) follows. □

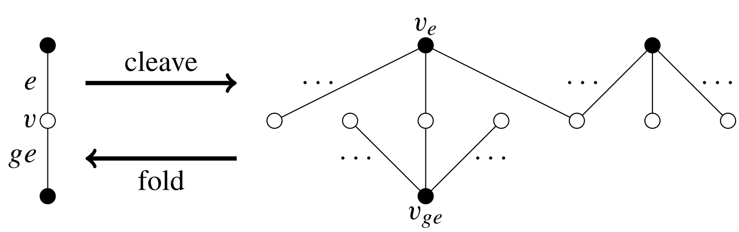

Figure 6. Equivariant collapsing free faces cleaves the leaf space of Z_C' to a tree $\breve{T}_v$ with infinite diameter.

4. Splittings of virtually free groups

Another way to use Theorem 3.1 is to obtain cleavings of G-trees whose edge and vertex groups are "smaller". This will be used as the inductive step in our proof of Theorem 1.6.

Corollary 4.1. *Let T be a G-tree in which the subgroups $\mathcal{H}$ act elliptically with infinite edge groups, and let G be many-ended relative to $\mathcal{H}$. Either some vertex*

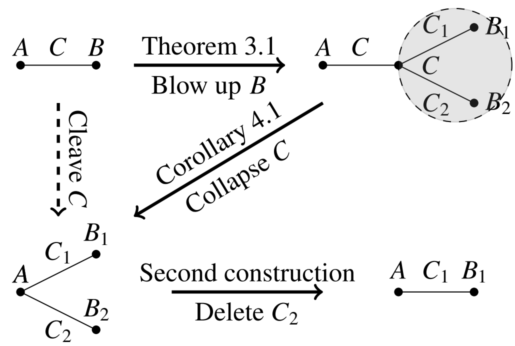

Figure 7. An example of the effects of Theorem 3.1, Corollary 4.1, and the second construction of the proof of Theorem 1.6 on a graph of groups. The vertices and edges are labeled by the corresponding vertex and edge groups. In all cases $B_i \prec B$ and $C_i \prec C$.

$v \in \text{Vertices}(T)$ *can be blown up to a tree with finite edge groups*; *or*, *there is an edge* $e \in \text{Edges}(T)$ *such that we can blow up* T, *relative to* $\mathcal{H}$, *to some tree* $\check{T}$, *and then collapse the edges in the orbit of* e *to points. The resulting tree* T' *can also be obtained from* T *by equivariantly cleaving some edge* e. *If* $e' \in \text{Edges}(T')$ *is a new edge obtained by a cleaving of* e, *then* $G_{e'} \prec G_e$. *Also, for each new vertex* $v' \in \text{Vertices}(T')$, *there is some* $v \in \text{Vertices}(T)$ *that got cleaved such that* $G_{v'} \prec G_v$.

Furthermore, *in passing from* T *to* T' *the number of edge orbits and the number of vertex orbits does not decrease and increases by at most* 1.

Proof. Suppose we are in case (2) of Theorem 3.1. Then some vertex v gets blown up to $\check{T}_v$ and some vertex stabilizer of $\check{T}_v$ coincides with G_e. Specifically $\check{T}$ can be obtained by deleting each blown up vertex v from T and then equivariantly reattaching every edge incident to v to the corresponding vertex in $\check{T}_v$.

In particular, if $e \in \text{Edges}(T)$ is an edge incident to v that satisfies (2) of Theorem 3.1, then it is attached to the vertex $v_e \in \text{Vertices}(\check{T}_v)$. We obtain T' by collapsing the G-orbits of e to points. This amounts to identifying the vertex v_e with the vertex $u_e \in \text{Vertices}(\check{T})$ that is the other endpoint of e. From Figure 6 it is clear that T' is obtained by cleaving T.

We finally note that in passing from T to $\check{T}$ and then from $\check{T}$ to T', the vertex and edge groups are nonincreasing. Otherwise, the required properties of T' are immediately satisfied by Theorem 3.1 (see Figure 7). □

Finally, we can give our description of the decompositions of virtually free groups as amalgamated free products or HNN extensions.

Proof of Theorem 1.6. We prove this result by successively applying Corollary 4.1 until some desirable terminating condition is met. Virtually free groups have no one-ended subgroups, so we will always be able to apply our corollary; furthermore, virtually free groups are finitely presented. It now follows by Dunwoody accessibility [Dunwoody 1985] that there are no infinite chains $C_1 \succ C_2 \succ \cdots$ of virtually free groups (recall Definition 1.3), and that all such chains must terminate with finite groups.

