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**ON THE ONE-ENDEDNESS OF GRAPHS OF GROUPS**

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# ON THE ONE-ENDEDNESS OF GRAPHS OF GROUPS

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**We give a technical result that implies a straightforward necessary and sufficient condition for a graph of groups with virtually cyclic edge groups to be one-ended. For arbitrary graphs of groups, we show that if their fundamental group is not one-ended, then we can blow up vertex groups to graphs of groups with simpler vertex and edge groups. As an application, we generalize a theorem of Swarup to decompositions of virtually free groups.**

## 1. Introduction

A finitely generated group  $G = \langle S \rangle$  is said to be *one-ended* if the corresponding Cayley graph  $\text{Cay}(G, S)$  cannot be separated into two or more infinite components by removing a finite subset. Otherwise  $G$  is said to be *many-ended*. It is a classical result due to Stallings [1971] that a many-ended group decomposes as either an amalgamated free product or an HNN extension over a finite group.

Given the Bass–Serre correspondence between group actions on simplicial trees and their decompositions, or splittings, as (fundamental groups of) graphs of groups (see [Serre 1980]), a finitely generated group  $G$  is many-ended if and only if it acts minimally, without inversions, and cocompactly on a simplicial tree  $T$  in which for some edge  $e$  the stabilizer  $G_e$  is finite.

It is often the case that a graph of groups with many-ended vertex groups is itself one-ended. For example, the fundamental group of a closed surface is one-ended but it is an amalgamated free product of free groups, which are many-ended. [Theorem 3.1](#), stated and proved in [Section 3](#), essentially characterizes one-ended graphs of groups. This result is rather technical, but has many “nontechnical” corollaries which we now present.

We say that  $G$  is *one-ended relative to a collection  $\mathcal{H}$  of subgroups* if for any minimal nontrivial  $G$ -tree  $T$  with finite edge stabilizers, there exists a subgroup  $H \in \mathcal{H}$  that acts without a global fixed point. Otherwise  $G$  is said to be *many-ended relative to  $\mathcal{H}$* . In this case,  $G$  admits a nontrivial splitting as a graph of groups relative to  $\mathcal{H}$  (i.e., groups in  $\mathcal{H}$  are conjugate into vertex groups) with finite edge groups.

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**Corollary 1.1.** *If  $G_1$  is one-ended relative to a collection  $\mathcal{H}_1 \cup \{C_1\}$ , and  $G_2$  is one-ended relative to  $\mathcal{H}_2 \cup \{C_2\}$  with  $C_1 \approx C_2$  virtually cyclic groups, then any free product with amalgamation of the form*

$$G_1 *_{C_1=C_2} G_2$$

*is one-ended relative to  $\mathcal{H}_1 \cup \mathcal{H}_2$ .*

In the case of graphs of free groups with cyclic edge groups, this corollary (actually its natural generalization, see [Corollary 1.5](#)) is proved in [[Wilton 2012](#), Theorem 18] and implied by results in [[Diao and Feighn 2005](#)]. [Corollary 1.1](#) is false if we do not require the amalgamating subgroups to be virtually cyclic or, synonymously, two-ended. Nonetheless, we can still understand the failure of one-endedness of general graphs of groups.

**Definition 1.2.** A  $G$ -equivariant map  $S \rightarrow T$  of simplicial  $G$ -trees is called a *collapse* if  $T$  is obtained by identifying some edge orbits of  $S$  to points. In this case we also say that  $S$  is obtained from  $T$  by a *blow up*. We call the preimage  $\check{T}_v \subset S$  of a vertex  $v \in T$  its *blowup*.

**Definition 1.3.** We write  $H \preccurlyeq G$  to signify that  $G$  splits essentially as a graph of groups with finite edge groups and  $H$  is a vertex group. A group  $G$  is *accessible* if it admits no infinite proper chains

$$G \succ G_1 \succ G_2 \succ \dots$$

For example, if  $F$  is a free group and  $H \preccurlyeq F$ , then  $H$  is a free factor of  $F$ . This next theorem, a formal consequence of [Theorem 3.1](#), states that if a graph of groups with finitely generated infinite edge group is not one-ended, then we can blow up some of its vertex groups.

**Theorem 1.4.** *Suppose that  $T$  is a  $G$ -tree (in which a collection of subgroups  $\mathcal{H}$  act elliptically) with infinite edge groups, and that  $G$  is not one-ended relative to  $\mathcal{H}$ . Then there is a vertex  $v \in \text{Vertices}(T)$  and an edge  $e \in \text{Edges}(T)$  with  $v \in e$  such that the orbit of  $v$  can be blown up with  $G_v$  acting minimally on the nontrivial blowups  $\check{T}_v$  satisfying the following properties:*

- $G_e \leq G_v$  is the stabilizer of a vertex in  $\check{T}_v$ .
- The edge groups of  $\check{T}_v$  are conjugate in  $G_v$  to the vertex groups of an essential amalgamated free product or HNN decomposition of  $G_e$  with a finite edge group.

*In particular, in the tree  $S$  obtained by blowing up the orbit of  $v$  in  $T$  to  $\check{T}_v$ , each vertex or edge stabilizer of  $S$  is  $\preccurlyeq$  a vertex or edge stabilizer of  $T$ , and at least one of these inclusions is strict. Furthermore the groups in  $\mathcal{H}$  act elliptically on  $S$ .*

We note that blowing up a  $G$ -tree is equivalent to *refining* a graph of groups. If  $G$  acts on a tree with accessible vertex and edge stabilizers then the order  $\prec$  actually tells us that the vertex groups of the blowup given by [Theorem 1.4](#) have lower complexity, in the sense that the process of successively blowing up vertex groups in this manner must terminate in finitely many steps.

Accessible groups, in turn, are abundant. Linnell [\[1983\]](#) showed that if there is a global bound on the order of finite order elements in a finitely generated group, then the group is accessible. Dunwoody [\[1985\]](#) showed that finitely presented groups are accessible. We now use [Theorem 1.4](#) to give a proof of [Corollary 1.1](#).

