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SMOOTH REPRESENTATIONS AND HECKE MODULES IN CHARACTERISTIC p

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Dedicated to the memory of Robert Steinberg.

Let G be a p-adic Lie group and $I \subseteq G$ be a compact open subgroup which is a torsionfree pro-p-group. Working over a coefficient field k of characteristic p we introduce a differential graded Hecke algebra for the pair (G, I) and show that the derived category of smooth representations of G in k-vector spaces is naturally equivalent to the derived category of differential graded modules over this Hecke DGA.

1. Background and motivation

Let G be a d-dimensional p-adic Lie group, and let k be any field. We denote by $\mathrm{Mod}_k(G)$ the category of smooth G-representations in k-vector spaces. It obviously has arbitrary direct sums.

Fix a compact open subgroup $I \subseteq G$. In $\mathrm{Mod}_k(G)$ we then have the representation

$$\mathrm{ind}_I^G(1) := \{k\text{-valued functions with finite support on } G/I\}$$

with G acting by left translations. Being generated by a single element, which is the characteristic function of the trivial coset, $\mathrm{ind}_I^G(1)$ is a compact object in $\mathrm{Mod}_k(G)$. It generates the full subcategory $\mathrm{Mod}_k^I(G)$ of all representations V in $\mathrm{Mod}_k(G)$ which are generated by their I-fixed vectors V^I. In general, $\mathrm{Mod}_k^I(G)$ is not an abelian category. The Hecke algebra of I by definition is the endomorphism ring

$$\mathcal{H}_I := \mathrm{End}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1))^{\mathrm{op}}.$$

We let $\mathrm{Mod}(\mathcal{H}_I)$ denote the category of left unital $\mathcal{H}_I$-modules. There is the pair of adjoint functors

$$\begin{aligned} H^0 : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V), \end{aligned}$$

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and

$$T_0 : \mathrm{Mod}(\mathcal{H}_I) \longrightarrow \mathrm{Mod}_k^I(G) \subseteq \mathrm{Mod}_k(G)$$
$$M \longmapsto \mathrm{ind}_I^G(1) \otimes_{\mathcal{H}_I} M.$$

If the characteristic of k does not divide the pro-order of I then the functor H^0 is exact. Then $\mathrm{ind}_I^G(1)$ is a projective compact object in $\mathrm{Mod}_k(G)$. Since it does not generate the full category $\mathrm{Mod}_k(G)$, one cannot apply the Gabriel–Popescu theorem (compare [Kashiwara and Schapira 2006, Theorem 8.5.8]) to the functor H^0. Nevertheless, in this case, one might hope for a close relation between the categories $\mathrm{Mod}_k^I(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$. This indeed happens, for example, for a connected reductive group G and its Iwahori subgroup I and the field $k = \mathbb{C}$; compare [Bernstein 1984, Corollary 3.9(ii)]. In addition, in this situation the algebra $\mathcal{H}_I$ turns out to be a generalized affine Hecke algebra so that its structure is explicitly known. Therefore, in characteristic zero, Hecke algebras have become one of the most important tools in the investigation of smooth G-representations.

In this light, it is a pressing question to better understand the relation between the two categories $\mathrm{Mod}_k(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$ in the opposite situation where k has characteristic p. Since p always will divide the pro-order of I, the functor H^0 certainly is no longer exact. Both functors H^0 and T_0 now have a very complicated behavior and little is known [Koziol 2014; Ollivier 2009; Ollivier and Schneider 2015]. This suggests that one should work in a derived framework which takes into account the higher cohomology of I.

This paper will demonstrate that by doing this — not in a naive way but in an appropriate differential graded context — the situation does improve drastically. We will show the somewhat surprising result that the object $\mathrm{ind}_I^G(1)$ becomes a compact generator of the full derived category of G provided I is a torsionfree pro-p-group.

The main result of this paper was proved already in 2007 but remained unpublished. At the time, we gave a somewhat ad hoc proof. Although the main arguments remain unchanged we now, by appealing to a general theorem of Keller, have arranged them in a way which makes the reasoning more transparent. In the context of the search for a p-adic local Langlands program, there is increasing interest in studying derived situations; see [Harris 2015]. We also have now [Ollivier and Schneider 2015] the first examples of explicit computations of the cohomology groups $H^i(I, \mathrm{ind}_I^G(1))$. I hope that these are sufficient reasons to finally publish the paper.

2. The unbounded derived category of G

We assume from now on throughout the paper that the field k has characteristic p and that I is a torsionfree pro-p-group. Let us first of all collect a few properties of the abelian category $\mathrm{Mod}_k(G)$.