First construction (pass to relatively one-ended vertex subgroups): Let T be a G-tree with one edge orbit Ge with G_e infinite. By accessibility, we may pass to a tree $T^{(2)}$ obtained by blowing up some vertices v of T to trees $\breve{T}_v$ such that the vertex groups of $\breve{T}_v$ are either finite or one-ended relative to the stabilizers G_f of the incident edges $f \ni v$. If possible, we take $T^{(1)} \subset T^{(2)}$ to be an infinite connected subtree obtained by deleting edges with finite stabilizers, and we set $G^{(1)} = G_{T^{(1)}}$, the setwise stabilizer. Note that the vertex groups of $T^{(1)}$ are $\preccurlyeq$ the vertex groups of T, and vertex groups are one-ended relative to the incident edge groups.

Second construction (pass to smaller edge groups): The second construction utilizes Corollary 4.1. If T_i is a G_i-tree with one edge orbit whose vertex groups are one-ended relative to the incident edge groups, we first apply Theorem 3.1 to blow up a vertex $v \in \text{Vertices}(T_i)$, and find ourselves in case (2) of the theorem. If $\breve{T}_v$ has a finite edge group then G_v is not one-ended relative to the incident edge groups, contradicting our assumption. By Corollary 4.1 we can collapse an edge of the blowup of T_i to get a cleaving T_i' that has at most two edge orbits, with edge groups $\prec$ the edge groups of T_i. The new vertex groups are also $\preccurlyeq$ the old vertex groups. If there are two edge orbits, then we obtain $T_{i+1} \subset T_i'$ as a maximal subtree containing only one edge orbit and set $G_{i+1} = (G_i)_{T_{i+1}}$, the setwise stabilizer. (See Figure 7.) If T' already has only one edge orbit then $T_{i+1} = T_i$ and $G_{i+1} = G_i$.

In both constructions, we pass to subgroups that split as graphs of groups such that the edge groups and vertex groups are $\preccurlyeq$ the edge and vertex groups of the original splitting of the overgroup.

We start with the amalgamated free product case. Let $T = T_0$ be the Bass–Serre tree corresponding to the splitting given in (1) of the statement of Theorem 1.6. Take the blowup $T_0^{(2)}$ obtained from the first construction. If one of the vertex groups of this blowup coincides with an incident edge group then we are done. Otherwise, we may pass to the $G^{(1)}$-tree $T_0^{(1)}$, which still has one edge orbit and two vertex orbits, and whose vertex groups are one-ended relative to the incident edge groups. Furthermore, because the new vertex groups are $\preccurlyeq$ the vertex groups of T, if the statement of the theorem holds for $G^{(1)}$ and the splitting corresponding to its action on $T_0^{(1)}$ (which is also an amalgamated free product), then the statement also holds for G and the splitting corresponding to its action on T.

We can now apply our second construction to the $G_0^{(1)}$-tree $T_0^{(1)}$ to obtain a G_1-tree T_1, which again must have one edge orbit and two vertex orbits. Furthermore, for the (conjugacy class of the) edge group, we have a proper containment $C_1 \prec C$. Again, because the vertex groups of T_1 are $\preccurlyeq$ the vertex groups of $T_0^{(1)}$, if the Theorem holds for this subgroup, it holds for G.

We repeatedly apply our construction, thus obtaining a sequence of groups that split as amalgamated free products. With each iteration of the second construction, we pass to a smaller edge group. Hence, by accessibility, eventually there is a subgroup G_i acting on $T_i^{(2)}$ (see the first construction) such that the vertex groups split as graphs of groups with finite edge groups and one of the incident edge groups coincides with the vertex group. Since $\preccurlyeq$ is transitive, (1) of Theorem 1.6 is satisfied.

We now tackle the HNN extension case. The proof proceeds in the same way. We repeatedly blow up, cleave, and pass to subtrees, the main difference being that the G-tree T has only one vertex orbit. If at some point one of the trees T_i or $T_i^{(1)}$ has two vertex orbits, then these vertex groups are vertex groups of a splitting of the vertex group of T_{i-1} with finite edge groups. It follows that if T_i satisfies (1) of Theorem 1.6, then T_{i-1} satisfies (2) of Theorem 1.6, and thus by transitivity of $\preccurlyeq$, so must our original splitting T. Otherwise, the proof goes through identically. □

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