*Proof of Corollary 1.1.* We show the contrapositive. Let  $T$  be the Bass–Serre tree dual to the splitting  $G = G_1 *_C G_2$ , and suppose that  $G$  is not one-ended relative to  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ . Note that any decomposition of a virtually cyclic group as an HNN extension or an essential amalgamated free product must have finite edge groups. It follows that in all cases, by [Theorem 1.4](#), some orbit of vertices  $Gv$  can be blown up to minimal  $gG_vg^{-1}$ -trees with finite edge groups. This implies that one of the vertex groups  $G_i$  fixing some vertex  $v \in \text{Vertices}(T)$  acts minimally on  $\check{T}_v$  with finite edge stabilizers, with

$$\mathcal{H}_i = \{H \in \mathcal{H} \mid H \cap G_i \neq \{1\}\}$$

and  $C_i = G_e$  for some  $v \in e \in \text{Edges}(T)$  acting elliptically. It follows that  $G_i$  is not one-ended relative to  $\mathcal{H}_i \cup \{C\}$ .  $\square$

This proof is easily adapted to give:

**Corollary 1.5.** *The fundamental group  $G$  of a graph of groups with two-ended edge groups is one-ended (relative to a collection  $\mathcal{H}$  of subgroups) if and only if every vertex group  $G_v$  is one-ended relative to the incident edge groups (and the collection  $\{H^g \cap G_v \mid g \in G, H \in \mathcal{H}\}$ ).*

Using the full strength of [Theorem 3.1](#), we also generalize a result of Swarup [\[1986\]](#) on the decomposition of free groups to virtually free groups. This result was already partially generalized by Cashen [\[2012\]](#) to decompositions of virtually free groups with virtually cyclic edge groups.

**Theorem 1.6.** *Let  $G$  be finitely generated and virtually free.*

- (1) *If  $G$  splits as an amalgamated free product  $G = A *_C B$  with  $C$  finitely generated and infinite, then there is some  $C_1 \preceq C$  such that  $C_1 \preceq A$  or  $C_1 \preceq B$ .*
- (2) *If  $G$  splits as an HNN extension  $G = A *_C,t$  with  $C$  finitely generated and infinite, then there is an infinite subgroup  $C_1 \preceq C$  and a splitting  $\Delta$  of  $A$  as a graph of groups with finite edge groups relative to  $\{C_1, t^{-1}C_1t\}$  such that either  $C_1$  or  $t^{-1}C_1t$  is a vertex group of  $\Delta$ .*

Unlike in Swarup’s proof, we do not use homological methods. Our proof is more along the lines of the geometric arguments found in [Wilton 2012; Louder 2008; Bestvina and Feighn 1994; Diao and Feighn 2005] using graphs of spaces  $X$  with  $\pi_1(X) = G$ . The presence of torsion, however, can make the attaching maps in the graphs of spaces difficult to describe. By using the more abstract  $G$ -cocompact core of the product of two  $G$ -trees [Guirardel 2005], we sidestep these difficulties. The core has been used before to study pairs of group splittings. In particular, Fujiwara and Papasoglu [2006] use it to show the existence of QH subgroups for one-ended groups that have hyperbolic-hyperbolic pairs of slender splittings; this is the main technicality in constructing group theoretical JSJ decompositions. Although it could be noted that the action of our group on the core gives rise to a  $G$ -orbihedron à la [Haefliger 1991], we will not need this machinery; in fact, modulo classical Bass–Serre theory and Guirardel’s Core Theorem for simplicial trees (Theorem 2.3, of which we sketch a proof), our argument is self-contained.

## 2. Preliminaries

**Group actions.** All group actions will be from the left. Let  $X$  be a  $G$ -set. If  $S \subset X$  is a subset, we denote by  $G_S$  the (setwise) stabilizer  $\{g \in G \mid gS = S\}$ . If  $S = \{x\}$  is a singleton, then we write  $G_x$  instead of  $G_{\{x\}}$ . We call a subset  $S \subset X$   $G$ -regular if for any  $x, y \in S$  in the same  $G$ -orbit, there is some  $g \in G_S$  such that  $gx = y$ . The following lemma is immediate.

**Lemma 2.1.** *Let  $X$  be a  $G$ -set. If  $S \subset X$  is  $G$ -regular, then we have an embedding*

$$G_S \backslash S \hookrightarrow G \backslash X.$$

In this paper, all trees will be simplicial. In particular we consider them to be topological spaces, equipped with a CW-structure, which also makes them into graphs. We further metrize these graphs by viewing edges as real intervals of length 1.

All  $G$ -trees  $T$  will be *without inversions*, meaning that for any edge  $e \in \text{Edges}(T)$ , if  $ge = e$  then  $g$  fixes  $e$  pointwise. Equivalently, if  $u, v \in \text{Vertices}(T)$  are the vertices at the ends of the edge  $e$ , then we have inclusions

$$G_u \geq G_e \leq G_v.$$

We call vertex stabilizers *vertex groups*, and edge stabilizers *edge groups*. We assume the reader is familiar with Bass–Serre theory and we switch freely between  $G$ -trees and splittings as graphs of groups, viewing the two as being equivalent.

A  $G$ -tree  $T$  is *essential* if every edge of  $T$  divides it into two infinite components.  $T$  is *minimal* if there are no proper subtrees  $S \subset T$  with  $G_S = G$ .  $T$  is *cocompact* if  $G \backslash T$  is compact. An element  $g$  or a subgroup  $H$  of  $G$  are said to *act elliptically* on  $T$  if the groups  $\langle g \rangle$  or  $H$  fix some  $v \in \text{Vertices}(T)$ .

**Products of trees, cores, and leaf spaces.** If  $T_1$  and  $T_2$  are  $G$ -trees, then we have a natural induced action  $G \curvearrowright T_1 \times T_2$ . Since the trees  $T_1, T_2$  are 1-dimensional CW complexes, we may consider their product  $T_1 \times T_2$  as a *square complex*, i.e., a 2-dimensional CW complex whose cells consist of vertices, edges, and squares. There are natural projections  $p_i : T_1 \times T_2 \rightarrow T_i$ . The following lemma is immediate.

**Lemma 2.2.** *If the actions  $G \curvearrowright T_1$  and  $G \curvearrowright T_2$  are without inversions, then so is the action  $G \curvearrowright T_1 \times T_2$ , i.e., if  $\sigma \supset \epsilon$  is an inclusion of cells (e.g., a square containing an edge), then  $G_\sigma \leq G_\epsilon$ .*

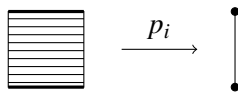
*If  $\mathcal{H}$  is a collection of subgroups acting elliptically on  $T_1$  and  $T_2$ , then each subgroup in  $\mathcal{H}$  fixes a vertex of  $T_1 \times T_2$ .*

The action  $G \curvearrowright T_1 \times T_2$  is not cocompact in general. It turns out, however, that we can extract a useful subset, namely Guirardel’s cocompact core. We state the special case of his result applied to simplicial trees.