Lemma 1. (i) $\mathrm{Mod}_k(G)$ *is* (AB5), *i.e., it has arbitrary colimits and filtered colimits are exact.*

(ii) $\mathrm{Mod}_k(G)$ *is* (AB3*), *i.e., it has arbitrary limits.*

(iii) $\mathrm{Mod}_k(G)$ *has enough injective objects.*

(iv) $\mathrm{Mod}_k(G)$ *is a Grothendieck category, i.e., it satisfies* (AB5) *and has a generator.*

(v) $V^I \neq 0$ *for any nonzero V in* $\mathrm{Mod}_k(G)$.

Proof. (i) This is obvious. (ii) Take the subspace of smooth vectors in the limit of k-vector spaces. (iii) This is shown in [Vignéras 1996, §I.5.9]. Alternatively, it is a consequence of (iv); compare [Kashiwara and Schapira 2006, Theorem 9.6.2]. (v) Since I is pro-p, where p is the characteristic of k, the only irreducible smooth representation of I is the trivial one.

(iv) Because of (i) it remains to exhibit a generator of $\mathrm{Mod}_k(G)$. We define

$$Y := \bigoplus_J \mathrm{ind}_J^G(1),$$

where J runs over all open subgroups in G. For any V in $\mathrm{Mod}_k(G)$, we have

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(Y, V) = \prod_J V^J.$$

Since $V = \bigcup_J V^J$, we easily deduce that Y is a generator of $\mathrm{Mod}_k(G)$. □

As usual, let $D(G) := D(\mathrm{Mod}_k(G))$ be the derived category of unbounded complexes in $\mathrm{Mod}_k(G)$.

Remark 2. $D(G)$ has arbitrary direct sums, which can be computed as direct sums of complexes.

Proof. See the first paragraph in [Kashiwara and Schapira 2006, §14.3]. □

According to [Lazard 1965, Théorème V.2.2.8; Serre 1965], the group I has cohomological dimension d. This means that the higher derived functors of the left exact functor

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Vec}_k \\ E &\longmapsto E^I \end{aligned}$$

into the category Vec_k of k-vector spaces are zero in degrees $> d$. On the other hand, the restriction functor

$$\begin{aligned} \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(I) \\ V &\longmapsto V|_I \end{aligned}$$

is exact and respects injective objects. The latter is a consequence of the fact that compact induction

$$\begin{aligned}\mathrm{Mod}_k(I) &\longrightarrow \mathrm{Mod}_k(G)\\ E &\longmapsto \mathrm{ind}_I^G(E)\end{aligned}$$

is an exact left adjoint; compare [Vignéras 1996, §I.5.7]. Hence the higher derived functors of the composed functor

$$\begin{aligned}H^0(I, \cdot\,) : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Vec}_k\\ V &\longmapsto V^I\end{aligned}$$

are given by $V \longmapsto H^i(I, V|_I)$ and vanish in degrees $> d$. It follows that the total right derived functor

$$RH^0(I, \cdot\,) : D(G) \longrightarrow D(\mathrm{Vec}_k)$$

between the corresponding (unbounded) derived categories exists [Hartshorne 1966, Corollary I.5.3].

To compute $RH^0(I, \cdot\,)$, we use the formalism of K-injective complexes as developed in [Spaltenstein 1988]. We let $C(\mathrm{Mod}_k(G))$ and $K(\mathrm{Mod}_k(G))$ denote the category of unbounded complexes in $\mathrm{Mod}_k(G)$ with chain maps and homotopy classes of chain maps, respectively, as morphisms. The K-injective complexes form a full triangulated subcategory $K_{\mathrm{inj}}(\mathrm{Mod}_k(G))$ of $K(\mathrm{Mod}_k(G))$. Exactly in the same way as [op. cit., Proposition 3.11] one can show that any complex in $C(\mathrm{Mod}_k(G))$ has a right K-injective resolution (recall from Lemma 1(ii) that the category $\mathrm{Mod}_k(G)$ has inverse limits). Alternatively, one may apply [Serpé 2003, Theorem 3.13] or [Kashiwara and Schapira 2006, Theorem 14.3.1] based upon Lemma 1(iv). The existence of K-injective resolutions means that the natural functor

$$K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \xrightarrow{\simeq} D(G)$$

is an equivalence of triangulated categories. We fix a quasi-inverse $\boldsymbol{i}$ of this functor. Then the derived functor $RH^0(I, \cdot\,)$ is naturally isomorphic to the composed functor

$$D(G) \xrightarrow{\;\boldsymbol{i}\;} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathrm{Vec}_k) \longrightarrow D(\mathrm{Vec}_k)$$

with the middle arrow given by

$$V^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet).$$

Explanation. Let $V^\bullet$ be a complex in $C(\mathrm{Mod}_k(G))$. To compute $RH^0(I, \cdot\,)$ according to [Hartshorne 1966], one chooses a quasi-isomorphism $V^\bullet \xrightarrow{\simeq} C^\bullet$ into a complex consisting of objects which are acyclic for the functor $H^0(I, \cdot\,)$. On the other hand, let $V^\bullet \xrightarrow{\simeq} A^\bullet$ be a quasi-isomorphism into a K-injective complex. By

