

*Pacific
Journal of
Mathematics*

**SMOOTH REPRESENTATIONS AND HECKE MODULES
IN CHARACTERISTIC p**

PETER SCHNEIDER

Volume 279 No. 1-2

December 2015

SMOOTH REPRESENTATIONS AND HECKE MODULES IN CHARACTERISTIC p

PETER SCHNEIDER

Dedicated to the memory of Robert Steinberg.

Let G be a p -adic Lie group and $I \subseteq G$ be a compact open subgroup which is a torsionfree pro- p -group. Working over a coefficient field k of characteristic p we introduce a differential graded Hecke algebra for the pair (G, I) and show that the derived category of smooth representations of G in k -vector spaces is naturally equivalent to the derived category of differential graded modules over this Hecke DGA.

1. Background and motivation

Let G be a d -dimensional p -adic Lie group, and let k be any field. We denote by $\text{Mod}_k(G)$ the category of smooth G -representations in k -vector spaces. It obviously has arbitrary direct sums.

Fix a compact open subgroup $I \subseteq G$. In $\text{Mod}_k(G)$ we then have the representation

$$\text{ind}_I^G(1) := \{k\text{-valued functions with finite support on } G/I\}$$

with G acting by left translations. Being generated by a single element, which is the characteristic function of the trivial coset, $\text{ind}_I^G(1)$ is a compact object in $\text{Mod}_k(G)$. It generates the full subcategory $\text{Mod}_k^I(G)$ of all representations V in $\text{Mod}_k(G)$ which are generated by their I -fixed vectors V^I . In general, $\text{Mod}_k^I(G)$ is not an abelian category. The Hecke algebra of I by definition is the endomorphism ring

$$\mathcal{H}_I := \text{End}_{\text{Mod}_k(G)}(\text{ind}_I^G(1))^{\text{op}}.$$

We let $\text{Mod}(\mathcal{H}_I)$ denote the category of left unital \mathcal{H}_I -modules. There is the pair of adjoint functors

$$\begin{aligned} H^0 : \text{Mod}_k(G) &\longrightarrow \text{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I = \text{Hom}_{\text{Mod}_k(G)}(\text{ind}_I^G(1), V), \end{aligned}$$

MSC2010: primary 16E45, 20C08, 22E50; secondary 16E35.

Keywords: differential graded Hecke algebra, smooth representation.

and

$$\begin{aligned} T_0 : \text{Mod}(\mathcal{H}_I) &\longrightarrow \text{Mod}_k^I(G) \subseteq \text{Mod}_k(G) \\ M &\longmapsto \text{ind}_I^G(1) \otimes_{\mathcal{H}_I} M. \end{aligned}$$

If the characteristic of k does not divide the pro-order of I then the functor H^0 is exact. Then $\text{ind}_I^G(1)$ is a projective compact object in $\text{Mod}_k(G)$. Since it does not generate the full category $\text{Mod}_k(G)$, one cannot apply the Gabriel–Popescu theorem (compare [Kashiwara and Schapira 2006, Theorem 8.5.8]) to the functor H^0 . Nevertheless, in this case, one might hope for a close relation between the categories $\text{Mod}_k^I(G)$ and $\text{Mod}(\mathcal{H}_I)$. This indeed happens, for example, for a connected reductive group G and its Iwahori subgroup I and the field $k = \mathbb{C}$; compare [Bernstein 1984, Corollary 3.9(ii)]. In addition, in this situation the algebra \mathcal{H}_I turns out to be a generalized affine Hecke algebra so that its structure is explicitly known. Therefore, in characteristic zero, Hecke algebras have become one of the most important tools in the investigation of smooth G -representations.

In this light, it is a pressing question to better understand the relation between the two categories $\text{Mod}_k(G)$ and $\text{Mod}(\mathcal{H}_I)$ in the opposite situation where k has characteristic p . Since p always will divide the pro-order of I , the functor H^0 certainly is no longer exact. Both functors H^0 and T_0 now have a very complicated behavior and little is known [Kozioł 2014; Ollivier 2009; Ollivier and Schneider 2015]. This suggests that one should work in a derived framework which takes into account the higher cohomology of I .

This paper will demonstrate that by doing this — not in a naive way but in an appropriate differential graded context — the situation does improve drastically. We will show the somewhat surprising result that the object $\text{ind}_I^G(1)$ becomes a compact generator of the full derived category of G provided I is a torsionfree pro- p -group.

The main result of this paper was proved already in 2007 but remained unpublished. At the time, we gave a somewhat ad hoc proof. Although the main arguments remain unchanged we now, by appealing to a general theorem of Keller, have arranged them in a way which makes the reasoning more transparent. In the context of the search for a p -adic local Langlands program, there is increasing interest in studying derived situations; see [Harris 2015]. We also have now [Ollivier and Schneider 2015] the first examples of explicit computations of the cohomology groups $H^i(I, \text{ind}_I^G(1))$. I hope that these are sufficient reasons to finally publish the paper.

2. The unbounded derived category of G

We assume from now on throughout the paper that the field k has characteristic p and that I is a torsionfree pro- p -group. Let us first of all collect a few properties of the abelian category $\text{Mod}_k(G)$.

Lemma 1. (i) $\text{Mod}_k(G)$ is (AB5), i.e., it has arbitrary colimits and filtered colimits are exact.

(ii) $\text{Mod}_k(G)$ is (AB3*), i.e., it has arbitrary limits.

(iii) $\text{Mod}_k(G)$ has enough injective objects.

(iv) $\text{Mod}_k(G)$ is a Grothendieck category, i.e., it satisfies (AB5) and has a generator.

(v) $V^I \neq 0$ for any nonzero V in $\text{Mod}_k(G)$.

Proof. (i) This is obvious. (ii) Take the subspace of smooth vectors in the limit of k -vector spaces. (iii) This is shown in [Vignéras 1996, §I.5.9]. Alternatively, it is a consequence of (iv); compare [Kashiwara and Schapira 2006, Theorem 9.6.2]. (v) Since I is pro- p , where p is the characteristic of k , the only irreducible smooth representation of I is the trivial one.

(iv) Because of (i) it remains to exhibit a generator of $\text{Mod}_k(G)$. We define

$$Y := \bigoplus_J \text{ind}_J^G(1),$$

where J runs over all open subgroups in G . For any V in $\text{Mod}_k(G)$, we have

$$\text{Hom}_{\text{Mod}_k(G)}(Y, V) = \prod_J V^J.$$

Since $V = \bigcup_J V^J$, we easily deduce that Y is a generator of $\text{Mod}_k(G)$. □

As usual, let $D(G) := D(\text{Mod}_k(G))$ be the derived category of unbounded complexes in $\text{Mod}_k(G)$.

Remark 2. $D(G)$ has arbitrary direct sums, which can be computed as direct sums of complexes.