**Theorem 2.3** (the Core Theorem, see [Guirardel 2005, Théorème principal and Corollaire 8.2]). *Let  $G \curvearrowright T_1, G \curvearrowright T_2$  be two minimal actions of a finitely generated group  $G$  on simplicial trees  $T_1, T_2$  with finitely generated edge stabilizers. Suppose furthermore that  $T_1, T_2$  do not equivariantly collapse to a common nontrivial tree.*

*Then there is a  $G$ -invariant subset  $\mathcal{C} \subset T_1 \times T_2$ , called the core of the action  $G \curvearrowright T_1 \times T_2$ , which is defined as the smallest connected  $G$ -invariant subset such that the restrictions of the projections  $p_i|_{\mathcal{C}} : \mathcal{C} \rightarrow T_i$  have connected fibers. The quotient  $\mathcal{S} = G \backslash \mathcal{C}$  is compact.*

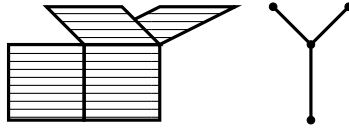
Suppose for the remainder of this section that  $T_1, T_2$  satisfy the hypotheses of Theorem 2.3. The restrictions of the projections  $p_i|_\sigma : \sigma \rightarrow T_i$  are well defined for each cell (i.e., a vertex, edge, square)  $\sigma \subset T_1 \times T_2$ . If  $\sigma$  is a square then the projection is onto an edge  $p_i(\sigma) \in \text{Edges}(T_i)$ . If  $\lambda_1, \lambda_2 \subset \sigma$  are two fibers of such a projection (see Figure 1), we can define a distance  $d_i^\sigma(\lambda_1, \lambda_2)$  to be the distance



**Figure 1.** The projection of a square on an edge and some of its fibers.

in  $p_i(\sigma)$  between the points  $p_i(\lambda_1), p_i(\lambda_2)$ , thus putting a metric  $d_i^\sigma$  on the set of  $p_i$ -fibers in a cell  $\sigma$ . We now define the  *$i$ -leaf space*  $\mathcal{L}_i$  of a subset  $Z \subset T_1 \times T_2$  to be the set of connected unions of  $p_i$ -fibers of cells in  $Z$ , called *leaves*, so that we see  $Z$  as being *foliated* by the leaves in  $\mathcal{L}_i$ .  $\mathcal{L}_i$  is a 1-complex with metrized edges; therefore we can endow  $\mathcal{L}_i$  with the path metric  $d_i$ . As a consequence of the direct product structure we have the following.

**Lemma 2.4.** *If  $Z \subset T_1 \times T_2$ , then the leaf spaces  $\mathcal{L}_i$  are forests (see Figure 2).*



**Figure 2.** The  $i$ -leaves in a square complex and the resulting leaf space, which is a tree.

If  $\mathcal{C} \subset T_1 \times T_2$  is a core then the leaf spaces  $\mathcal{L}_i$  are homeomorphic to the trees  $T_i$ . Later, however, we will be performing operations that will alter the leaf spaces.

**Induced splittings.** Let  $v \in \text{Vertices}(T_i)$ ,  $e \in \text{Edges}(T_i)$  and let  $m_e$  be the midpoint of  $e$ . Let  $\tau_v = p_i^{-1}(\{v\}) \cap \mathcal{C}$  and  $\tau_e = p_i^{-1}(\{m_e\}) \cap \mathcal{C}$ . By [Theorem 2.3](#) the preimages  $\tau_v, \tau_e$  are connected and are therefore leaves in  $\mathcal{L}_i$ .

Since we have an action  $G \curvearrowright \mathcal{C}$ , since  $\tau_v, \tau_e$  are defined as  $T_i$ -point preimages via a  $G$ -equivariant map, and since  $G_v, G_e$  are exactly the stabilizers of these points  $v, m_e$ , the subsets  $\tau_v, \tau_e \leq \mathcal{C}$  are  $G$ -regular. So, by [Lemma 2.1](#) we have embeddings

$$G_v \backslash \tau_v \hookrightarrow G \backslash \mathcal{C} \hookleftarrow G_e \backslash \tau_e.$$

By [Theorem 2.3](#),  $G \backslash \mathcal{C}$  is compact so the quotients  $G_v \backslash \tau_v, G_e \backslash \tau_e$  must be as well. Moreover, because  $\tau_v, \tau_e$  are contained in  $p_i$ -fibers, for  $j \neq i$  the restrictions

$$p_j|_{\tau_v} : \tau_v \rightarrow T_j, \quad p_j|_{\tau_e} : \tau_e \rightarrow T_j$$

are injective. Finally, the projection  $p_j|_{\mathcal{C}} : \mathcal{C} \rightarrow T_j$  is  $G$ -equivariant. We have shown the following.

**Lemma 2.5.** *If  $v \in \text{Vertices}(T_i)$ ,  $e \in \text{Edges}(T_i)$ ,  $j \neq i$ , then the fibers  $\tau_v, \tau_e$  are mapped injectively via  $p_j$  to subtrees that are  $G_v, G_e$ -invariant, respectively. Viewed as subsets of the core  $\mathcal{C} \subset T_1 \times T_2$ ,  $\tau_v$  and  $\tau_e$  coincide with their  $j$ -leaf spaces.*

*The actions  $G_e \curvearrowright \tau_e, G_v \curvearrowright \tau_v$  are cocompact. Moreover  $\tau_v, \tau_e$  are infinite if and only if the actions of the subgroups  $G_v \curvearrowright T_j, G_e \curvearrowright T_j$  are without global fixed points.*

The  $G_v, G_e$ -trees  $\tau_v, \tau_e$  give splittings induced by the action on  $T_j$ . The blowups of [Theorem 3.1](#) will be obtained by modifying the trees  $\tau_v$ . For aficionados of CAT(0) cube complexes, it is worth remarking that the core  $\mathcal{C}$  is a CAT(0) square complex, in fact a  $\mathcal{V}\mathcal{H}$ -complex, and that the set of fibers  $\tau_e, e \in \text{Edges}(T_i)$  is the set of hyperplanes.

**Spurs, free faces, and cleavings.** In the previous subsection we obtained cocompact  $G_v, G_e$ -trees  $\tau_v, \tau_e$ . We say a tree has a *spur* if it has a vertex of degree 1. An edge adjacent to a spur is called a *hair*. We now give a shaving process.

**Lemma 2.6.** *Let  $T$  be a cocompact  $G$ -tree.  $T$  is minimal if  $T$  doesn't have any spurs. If  $T$  is not minimal, then we can obtain the minimal subtree  $T(G)$  as the final term of a finite sequence*

$$T = T_0, \dots, T_k = T(G),$$

where  $T_{i+1}$  is obtained from  $T_i$  by  $G$ -equivariantly contracting one  $G$ -orbit of hairs to points.

