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ACTION OF LONGEST ELEMENT ON A HECKE ALGEBRA CELL MODULE

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Dedicated to the memory of Robert Steinberg

By a result of Mathas, the basis element T_{w_0} of the Hecke algebra of a finite Coxeter group acts in the canonical basis of a cell module as a permutation matrix times plus or minus a power of v. We generalize this result to the unequal parameter case. We also show that the image of T_{w_0} in the corresponding asymptotic Hecke algebra is given by a simple formula.

Introduction

0.1. The Hecke algebra $\mathcal{H}$ (over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, v an indeterminate) of a finite Coxeter group W has two bases as an $\mathcal{A}$-module: the standard basis $\{T_x; x \in W\}$ and the basis $\{C_x; x \in W\}$ introduced in [Kazhdan and Lusztig 1979]. The second basis determines a decomposition of W into two-sided cells and a partial order for the set of two-sided cells, see [Kazhdan and Lusztig 1979]. Let $l \to \mathbb{N}$ be the length function, let w_0 be the longest element of W and let $\boldsymbol{c}$ be a two-sided cell. Let a (resp. a') be the value of the $\boldsymbol{a}$-function [Lusztig 2003, 13.4] on $\boldsymbol{c}$ (resp. on $w_0\boldsymbol{c}$). The following result was proved by Mathas [1996].

(a) *There exists a unique permutation $u \mapsto u^*$ of $\boldsymbol{c}$ such that for any $u \in \boldsymbol{c}$ we have $T_{w_0}(-1)^{l(u)}C_u = (-1)^{l(w_0)+a'}v^{-a+a'}(-1)^{l(u^*)}C_{u^*}$ plus an $\mathcal{A}$-linear combination of elements $C_{u'}$ with u' in a two-sided cell strictly smaller than $\boldsymbol{c}$. Moreover, for any $u \in \boldsymbol{c}$ we have $(u^*)^* = u$.*

A related (but weaker) result appears in [Lusztig 1984, (5.12.2)]. A result similar to (a) which concerns canonical bases in representations of quantum groups appears in [Lusztig 1990, Corollary 5.9]; now, in the case where W is of type A, (a) can be deduced from [loc. cit.] using the fact that irreducible representations of the Hecke algebra of type A (with their canonical bases) can be realized as 0-weight spaces of certain irreducible representations of a quantum group with their canonical bases.

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As R. Bezrukavnikov pointed out to the author, (a) specialized for $v = 1$ (in the group algebra of W instead of $\mathcal{H}$) and assuming that W is crystallographic can be deduced from [Bezrukavnikov et al. 2012, Proposition 4.1] (a statement about Harish-Chandra modules), although it is not explicitly stated there.

In this paper we shall prove a generalization of (a) which applies to the Hecke algebra associated to W and any weight function assumed to satisfy the properties P1–P15 in [Lusztig 2003, §14], see Theorem 2.3; (a) corresponds to the special case where the weight function is equal to the length function. As an application we show that the image of T_{w_0} in the asymptotic Hecke algebra is given by a simple formula (see Corollary 2.8).

0.2. ***Notation.*** W is a finite Coxeter group; the set of simple reflections is denoted by S. We shall adopt many notations of [Lusztig 2003]. Let $\le$ be the standard partial order on W. Let $l \to \mathbb{N}$ be the length function of W and let $L \to \mathbb{N}$ be a weight function (see [Lusztig 2003, 3.1]), that is, a function such that $L(ww') = L(w) + L(w')$ for any w, w' in W such that $l(ww') = l(w) + l(w')$; we assume that $L(s) > 0$ for any $s \in S$. Let $w_0, \mathcal{A}$ be as in Section 0.1 and let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ associated to W, L as in [Lusztig 2003, 3.2]; we shall assume that properties P1–P15 in [Lusztig 2003, §14] are satisfied. (This holds automatically if $L = l$ by [Lusztig 2003, §15] using the results of [Elias and Williamson 2014]. This also holds in the quasisplit case, see [Lusztig 2003, §16].) We have $\mathcal{A} \subset \mathcal{A}' \subset K$ where $\mathcal{A}' = \mathbb{C}[v, v^{-1}]$, $K = \mathbb{C}(v)$. Let $\mathcal{H}_K = K \otimes_{\mathcal{A}} \mathcal{H}$ (a K-algebra). Recall that $\mathcal{H}$ has an $\mathcal{A}$-basis $\{T_x; x \in W\}$, see [Lusztig 2003, 3.2] and an $\mathcal{A}$-basis $\{c_x; x \in W\}$, see [Lusztig 2003, 5.2]. For $x \in W$ we have $c_x = \sum_{y \in W} p_{y,x} T_y$ and $T_x = \sum_{y \in W} (-1)^{l(xy)} p_{w_0 x, w_0 y} c_y$ (see [Lusztig 2003, 11.4]) where $p_{x,x} = 1$ and $p_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $y \ne x$. We define preorders $\le_{\mathcal{L}}, \le_{\mathcal{R}}, \le_{\mathcal{LR}}$ on W in terms of $\{c_x; x \in W\}$ as in [Lusztig 2003, 8.1]. Let $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ be the corresponding equivalence relations on W, see [Lusztig 2003, 8.1] (the equivalence classes are called left cells, right cells, two-sided cells). Let $\bar{\ }: \mathcal{A} \to \mathcal{A}$ be the ring involution such that $\overline{v^n} = v^{-n}$ for $n \in \mathbb{Z}$. Let $\bar{\ }: \mathcal{H} \to \mathcal{H}$ be the ring involution such that $\overline{f T_x} = \bar{f} T_{x^{-1}}^{-1}$ for $x \in W$, $f \in \mathcal{A}$. For $x \in W$ we have $\overline{c_x} = c_x$. Let $h \mapsto h^\dagger$ be the algebra automorphism of $\mathcal{H}$ or of $\mathcal{H}_K$ given by $T_x \mapsto (-1)^{l(x)} T_{x^{-1}}^{-1}$ for all $x \in W$, see [Lusztig 2003, 3.5]. Then the basis $\{c_x^\dagger; x \in W\}$ of $\mathcal{H}$ is defined. (In the case where $L = l$, for any x we have $c_x^\dagger = (-1)^{l(x)} C_x$ where C_x is as in Section 0.1.) Let $h \mapsto h^\flat$ be the algebra antiautomorphism of $\mathcal{H}$ given by $T_x \mapsto T_{x^{-1}}$ for all $x \in W$, see [Lusztig 2003, 3.5]; for $x \in W$ we have $c_x^\flat = c_{x^{-1}}$, see [Lusztig 2003, 5.8]. For $x, y \in W$ we have $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$, $c_x^\dagger c_y^\dagger = \sum_{z \in W} h_{x,y,z} c_z^\dagger$, where $h_{x,y,z} \in \mathcal{A}$. For any $z \in W$ there is a unique number $\boldsymbol{a}(z) \in \mathbb{N}$ such that for any x, y in W we have

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\boldsymbol{a}(z)} + \text{strictly smaller powers of } v,$$

where $g_{x,y,z^{-1}} \in \mathbb{Z}$ and $g_{x,y,z^{-1}} \neq 0$ for some x, y in W. We have also

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{-\boldsymbol{a}(z)} + \text{strictly larger powers of } v.$$

Moreover $z \mapsto \boldsymbol{a}(z)$ is constant on any two-sided cell. The free abelian group J with basis $\{t_w;\, w \in W\}$ has an associative ring structure given by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$; it has a unit element of the form $\sum_{d \in \mathcal{D}} n_d t_d$ where $\mathcal{D}$ is a subset of W consisting of certain elements with square 1 and $n_d = \pm 1$. Moreover for $d \in \mathcal{D}$ we have $n_d = \gamma_{d,d,d}$.