Proof. See the first paragraph in [Kashiwara and Schapira 2006, §14.3]. □

According to [Lazard 1965, Théorème V.2.2.8; Serre 1965], the group I has cohomological dimension d . This means that the higher derived functors of the left exact functor

$$\begin{aligned} \text{Mod}_k(I) &\longrightarrow \text{Vec}_k \\ E &\longmapsto E^I \end{aligned}$$

into the category Vec_k of k -vector spaces are zero in degrees $> d$. On the other hand, the restriction functor

$$\begin{aligned} \text{Mod}_k(G) &\longrightarrow \text{Mod}_k(I) \\ V &\longmapsto V|_I \end{aligned}$$

is exact and respects injective objects. The latter is a consequence of the fact that compact induction

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Mod}_k(G) \\ E &\longmapsto \mathrm{ind}_I^G(E) \end{aligned}$$

is an exact left adjoint; compare [Vignéras 1996, §I.5.7]. Hence the higher derived functors of the composed functor

$$\begin{aligned} H^0(I, \cdot) : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Vec}_k \\ V &\longmapsto V^I \end{aligned}$$

are given by $V \longmapsto H^i(I, V|_I)$ and vanish in degrees $> d$. It follows that the total right derived functor

$$RH^0(I, \cdot) : D(G) \longrightarrow D(\mathrm{Vec}_k)$$

between the corresponding (unbounded) derived categories exists [Hartshorne 1966, Corollary I.5.3].

To compute $RH^0(I, \cdot)$, we use the formalism of K -injective complexes as developed in [Spaltenstein 1988]. We let $C(\mathrm{Mod}_k(G))$ and $K(\mathrm{Mod}_k(G))$ denote the category of unbounded complexes in $\mathrm{Mod}_k(G)$ with chain maps and homotopy classes of chain maps, respectively, as morphisms. The K -injective complexes form a full triangulated subcategory $K_{\mathrm{inj}}(\mathrm{Mod}_k(G))$ of $K(\mathrm{Mod}_k(G))$. Exactly in the same way as [op. cit., Proposition 3.11] one can show that any complex in $C(\mathrm{Mod}_k(G))$ has a right K -injective resolution (recall from Lemma 1(ii) that the category $\mathrm{Mod}_k(G)$ has inverse limits). Alternatively, one may apply [Serpé 2003, Theorem 3.13] or [Kashiwara and Schapira 2006, Theorem 14.3.1] based upon Lemma 1(iv). The existence of K -injective resolutions means that the natural functor

$$K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \xrightarrow{\cong} D(G)$$

is an equivalence of triangulated categories. We fix a quasi-inverse i of this functor. Then the derived functor $RH^0(I, \cdot)$ is naturally isomorphic to the composed functor

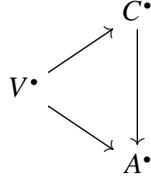
$$D(G) \xrightarrow{i} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathrm{Vec}_k) \longrightarrow D(\mathrm{Vec}_k)$$

with the middle arrow given by

$$V^\bullet \mapsto \mathrm{Hom}_{\mathrm{Mod}_k(G)}^\bullet(\mathrm{ind}_I^G(1), V^\bullet).$$

Explanation. Let V^\bullet be a complex in $C(\mathrm{Mod}_k(G))$. To compute $RH^0(I, \cdot)$ according to [Hartshorne 1966], one chooses a quasi-isomorphism $V^\bullet \xrightarrow{\cong} C^\bullet$ into a complex consisting of objects which are acyclic for the functor $H^0(I, \cdot)$. On the other hand, let $V^\bullet \xrightarrow{\cong} A^\bullet$ be a quasi-isomorphism into a K -injective complex. By

[Spaltenstein 1988, Proposition 1.5(c)] we then have, in $K(\text{Mod}_k(G))$, a unique commutative diagram:



We claim that the induced map

$$(C^\bullet)^I \xrightarrow{\cong} (A^\bullet)^I$$

is a quasi-isomorphism. Choose quasi-isomorphisms

$$A^\bullet \xrightarrow{\cong} \tilde{C}^\bullet \xrightarrow{\cong} \tilde{A}^\bullet$$

where \tilde{C}^\bullet consists of $H^0(I, \cdot)$ -acyclic objects and \tilde{A}^\bullet is K -injective. By [Spaltenstein 1988, Proposition 1.5(b)], the composite is an isomorphism in $K(\text{Mod}_k(G))$ and hence induces a quasi-isomorphism $(A^\bullet)^I \xrightarrow{\cong} (\tilde{A}^\bullet)^I$. But by [Hartshorne 1966, Theorem I.5.1 and Corollary I.5.3(γ)], the composite $C^\bullet \xrightarrow{\cong} A^\bullet \xrightarrow{\cong} \tilde{C}^\bullet$ also induces a quasi-isomorphism $(C^\bullet)^I \xrightarrow{\cong} (\tilde{C}^\bullet)^I$.

Lemma 3. *The (hyper)cohomology functor $H^\ell(I, \cdot)$, for any $\ell \in \mathbb{Z}$, commutes with arbitrary direct sums in $D(G)$.*

Proof. First of all we observe that the cohomology functor $H^\ell(I, \cdot)$ commutes with arbitrary direct sums in $\text{Mod}_k(G)$ [Serre 1994, §I.2.2, Proposition 8]. This, in particular, implies that arbitrary direct sums of $H^0(I, \cdot)$ -acyclic objects in $\text{Mod}_k(G)$ again are $H^0(I, \cdot)$ -acyclic. Now let $(V_j^\bullet)_{j \in J}$ be a family of objects in $D(G)$, where we view each V_j^\bullet as an actual complex. Then, according to Remark 2, the direct sum of the V_j^\bullet in $D(G)$ is represented by the direct sum complex $\bigoplus_j V_j^\bullet$. Now we choose quasi-isomorphisms $V_j^\bullet \xrightarrow{\cong} C_j^\bullet$ in $C(\text{Mod}_k(G))$, where all representations C_j^m are $H^0(I, \cdot)$ -acyclic. By the preliminary observation, the direct sum map

$$\bigoplus_j V_j^\bullet \xrightarrow{\cong} C^\bullet := \bigoplus_j C_j^\bullet$$

again is a quasi-isomorphism where all terms of the target complex are $H^0(I, \cdot)$ -acyclic. We therefore obtain

$$H^\ell(I, \bigoplus_j V_j^\bullet) = h^\ell((C^\bullet)^I) = \bigoplus_j h^\ell((C_j^\bullet)^I) = \bigoplus_j H^\ell(I, V_j^\bullet). \quad \square$$

As usual, we view $\text{Mod}_k(G)$ as the full subcategory of those complexes in $D(G)$ which have zero terms outside of degree zero.

Lemma 4. $\text{ind}_I^G(1)$ is a compact object in $D(G)$.