*Proof.* Let  $v \in \text{Vertices}(T)$  be a spur adjacent to an edge  $e \in \text{Edges}(T)$  and let  $u \in \text{Vertices}(T)$  be the other endpoint of  $e$ . The map  $T \rightarrow T$  obtained by  $G$ -equivariantly collapsing  $ge$  onto  $gu$  for  $g \in G$  is a deformation retraction onto a proper  $G$ -invariant subtree, so  $T$  is not minimal.

Suppose now that  $T$  is not minimal. Then there is some proper  $G$ -invariant subtree  $S \subset T$ . Let  $K$  be the closure of some connected component of  $T \setminus S$ . Then  $K \cap S = \{v\}$  for some  $v \in \text{Vertices}(S)$ . Since  $S$  is  $G$ -invariant and connected, we must have  $G_K \leq G_v$ . It follows that for any  $w \in \text{Vertices}(K)$  and any  $g \in G_K$  the distance  $d_T(w, v) = d_T(gw, v)$ , i.e., the action of  $G_K$  on  $K$  is the action on a rooted tree with root  $v$ . Since  $K$  is  $G$ -regular, we have an embedding  $G_K \backslash K \hookrightarrow G \backslash T$  which is compact; thus  $K$  must have finite radius since  $G_K$  preserves distances from the root.

Since  $K$  is a rooted tree with finite diameter it must have a nonroot vertex of valence 1. By the argument at the beginning of the proof, we can  $G_K$ -equivariantly collapse hairs, and since  $G_K \curvearrowright K$  is cocompact, after finitely many collapses we will have collapsed  $K$  to  $v$ . Again since  $G \curvearrowright T$  is cocompact, there are only finitely many orbits of connected components of  $T \setminus S$ , so the result follows.  $\square$

If  $\sigma$  is a square in some  $Z \subset T_1 \times T_2$ , then we say an edge  $e \subset \sigma$  is a *free face* if it is only contained in one square. The following terminology is due to Wise [2004].

**Definition 2.7.** Let  $e \in \text{Edges}(T_i)$  and let  $\tau_e \subset \mathcal{C}$  be the fiber of  $e$  as in Lemma 2.5. The *hypercarrier*  $\mathcal{H}_{\mathcal{C}}(\tau_e)$  is the union of squares of  $\mathcal{C}$  intersecting  $\tau_e$  nontrivially.

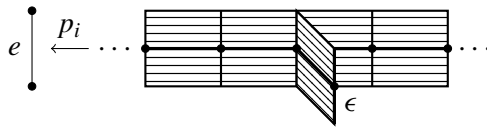
We note that for  $e \in T_i$ , a hypercarrier is mapped to an edge of  $T_i$  and that  $\mathcal{H}_{\mathcal{C}}(\tau_e)$  is homeomorphic to  $\tau_e \times [-1, 1]$ .

**Definition 2.8.** We say an edge  $e$  in some  $Z \subset T_1 \times T_2$  is  $i$ -transverse if it coincides with its  $i$ -leaf space, or equivalently if it is mapped monomorphically via  $p_i|_e$ , or equivalently if it is contained in a  $j$ -leaf.

An immediate consequence of Lemma 2.6 and Figure 3 is the following.

**Lemma 2.9.** *Let  $e \in \text{Edges}(T_i)$ . If  $G_e \curvearrowright \tau_e$  is not minimal then  $\mathcal{H}_{\mathcal{C}}(\tau_e)$  has a square  $\sigma$  containing an  $i$ -transverse free face  $e$ .*





**Figure 3.** A spur of  $\tau_e$  and the corresponding free face  $\epsilon$  in the hypercarrier  $\mathcal{H}_\epsilon(\tau_e)$ .

We now borrow some terminology from [Diao and Feighn 2005].

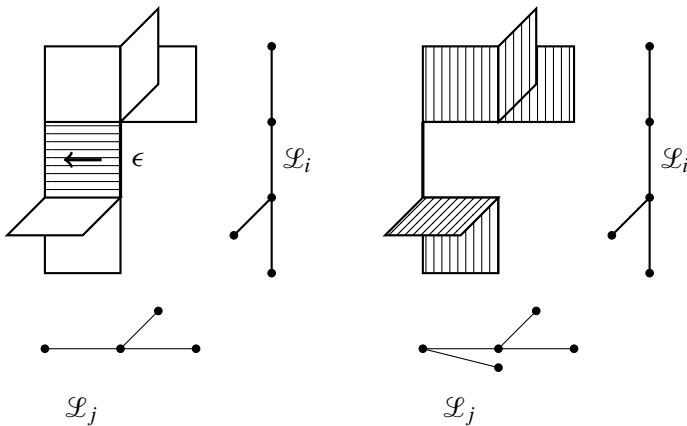
**Definition 2.10.** A simplicial map  $S \rightarrow T$  between two trees that is obtained by identifying edges sharing a common vertex is called a *folding*. If  $T$  is obtained from  $S$  by a folding, then we say  $S$  is obtained from  $T$  by a *cleaving*.

The next lemma is now immediate (see Figure 4).

**Lemma 2.11.** Let  $\epsilon \subset Z \subset T_1 \times T_2$  be an  $i$ -transverse free face in a square  $\sigma$ . If we collapse  $\sigma$  onto the face opposite to  $\epsilon$ , then the leaf space  $\mathcal{L}_i$  is unchanged and the leaf space  $\mathcal{L}_j$  gets cleaved.

In fact this lemma can be used backwards to give a proof of Theorem 2.3. We will sketch it, leaving the details to an interested reader familiar with folding sequences [Bestvina and Feighn 1991; Stallings 1991; Dunwoody 1998; Kapovich et al. 2005].