For any $x \in W$ there is a unique element $d_x \in \mathcal{D}$ such that $x \sim_{\mathcal{L}} d_x$. For a commutative ring R with 1 we set $J_R = R \otimes J$ (an R-algebra).

There is a unique $\mathcal{A}$-algebra homomorphism $\phi : \mathcal{H} \to J_{\mathcal{A}}$ such that $\phi(c_x^\dagger) = \sum_{d \in \mathcal{D}, z \in W; d_z = d} h_{x,d,z} n_d t_z$ for any $x \in W$. After applying $\mathbb{C} \otimes_{\mathcal{A}}$ to ϕ (we regard $\mathbb{C}$ as an $\mathcal{A}$-algebra via $v \mapsto 1$), ϕ becomes a $\mathbb{C}$-algebra isomorphism $\phi_{\mathbb{C}} : \mathbb{C}[W] \xrightarrow{\sim} J_{\mathbb{C}}$ (see [Lusztig 2003, 20.1(e)]). After applying $K \otimes_{\mathcal{A}}$ to ϕ, ϕ becomes a K-algebra isomorphism $\phi_K : \mathcal{H}_K \xrightarrow{\sim} J_K$ (see [Lusztig 2003, 20.1(d)]).

For any two-sided cell $\boldsymbol{c}$ let $\mathcal{H}^{\leq \boldsymbol{c}}$ (resp. $\mathcal{H}^{< \boldsymbol{c}}$) be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{c_x^\dagger,\, x \in W,\, x \leq_{\mathcal{LR}} x' \text{ for some } x' \in \boldsymbol{c}\}$ (resp. $\{c_x^\dagger,\, x \in W,\, x <_{\mathcal{LR}} x' \text{ for some } x' \in \boldsymbol{c}\}$). Note that $\mathcal{H}^{\leq \boldsymbol{c}}$, $\mathcal{H}^{< \boldsymbol{c}}$ are two-sided ideals in $\mathcal{H}$. Hence $\mathcal{H}^{\boldsymbol{c}} := \mathcal{H}^{\leq \boldsymbol{c}} / \mathcal{H}^{< \boldsymbol{c}}$ is an $(\mathcal{H}, \mathcal{H})$-bimodule. It has an $\mathcal{A}$-basis $\{c_x^\dagger,\, x \in \boldsymbol{c}\}$. Let $J^{\boldsymbol{c}}$ be the subgroup of J spanned by $\{t_x;\, x \in \boldsymbol{c}\}$. This is a two-sided ideal of J. Similarly, $J_{\mathbb{C}}^{\boldsymbol{c}} := \mathbb{C} \otimes J^{\boldsymbol{c}}$ is a two-sided ideal of $J_{\mathbb{C}}$ and $J_K^{\boldsymbol{c}} := K \otimes J^{\boldsymbol{c}}$ is a two-sided ideal of J_K.

We write $E \in \operatorname{Irr} W$ whenever E is a simple $\mathbb{C}[W]$-module. We can view E as a (simple) $J_{\mathbb{C}}$-module $E_\spadesuit$ via the isomorphism $\phi_{\mathbb{C}}^{-1}$. Then the (simple) J_K-module $K \otimes_{\mathbb{C}} E_\spadesuit$ can be viewed as a (simple) $\mathcal{H}_K$-module E_v via the isomorphism ϕ_K. Let $E^\dagger$ be the simple $\mathbb{C}[W]$-module which coincides with E as a $\mathbb{C}$-vector space but with the w action on $E^\dagger$ (for $w \in W$) being $(-1)^{l(w)}$ times the w-action on E. Let $\boldsymbol{a}_E \in \mathbb{N}$ be as in [Lusztig 2003, 20.6(a)].

1. Preliminaries

1.1. Let $\sigma : W \to W$ be the automorphism given by $w \mapsto w_0 w w_0$; it satisfies $\sigma(S) = S$ and it extends to a $\mathbb{C}$-algebra isomorphism $\sigma : \mathbb{C}[W] \to \mathbb{C}[W]$. For $s \in S$ we have $l(w_0) = l(w_0 s) + l(s) = l(\sigma(s)) + l(\sigma(s) w_0)$ hence $L(w_0) = L(w_0 s) + L(s) = L(\sigma(s)) + L(\sigma(s) w_0) = L(\sigma(s)) + L(w_0 s)$ so that $L(\sigma(s)) = L(s)$. It follows that $L(\sigma(w)) = L(w)$ for all $w \in W$ and that we have an $\mathcal{A}$-algebra automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ where $\sigma(T_w) = T_{\sigma(w)}$ for any $w \in W$. This extends to a K-algebra isomorphism $\sigma : \mathcal{H}_K \to \mathcal{H}_K$. We have $\sigma(c_w) = c_{\sigma(w)}$ for any $w \in W$. For any $h \in \mathcal{H}$ we have $\sigma(h^\dagger) = (\sigma(h))^\dagger$. Hence we have $\sigma(c_w^\dagger) = c_{\sigma(w)}^\dagger$ for any $w \in W$. We have $h_{\sigma(x),\sigma(y),\sigma(z)} = h_{x,y,z}$ for all $x, y, z \in W$. It follows that $\boldsymbol{a}(\sigma(w)) = \boldsymbol{a}(w)$ for all $w \in W$ and $\gamma_{\sigma(x),\sigma(y),\sigma(z)} = \gamma_{x,y,z}$ for all $x, y, z \in W$ so that we have a ring isomorphism $\sigma : J \to J$ where $\sigma(t_w) = t_{\sigma(w)}$ for any $w \in W$. This extends to an $\mathcal{A}$-algebra isomorphism $\sigma : J_{\mathcal{A}} \to J_{\mathcal{A}}$, to a $\mathbb{C}$-algebra isomorphism $\sigma : J_{\mathbb{C}} \to J_{\mathbb{C}}$ and to a K-algebra isomorphism $\sigma : J_K \to J_K$. From the definitions we see that $\phi : \mathcal{H} \to J_{\mathcal{A}}$ (see Section 0.2) satisfies $\phi\sigma = \sigma\phi$. Hence $\phi_{\mathbb{C}}$ satisfies $\phi_{\mathbb{C}}\sigma = \sigma\phi_{\mathbb{C}}$ and ϕ_K satisfies $\phi_K\sigma = \sigma\phi_K$. We show:

(a) *For $h \in \mathcal{H}$ we have $\sigma(h) = T_{w_0} h T_{w_0}^{-1}$.*

It is enough to show this for h running through a set of algebra generators of $\mathcal{H}$. Thus we can assume that $h = T_s^{-1}$ with $s \in S$. We must show that $T_{\sigma(s)}^{-1} T_{w_0} = T_{w_0} T_s^{-1}$: both sides are equal to $T_{\sigma(s)w_0} = T_{w_0 s}$.

Lemma 1.2. *For any $x \in W$ we have $\sigma(x) \sim_{\mathcal{LR}} x$.*

From 1.1(a) we deduce that $T_{w_0} c_x T_{w_0}^{-1} = c_{\sigma(x)}$. In particular, $\sigma(x) \leq_{\mathcal{LR}} x$. Replacing x by $\sigma(x)$ we obtain $x \leq_{\mathcal{LR}} \sigma(x)$. The lemma follows.