Proof. We have to show that the functor $\text{Hom}_{D(G)}(\text{ind}_I^G(1), \cdot)$ commutes with arbitrary direct sums in $D(G)$. For any V^\bullet in $D(G)$, we compute

$$(1) \quad \begin{aligned} \text{Hom}_{D(G)}(\text{ind}_I^G(1), V^\bullet) &= \text{Hom}_{K(\text{Mod}_k(G))}(\text{ind}_I^G(1), \mathbf{i}(V^\bullet)) \\ &= h^0(\mathbf{i}(V^\bullet)^I) = H^0(I, V^\bullet), \end{aligned}$$

where the first identity uses [Spaltenstein 1988, Proposition 1.5(b)]. The claim therefore follows from Lemma 3. \square

Proposition 5. *Let E^\bullet be in $D(I)$. Then $E^\bullet = 0$ if and only if $H^j(I, E^\bullet) = 0$ for any $j \in \mathbb{Z}$.*

Proof. The completed group ring $\Omega := \varprojlim_N k[I/N]$ of I over k , where N runs over all open normal subgroups of I , is a pseudocompact local ring; compare [Schneider 2011, §19]. If $\mathfrak{m} \subseteq \Omega$ denotes the maximal ideal, then $\Omega/\mathfrak{m} = k$. Since Ω is noetherian — [Lazard 1965, Proposition V.2.2.4] for $k = \mathbb{F}_p$ and [Schneider 2011, Theorem 33.4] together with [Bourbaki 2006, Chapitre IX, §2.3, Proposition 5] in general — its pseudocompact topology coincides with the \mathfrak{m} -adic topology [Schneider 2011, Lemma 19.8]. This implies that:

- Ω/\mathfrak{m}^j lies in $\text{Mod}_k(I)$ for any $j \in \mathbb{N}$.
- For any E in $\text{Mod}_k(I)$, we have

$$E = \bigcup_{j \in \mathbb{N}} E^{\mathfrak{m}^j=0} \quad \text{where } E^{\mathfrak{m}^j=0} := \{v \in E : \mathfrak{m}^j v = 0\}.$$

Because of

$$E^{\mathfrak{m}^j=0} = \text{Hom}_{\text{Mod}_k(I)}(\Omega/\mathfrak{m}^j, E),$$

we need to consider the left exact functors $\text{Hom}_{\text{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ on $\text{Mod}_k(I)$. Their right derived functors, of course, are $\text{Ext}_{\text{Mod}_k(I)}^i(\Omega/\mathfrak{m}^j, \cdot)$. In particular,

$$\text{Ext}_{\text{Mod}_k(I)}^i(\Omega/\mathfrak{m}, \cdot) = H^i(I, \cdot).$$

For any $j \in \mathbb{N}$, we have the short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0$$

in $\text{Mod}_k(I)$. Moreover, $\mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{n(j)}$ for some $n(j) \geq 0$ since Ω is noetherian. The associated long exact Ext-sequence therefore reads

$$\cdots \longrightarrow \text{Ext}_{\text{Mod}_k(I)}^i(\Omega/\mathfrak{m}^j, \cdot) \longrightarrow \text{Ext}_{\text{Mod}_k(I)}^i(\Omega/\mathfrak{m}^{j+1}, \cdot) \longrightarrow H^i(I, \cdot)^{n(j)} \longrightarrow \cdots$$

By induction with respect to j , we deduce that:

- Each functor $\text{Hom}_{\text{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ has cohomological dimension $\leq d$.

- Each $H^0(I, \cdot)$ -acyclic object in $\text{Mod}_k(I)$ is $\text{Hom}_{\text{Mod}_k(I)}(\Omega/m^j, \cdot)$ -acyclic for any $j \geq 1$.

It follows that the total right derived functors $R\text{Hom}_{\text{Mod}_k(I)}(\Omega/m^j, \cdot)$ on $D(I)$ exist. More explicitly, let E^\bullet be any complex in $D(I)$ and choose a quasi-isomorphism $E^\bullet \xrightarrow{\cong} C^\bullet$ into a complex consisting of $H^0(I, \cdot)$ -acyclic objects. It then follows that we have the short exact sequence of complexes

$$0 \rightarrow \text{Hom}_{\text{Mod}_k(I)}^\bullet(\Omega/m^j, C^\bullet) \rightarrow \text{Hom}_{\text{Mod}_k(I)}^\bullet(\Omega/m^{j+1}, C^\bullet) \rightarrow ((C^\bullet)^I)^{n(j)} \rightarrow 0.$$

Suppose now that $RH^0(I, E^\bullet) = 0$. This means that the complex $(C^\bullet)^I$ is exact. By induction with respect to j , we obtain the exactness of the complex

$$\text{Hom}_{\text{Mod}_k(I)}^\bullet(\Omega/m^j, C^\bullet) = (C^\bullet)^{m^j=0}$$

for any $j \in \mathbb{N}$. Hence C^\bullet and E^\bullet are exact. □

Proposition 6. *$\text{ind}_I^G(1)$ is a generator of the triangulated category $D(G)$ in the sense that any strictly full triangulated subcategory of $D(G)$, closed under all direct sums, which contains $\text{ind}_I^G(1)$, coincides with $D(G)$.*

Proof. By (1) we have

$$\begin{aligned} \text{Hom}_{D(G)}(\text{ind}_I^G(1)[j], V^\bullet) &= \text{Hom}_{D(G)}(\text{ind}_I^G(1), V^\bullet[-j]) \\ &= H^0(I, V^\bullet[-j]) = H^{-j}(I, V^\bullet) \end{aligned}$$

for any V^\bullet in $D(G)$. Hence, Proposition 5 implies that the family of shifts $\{\text{ind}_I^G(1)[j]\}_{j \in \mathbb{Z}}$ is a generating set of $D(G)$ in the sense of Neeman [2001, Definition 8.1.1]. On the other hand, by Lemma 4, each shift $\text{ind}_I^G(1)[j]$ is a compact object. In Neeman’s language this means that $\{\text{ind}_I^G(1)[j]\}_{j \in \mathbb{Z}}$ is an \aleph_0 -perfect class consisting of \aleph_0 -small objects [Neeman 2001, Remark 4.2.6 and Definition 4.2.7]. According to Neeman’s Lemma 4.2.1, the class $\{\text{ind}_I^G(1)[j]\}_{j \in \mathbb{Z}}$ then is β -perfect for any infinite cardinal β . Hence Neeman’s Theorem 8.3.3 applies and shows (see the explanations in §3.2.6–3.2.8 of that same reference) that any strictly full triangulated subcategory of $D(G)$ closed under all direct sums which contains $\text{ind}_I^G(1)$, and therefore the whole class $\{\text{ind}_I^G(1)[j]\}_{j \in \mathbb{Z}}$, coincides with $D(G)$. □

3. The Hecke DGA

In order to also “derive” the picture on the Hecke algebra side we fix an injective resolution $\text{ind}_I^G(1) \xrightarrow{\cong} \mathcal{I}^\bullet$ in $C(\text{Mod}_k(G))$ and introduce the differential graded algebra

$$\mathcal{H}_I^\bullet := \text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet)^{\text{op}}$$

over k . We recall that

$$\mathcal{H}_I^n = \prod_{q \in \mathbb{Z}} \text{Hom}_{\text{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n})$$

with differential

$$(da)_q(x) = d(a_q(x)) - (-1)^n a_{q+1}(dx)$$

for $a = (a_q) \in \mathcal{H}_I^n$ and multiplication

$$(ba)_q := (-1)^{mn} a_{q+m} \circ b_q$$

for $a = (a_q) \in \mathcal{H}_I^n$ and $b = (b_q) \in \mathcal{H}_I^m$. The cohomology of \mathcal{H}_I^\bullet is given by

$$h^*(\mathcal{H}_I^\bullet) = \text{Ext}_{\text{Mod}_k(G)}^*(\text{ind}_I^G(1), \text{ind}_I^G(1));$$

compare [Hartshorne 1966, §I.6]. In particular,

$$h^0(\mathcal{H}_I^\bullet) = \mathcal{H}_I.$$

Remark 7. $h^*(\mathcal{H}_I^\bullet) = H^*(I, \text{ind}_I^G(1))$ and, in particular, $h^i(\mathcal{H}_I^\bullet) = 0$ for $i > d$.