*Sketch of the proof of Theorem 2.3.* Pick some vertex  $v \in T_1 \times T_2$  and consider its  $G$ -orbit. We can add finitely many connected  $G$ -orbits of edges to get a connected  $G$ -complex  $Gv \subset \mathcal{C}_1 \subset T_1 \times T_2$ .  $\mathcal{C}_1$  has leaf spaces  $\mathcal{L}_1, \mathcal{L}_2$  which project onto  $T_1, T_2$ . The disconnectedness of the fibers of the projections  $p_i|_{\mathcal{C}_1} : \mathcal{C}_1 \rightarrow T_i$  coincides with



**Figure 4.** The effects of collapsing an  $i$ -transverse free face  $\epsilon$ : the leaf space  $\mathcal{L}_j$  gets cleaved,  $\mathcal{L}_i$  remains unchanged. On the right the  $j$ -leaves are drawn.

the failure of injectivity of the projections  $\mathcal{L}_i \rightarrow T_i$ . By [Lemma 2.11](#) (backwards) adding a square can give a folding of one of the leaf spaces. Since the edge groups of  $T_1, T_2$  are finitely generated, and because adding all the squares of  $T_1 \times T_2$  folds  $\mathcal{L}_i$  to  $T_i$ , it follows that the leaf spaces  $\mathcal{L}_i$  can be made to coincide with  $T_i$  after adding finitely many  $G$ -orbits of squares.  $\square$

### 3. The statement and proof of the main theorem

For this section we fix a collection  $\mathcal{H}$  of subgroups of  $G$ . We let  $T_\infty$  and  $T_{\mathcal{F}}$  be cocompact, minimal  $G$ -trees in which the subgroups in  $\mathcal{H}$  act elliptically. We further require that edge groups of  $T_\infty$  are infinite and finitely generated and that edge groups of  $T_{\mathcal{F}}$  are finite. Note that any nontrivial tree obtained by a collapse of  $T_\infty$  has infinite edge groups whereas any collapse of  $T_{\mathcal{F}}$  has finite edge groups. It follows that  $T_\infty$  and  $T_{\mathcal{F}}$ , having no nontrivial common collapses, satisfy the hypotheses of [Theorem 2.3](#).

**Theorem 3.1** (Main Theorem). *Let  $\mathcal{H}$  be a collection of subgroups of  $G$  and let  $T_\infty$  and  $T_{\mathcal{F}}$  be cocompact, minimal  $G$ -trees in which the subgroups in  $\mathcal{H}$  act elliptically. Suppose furthermore that the edge groups of  $T_{\mathcal{F}}$  are finite and that the edge groups of  $T_\infty$  are infinite. Then there exists a vertex  $v \in \text{Vertices}(T_\infty)$  and a nontrivial, cocompact, minimal  $G_v$ -tree  $\check{T}_v$  such that*

- (i) *for every  $f \in \text{Edges}(T_\infty)$  incident to  $v$  the subgroups  $G_f \leq G_v$  act elliptically on  $\check{T}_v$ , and*
- (ii) *for every  $H \in \mathcal{H}$  and  $g \in G$  the subgroup  $H^g \cap G_v \leq G_v$  acts elliptically on  $\check{T}_v$ .*

Moreover, either

- (1) *every edge group of  $\check{T}_v$  is finite, or*
- (2) *there is some edge  $e \in \text{Edges}(T_\infty)$ , incident to  $v$ , that not only satisfies (i), but also satisfies the following:*
  - (a)  *$G_e$  splits essentially as an amalgamated free product or an HNN extension with finite edge group.*
  - (b)  *$G_e = G_{v_e}$  for some vertex  $v_e \in \text{Vertices}(\check{T}_v)$ .*
  - (c) *The edge stabilizers of  $\check{T}_v$  are conjugate in  $G_v$  to the vertex group(s) of the splitting of  $G_e$  found in (a); in particular, the edge groups of  $\check{T}_v$  are  $\prec G_e$ .*
  - (d) *The vertex groups of  $\check{T}_v$  that are not conjugate in  $G_v$  to  $G_e$  are also vertex groups of a one-edge splitting of  $G_v$  with a finite edge group; in particular these vertex groups of  $\check{T}_v$  are  $\prec G_v$ .*

An example of what happens in situation (2) is shown in [Figure 7](#).

*Proof.* Let  $\mathcal{C}$  be the core of  $T_\infty \times T_{\mathcal{F}}$ . The  $\infty$ -leaf space  $\mathcal{L}_\infty$  is the tree  $T_\infty$ , and we can see  $\mathcal{C}$  as a tree of spaces (see [\[Scott and Wall 1979\]](#) for details) which is a union

of vertex spaces  $\tau_v$  for  $v \in \text{Vertices}(T_\infty)$  and edge spaces  $\mathcal{H}_e(\tau_e) = \tau_e \times [-1, 1]$  for  $e \in \text{Edges}(T_\infty)$  attached to the  $\tau_v$  along the subspaces  $\tau_e \times \{\pm 1\}$ .

It may be that for some  $e \in \text{Edges}(T_\infty)$ , the  $G_e$ -trees  $\tau_e$  are not minimal. By Lemmas 2.9, 2.6, and 2.11, we can repeatedly  $G$ -equivariantly collapse  $\infty$ -transverse free faces, so that after finitely many steps we obtain a *shaved core*  $\mathcal{C}'_s$  such that the  $\tau_e \cap \mathcal{C}'_s$  are minimal  $G_e$ -trees. Although the  $\mathcal{F}$ -leaf space was cleaved repeatedly in the shaving process given by Lemma 2.6, the  $\infty$ -leaf space is unchanged. We still write  $\mathcal{L}_\infty = T_\infty$ .

We now construct a complex  $\mathcal{C}_s \subset \mathcal{C}'_s \subset \mathcal{C}$ , called the  $\infty$ -minimal core. Its principal feature is that for every  $v \in \text{Vertices}(T_\infty)$  and  $e \in \text{Edges}(T_\infty)$ , the trees  $\tau_v \cap \mathcal{C}_s$  and  $\tau_e \cap \mathcal{C}_s$  are minimal  $G_v$ - and  $G_e$ -trees, respectively. Define  $\mathcal{H}_{e'_s}(\tau_e) = \mathcal{H}_e(\tau_e) \cap \mathcal{C}'_s$ . We call  $\mathcal{H}_{e'_s}(\tau_e)$  the  $\mathcal{C}'_s$ -hypercarrier attached to a vertex space  $\tau_v$  in  $\mathcal{C}'_s$ . Note that  $\tau_e \cap \mathcal{C}'_s$  naturally projects injectively into  $\tau_v$  as a minimal  $G_e$ -invariant subtree where  $G_e \leq G_v$ . If  $T$  is a  $G$ -tree and  $H \leq G$ , denoting by  $T(S)$  the minimal  $S$ -invariant subtree, we have  $T(H) \subset T(G)$ . It therefore follows that all the  $\mathcal{C}'_s$ -hypercarriers attached to  $\tau_v$  are actually attached to the minimal  $G_v$ -invariant subtree of  $\tau_v$ . By Lemma 2.6, after finitely many equivariant spur collapses we can make the vertex spaces  $\tau_v$  into minimal  $G_v$ -trees. None of these collapses will affect the attached  $\mathcal{C}'_s$ -hypercarriers  $\mathcal{H}_{e'_s}(\tau_e)$ , and the leaf space  $\mathcal{L}_\infty = T_\infty$  is preserved. We have therefore constructed  $\mathcal{C}_s$ , the  $\infty$ -minimal core. Denote  $\mathcal{H}_{e_s}(\tau_e) = \mathcal{H}_{e'_s}(\tau_e) \cap \mathcal{C}_s$ . By what was written above,  $\mathcal{H}_{e_s}(\tau_e) = \mathcal{H}_{e'_s}(\tau_e)$ , and we now call  $\mathcal{H}_{e_s}(\tau_e)$  a  $\mathcal{C}_s$ -hypercarrier.