[Spaltenstein 1988, Proposition 1.5(c)] we then have, in $K(\mathrm{Mod}_k(G))$, a unique commutative diagram:

$$\begin{array}{ccc} & & C^\bullet \\ & \nearrow & \big\downarrow \\ V^\bullet & & \\ & \searrow & \big\downarrow \\ & & A^\bullet \end{array}$$

We claim that the induced map

$$(C^\bullet)^I \xrightarrow{\simeq} (A^\bullet)^I$$

is a quasi-isomorphism. Choose quasi-isomorphisms

$$A^\bullet \xrightarrow{\simeq} \tilde{C}^\bullet \xrightarrow{\simeq} \tilde{A}^\bullet$$

where $\tilde{C}^\bullet$ consists of $H^0(I, \cdot\,)$-acyclic objects and $\tilde{A}^\bullet$ is K-injective. By [Spaltenstein 1988, Proposition 1.5(b)], the composite is an isomorphism in $K(\mathrm{Mod}_k(G))$ and hence induces a quasi-isomorphism $(A^\bullet)^I \xrightarrow{\simeq} (\tilde{A}^\bullet)^I$. But by [Hartshorne 1966, Theorem I.5.1 and Corollary I.5.3(γ)], the composite $C^\bullet \xrightarrow{\simeq} A^\bullet \xrightarrow{\simeq} \tilde{C}^\bullet$ also induces a quasi-isomorphism $(C^\bullet)^I \xrightarrow{\simeq} (\tilde{C}^\bullet)^I$.

Lemma 3. *The* (*hyper*)*cohomology functor* $H^\ell(I, \cdot\,)$, *for any* $\ell \in \mathbb{Z}$, *commutes with arbitrary direct sums in* $D(G)$.

Proof. First of all we observe that the cohomology functor $H^\ell(I, \cdot\,)$ commutes with arbitrary direct sums in $\mathrm{Mod}_k(G)$ [Serre 1994, §I.2.2, Proposition 8]. This, in particular, implies that arbitrary direct sums of $H^0(I, \cdot\,)$-acyclic objects in $\mathrm{Mod}_k(G)$ again are $H^0(I, \cdot\,)$-acyclic. Now let $(V_j^\bullet)_{j\in J}$ be a family of objects in $D(G)$, where we view each $V_j^\bullet$ as an actual complex. Then, according to Remark 2, the direct sum of the $V_j^\bullet$ in $D(G)$ is represented by the direct sum complex $\bigoplus_j V_j^\bullet$. Now we choose quasi-isomorphisms $V_j^\bullet \xrightarrow{\simeq} C_j^\bullet$ in $C(\mathrm{Mod}_k(G))$, where all representations C_j^m are $H^0(I, \cdot\,)$-acyclic. By the preliminary observation, the direct sum map

$$\bigoplus_j V_j^\bullet \xrightarrow{\simeq} C^\bullet := \bigoplus_j C_j^\bullet$$

again is a quasi-isomorphism where all terms of the target complex are $H^0(I, \cdot\,)$-acyclic. We therefore obtain

$$H^\ell\big(I, \bigoplus_j V_j^\bullet\big) = h^\ell((C^\bullet)^I) = \bigoplus_j h^\ell((C_j^\bullet)^I) = \bigoplus_j H^\ell(I, V_j^\bullet). \qquad \square$$

As usual, we view $\mathrm{Mod}_k(G)$ as the full subcategory of those complexes in $D(G)$ which have zero terms outside of degree zero.

Lemma 4. $\mathrm{ind}_I^G(1)$ *is a compact object in* $D(G)$.