Proof. We compute

$$\begin{aligned} h^*(\mathcal{H}_I^\bullet) &= \text{Ext}_{\text{Mod}_k(G)}^*(\text{ind}_I^G(1), \text{ind}_I^G(1)) \\ &= h^*(\text{Hom}_{\text{Mod}_k(G)}(\text{ind}_I^G(1), \mathcal{I}^\bullet)) \\ &= h^*((\mathcal{I}^\bullet)^I) = H^*(I, \text{ind}_I^G(1)). \end{aligned}$$

□

Let $D(\mathcal{H}_I^\bullet)$ be the derived category of differential graded left \mathcal{H}_I^\bullet -modules. Note that \mathcal{H}_I^\bullet is a compact generator of $D(\mathcal{H}_I^\bullet)$ [Keller 1998, §2.5]. It is well known that \mathcal{H}_I^\bullet and $D(\mathcal{H}_I^\bullet)$ do not depend, up to quasi-isomorphism and equivalence, respectively, on the choice of the injective resolution \mathcal{I}^\bullet . For the convenience of the reader, we briefly recall the argument. Let $\text{ind}_I^G(1) \xrightarrow{\simeq} \mathcal{J}^\bullet$ be a second injective resolution in $C(\text{Mod}_k(G))$, and let $f : \mathcal{I}^\bullet \rightarrow \mathcal{J}^\bullet$ be a homotopy equivalence inducing the identity on $\text{ind}_I^G(1)$ with homotopy inverse g . We form the differential graded algebra

$$\mathcal{A}^\bullet := \{ (a, b) \in \text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{J}^\bullet)^{\text{op}} \times \text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet)^{\text{op}} : a \circ f = f \circ a \}$$

(with respect to componentwise multiplication) and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^\bullet & \xrightarrow{\text{pr}_2} & \text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet)^{\text{op}} \\ \text{pr}_1 \downarrow & & \downarrow b \mapsto f \circ b \\ \text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{J}^\bullet)^{\text{op}} & \xrightarrow{a \mapsto a \circ f} & \text{Hom}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet, \mathcal{J}^\bullet). \end{array}$$

Obviously, the maps pr_i are homomorphisms of differential graded algebras (and the bottom horizontal and right perpendicular arrows are homotopy equivalences of complexes). By direct inspection, one checks that the pr_i , in fact, are quasi-isomorphisms. Hence the differential graded algebras $\text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet)^{\text{op}}$ and $\text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{J}^\bullet)^{\text{op}}$ are naturally quasi-isomorphic to each other. Moreover, by appealing to [Bernstein and Lunts 1994, Theorem 10.12.5.1], we see that the functors

$$D(\text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet)^{\text{op}}) \xrightarrow[\sim]{(\text{pr}_2)_*} D(\mathcal{A}^\bullet) \xleftarrow[\sim]{(\text{pr}_1)_*} D(\text{End}_{\text{Mod}_k(G)}^\bullet(\mathcal{J}^\bullet)^{\text{op}})$$

are equivalences of triangulated categories.

There is the following pair of adjoint functors

$$H : D(G) \longrightarrow D(\mathcal{H}_I^\bullet) \quad \text{and} \quad T : D(\mathcal{H}_I^\bullet) \longrightarrow D(G).$$

For any K -injective complex V^\bullet in $\text{Mod}_k(G)$, the natural chain map

$$\text{Hom}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet, V^\bullet) \xrightarrow{\simeq} \text{Hom}_{\text{Mod}_k(G)}^\bullet(\text{ind}_I^G(1), V^\bullet)$$

is a quasi-isomorphism. But the left hand term is a differential graded left \mathcal{H}_I^\bullet -module in a natural way. In fact, we have the functor

$$\begin{aligned} K_{\text{inj}}(\text{Mod}_k(G)) &\longrightarrow K(\mathcal{H}_I^\bullet) \\ V^\bullet &\longmapsto \text{Hom}_{\text{Mod}_k(G)}^\bullet(\mathcal{I}^\bullet, V^\bullet) \end{aligned}$$

into the homotopy category $K(\mathcal{H}_I^\bullet)$ of differential graded left \mathcal{H}_I^\bullet -modules, which allows us to define the composed functor

$$H : D(G) \xrightarrow{i} K_{\text{inj}}(\text{Mod}_k(G)) \longrightarrow K(\mathcal{H}_I^\bullet) \longrightarrow D(\mathcal{H}_I^\bullet).$$

The diagram

$$(2) \quad \begin{array}{ccc} D(G) & \xrightarrow{H} & D(\mathcal{H}_I^\bullet) \\ & \searrow \text{RH}^0(I, \cdot) & \downarrow \text{forget} \\ & & D(\text{Vec}_k) \end{array}$$

then is commutative up to natural isomorphism.

For the functor T in the opposite direction we first note that \mathcal{I}^\bullet is naturally a differential graded right \mathcal{H}_I^\bullet -module so that we can form the graded tensor product $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} M^\bullet$ with any differential graded left \mathcal{H}_I^\bullet -module M^\bullet . This tensor product is naturally a complex in $C(\text{Mod}_k(G))$. We now define T to be the composite

$$T : D(\mathcal{H}_I^\bullet) \xrightarrow{p} K_{\text{pro}, \mathcal{H}_I^\bullet} \xrightarrow{\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet}} K(\text{Mod}_k(G)) \longrightarrow D(G).$$

Here $K_{\text{pro}, \mathcal{H}_I^\bullet}$ denotes the full triangulated subcategory of $K(\mathcal{H}_I^\bullet)$ consisting of K -projective modules and \mathbf{p} is a quasi-inverse of the equivalence of triangulated categories $K_{\text{pro}, \mathcal{H}_I^\bullet} \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet)$; compare [Bernstein and Lunts 1994, Corollary 10.12.2.9].

The usual standard computation shows that T is left adjoint to H .

4. The main theorem

We need one more property of the derived category $D(G)$.

Lemma 8. *The triangulated category $D(G)$ is algebraic.*

Proof. The composite functor

$$D(G) \xrightarrow{i} K_{\text{inj}}(\text{Mod}_k(G)) \xrightarrow{\subseteq} K(\text{Mod}_k(G))$$

is a fully faithful exact functor between triangulated categories. Hence, the assertion follows from [Krause 2007, Lemma 7.5]. □

In view of Lemmas 4 and 8 and Proposition 6, all assumptions of Keller’s theorem [1994, Theorem 4.3; 1998, Theorem 3.3(a)] (compare also [Bondal and van den Bergh 2003, Theorem 3.1.7]) are satisfied and we obtain our main result.