For every  $k \in \text{Edges}(T_{\mathcal{F}})$ ,  $G_k$  is finite, therefore a minimal  $G_k$ -tree is a point; thus, by cocompactness and regularity, the trees  $\tau_k \in \mathcal{C}$  have finite diameter and the same must be true of every connected component of  $\tau_k \cap \mathcal{C}_s$ . So, every connected component of  $\tau_k \cap \mathcal{C}_s$  has a spur. It therefore follows that  $\mathcal{C}_s$  must have an  $\mathcal{F}$ -transverse free face  $\epsilon$  containing a spur of some connected component of  $\tau_k \cap \mathcal{C}_s$  for some  $k \in \text{Edges}(T_{\mathcal{F}})$ . Furthermore, the stabilizer  $G_\epsilon \leq G_{p_{\mathcal{F}}(\epsilon)}$  is an edge stabilizer of  $T_{\mathcal{F}}$ , and therefore finite. This  $\mathcal{F}$ -transverse free face  $\epsilon$  must be contained in some  $\tau_v \cap \mathcal{C}_s$  for  $v \in \text{Vertices}(T_\infty)$ . Suppose first that  $\epsilon$  was not contained in any  $\mathcal{C}_s$ -hypercarrier attached to  $\tau_v \cap \mathcal{C}_s$ . Then for every  $e \ni v$  in  $\text{Edges}(T_\infty)$ ,  $G_e$  fixes some  $\mathcal{C}_s$ -hypercarrier  $\mathcal{H}_{e_s}(\tau_e)$  such that  $\mathcal{H}_{e_s}(\tau_e) \cap \tau_v = \tau_e^+$  is contained in the complement  $(\tau_v \cap \mathcal{C}_s) \setminus G_v \epsilon$ .

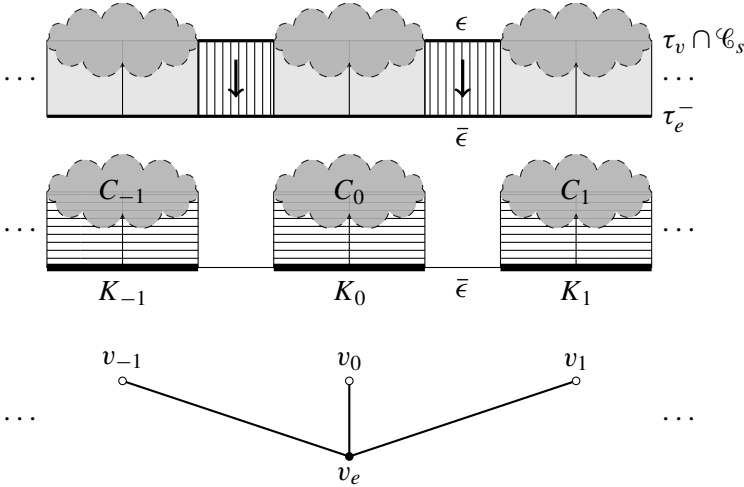
**Definition 3.2.** Let  $T$  be a minimal  $G$ -tree and  $e \in \text{Edges}(T)$ . We denote by  $C(T, e)$  the *non- $e$ -collapse* of  $T$ , the tree whose edges are the orbit  $Ge \subset T$  and whose vertices are the closures of the connected components of  $T \setminus Ge$ , with a vertex  $v$  adjacent to an edge  $e'$  in  $C(T, e)$  if and only if, viewed as subsets of  $T$ ,  $e' \cap v \neq \emptyset$ .

It therefore follows that  $\tilde{T}_v = C(\tau_v \cap \mathcal{C}_s, \epsilon)$  is a tree with finite edge groups, in which each  $G_e \leq G_v$  ( $e \in \text{Edges}(T_\infty)$ ) acts elliptically, and also conjugates of groups in  $\mathcal{H}$  intersecting  $G_v$  act elliptically. Thus (i), (ii) and (1) are satisfied.

Otherwise, the free face  $\epsilon \subset \tau_v \cap \mathcal{C}_s$  is, by definition of a free face, contained in *exactly one*  $\mathcal{C}_s$ -hypercarrier  $\mathcal{H}_{\mathcal{C}_s}(\tau_e)$ . We now construct the  $G_v$ -tree  $\check{T}_v$  satisfying (2). This construction is illustrated in Figure 5. We first take the subset

$$Z = \left( \tau_v \cup \bigcup_{e \ni v} \mathcal{H}_{\mathcal{C}_s}(\tau_e) \right) \cap \mathcal{C}_s,$$

i.e.,  $\tau_v \cap \mathcal{C}_s$  to which we attach all adjacent  $\mathcal{C}_s$ -hypercarriers. Now the  $G_v$ -translates of  $\epsilon$  are contained in the  $\mathcal{C}_s$ -hypercarriers  $\mathcal{H}_{\mathcal{C}_s}(\tau_{ge})$  for  $g \in G_v$ . For each such  $\mathcal{C}_s$ -hypercarrier we denote by  $\tau_{ge}^-$  the connected component of  $\tau_e \times \{\pm 1\} \subset \mathcal{H}_{\mathcal{C}_s}(\tau_{ge})$  not contained in  $\tau_v \cap \mathcal{C}_s$  (see the top of Figure 5).



**Figure 5.** Constructing  $\check{T}_v$ . The top shows a portion of  $Z$ , the middle shows the result of equivariantly collapsing the free face  $\epsilon$ , and the bottom shows the corresponding  $\infty$ -leaf space.

We now  $G_v$ -equivariantly collapse the square  $\sigma \supset \epsilon$  onto the opposite side  $\bar{\epsilon}$ , obtaining a connected  $G_v$ -subset  $Z_c \subset Z$  (see the middle of Figure 5). The resulting intersection  $\tau_v \cap Z_c$  consists of a collection of connected components  $\{C_i \mid i \in I\}$ . Similarly, the  $G_e$ -translates of  $\bar{\epsilon}$  give connected components  $\{K_i \mid i \in I\}$  of  $\tau_e \setminus G_e \bar{\epsilon}$ . Because  $G_e$  acts on  $C(\tau_e^-, \bar{\epsilon})$ , and by minimality of  $\tau_e \cap \mathcal{C}_s$ , this action is also minimal with one edge orbit. This gives us (a).