1.3. Let $E \in \mathrm{Irr} W$. We define $\sigma_E : E \to E$ by $\sigma_E(e) = w_0 e$ for $e \in E$. We have $\sigma_E^2 = 1$. For $e \in E$, $w \in W$, we have $\sigma_E(we) = \sigma(w)\sigma_E(e)$. We can view σ_E as a vector space isomorphism $E_\spadesuit \xrightarrow{\sim} E_\spadesuit$. For $e \in E_\spadesuit$, $w \in W$ we have $\sigma_E(t_w e) = t_{\sigma(w)}\sigma_E(e)$. Now $\sigma_E : E_\spadesuit \to E_\spadesuit$ defines by extension of scalars a vector space isomorphism $E_v \to E_v$ denoted again by σ_E. It satisfies $\sigma_E^2 = 1$. For $e \in E_v$, $w \in W$ we have $\sigma_E(T_w e) = T_{\sigma(w)}\sigma_E(e)$.

Lemma 1.4. *Let $E \in \mathrm{Irr} W$. There is a unique (up to multiplication by a scalar in $K - \{0\}$) vector space isomorphism $g : E_v \to E_v$ such that $g(T_w e) = T_{\sigma(w)} g(e)$ for all $w \in W$, $e \in E_v$. We can take for example $g = T_{w_0} : E_v \to E_v$ or $g = \sigma_E : E_v \to E_v$. Hence $T_{w_0} = \lambda_E \sigma_E : E_v \to E_v$ where $\lambda_E \in K - \{0\}$.*

The existence of g is clear from the second sentence of the lemma. If g' is another isomorphism $g' : E_v \to E_v$ such that $g'(T_w e) = T_{\sigma(w)} g'(e)$ for all $w \in W$, $e \in E_v$, then for any $e \in E_v$ we have $g^{-1} g'(T_w e) = g^{-1} T_{\sigma(w)} g'(e) = T_w g^{-1} g'(e)$ and using Schur's lemma we see that $g^{-1} g'$ is a scalar. This proves the first sentence of the lemma hence the third sentence of the lemma.

1.5. Let $E \in \mathrm{Irr} W$. We have

(a) $$\sum_{x \in W} \mathrm{tr}(T_x, E_v)\, \mathrm{tr}(T_{x^{-1}}, E_v) = f_{E_v} \dim(E)$$

where $f_{E_v} \in \mathcal{A}'$ is of the form

(b) $$f_{E_v} = f_0 v^{-2\boldsymbol{a}_E} + \text{strictly higher powers of } v$$

and $f_0 \in \mathbb{C} - \{0\}$. (See [Lusztig 2003, 19.1(e), 20.1(c), 20.7].)

From Lemma 1.4 we see that $\lambda_E^{-1}T_{w_0}$ acts on E_v as σ_E. Using [Lusztig 2005, 34.14(e)] with $c=\lambda_E^{-1}T_{w_0}$ (an invertible element of $\mathcal{H}_K$) we see that

(c) $$\sum_{x\in W}\operatorname{tr}(T_x\sigma_E, E_v)\operatorname{tr}(\sigma_E^{-1}T_{x^{-1}}, E_v)=f_{E_v}\dim(E).$$

Lemma 1.6. *Let $E\in\operatorname{Irr}W$. We have $\lambda_E=v^{n_E}$ for some $n_E\in\mathbb{Z}$.*

For any $x\in W$ we have

$$\operatorname{tr}(\sigma_E c_x^\dagger, E_v)=\sum_{d\in\mathcal{D},z\in W;d=d_z}h_{x,d,z}n_d\operatorname{tr}(\sigma_E t_z, E_\spadesuit)\in\mathcal{A}'$$

since $\operatorname{tr}(\sigma_E t_z, E_\spadesuit)\in\mathbb{C}$. It follows that $\operatorname{tr}(\sigma_E h, E_v)\in\mathcal{A}'$ for any $h\in\mathcal{H}$. In particular, both $\operatorname{tr}(\sigma_E T_{w_0}, E_v)$ and $\operatorname{tr}(T_{w_0}^{-1}\sigma_E, E_v)$ belong to $\mathcal{A}'$. Thus $\lambda_E\dim E$ and $\lambda_E^{-1}\dim E$ belong to $\mathcal{A}'$ so that $\lambda_E=bv^n$ for some $b\in\mathbb{C}-\{0\}$ and $n\in\mathbb{Z}$. From the definitions we have $\lambda_E|_{v=1}=1$ (for $v=1$, T_{w_0} becomes w_0) hence $b=1$. The lemma is proved.

Lemma 1.7. *Let $E\in\operatorname{Irr}W$. There exists $\epsilon_E\in\{1,-1\}$ such that for any $x\in W$ we have*

(a) $$\operatorname{tr}(\sigma_{E^\dagger}T_x, (E^\dagger)_v)=\epsilon_E(-1)^{l(x)}\operatorname{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v).$$

Let $(E_v)^\dagger$ be the $\mathcal{H}_K$-module with underlying vector space E_v such that the action of $h\in\mathcal{H}_K$ on $(E_v)^\dagger$ is the same as the action of $h^\dagger$ on E_v. From the proof in [Lusztig 2003, 20.9] we see that there exists an isomorphism of $\mathcal{H}_K$-modules $b:(E_v)^\dagger\xrightarrow{\sim}(E^\dagger)_v$. Let $\iota:(E_v)^\dagger\to(E_v)^\dagger$ be the vector space isomorphism which corresponds under b to $\sigma_{E^\dagger}:(E^\dagger)_v\to(E^\dagger)_v$. Then we have $\operatorname{tr}(\sigma_{E^\dagger}T_x,(E^\dagger)_v)=\operatorname{tr}(\iota T_x,(E_v)^\dagger)$. It is enough to prove that $\iota=\pm\sigma_E$ as a K-linear map of the vector space $E_v=(E_v)^\dagger$ into itself. From the definition we have $\iota(T_w e)=T_{\sigma(w)}\iota(e)$ for all $w\in W, e\in(E_v)^\dagger$. Hence $(-1)^{l(w)}\iota(T_{w^{-1}}^{-1}e)=(-1)^{l(w)}T_{\sigma(w^{-1})}^{-1}\iota(e)$ for all $w\in W, e\in E_v$. It follows that $\iota(he)=(-1)^{l(w)}T_{\sigma(h)}\iota(e)$ for all $h\in\mathcal{H}, e\in E_v$. Hence $\iota(T_w e)=T_{\sigma(w)}\iota(e)$ for all $w\in W, e\in E_v$. By the uniqueness in Lemma 1.4 we see that $\iota=\epsilon_E\sigma_E:E_v\to E_v$ where $\epsilon_E\in K-\{0\}$. Since $\iota^2=1$, $\sigma_E^2=1$, we see that $\epsilon_E=\pm1$. The lemma is proved.