Theorem 9. *The functor H is an equivalence between triangulated categories*

$$D(G) \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet).$$

Of course, it follows formally that the adjoint functor T is a left inverse of H .

Remark 10. The full subcategory $D(G)^c$ of all compact objects in $D(G)$ is the smallest strictly full triangulated subcategory closed under direct summands which contains $\text{ind}_I^G(1)$.

Proof. In view of Lemma 4 and Proposition 6 this follows from [Neeman 1992, Lemma 2.2]. □

The subcategory $D(G)^c$ should be viewed as the analog of the subcategory of perfect complexes in the derived category of a ring; compare [Keller 1998, Lemma 1.4].

Another important subcategory of $D(G)$ is the bounded derived category

$$D^b(G) := D^b(\text{Mod}_k(G)).$$

Correspondingly we have the full subcategory $D^b(\mathcal{H}_I^\bullet)$ of all differential graded modules M^\bullet in $D(\mathcal{H}_I^\bullet)$ such that $h^j(M^\bullet) = 0$ for all but finitely many $j \in \mathbb{Z}$. Since I has finite cohomological dimension, the commutative diagram (2) shows that H restricts to a fully faithful functor

$$D^b(G) \longrightarrow D^b(\mathcal{H}_I^\bullet).$$

On the other hand, the behavior of the functor T is controlled by an Eilenberg–Moore spectral sequence

$$E_2^{r,s} = \text{Tor}_{-r}^{h^*(\mathcal{H}_I^*)}(\mathcal{H}_I, h^*(M^\bullet))^s \implies h^{r+s}(T(M^\bullet))$$

[May, Theorem 4.1]. This suggests that, except in very special cases, the functor T will not preserve the bounded subcategories.

5. Complements

5.1. The top cohomology. A first step in the investigation of the DGA \mathcal{H}_I^* might be the computation of its cohomology algebra $h^*(\mathcal{H}_I^*)$. By Remark 7, the latter is concentrated in degrees 0 to d . Of course the usual Hecke algebra $\mathcal{H}_I = h^0(\mathcal{H}_I^*)$ is a subalgebra of $h^*(\mathcal{H}_I^*)$. We determine here the top cohomology $h^d(\mathcal{H}_I^*)$ as a right \mathcal{H}_I -module.

Using the I -equivariant linear map

$$\begin{aligned} \pi_I : \text{ind}_I^G(1) &\longrightarrow \text{ind}_I^G(1)^I = \mathcal{H}_I \\ \phi &\longmapsto \left[h \longmapsto \sum_{g \in I/I \cap hIh^{-1}} \phi(gh) \right] \end{aligned}$$

we obtain the map

$$\pi_I^* : h^*(\mathcal{H}_I^*) = H^*(I, \text{ind}_I^G(1)) \xrightarrow{H^*(I, \pi_I)} H^*(I, \mathcal{H}_I) = H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I.$$

The last equality in this chain comes from the universal coefficient theorem, which is applicable since I as a Poincaré group [Lazard 1965, Théorème V.2.5.8] has finite cohomology $H^*(I, \mathbb{F}_p)$. Of course, as a ring \mathcal{H}_I is a right module over itself. For our purposes, we have to consider a modification of this module structure which is specific to characteristic p .

As a k -vector space $\text{ind}_I^G(1)^I = \mathcal{H}_I$ has the basis $\{\chi_{IxI}\}_{x \in I \backslash G/I}$ consisting of the characteristic functions of the double cosets IxI . If we denote the multiplication in the algebra \mathcal{H}_I , as usual, by the symbol “ $*$ ” for convolution, then in this basis it is given by the formula

$$\chi_{IxI} * \chi_{IhI} = \sum_{y \in I \backslash G/I} c_{x,y;h} \chi_{IyI},$$

where the coefficients are

$$c_{x,y;h} = (\chi_{IxI} * \chi_{IhI})(y) = \sum_{g \in G/I} \chi_{IxI}(g) \chi_{IhI}(g^{-1}y) = |IxI \cap yIh^{-1}I/I| \cdot 1_k,$$

with 1_k denoting the unit element in the field k . Of course, for fixed x and h we have $c_{x,y;h} = 0$ for all but finitely many $y \in I \backslash G/I$. But $IxI \cap yIh^{-1}I \neq \emptyset$ implies

$IxI \subseteq IyIh^{-1}I$; by compactness, the latter is a finite union of double cosets. Hence, also for fixed y and h , we have $c_{x,y;h} \neq 0$ for at most finitely many $x \in I \backslash G/I$. It follows that by combining the transpose of these coefficient matrices with the antiautomorphism

$$\begin{aligned} \mathcal{H}_I &\longrightarrow \mathcal{H}_I \\ \chi &\longmapsto \chi^*(g) := \chi(g^{-1}), \end{aligned}$$

we obtain through the formula

$$\chi_{IxI} *_{\tau} \chi_{IhI} := \sum_{y \in I \backslash G/I} c_{y,x;h^{-1}} \chi_{IyI}$$

a new right action of \mathcal{H}_I on itself. We denote this new module by \mathcal{H}_I^{τ} .

Remark. We compute

$$\begin{aligned} |IyI/I| \cdot c_{x,y;h} &= |IyI/I| \cdot (\chi_{IxI} * \chi_{IhI})(y) \\ &= \sum_{z \in G/I} \chi_{IyI}(z) (\chi_{IxI} * \chi_{Ih^{-1}I}^*)(z) \\ &= (\chi_{IyI} * (\chi_{IxI} * \chi_{Ih^{-1}I}^*)) (1) \\ &= ((\chi_{IyI} * \chi_{Ih^{-1}I}) * \chi_{IxI}^*) (1) \\ &= \sum_{z \in G/I} (\chi_{IyI} * \chi_{Ih^{-1}I})(z) \chi_{IxI}(z) \\ &= |IxI/I| \cdot (\chi_{IyI} * \chi_{Ih^{-1}I})(x) \\ &= |IxI/I| \cdot c_{y,x;h^{-1}}. \end{aligned}$$

This, of course, is valid with integral coefficients (instead of k). Moreover, $|IxI/I|$ is always a power of p . It follows that over any field of characteristic different from p one has $\mathcal{H}_I^{\tau} \cong \mathcal{H}_I$. It also follows that $c_{x,y;h} = c_{y,x;h^{-1}}$ whenever both are nonzero.

It is straightforward to check that

$$\pi_I(\phi) *_{\tau} \chi_{IhI} = \pi_I(\phi * \chi_{IhI})$$

holds true for any $\phi \in \text{ind}_I^G(1)$ and any $h \in G$. Hence,

$$\pi_I : \text{ind}_I^G(1) \longrightarrow \mathcal{H}_I^{\tau} \quad \text{and} \quad \pi_I^* : h^*(\mathcal{H}_I^{\tau}) \longrightarrow H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^{\tau}$$

are maps of right \mathcal{H}_I -modules.