For every edge  $f \in \text{Edges}(T_\infty)$  incident to  $v$  that is not in the  $G_v$ -orbit of  $e$ , the orbit  $G_v \epsilon$  does not intersect  $\mathcal{H}_{\mathcal{C}_s}(\tau_f) \cap \tau_v$ . It follows that each such  $G_f \leq G_v$  stabilizes some component  $C_i$ . We now detach from  $Z_c$  all  $\mathcal{C}_s$ -hypercarriers not stabilized by a  $G_v$ -conjugate of  $G_e$ , producing a  $G_v$ -complex  $Z'_c \subset Z_c$ , specifically

$$Z'_c = Z_c \cap \left( \tau_v \cup \bigcup_{g \in G_v} \mathcal{H}_{\mathcal{C}_s}(\tau_{ge}) \right).$$

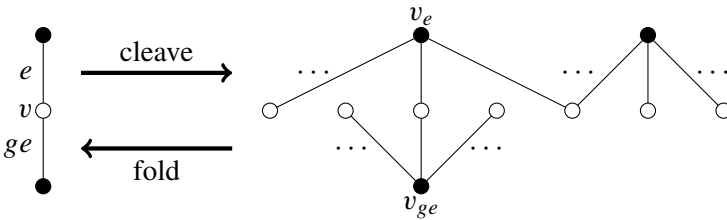
Next, to get the  $G_v$ -tree  $T_v$ , we collapse each  $G_v$ -translate of  $\tau_e^-$  to a vertex  $v_e$ , collapse each component  $C_i$  to a vertex  $v_i$ , and collapse each connected component of  $G_v$ -translates of  $\mathcal{H}_{\mathcal{C}_S}(\tau_e) \cap Z'_C$  onto an edge connecting  $v_e$  and the corresponding vertex  $v_i$ . This is illustrated at the bottom of [Figure 5](#).

Equivalently, if we consider the  $\infty$ -leaf space corresponding to the union of the  $\mathcal{C}_S$ -hypercarriers  $g\mathcal{H}_{\mathcal{C}_S}(\tau_e)$  attached to  $\tau_v \cap \mathcal{C}_S$  for  $g \in G_v$ , then we have a tree of radius 1, which is  $G_v$ -isomorphic to  $\{v\} \cup (\bigcup_{g \in G_v} ge) \subset T_\infty$ . After equivariantly collapsing the free face  $\epsilon$ , [Lemma 2.11](#) gives us a cleaving of this radius 1 subtree to the infinite tree  $\check{T}_v$  constructed above. See [Figure 6](#). We note that if we took the  $\infty$ -leaf space of  $Z_c$ , i.e., had we not detached the other hypercarriers, the resulting leaf space would be a tree with many spurs. The tree  $\check{T}_v$  we obtain is a minimal  $G_v$ -tree that satisfies (b) and (i).

Moreover, we note that by construction, every subgroup  $H^g \cap G_v$ , for  $g \in G$  and  $H \in \mathcal{H}$ , acts elliptically on  $\check{T}_v$ . So (ii) is satisfied as well.

Since  $\tau_e^-$  is  $G_v$ -regular, the vertex stabilizers of  $C(\tau_e^-, \bar{\epsilon})$  coincide with the component stabilizers  $(G_e)_{K_i} = (G_v)_{K_i}$ . We also have  $(G_v)_{C_i} \cap (G_v)_{\tau_e^-} = (G_v)_{K_i}$  (again referring to the middle of [Figure 5](#)). It now follows that the edge stabilizers of  $\check{T}_v$  satisfy (c).

Finally note that the vertex groups of  $\check{T}_v$  that are not stabilized by  $G_v$ -conjugates of  $G_e$  are also the vertex groups of  $C(\tau_v, \epsilon)$  (see the top of [Figure 5](#)). Finally, since  $G_\epsilon$  is finite, (d) follows. □

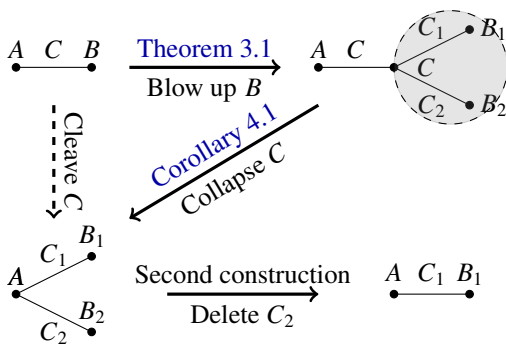


**Figure 6.** Equivariant collapsing free faces cleaves the leaf space of  $Z'_C$  to a tree  $\check{T}_v$  with infinite diameter.

### 4. Splittings of virtually free groups

Another way to use [Theorem 3.1](#) is to obtain cleavings of  $G$ -trees whose edge and vertex groups are “smaller”. This will be used as the inductive step in our proof of [Theorem 1.6](#).

**Corollary 4.1.** *Let  $T$  be a  $G$ -tree in which the subgroups  $\mathcal{H}$  act elliptically with infinite edge groups, and let  $G$  be many-ended relative to  $\mathcal{H}$ . Either some vertex*



**Figure 7.** An example of the effects of [Theorem 3.1](#), [Corollary 4.1](#), and the second construction of the proof of [Theorem 1.6](#) on a graph of groups. The vertices and edges are labeled by the corresponding vertex and edge groups. In all cases  $B_i < B$  and  $C_i < C$ .

$v \in \text{Vertices}(T)$  can be blown up to a tree with finite edge groups; or, there is an edge  $e \in \text{Edges}(T)$  such that we can blow up  $T$ , relative to  $\mathcal{H}$ , to some tree  $\check{T}$ , and then collapse the edges in the orbit of  $e$  to points. The resulting tree  $T'$  can also be obtained from  $T$  by equivariantly cleaving some edge  $e$ . If  $e' \in \text{Edges}(T')$  is a new edge obtained by a cleaving of  $e$ , then  $G_{e'} < G_e$ . Also, for each new vertex  $v' \in \text{Vertices}(T')$ , there is some  $v \in \text{Vertices}(T)$  that got cleaved such that  $G_{v'} < G_v$ .