Lemma 1.8. *Let $E\in\operatorname{Irr}W$. We have $n_E=-\boldsymbol{a}_E+\boldsymbol{a}_{E^\dagger}$.*

For $x\in W$ we have (using Lemmas 1.4 and 1.6)

(a) $$\operatorname{tr}(T_{w_0x}, E_v)=\operatorname{tr}(T_{w_0}T_{x^{-1}}^{-1}, E_v)=v^{n_E}\operatorname{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v).$$

Making a change of variable $x \mapsto w_0 x$ in 1.5(a) and using that $T_{x^{-1}w_0} = T_{w_0\sigma(x)^{-1}}$ we obtain

$$\begin{aligned} f_{E_v} \dim(E) &= \sum_{x\in W} \operatorname{tr}(T_{w_0 x}, E_v) \operatorname{tr}(T_{w_0\sigma(x)^{-1}}, E_v) \\ &= v^{2n_E} \sum_{x\in W} \operatorname{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v) \operatorname{tr}(\sigma_E T_{\sigma(x)}^{-1}, E_v). \end{aligned}$$

Using now Lemma 1.7 and the equality $l(x) = l(\sigma(x^{-1}))$ we obtain

$$\begin{aligned} f_{E_v} \dim(E) &= v^{2n_E} \sum_{x\in W} \operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) \operatorname{tr}(\sigma_{E^\dagger} T_{\sigma(x^{-1})}, (E^\dagger)_v) \\ &= v^{2n_E} \sum_{x\in W} \operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) \operatorname{tr}(T_{\xi^{-1}} \sigma_{E^\dagger}, (E^\dagger)_v) \\ &= v^{2n_E} f_{(E^\dagger)_v} \dim(E^\dagger). \end{aligned}$$

(The last step uses 1.5(c) for $E^\dagger$ instead of E.) Thus we have $f_{E_v} = v^{2n_E} f_{(E^\dagger)_v}$. The left-hand side is as in 1.5(b) and similarly the right-hand side of the form

$$f'_0 v^{2n_E - 2\boldsymbol{a}_{E^\dagger}} + \text{strictly higher powers of } v$$

where $f_0, f'_0 \in \mathbb{C} - \{0\}$. It follows that $-2\boldsymbol{a}_E = 2n_E - 2\boldsymbol{a}_{E^\dagger}$. The lemma is proved.

Lemma 1.9. *Let $E \in \operatorname{Irr} W$ and let $x \in W$. We have*

(a) $$\operatorname{tr}(T_x, E_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(t_x, E_\spadesuit) \mod v^{-\boldsymbol{a}_E+1}\mathbb{C}[v],$$

(b) $$\operatorname{tr}(\sigma_E T_x, E_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(\sigma_E t_x, E_\spadesuit) \mod v^{-\boldsymbol{a}_E+1}\mathbb{C}[v].$$

For a proof of (a), see [Lusztig 2003, 20.6(b)]. We now give a proof of (b) along the same lines as that of (a). There is a unique two sided cell $\boldsymbol{c}$ such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}$. Let $a = \boldsymbol{a}(z)$ for all $z \in \boldsymbol{c}$. By [Lusztig 2003, 20.6(c)] we have $a = \boldsymbol{a}_E$. From the definition of c_x we see that $T_x = \sum_{y\in W} f_y c_y$, where $f_x = 1$ and $f_y \in v^{-1}\mathbb{Z}[v^{-1}]$ for $y \neq x$. Applying $\dagger$ we obtain $(-1)^{l(x)} T_{x^{-1}}^{-1} = \sum_{y\in W} f_y c_y^\dagger$; applying $\bar{\ }$ we obtain $(-1)^{l(x)} T_x = \sum_{y\in W} \bar{f}_y c_y^\dagger$. Thus we have

$$(-1)^{l(x)} \operatorname{tr}(\sigma_E T_x, E_v) = \sum_{y\in W} \bar{f}_y \operatorname{tr}(\sigma_E c_y^\dagger, E_v) = \sum_{\substack{y,z\in W \\ d\in\mathcal{D};\, d=d_z}} \bar{f}_y h_{y,d,z} n_d \operatorname{tr}(\sigma_E t_z, E_\spadesuit).$$

In the last sum we can assume that $z \in \boldsymbol{c}$ and $d \in \boldsymbol{c}$ so that $h_{y,d,z} = \gamma_{y,d,z^{-1}} v^{-a}$ mod $v^{-a+1}\mathbb{Z}[v]$. Since $\bar{f}_x = 1$ and $\bar{f}_y \in v\mathbb{Z}[v]$ for all $y \neq x$ we see that

$$(-1)^{l(x)} \operatorname{tr}(\sigma_E T_x, E_v) = \sum_{\substack{z\in\boldsymbol{c} \\ d\in\mathcal{D}\cap\boldsymbol{c}}} \gamma_{x,d,z^{-1}} n_d v^{-a} \operatorname{tr}(\sigma_E t_z, E_\spadesuit) \mod v^{-a+1}\mathbb{C}[v].$$

If $x \notin \boldsymbol{c}$ then $\gamma_{x,d,z^{-1}} = 0$ for all d, z in the sum so that $\operatorname{tr}(\sigma_E T_x, E_v) = 0$; we have also $\operatorname{tr}(\sigma_E t_x, E_\spadesuit) = 0$ and the desired formula follows. We now assume that $x \in \boldsymbol{c}$. Then for d, z as above we have $\gamma_{x,d,z^{-1}} = 0$ unless $x = z$ and $d = d_x$ in which case $\gamma_{x,d,z^{-1}} n_d = 1$. Thus (b) holds again. The lemma is proved.

Lemma 1.10. *Let $E \in \operatorname{Irr} W$. Let $\boldsymbol{c}$ be the unique two sided cell such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}$. Let $\boldsymbol{c}'$ be the unique two sided cell such that $t_z|_{(E^\dagger)_\spadesuit} = 0$ for $z \in W - \boldsymbol{c}'$. We have $\boldsymbol{c}' = w_0 \boldsymbol{c}$.*

Using 1.8(a) and 1.7(a) we have

(a) $$\operatorname{tr}(T_{w_0 x}, E_v) = v^{n_E} \operatorname{tr}(\sigma_E T^{-1}_{x^{-1}}, E_v) = v^{n_E} \epsilon_E (-1)^{l(x)} \operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v).$$

Using 1.9(a) for E and 1.9(b) for $E^\dagger$ we obtain

$$\operatorname{tr}(T_{w_0 x}, E_v) = (-1)^{l(w_0 x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) \mod v^{-\boldsymbol{a}_E + 1}\mathbb{C}[v],$$
$$\operatorname{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) = (-1)^{l(x)} v^{-\boldsymbol{a}_{E^\dagger}} \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit) \mod v^{-\boldsymbol{a}_{E^\dagger}+1}\mathbb{C}[v].$$

Combining with (a) we obtain

$$\begin{aligned}(-1)^{l(w_0 x)} v^{-\boldsymbol{a}_E} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) &+ \text{strictly higher powers of } v\\ = v^{n_E} \epsilon_E v^{-\boldsymbol{a}_{E^\dagger}} \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit) &+ \text{strictly higher powers of } v.\end{aligned}$$

Using the equality $n_E = -\boldsymbol{a}_E + \boldsymbol{a}_{E^\dagger}$ (see Lemma 1.8) we deduce

$$(-1)^{l(w_0 x)} \operatorname{tr}(t_{w_0 x}, E_\spadesuit) = \epsilon_E \operatorname{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit).$$

Now we can find $x \in W$ such that $\operatorname{tr}(t_{w_0 x}, E_\spadesuit) \neq 0$ and the previous equality shows that $t_x|_{(E^\dagger)_\spadesuit} \neq 0$. Moreover from the definition we have $w_0 x \in \boldsymbol{c}$ and $x \in \boldsymbol{c}'$ so that $w_0 \boldsymbol{c} \cap \boldsymbol{c}' \neq \varnothing$. Since $w_0 \boldsymbol{c}$ is a two-sided cell (see [Lusztig 2003, 11.7(d)]) it follows that $w_0 \boldsymbol{c} = \boldsymbol{c}'$. The lemma is proved.