Proposition 11. *The map π_I^d is an isomorphism*

$$h^d(\mathcal{H}_I^{\tau}) \xrightarrow{\cong} H^d(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^{\tau}$$

of right \mathcal{H}_I -modules. By fixing a basis of the one dimensional \mathbb{F}_p -vector space $H^d(I, \mathbb{F}_p)$, we therefore obtain $h^d(\mathcal{H}_I^\bullet) \cong \mathcal{H}_I^\tau$ as right \mathcal{H}_I -modules.

Proof. It remains to show that π_I^d is bijective. We have the I -equivariant decomposition

$$\text{ind}_I^G(1) = \bigoplus_{x \in I \backslash G / I} \text{ind}_{I \cap xIx^{-1}}^I(1).$$

The map π_I restricts to

$$\begin{aligned} \pi_I : \text{ind}_{I \cap xIx^{-1}}^I(1) &\longrightarrow k \cdot \chi_{IxI} \subseteq \mathcal{H}_I \\ \phi &\longmapsto \left(\sum_{y \in I / I \cap xIx^{-1}} \phi(y) \right) \cdot \chi_{IxI}. \end{aligned}$$

Since $H^*(I, \cdot)$ commutes with arbitrary direct sums it therefore suffices to show that the map

$$H^d \left(I, \phi \longmapsto \sum_{y \in I / I \cap xIx^{-1}} \phi(y) \right) : H^d(I, \text{ind}_{I \cap xIx^{-1}}^I(1_{\mathbb{F}_p})) \longrightarrow H^d(I, \mathbb{F}_p)$$

is bijective. Using Shapiro’s lemma this latter map identifies (compare [Serre 1994, §I.2.5]) with the corestriction map

$$\text{Cor} : H^d(I \cap xIx^{-1}, \mathbb{F}_p) \longrightarrow H^d(I, \mathbb{F}_p),$$

which for Poincaré groups of dimension d is an isomorphism of one dimensional vector spaces [op. cit., (4) on p. 37]. □

5.2. The easiest example. As an example, we will make explicit the case where $G = I = \mathbb{Z}_p$ is the additive group of p -adic integers, which we nevertheless write multiplicatively with unit element e . In order to distinguish it from the unit element $1 \in k$ we will denote the multiplicative unit in \mathbb{Z}_p by γ . Let Ω denote the completed group ring of \mathbb{Z}_p over k . We have:

- (a) The category $\text{Mod}_k(G)$ coincides with the category of torsion Ω -modules.
- (b) Sending $\gamma - 1$ to t defines an isomorphism of k -algebras $\Omega \cong k[[t]]$ between Ω and the formal power series ring in one variable t over k .

For any V in $\text{Mod}_k(G)$ we have the smooth G -representation $C^\infty(G, V)$ of all V -valued locally constant functions on G , where $g \in G$ acts on $f \in C^\infty(G, V)$ by ${}^g f(h) := g(f(g^{-1}h))$. One easily checks:

- (c) $C^\infty(G, V) = C^\infty(G, k) \otimes_k V$ with the diagonal G -action on the right hand side.
- (d) The map $\text{Hom}_{\text{Mod}_k(G)}(W, C^\infty(G, V)) \xrightarrow{\cong} \text{Hom}_k(W, V)$ sending F to $[w \mapsto F(w)(e)]$ is an isomorphism for any W in $\text{Mod}_k(G)$. It follows that $C^\infty(G, V)$ is an injective object in $\text{Mod}_k(G)$.

(e) The short exact sequence

$$(3) \quad 0 \longrightarrow V \longrightarrow C^\infty(G, k) \otimes_k V \xrightarrow{\gamma_* - 1 \otimes \text{id}} C^\infty(G, k) \otimes_k V \longrightarrow 0,$$

where $\gamma_*(\phi)(h) = \phi(h\gamma)$ is an injective resolution of V in $\text{Mod}_k(G)$.

(f) For any $g \in G$ define the map $F_g : C^\infty(G, k) \rightarrow C^\infty(G, k)$ by $F_g(\phi)(h) := \phi(hg)$. In particular, $F_\gamma = \gamma_*$. Sending g to F_g defines an isomorphism of k -algebras

$$\Omega \xrightarrow{\cong} \text{End}_{\text{Mod}_k(G)}(C^\infty(G, k)).$$

Obviously $\text{ind}_I^G(1) = k$ is the trivial G -representation. By (3) we may take for \mathcal{I}^\bullet the injective resolution

$$C^\infty(G, k) \xrightarrow{\gamma_* - 1} C^\infty(G, k) \longrightarrow 0 \longrightarrow \dots$$

Using (f) we deduce that \mathcal{H}_I^* is

$$\dots \longrightarrow \mathcal{H}_I^{-1} = \Omega \xrightarrow{d^{-1}} \mathcal{H}_I^0 = \Omega \times \Omega \xrightarrow{d^0} \mathcal{H}_I^1 = \Omega \longrightarrow \dots$$

with

$$d^{-1}a = ((\gamma - 1)a, (\gamma - 1)a) \quad \text{and} \quad d^0(a, b) = (\gamma - 1)(a - b)$$

and multiplication

$$\begin{aligned} (a_{-1}, (a_0, b_0), a_1) \cdot (a'_{-1}, (a'_0, b'_0), a'_1) \\ = (a'_0 a_{-1} + a'_{-1} b_0, (a'_0 a_0 - a'_{-1} a_1, b'_0 b_0 - a'_1 a_{-1}), a'_1 a_0 + b'_0 a_1). \end{aligned}$$

Using (b) we then identify \mathcal{H}_I^* with the upper row in the commutative diagram

$$\begin{array}{ccccc} k[[t]] & \xrightarrow{a \mapsto (ta, ta)} & k[[t]] \times k[[t]] & \xrightarrow{(a, b) \mapsto t(a-b)} & k[[t]] \\ \uparrow & & \uparrow a \mapsto (a, a) & & \uparrow \subseteq \\ 0 & \longrightarrow & k & \xrightarrow{0} & k \end{array}$$

We view the bottom row as the differential graded algebra of dual numbers $k[\epsilon]/(\epsilon^2)$ in degrees 0 and 1 with the zero differential. It is easy to check that the vertical arrows in the above diagram constitute a quasi-isomorphism of differential graded algebras. In particular, this says that \mathcal{H}_I^* is quasi-isomorphic to its cohomology algebra with zero differential (ϵ corresponds to the projection map $G = \mathbb{Z}_p \rightarrow \mathbb{F}_p \subseteq k$, as a generator of $H^1(G, k) = \text{Hom}^{\text{cont}}(\mathbb{Z}_p, k)$). According to our Theorem 9, we therefore obtain that H composed with the pullback along the above quasi-isomorphism is an equivalence of triangulated categories

$$(4) \quad D(\mathbb{Z}_p) \xrightarrow{\cong} D(k[\epsilon]/(\epsilon^2)).$$

We finish by determining this functor explicitly. Let V be an object in $\text{Mod}_k(G)$. Using the injective resolution (3) we can represent $H(V)$ by the complex

$$\text{Hom}_{\text{Mod}_k(G)}^\bullet([C^\infty(G, k) \xrightarrow{\gamma_*^{-1}} C^\infty(G, k)], [C^\infty(G, k) \otimes_k V \xrightarrow{\gamma_*^{-1} \otimes \text{id}} C^\infty(G, k) \otimes_k V]).$$