Furthermore, in passing from  $T$  to  $T'$  the number of edge orbits and the number of vertex orbits does not decrease and increases by at most 1.

*Proof.* Suppose we are in case (2) of [Theorem 3.1](#). Then some vertex  $v$  gets blown up to  $\check{T}_v$  and some vertex stabilizer of  $\check{T}_v$  coincides with  $G_e$ . Specifically  $\check{T}$  can be obtained by deleting each blown up vertex  $v$  from  $T$  and then equivariantly reattaching every edge incident to  $v$  to the corresponding vertex in  $\check{T}_v$ .

In particular, if  $e \in \text{Edges}(T)$  is an edge incident to  $v$  that satisfies (2) of [Theorem 3.1](#), then it is attached to the vertex  $v_e \in \text{Vertices}(\check{T}_v)$ . We obtain  $T'$  by collapsing the  $G$ -orbits of  $e$  to points. This amounts to identifying the vertex  $v_e$  with the vertex  $u_e \in \text{Vertices}(\check{T})$  that is the other endpoint of  $e$ . From [Figure 6](#) it is clear that  $T'$  is obtained by cleaving  $T$ .

We finally note that in passing from  $T$  to  $\check{T}$  and then from  $\check{T}$  to  $T'$ , the vertex and edge groups are nonincreasing. Otherwise, the required properties of  $T'$  are immediately satisfied by [Theorem 3.1](#) (see [Figure 7](#)). □

Finally, we can give our description of the decompositions of virtually free groups as amalgamated free products or HNN extensions.

*Proof of Theorem 1.6.* We prove this result by successively applying [Corollary 4.1](#) until some desirable terminating condition is met. Virtually free groups have no

one-ended subgroups, so we will always be able to apply our corollary; furthermore, virtually free groups are finitely presented. It now follows by Dunwoody accessibility [Dunwoody 1985] that there are no infinite chains  $C_1 \succ C_2 \succ \cdots$  of virtually free groups (recall Definition 1.3), and that all such chains must terminate with finite groups.

**First construction** (pass to relatively one-ended vertex subgroups): Let  $T$  be a  $G$ -tree with one edge orbit  $Ge$  with  $G_e$  infinite. By accessibility, we may pass to a tree  $T^{(2)}$  obtained by blowing up some vertices  $v$  of  $T$  to trees  $\check{T}_v$  such that the vertex groups of  $\check{T}_v$  are either finite or one-ended relative to the stabilizers  $G_f$  of the incident edges  $f \ni v$ . If possible, we take  $T^{(1)} \subset T^{(2)}$  to be an infinite connected subtree obtained by deleting edges with finite stabilizers, and we set  $G^{(1)} = G_{T^{(1)}}$ , the setwise stabilizer. Note that the vertex groups of  $T^{(1)}$  are  $\preceq$  the vertex groups of  $T$ , and vertex groups are one-ended relative to the incident edge groups.

**Second construction** (pass to smaller edge groups): The second construction utilizes Corollary 4.1. If  $T_i$  is a  $G_i$ -tree with one edge orbit whose vertex groups are one-ended relative to the incident edge groups, we first apply Theorem 3.1 to blow up a vertex  $v \in \text{Vertices}(T_i)$ , and find ourselves in case (2) of the theorem. If  $\check{T}_v$  has a finite edge group then  $G_v$  is not one-ended relative to the incident edge groups, contradicting our assumption. By Corollary 4.1 we can collapse an edge of the blowup of  $T_i$  to get a cleaving  $T'_i$  that has at most two edge orbits, with edge groups  $\prec$  the edge groups of  $T_i$ . The new vertex groups are also  $\preceq$  the old vertex groups. If there are two edge orbits, then we obtain  $T_{i+1} \subset T'_i$  as a maximal subtree containing only one edge orbit and set  $G_{i+1} = (G_i)_{T_{i+1}}$ , the setwise stabilizer. (See Figure 7.) If  $T'$  already has only one edge orbit then  $T_{i+1} = T_i$  and  $G_{i+1} = G_i$ .

In both constructions, we pass to subgroups that split as graphs of groups such that the edge groups and vertex groups are  $\preceq$  the edge and vertex groups of the original splitting of the overgroup.

We start with the amalgamated free product case. Let  $T = T_0$  be the Bass–Serre tree corresponding to the splitting given in (1) of the statement of Theorem 1.6. Take the blowup  $T_0^{(2)}$  obtained from the first construction. If one of the vertex groups of this blowup coincides with an incident edge group then we are done. Otherwise, we may pass to the  $G^{(1)}$ -tree  $T_0^{(1)}$ , which still has one edge orbit and two vertex orbits, and whose vertex groups are one-ended relative to the incident edge groups. Furthermore, because the new vertex groups are  $\preceq$  the vertex groups of  $T$ , if the statement of the theorem holds for  $G^{(1)}$  and the splitting corresponding to its action on  $T_0^{(1)}$  (which is also an amalgamated free product), then the statement also holds for  $G$  and the splitting corresponding to its action on  $T$ .

We can now apply our second construction to the  $G_0^{(1)}$ -tree  $T_0^{(1)}$  to obtain a  $G_1$ -tree  $T_1$ , which again must have one edge orbit and two vertex orbits. Furthermore,

for the (conjugacy class of the) edge group, we have a proper containment  $C_1 < C$ . Again, because the vertex groups of  $T_1$  are  $\preceq$  the vertex groups of  $T_0^{(1)}$ , if the Theorem holds for this subgroup, it holds for  $G$ .

We repeatedly apply our construction, thus obtaining a sequence of groups that split as amalgamated free products. With each iteration of the second construction, we pass to a smaller edge group. Hence, by accessibility, eventually there is a subgroup  $G_i$  acting on  $T_i^{(2)}$  (see the first construction) such that the vertex groups split as graphs of groups with finite edge groups and one of the incident edge groups coincides with the vertex group. Since  $\preceq$  is transitive, (1) of Theorem 1.6 is satisfied.

We now tackle the HNN extension case. The proof proceeds in the same way. We repeatedly blow up, cleave, and pass to subtrees, the main difference being that the  $G$ -tree  $T$  has only one vertex orbit. If at some point one of the trees  $T_i$  or  $T_i^{(1)}$  has two vertex orbits, then these vertex groups are vertex groups of a splitting of the vertex group of  $T_{i-1}$  with finite edge groups. It follows that if  $T_i$  satisfies (1) of Theorem 1.6, then  $T_{i-1}$  satisfies (2) of Theorem 1.6, and thus by transitivity of  $\preceq$ , so must our original splitting  $T$ . Otherwise, the proof goes through identically.  $\square$

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