Lemma 1.11. *Let $\boldsymbol{c}$ be a two-sided cell of W. Let $\boldsymbol{c}'$ be the two-sided cell $w_0 \boldsymbol{c} = \boldsymbol{c} w_0$ (see Lemma 1.2). Let $a = \boldsymbol{a}(x)$ for any $x \in \boldsymbol{c}$; let $a' = \boldsymbol{a}(x')$ for any $x' \in \boldsymbol{c}'$. The K-linear map $J^{\boldsymbol{c}}_K \to J^{\boldsymbol{c}}_K$ given by $\xi \mapsto \phi(v^{a-a'} T_{w_0})\xi$ (left multiplication in J_K) is obtained from a $\mathbb{C}$-linear map $J^{\boldsymbol{c}}_{\mathbb{C}} \to J^{\boldsymbol{c}}_{\mathbb{C}}$ (with square 1) by extension of scalars from $\mathbb{C}$ to K.*

We can find a direct sum decomposition $J^{\boldsymbol{c}}_{\mathbb{C}} = \oplus_{i=1}^m E^i$ where E^i are simple left ideals of $J_{\mathbb{C}}$ contained in $J^{\boldsymbol{c}}_{\mathbb{C}}$. We have $J^{\boldsymbol{c}}_K = \oplus_{i=1}^m K \otimes E^i$. It is enough to show that for any i, the K-linear map $K \otimes E^i \to K \otimes E^i$ given by the action of $\phi(v^{a-a'} T_{w_0})$ in the left J_K-module structure of $K \otimes E^i$ is obtained from a $\mathbb{C}$-linear map $E^i \to E^i$ (with square 1) by extension of scalars from $\mathbb{C}$ to K. We can find $E \in \operatorname{Irr} W$ such that E^i is isomorphic to $E_\spadesuit$ as a $J_{\mathbb{C}}$-module. It is then enough to show that the action of $v^{a-a'} T_{w_0}$ in the left $\mathcal{H}_K$-module structure of E_v is obtained from the map $\sigma_E : E \to E$ by extension of scalars from $\mathbb{C}$ to K. This follows from the equality

$v^{a-a'}T_{w_0} = \sigma_E : E_v \to E_v$ (since σ_E is obtained by extension of scalars from a $\mathbb{C}$-linear map $E \to E$ with square 1) provided that we show that $-n_E = a - a'$. Since $n_E = -\boldsymbol{a}_E + \boldsymbol{a}_{E^\dagger}$ (see Lemma 1.8) it is enough to show that $a = \boldsymbol{a}_E$ and $a' = \boldsymbol{a}_{E^\dagger}$. The equality $a = \boldsymbol{a}_E$ follows from [Lusztig 2003, 20.6(c)]. The equality $a' = \boldsymbol{a}_{E^\dagger}$ also follows from [Lusztig 2003, 20.6(c)] applied to $E^\dagger$, $\boldsymbol{c}' = w_0\boldsymbol{c}$ instead of $E, \boldsymbol{c}$ (see Lemma 1.10). The lemma is proved.

Lemma 1.12. *In the setup of Lemma 1.11 we have*

(a) $$\phi(v^{a-a'}T_{w_0})t_x = \sum_{x' \in \boldsymbol{c}} m_{x',x} t_{x'}$$

and

(b) $$\phi(v^{2a-2a'}T_{w_0}^2)t_x = t_x$$

for any $x \in \boldsymbol{c}$, where $m_{x',x} \in \mathbb{Z}$.

Now (b) and the fact that (a) holds with $m_{x',x} \in \mathbb{C}$ is just a restatement of Lemma 1.11. Since $\phi(v^{a-a'}T_{w_0}) \in J_{\mathcal{A}}$ we have also $m_{x',x} \in \mathcal{A}$. We now use that $\mathcal{A} \cap \mathbb{C} = \mathbb{Z}$ and the lemma follows.

Lemma 1.13. *In the setup of Lemma 1.11 we have for any $x \in \boldsymbol{c}$ the equalities*

(a) $$v^{a-a'}T_{w_0}c_x^\dagger = \sum_{x' \in \boldsymbol{c}} m_{x',x} c_{x'}^\dagger$$

and

(b) $$v^{2a-2a'}T_{w_0}^2 c_x^\dagger = c_x^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$, where $m_{x',x} \in \mathbb{Z}$ are the same as in Lemma 1.12. Moreover, if $m_{x',x} \neq 0$ then $x' \sim_{\mathcal{L}} x$.

The first sentence follows from Lemma 1.12 using [Lusztig 2003, 18.10(a)]. Clearly, if $m_{x',x} \neq 0$ then $x' \leq_{\mathcal{L}} x$, which together with $x' \sim_{\mathcal{LR}} x$ implies $x' \sim_{\mathcal{L}} x$.

2. The main results

2.1. In this section we fix a two-sided cell $\boldsymbol{c}$ of W; a, a' are as in Lemma 1.11. We define an $\mathcal{A}$-linear map $\theta : \mathcal{H}^{\leq \boldsymbol{c}} \to \mathcal{A}$ by $\theta(c_x^\dagger) = 1$ if $x \in \mathcal{D} \cap \boldsymbol{c}$, $\theta(c_x^\dagger) = 0$ if $x \leq_{\mathcal{LR}} x'$ for some $x' \in \boldsymbol{c}$ and $x \notin \mathcal{D} \cap \boldsymbol{c}$. Note that θ is zero on $\mathcal{H}^{<\boldsymbol{c}}$ hence it can be viewed as an $\mathcal{A}$-linear map $\mathcal{H}^{\boldsymbol{c}} \to \mathcal{A}$.

Lemma 2.2. *Let $x, x' \in \boldsymbol{c}$. We have*

(a) $$\theta(c_{x^{-1}}^\dagger c_{x'}^\dagger) = n_{d_x}\delta_{x,x'}v^a + \text{strictly lower powers of } v.$$

The left-hand side of (a) is

$$\sum_{d\in\mathcal{D}\cap\boldsymbol{c}} h_{x^{-1},x',d} = \sum_{d\in\mathcal{D}\cap\boldsymbol{c}} \gamma_{x^{-1},x',d} v^a + \text{strictly lower powers of } v$$
$$= n_{d_x}\delta_{x,x'} v^a + \text{strictly lower powers of } v.$$

The lemma is proved.

We now state one of the main results of this paper.

Theorem 2.3. *There exists a unique permutation $u \mapsto u^*$ of $\boldsymbol{c}$ (with square 1) such that for any $u \in \boldsymbol{c}$ we have*

(a) $$v^{a-a'} T_{w_0} c_u^\dagger = \epsilon_u c_{u^*}^\dagger \mod \mathcal{H}^{<\boldsymbol{c}}$$

where $\epsilon_u = \pm 1$. For any $u \in \boldsymbol{c}$ we have $\epsilon_{u^{-1}} = \epsilon_u = \epsilon_{\sigma(u)} = \epsilon_{u^*}$ and $\sigma(u^*) = (\sigma(u))^* = ((u^{-1})^*)^{-1}$.