Furthermore, using the identifications in (c) and (d), this latter complex can be computed to be the complex

$$\begin{aligned} \text{Hom}_k(C^\infty(G, k), V) &\xrightarrow{d^{-1}} \text{Hom}_k(C^\infty(G, k), V) \times \text{Hom}_k(C^\infty(G, k), V) \\ &\xrightarrow{d^0} \text{Hom}_k(C^\infty(G, k), V) \end{aligned}$$

in degrees $-1, 0$, and 1 with the differentials

$$\begin{aligned} d^{-1}f &= (f \circ (\gamma_* - 1), f \circ (\gamma_* - 1) + (\gamma - 1) \circ f \circ \gamma_*) \quad \text{and} \\ d^0(f_0, f_1) &= (\gamma - 1) \circ f_0 \circ \gamma_* + (f_0 - f_1) \circ (\gamma_* - 1). \end{aligned}$$

Let $\delta_e \in \text{Hom}_k(C^\infty(G, k), k)$ denote the ‘‘Dirac distribution’’ $\delta_e(\phi) := \phi(e)$ in the unit element. The diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \text{Hom}_k(C^\infty(G, k), V) \\ \downarrow & & \downarrow d^{-1} \\ V & \xrightarrow{v \mapsto (\delta_e(\cdot)v, \delta_e(\cdot)\gamma(v))} & \text{Hom}_k(C^\infty(G, k), V) \times \text{Hom}_k(C^\infty(G, k), V) \\ \downarrow \gamma^{-1} & & \downarrow d^0 \\ V & \xrightarrow{v \mapsto \delta_e(\cdot)v} & \text{Hom}_k(C^\infty(G, k), V) \end{array}$$

is commutative. We claim that the horizontal arrows form a quasi-isomorphism α^\bullet . In order to define a map in the opposite direction we let $\phi_1 \in C^\infty(G, k)$ denote the constant function with value 1. Using that $\gamma_*(\phi_1) = \phi_1$, one checks that the diagram

$$\begin{array}{ccc} \text{Hom}_k(C^\infty(G, k), V) & \longrightarrow & 0 \\ \downarrow d^{-1} & & \downarrow \\ \text{Hom}_k(C^\infty(G, k), V) \times \text{Hom}_k(C^\infty(G, k), V) & \xrightarrow{(f_0, f_1) \mapsto f_0(\phi_1)} & V \\ \downarrow d^0 & & \downarrow \gamma^{-1} \\ \text{Hom}_k(C^\infty(G, k), V) & \xrightarrow{f \mapsto f(\phi_1)} & V \end{array}$$

is commutative. Hence the horizontal arrows define a homomorphism of complexes β^\bullet such that $\beta^\bullet \circ \alpha^\bullet = \text{id}$. Applying $\text{Hom}_k(\cdot, V)$ to our injective resolution of k , we obtain the short exact sequence

$$0 \longrightarrow \text{Hom}_k(C^\infty(G, k), V) \xrightarrow{f \mapsto f \circ (\gamma_*^{-1})} \text{Hom}_k(C^\infty(G, k), V) \xrightarrow{\beta^1} V \longrightarrow 0.$$

This implies that d^{-1} is injective and that $\text{im}(d^0) \supseteq \ker(\beta^1)$. The former says that the cohomology in degree -1 is zero. Because of

$$(5) \quad \text{Hom}_k(C^\infty(G, k), V) = \ker(\beta^1) \oplus \text{im}(\alpha^1),$$

the latter shows the surjectivity of $h^1(\alpha^*)$. Hence $h^1(\alpha^*)$ is bijective. A pair (f_0, f_1) represents a class in $\ker(h^0(\beta^*))$ if and only if $d^0(f_0, f_1) = 0$ and $\beta^0(f_0, f_1) = 0$. The first condition implies that

$$f_1 \circ (\gamma_* - 1) = (\gamma - 1) \circ f_0 \circ \gamma_* + f_0 \circ (\gamma_* - 1).$$

By (5) the second condition says that we may write $f_0 = \delta_e(\cdot)v + f \circ (\gamma_* - 1)$ for $v := f_0(\phi_1) \in V$ and some $f \in \text{Hom}_k(C^\infty(G, k), V)$. Inserting this into the above equation we obtain

$$f_1 \circ (\gamma_* - 1) = \delta_e(\cdot)(\gamma(v) - v) + (\gamma \circ f \circ \gamma_* - f) \circ (\gamma_* - 1).$$

It follows that

$$\gamma(v) = v \quad \text{and} \quad f_1 = (\gamma \circ f \circ \gamma_* - f).$$

Using this last identity one checks that $(f_0, f_1) = d^{-1}f + (\delta_e(\cdot)v, 0)$. But we have $0 = d^0(\delta_e(\cdot)v, 0) = \delta_e(\gamma_* \cdot)(\gamma - 1)(v) + \delta_e((\gamma_* - 1) \cdot)v = \delta_e((\gamma_* - 1) \cdot)v$, which implies that $v = 0$. We conclude that $h^0(\beta^*)$ is injective and hence bijective and that therefore $h^0(\alpha^*)$ is bijective.

A differential graded $k[\epsilon]/(\epsilon^2)$ -module is the same as a graded k -vector space with two anticommuting differentials ϵ and d of degree 1. Given the smooth G -representation V , we form the graded $k[\epsilon]/(\epsilon^2)$ -module $k[\epsilon]/(\epsilon^2) \otimes_k V$ (sitting in degrees 0 and 1) and equip it with the differential $d_V(v_0 + v_1\epsilon) := (\gamma - 1)(v_0)\epsilon$. The above computations together with the fact that ϵ corresponds to the identity in $\mathcal{H}_I^1 = \text{Hom}_{\text{Mod}_k(G)}(\mathcal{I}^0, \mathcal{I}^1) = \text{End}_{\text{Mod}_k(G)}(C^\infty(G, k))$ proves the following:

Proposition 12. *The equivalence (4) sends V in $\text{Mod}_k(G)$ to the differential graded module $(k[\epsilon]/(\epsilon^2) \otimes_k V, d_V)$.*

Acknowledgement

I thank W. Soergel for a very inspiring discussion about injective resolutions and DGAs.