Let $u \in \boldsymbol{c}$. We set $Z = \theta((v^{a-a'} T_{w_0} c_u^\dagger)^\flat v^{a-a'} T_{w_0} c_u^\dagger)$. We compute Z in two ways, using Lemma 2.2 and Lemma 1.13. We have

$$Z = \theta(c_{u^{-1}}^\dagger v^{2a-2a'} T_{w_0}^2 c_u^\dagger) = \theta(c_{u^{-1}}^\dagger c_u^\dagger) = n_{d_u} v^a + \text{strictly lower powers of } v,$$

$$\begin{aligned} Z &= \theta\Biggl(\Bigl(\sum_{y\in\boldsymbol{c}} m_{y,u} c_y^\dagger\Bigr)^\flat \sum_{y'\in\boldsymbol{c}} m_{y',u} c_{y'}^\dagger\Biggr) = \sum_{y,y'\in\boldsymbol{c}} m_{y,u} m_{y',u} \theta(c_{y^{-1}}^\dagger c_{y'}^\dagger) \\ &= \sum_{y,y'\in\boldsymbol{c}} m_{y,u} m_{y',u} n_{d_y} \delta_{y,y'} v^a + \text{strictly lower powers of } v \\ &= \sum_{y\in\boldsymbol{c}} n_{d_y} m_{y,u}^2 v^a + \text{strictly lower powers of } v \\ &= \sum_{y\in\boldsymbol{c}} n_{d_u} m_{y,u}^2 v^a + \text{strictly lower powers of } v \end{aligned},$$

where $m_{y,u} \in \mathbb{Z}$ is zero unless $y \sim_{\mathcal{L}} u$ (see Lemma 1.13), in which case we have $d_y = d_u$. We deduce that $\sum_{y\in\boldsymbol{c}} m_{y,u}^2 = 1$, so that we have $m_{y,u} = \pm 1$ for a unique $y \in \boldsymbol{c}$ (denoted by u^*) and $m_{y,u} = 0$ for all $y \in \boldsymbol{c} - \{u^*\}$. Then 2.3(a) holds. Using 2.3(a) and Lemma 1.13(b) we see that $u \mapsto u^*$ has square 1 and that $\epsilon_u \epsilon_{u^*} = 1$.

The automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ (see Section 1.1) satisfies the equality $\sigma(c_u^\dagger) = c_{\sigma(u)}^\dagger$ for any $u \in W$; note also that $w \in \boldsymbol{c} \leftrightarrow \sigma(w) \in \boldsymbol{c}$ (see Lemma 1.2). Applying σ to 2.3(a) we obtain

$$v^{a-a'} T_{w_0} c_{\sigma(u)}^\dagger = \epsilon_u c_{\sigma(u^*)}^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$. By 2.3(a) we have also $v^{a-a'} T_{w_0} c_{\sigma(u)}^\dagger = \epsilon_{\sigma(u)} c_{(\sigma(u))^*}^\dagger$ in $\mathcal{H}^{\boldsymbol{c}}$. It follows that $\epsilon_u c_{\sigma(u^*)}^\dagger = \epsilon_{\sigma(u)} c_{(\sigma(u))^*}^\dagger$ hence $\epsilon_u = \epsilon_{\sigma(u)}$ and $\sigma(u^*) = (\sigma(u))^*$.

Applying $h \mapsto h^\flat$ to 2.3(a) we obtain

$$v^{a-a'} c_{u^{-1}}^\dagger T_{w_0} = \epsilon_u c_{(u^*)^{-1}}^\dagger$$

in $\mathcal{H}^{\boldsymbol{c}}$. By 2.3(a) we have also

$$v^{a-a'} c^{\dagger}_{u^{-1}} T_{w_0} = v^{a-a'} T_{w_0} c^{\dagger}_{\sigma(u^{-1})} = \epsilon_{\sigma(u^{-1})} c^{\dagger}_{(\sigma(u^{-1}))^*}$$

in $\mathcal{H}^{\boldsymbol{c}}$. It follows that $\epsilon_u c^{\dagger}_{(u^*)^{-1}} = \epsilon_{\sigma(u^{-1})} c^{\dagger}_{(\sigma(u^{-1}))^*}$ hence $\epsilon_u = \epsilon_{\sigma(u^{-1})}$ and $(u^*)^{-1} = (\sigma(u^{-1}))^*$. Since $\epsilon_{\sigma(u^{-1})} = \epsilon_{u^{-1}}$, we see that $\epsilon_u = \epsilon_{u^{-1}}$. Replacing u by u^{-1} in $(u^*)^{-1} = (\sigma(u^{-1}))^*$ we obtain $((u^{-1})^*)^{-1} = (\sigma(u))^*$ as required. The theorem is proved.

2.4. For $u \in \boldsymbol{c}$ we have

(a) $$u \sim_{\mathcal{L}} u^*,$$

(b) $$\sigma(u) \sim_{\mathcal{R}} u^*.$$

Indeed, (a) follows from Lemma 1.13. To prove (b) it is enough to show that $\sigma(u)^{-1} \sim_{\mathcal{L}} (u^*)^{-1}$. Using (a) for $\sigma(u)^{-1}$ instead of u we see that it is enough to show that $(\sigma(u^{-1}))^* = (u^*)^{-1}$; this follows from Theorem 2.3.

If we assume that

(c) *any left cell in* $\boldsymbol{c}$ *intersects any right cell in* $\boldsymbol{c}$ *in exactly one element*

then by (a), (b), for any $u \in \boldsymbol{c}$,

(d) u^* *is the unique element of* $\boldsymbol{c}$ *in the intersection of the left cell of* u *with right cell of* $\sigma(u)$.

Note that condition (c) is satisfied for any $\boldsymbol{c}$ if W is of type A_n or if W is of type B_n ($n \geq 2$) with $L(s) = 2$ for all but one $s \in S$ and $L(s) = 1$ or 3 for the remaining $s \in S$. (In this last case we are in the quasisplit case and we have $\sigma = 1$ hence $u^* = u$ for all u.)

Theorem 2.5. *For any* $x \in W$ *we set* $\vartheta(x) = \gamma_{w_0 d_{w_0 x^{-1}}, x, (x^*)^{-1}}$.

(a) *If* $d \in \mathcal{D}$ *and* $x, y \in \boldsymbol{c}$ *satisfy* $\gamma_{w_0 d, x, y} \neq 0$ *then* $y = (x^*)^{-1}$.

(b) *If* $x \in \boldsymbol{c}$ *then there is a unique* $d \in \mathcal{D} \cap w_0 \boldsymbol{c}$ *such that* $\gamma_{w_0 d, x, (x^*)^{-1}} \neq 0$, *namely* $d = d_{w_0 x^{-1}}$. *Moreover we have* $\vartheta(x) = \pm 1$.

(c) *For* $u \in \boldsymbol{c}$ *we have* $\epsilon_u = (-1)^{l(w_0 d)} n_d \vartheta(u)$ *where* $d = d_{w_0 u^{-1}}$.

Applying $h \mapsto h^{\dagger}$ to 2.3(a) we obtain for any $u \in \boldsymbol{c}$:

(d) $$v^{a-a'} (-1)^{l(w_0)} \overline{T_{w_0}} c_u = \sum_{z \in \boldsymbol{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \boldsymbol{c}} \mathcal{A} c_{z'}.$$

We have $T_{w_0} = \sum_{y \in W} (-1)^{l(w_0 y)} p_{1, w_0 y} c_y$ hence $\overline{T_{w_0}} = \sum_{y \in W} (-1)^{l(w_0 y)} \overline{p_{1, w_0 y}} c_y$. Introducing this in (d) we obtain

$$v^{a-a'} \sum_{y \in W} (-1)^{l(y)} \overline{p_{1, w_0 y}} c_y c_u = \sum_{z \in \boldsymbol{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \boldsymbol{c}} \mathcal{A} c_{z'},$$

that is,

$$v^{a-a'}\sum_{y,z\in W}(-1)^{l(y)}\overline{p_{1,w_0y}}h_{y,u,z}c_z=\sum_{z\in\boldsymbol{c}}\delta_{z,u^*}\epsilon_u c_z \mod \sum_{z'\in W-\boldsymbol{c}}\mathcal{A}c_{z'}.$$

Thus, for $z\in\boldsymbol{c}$ we have

(e) $$v^{a-a'}\sum_{y\in W}(-1)^{l(y)}\overline{p_{1,w_0y}}h_{y,u,z}=\delta_{z,u^*}\epsilon_u.$$

Here we have $h_{y,u,z}=\gamma_{y,u,z^{-1}}v^{-a} \mod v^{-a+1}\mathbb{Z}[v]$ and we can assume than $z\le_{\mathcal{R}} y$ so that $w_0y\le_{\mathcal{R}} w_0z$ and $\boldsymbol{a}(w_0y)\ge\boldsymbol{a}(w_0z)=a'$.

For $w\in W$ we set $s_w=n_w$ if $w\in\mathcal{D}$ and $s_w=0$ if $w\notin\mathcal{D}$. By [Lusztig 2003, 14.1] we have $p_{1,w}=s_wv^{-\boldsymbol{a}(w)} \mod v^{-\boldsymbol{a}(w)-1}\mathbb{Z}[v^{-1}]$ hence $\overline{p_{1,w}}=s_wv^{\boldsymbol{a}(w)}$ mod $v^{\boldsymbol{a}(w)+1}\mathbb{Z}[v]$. Hence for y in the sum above we have $\overline{p_{1,w_0y}}=s_{w_0y}v^{\boldsymbol{a}(w_0y)}$ mod $v^{\boldsymbol{a}(w_0y)+1}\mathbb{Z}[v]$. Thus (e) gives

$$v^{a-a'}\sum_{y\in\boldsymbol{c}}(-1)^{l(y)}s_{w_0y}\gamma_{y,u,z^{-1}}v^{\boldsymbol{a}(w_0y)-a}-\delta_{z,u^*}\epsilon_u\in v\mathbb{Z}[v]$$

and using $\boldsymbol{a}(w_0y)=a'$ for $y\in\boldsymbol{c}$ we obtain

$$\sum_{y\in\boldsymbol{c}}(-1)^{l(y)}s_{w_0y}\gamma_{y,u,z^{-1}}=\delta_{z,u^*}\epsilon_u.$$

Using the definition of s_{w_0y} we obtain

(f) $$\sum_{d\in\mathcal{D}\cap w_0\boldsymbol{c}}(-1)^{l(w_0d)}n_d\gamma_{w_0d,u,z^{-1}}=\delta_{z,u^*}\epsilon_u.$$

Next we note that

(g) *if $d\in\mathcal{D}$ and $x,y\in\boldsymbol{c}$ satisfy $\gamma_{w_0d,x,y}\ne0$ then $d=d_{w_0x^{-1}}$.*

Indeed from [Lusztig 2003, §14, P8] we deduce $w_0d\sim_{\mathcal{L}}x^{-1}$. Using [Lusztig 2003, 11.7] we deduce $d\sim_{\mathcal{L}}w_0x^{-1}$ so that $d=d_{w_0^{-1}x^{-1}}$. This proves (g).

Using (g) we can rewrite (f) as follows.

(h) $$(-1)^{l(w_0)}(-1)^{l(d)}n_d\gamma_{w_0d,u,z^{-1}}=\delta_{z,u^*}\epsilon_u$$

where $d=d_{w_0u^{-1}}$.

We prove (a). Assume that $d\in\mathcal{D}$ and $x,y\in\boldsymbol{c}$ satisfy $\gamma_{w_0d,x,y}\ne0$, $y\ne(x^*)^{-1}$. Using (g) we have $d=d_{w_0x^{-1}}$. Using (h) with $u=x$, $z=y^{-1}$ we see that $\gamma_{w_0d,x,y}=0$, a contradiction. This proves (a).

We prove (b). Using (h) with $u=x$, $z=x^*$ we see that

(i) $$(-1)^{l(w_0d)}n_d\gamma_{w_0d,x,(x^*)^{-1}}=\epsilon_u$$

where $d = d_{w_0 x^{-1}}$. Hence the existence of d in (b) and the equality $\vartheta(x) = \pm 1$ follow; the uniqueness of d follows from (g).

Now (c) follows from (i). This completes the proof of the theorem.

2.6. In the case where $L = l$, $\vartheta(u)$ (in 2.5(c)) is ≥ 0 and ± 1 hence 1; moreover, $n_d = 1$, $(-1)^{l(d)} = (-1)^{a'}$ for any $d \in \mathcal{D} \cap w_0\boldsymbol{c}$ (by the definition of $\mathcal{D}$). Hence we have $\epsilon_u = (-1)^{l(w_0)+a'}$ for any $u \in \boldsymbol{c}$, a result of Mathas [1996].

Now Theorem 2.5 also gives a characterization of u^* for $u \in \boldsymbol{c}$; it is the unique element $u' \in \boldsymbol{c}$ such that $\gamma_{w_0 d, u, u'^{-1}} \neq 0$ for some $d \in \mathcal{D} \cap w_0\boldsymbol{c}$.

We will show:

(a) *The subsets* $X = \{d^*; d \in \mathcal{D} \cap \boldsymbol{c}\}$ *and* $X' = \{w_0 d'; d' \in \mathcal{D} \cap w_0\boldsymbol{c}\}$ *of* $\boldsymbol{c}$ *coincide.*

Let $d \in \mathcal{D} \cap \boldsymbol{c}$. By 2.5(b) we have $\gamma_{w_0 d', d, (d^*)^{-1}} = \pm 1$ for some $d' \in \mathcal{D} \cap w_0\boldsymbol{c}$. Hence $\gamma_{(d^*)^{-1}, w_0 d', d} = \pm 1$. Using [Lusztig 2003, 14.2, P2] we deduce $d^* = w_0 d'$. Thus $X \subset X'$. Let Y (resp. Y') be the set of left cells contained in $\boldsymbol{c}$ (resp. $w_0\boldsymbol{c}$). We have $\sharp(X) = \sharp(Y)$ and $\sharp(X') = \sharp(Y')$. By [Lusztig 2003, 11.7(c)] we have $\sharp(Y) = \sharp(Y')$. It follows that $\sharp(X) = \sharp(X')$. Since $X \subset X'$, we must have $X = X'$. This proves (a).