References

[Bernstein 1984] J. N. Bernstein, “Le “centre” de Bernstein”, pp. 1–32 in *Représentations des groupes réductifs sur un corps local*, edited by P. Deligne, Hermann, Paris, 1984. MR 86e:22028 Zbl 0599.22016

[Bernstein and Lunts 1994] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics **1578**, Springer, Berlin, 1994. MR 95k:55012 Zbl 0808.14038

- [Bondal and van den Bergh 2003] A. Bondal and M. van den Bergh, “Generators and representability of functors in commutative and noncommutative geometry”, *Mosc. Math. J.* **3**:1 (2003), 1–36, 258. MR 2004h:18009 Zbl 1135.18302
- [Bourbaki 2006] N. Bourbaki, *Algèbre commutative: Chapitres 8 et 9*, Springer, Berlin, 2006. MR 2007h:13001 Zbl 1103.13003
- [Harris 2015] M. Harris, “Speculations on the mod p representation theory of p -adic groups”, preprint, Institut de mathématiques de Jussieu, 2015.
- [Hartshorne 1966] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics **20**, Springer, New York, 1966. MR 36 #5145 Zbl 0212.26101
- [Kashiwara and Schapira 2006] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften **332**, Springer, Berlin, 2006. MR 2006k:18001 Zbl 1118.18001
- [Keller 1994] B. Keller, “Deriving DG categories”, *Ann. Sci. École Norm. Sup. (4)* **27**:1 (1994), 63–102. MR 95e:18010 Zbl 0799.18007
- [Keller 1998] B. Keller, “On the construction of triangle equivalences”, pp. 155–176 in *Derived equivalences for group rings*, edited by S. König and A. Zimmermann, Lecture Notes in Math. **1685**, Springer, Berlin, 1998. MR 1649844
- [Koziol 2014] K. Koziol, *Pro- p -Iwahori–Hecke Algebras in the mod- p Local Langlands Program*, Ph.D. thesis, Columbia University, 2014, Available at <http://search.proquest.com/docview/1537396486>. MR 3232375
- [Krause 2007] H. Krause, “Derived categories, resolutions, and Brown representability”, pp. 101–139 in *Interactions between homotopy theory and algebra* (Chicago, 2004), edited by L. L. Avramov et al., Contemp. Math. **436**, Amer. Math. Soc., Providence, RI, 2007. MR 2008h:18012a Zbl 1132.18005
- [Lazard 1965] M. Lazard, “Groupes analytiques p -adiques”, *Inst. Hautes Études Sci. Publ. Math.* **26** (1965), 389–603. MR 35 #188 Zbl 0139.02302
- [May] J. P. May, “Derived categories from a topological point of view”, lecture notes, University of Chicago, ca. 1995.
- [Neeman 1992] A. Neeman, “The connection between the K -theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel”, *Ann. Sci. École Norm. Sup. (4)* **25**:5 (1992), 547–566. MR 93k:18015 Zbl 0868.19001
- [Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, 2001. MR 2001k:18010 Zbl 0974.18008
- [Ollivier 2009] R. Ollivier, “Le foncteur des invariants sous l’action du pro- p -Iwahori de $GL_2(F)$ ”, *J. Reine Angew. Math.* **635** (2009), 149–185. MR 2011c:22031 Zbl 1181.22017
- [Ollivier and Schneider 2015] R. Ollivier and P. Schneider, “A canonical torsion theory for pro- p Iwahori–Hecke modules”, preprint, 2015.
- [Schneider 2011] P. Schneider, *p -adic Lie groups*, Grundlehren der Mathematischen Wissenschaften **344**, Springer, Heidelberg, 2011. MR 2012h:22010 Zbl 1223.22008
- [Serpé 2003] C. Serpé, “Resolution of unbounded complexes in Grothendieck categories”, *J. Pure Appl. Algebra* **177**:1 (2003), 103–112. MR 2003j:18015 Zbl 1033.18007
- [Serre 1965] J.-P. Serre, “Sur la dimension cohomologique des groupes profinis”, *Topology* **3** (1965), 413–420. MR 31 #4853 Zbl 0136.27402
- [Serre 1994] J.-P. Serre, *Cohomologie Galoisienne*, 5ème ed., Lecture Notes in Mathematics **5**, Springer, Berlin, 1994. MR 96b:12010 Zbl 0812.12002
- [Spaltenstein 1988] N. Spaltenstein, “Resolutions of unbounded complexes”, *Compositio Math.* **65**:2 (1988), 121–154. MR 89m:18013 Zbl 0636.18006

[Vignéras 1996] M.-F. Vignéras, *Représentations l -modulaires d'un groupe réductif p -adique avec $l \neq p$* , Progress in Mathematics **137**, Birkhäuser, Boston, MA, 1996. MR 97g:22007 Zbl 0859.22001

Received May 28, 2015.

PETER SCHNEIDER
MATHEMATISCHES INSTITUT
UNIVERSITÄT MÜNSTER
EINSTEINSTR. 62
48149 MÜNSTER
GERMANY
pschnei@uni-muenster.de

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US \$420/year for the electronic version, and \$570/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 279 No. 1-2 December 2015

In memoriam: Robert Steinberg

Robert Steinberg (1922–2014): In memoriam V. S. VARADARAJAN	1
Cellularity of certain quantum endomorphism algebras HENNING H. ANDERSEN, GUSTAV I. LEHRER and RUIBIN ZHANG	11
Lower bounds for essential dimensions in characteristic 2 via orthogonal representations ANTONIO BABIC and VLADIMIR CHERNOUSOV	37
Cocharacter-closure and spherical buildings MICHAEL BATE, SEBASTIAN HERPEL, BENJAMIN MARTIN and GERHARD RÖHRLE	65
Embedding functor for classical groups and Brauer–Manin obstruction EVA BAYER-FLUCKIGER, TING-YU LEE and RAMAN PARIMALA	87
On maximal tori of algebraic groups of type G_2 CONSTANTIN BELI, PHILIPPE GILLE and TING-YU LEE	101
On extensions of algebraic groups with finite quotient MICHEL BRION	135
Essential dimension and error-correcting codes SHANE CERNELE and ZINOVY REICHSTEIN	155
Notes on the structure constants of Hecke algebras of induced representations of finite Chevalley groups CHARLES W. CURTIS	181
Complements on disconnected reductive groups FRANÇOIS DIGNE and JEAN MICHEL	203
Extending Hecke endomorphism algebras JIE DU, BRIAN J. PARSHALL and LEONARD L. SCOTT	229
Products of partial normal subgroups ELLEN HENKE	255
Lusztig induction and ℓ -blocks of finite reductive groups RADHA KESSAR and GUNTER MALLE	269
Free resolutions of some Schubert singularities MANOJ KUMMINI, VENKATRAMANI LAKSHMIBAI, PRAMATHANATH SASTRY and C. S. SESHADRI	299
Free resolutions of some Schubert singularities in the Lagrangian Grassmannian VENKATRAMANI LAKSHMIBAI and REUVEN HODGES	329
Distinguished unipotent elements and multiplicity-free subgroups of simple algebraic groups MARTIN W. LIEBECK, GARY M. SEITZ and DONNA M. TESTERMAN	357
Action of longest element on a Hecke algebra cell module GEORGE LUSZTIG	383
Generic stabilisers for actions of reductive groups BENJAMIN MARTIN	397
On the equations defining affine algebraic groups VLADIMIR L. POPOV	423
Smooth representations and Hecke modules in characteristic p PETER SCHNEIDER	447
On CRDAHA and finite general linear and unitary groups BHAMA SRINIVASAN	465
Weil representations of finite general linear groups and finite special linear groups PHAM HUU TIEP	481
The pro- p Iwahori Hecke algebra of a reductive p -adic group, V (parabolic induction) MARIE-FRANCE VIGNÉRAS	499
Acknowledgement	531



0030-8730(2015)279:1;1-1