Theorem 2.7. *We have*

$$\phi(v^{a-a'} T_{w_0}) = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d) \epsilon_d t_{d^*} \mod \sum_{u \in W - \boldsymbol{c}} \mathcal{A} t_u.$$

We set $\phi(v^{a-a'} T_{w_0}) = \sum_{u \in W} p_u t_u$ where $p_u \in \mathcal{A}$. Combining 1.12a, 1.13a, 2.3(a) we see that for any $x \in \boldsymbol{c}$ we have

$$\phi(v^{a-a'} T_{w_0}) t_x = \epsilon_x t_{x^*},$$

hence

$$\epsilon_x t_{x^*} = \sum_{u \in \boldsymbol{c}} p_u t_u t_x = \sum_{u, y \in \boldsymbol{c}} p_u \gamma_{u, x, y^{-1}} t_y.$$

It follows that for any $x, y \in \boldsymbol{c}$ we have

$$\sum_{u \in \boldsymbol{c}} p_u \gamma_{u, x, y^{-1}} = \delta_{y, x^*} \epsilon_x.$$

Taking $x = w_0 d$ where $d = d_{w_0 y} \in \mathcal{D} \cap w_0\boldsymbol{c}$ we obtain

$$\sum_{u \in \boldsymbol{c}} p_u \gamma_{w_0 d_{w_0 y}, y^{-1}, u} = \delta_{y, (w_0 d_{w_0 y})^*} \epsilon_{w_0 d_{w_0 y}}$$

which, by Theorem 2.5, can be rewritten as

$$p_{((y^{-1})^*)^{-1}} \vartheta(y^{-1}) = \delta_{y, (w_0 d_{w_0 y})^*} \epsilon_{w_0 d_{w_0 y}}.$$

We see that for any $y \in \boldsymbol{c}$ we have

$$p_{\sigma(y^*)} = \delta_{y,(w_0 d_{w_0 y})^*} \vartheta(y^{-1}) \epsilon_{w_0 d_{w_0 y}}.$$

In particular we have $p_{\sigma(y^*)} = 0$ unless $y = (w_0 d_{w_0 y})^*$ in which case

$$p_{\sigma(y^*)} = p_{(\sigma(y))^*)} = \vartheta(y^{-1}) \epsilon_y.$$

(We use that $\epsilon_{y^*} = \epsilon_y$.) If $y = (w_0 d_{w_0 y})^*$ then $y^* \in X'$ hence by 2.6(a), $y^* = d^*$ that is $y = d$ for some $d \in \mathcal{D}$. Conversely, if $y \in \mathcal{D}$ then $w_0 y^* \in \mathcal{D}$ (by 2.6(a)) and $w_0 y^* \sim_{\mathcal{L}} w_0 y$ (since $y^* \sim_{\mathcal{L}} y$) hence $d_{w_0 y} = w_0 y^*$. We see that $y = (w_0 d_{w_0 y})^*$ if and only if $y \in \mathcal{D}$. We see that

$$\phi(v^{a-a'} T_{w_0}) = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d^{-1}) \epsilon_d t_{(\sigma(d))^*} + \sum_{u \in W - \boldsymbol{c}} p_u t_u.$$

Now $d \mapsto \sigma(d)$ is a permutation of $\mathcal{D} \cap \boldsymbol{c}$ and $\vartheta(d^{-1}) = \vartheta(d) = \vartheta(\sigma(d))$, $\epsilon_{\sigma(d)} = \epsilon_d$. The theorem follows.

Corollary 2.8. $$\phi(T_{w_0}) = \sum_{d \in \mathcal{D}} \vartheta(d) \epsilon_d v^{-\boldsymbol{a}(d) + \boldsymbol{a}(w_0 d)} t_{d^*} \in J_{\mathcal{A}}.$$

2.9. We set $\mathfrak{T}_{\boldsymbol{c}} = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} \vartheta(d) \epsilon_d t_{d^*} \in J^{\boldsymbol{c}}$. We show:

(a) $\mathfrak{T}_{\boldsymbol{c}}^2 = \sum_{d \in \mathcal{D} \cap \boldsymbol{c}} n_d t_d$.

(b) $t_x \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} t_{\sigma(x)}$ for any $x \in W$.

By Theorem 2.7 we have $\phi(v^{a-a'} T_{w_0}) = \mathfrak{T}_{\boldsymbol{c}} + \xi$ where $\xi \in J_K^{W-\boldsymbol{c}} := \sum_{u \in W - \boldsymbol{c}} K t_u$. Since $J_K^{\boldsymbol{c}}$, $J_K^{W-\boldsymbol{c}}$ are two-sided ideals of J_K with intersection zero and $\phi_K : \mathcal{H}_K \to J_K$ is an algebra homomorphism, it follows that

$$\phi(v^{2a-2a'} T_{w_0}^2) = (\phi(v^{a-a'} T_{w_0}))^2 = (\mathfrak{T}_{\boldsymbol{c}} + \xi)^2 = \mathfrak{T}_{\boldsymbol{c}}^2 + \xi'$$

where $\xi' \in J_K^{W-\boldsymbol{c}}$. Hence, for any $x \in \boldsymbol{c}$ we have $\phi(v^{2a-2a'} T_{w_0}^2) t_x = \mathfrak{T}_{\boldsymbol{c}}^2 t_x$ so that (using 1.12b): $t_x = \mathfrak{T}_{\boldsymbol{c}}^2 t_x$. We see that $\mathfrak{T}_{\boldsymbol{c}}^2$ is the unit element of the ring $J_K^{\boldsymbol{c}}$. Thus (a) holds.

We prove (b). For any $y \in W$ we have $T_y T_{w_0} = T_{w_0} T_{\sigma(y)}$ hence, applying ϕ_K,

$$\phi(T_y) \phi(v^{a-a'} T_{w_0}) = \phi(v^{a-a'} T_{w_0}) \phi(T_{\sigma(y)}),$$

that is, $\phi(T_y)(\mathfrak{T}_{\boldsymbol{c}} + \xi) = (\mathfrak{T}_{\boldsymbol{c}} + \xi) \phi(T_{\sigma(y)})$. Thus, $\phi(T_y) \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} \phi(T_{\sigma(y)}) + \xi_1$ where $\xi_1 \in J_K^{W-\boldsymbol{c}}$. Since ϕ_K is an isomorphism, it follows that for any $x \in W$ we have $t_x \mathfrak{T}_{\boldsymbol{c}} = \mathfrak{T}_{\boldsymbol{c}} t_{\sigma(x)} \mod J_K^{W-\boldsymbol{c}}$. Thus (b) holds.

2.10. In this subsection we assume that $L = l$. In this case Corollary 2.8 becomes

$$\phi(T_{w_0}) = \sum_{d\in\mathcal{D}} (-1)^{l(w_0)+\boldsymbol{a}(w_0 d)} v^{-\boldsymbol{a}(d)+\boldsymbol{a}(w_0 d)} t_{d^*} \in J_{\mathcal{A}}.$$

(We use that $\vartheta(d) = 1$.)

For any left cell Γ contained in $\boldsymbol{c}$ let n_Γ be the number of fixed points of the permutation $u \mapsto u^*$ of Γ. Now Γ carries a representation $[\Gamma]$ of W and from Theorem 2.3 we see that $\operatorname{tr}(w_0, [\Gamma]) = \pm n_\Gamma$. Thus n_Γ is the absolute value of the integer $\operatorname{tr}(w_0, [\Gamma])$. From this the number n_Γ can be computed for any Γ. In this way we see for example that if W is of type E_7 or E_8 and $\boldsymbol{c}$ is not an exceptional two-sided cell, then $n_\Gamma > 0$.

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