Proof. We have to show that the functor $\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), \cdot)$ commutes with arbitrary direct sums in $D(G)$. For any $V^\bullet$ in $D(G)$, we compute

$$\text{(1)} \qquad \mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet) = \mathrm{Hom}_{K(\mathrm{Mod}_k(G))}(\mathrm{ind}_I^G(1), \boldsymbol{i}(V^\bullet)) = h^0(\boldsymbol{i}(V^\bullet)^I) = H^0(I, V^\bullet),$$

where the first identity uses [Spaltenstein 1988, Proposition 1.5(b)]. The claim therefore follows from Lemma 3. □

Proposition 5. *Let $E^\bullet$ be in $D(I)$. Then $E^\bullet = 0$ if and only if $H^j(I, E^\bullet) = 0$ for any $j \in \mathbb{Z}$.*

Proof. The completed group ring $\Omega := \varprojlim_N k[I/N]$ of I over k, where N runs over all open normal subgroups of I, is a pseudocompact local ring; compare [Schneider 2011, §19]. If $\mathfrak{m} \subseteq \Omega$ denotes the maximal ideal, then $\Omega/\mathfrak{m} = k$. Since Ω is noetherian — [Lazard 1965, Proposition V.2.2.4] for $k = \mathbb{F}_p$ and [Schneider 2011, Theorem 33.4] together with [Bourbaki 2006, Chapitre IX, §2.3, Proposition 5] in general — its pseudocompact topology coincides with the $\mathfrak{m}$-adic topology [Schneider 2011, Lemma 19.8]. This implies that:

- $\Omega/\mathfrak{m}^j$ lies in $\mathrm{Mod}_k(I)$ for any $j \in \mathbb{N}$.
- For any E in $\mathrm{Mod}_k(I)$, we have

$$E = \bigcup_{j \in \mathbb{N}} E^{\mathfrak{m}^j = 0} \quad \text{where } E^{\mathfrak{m}^j=0} := \{v \in E : \mathfrak{m}^j v = 0\}.$$

Because of

$$E^{\mathfrak{m}^j=0} = \mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, E),$$

we need to consider the left exact functors $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ on $\mathrm{Mod}_k(I)$. Their right derived functors, of course, are $\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$. In particular,

$$\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}, \cdot) = H^i(I, \cdot).$$

For any $j \in \mathbb{N}$, we have the short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0$$

in $\mathrm{Mod}_k(I)$. Moreover, $\mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{n(j)}$ for some $n(j) \geq 0$ since Ω is noetherian. The associated long exact Ext-sequence therefore reads

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, \cdot) \longrightarrow H^i(I, \cdot)^{n(j)} \longrightarrow \cdots$$

By induction with respect to j, we deduce that:

- Each functor $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, \cdot)$ has cohomological dimension $\leq d$.

– Each $H^0(I,\cdot)$-acyclic object in $\mathrm{Mod}_k(I)$ is $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,\cdot)$-acyclic for any $j\geq 1$.

It follows that the total right derived functors $R\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,\cdot)$ on $D(I)$ exist. More explicitly, let $E^\bullet$ be any complex in $D(I)$ and choose a quasi-isomorphism $E^\bullet \xrightarrow{\simeq} C^\bullet$ into a complex consisting of $H^0(I,\cdot)$-acyclic objects. It then follows that we have the short exact sequence of complexes

$$0\to \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet)\to \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, C^\bullet)\to ((C^\bullet)^I)^{n(j)}\to 0.$$

Suppose now that $RH^0(I,E^\bullet)=0$. This means that the complex $(C^\bullet)^I$ is exact. By induction with respect to j, we obtain the exactness of the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet)=(C^\bullet)^{\mathfrak{m}^j=0}$$

for any $j\in\mathbb{N}$. Hence $C^\bullet$ and $E^\bullet$ are exact. □

Proposition 6. $\mathrm{ind}_I^G(1)$ *is a generator of the triangulated category* $D(G)$ *in the sense that any strictly full triangulated subcategory of* $D(G)$, *closed under all direct sums*, *which contains* $\mathrm{ind}_I^G(1)$, *coincides with* $D(G)$.

Proof. By (1) we have

$$\begin{aligned}\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1)[j], V^\bullet) &= \mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet[-j])\\ &= H^0(I, V^\bullet[-j]) = H^{-j}(I, V^\bullet)\end{aligned}$$

for any $V^\bullet$ in $D(G)$. Hence, Proposition 5 implies that the family of shifts $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is a generating set of $D(G)$ in the sense of Neeman [2001, Definition 8.1.1]. On the other hand, by Lemma 4, each shift $\mathrm{ind}_I^G(1)[j]$ is a compact object. In Neeman's language this means that $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is an $\aleph_0$-perfect class consisting of $\aleph_0$-small objects [Neeman 2001, Remark 4.2.6 and Definition 4.2.7]. According to Neeman's Lemma 4.2.1, the class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ then is β-perfect for any infinite cardinal β. Hence Neeman's Theorem 8.3.3 applies and shows (see the explanations in §3.2.6–3.2.8 of that same reference) that any strictly full triangulated subcategory of $D(G)$ closed under all direct sums which contains $\mathrm{ind}_I^G(1)$, and therefore the whole class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$, coincides with $D(G)$. □

3. The Hecke DGA

In order to also "derive" the picture on the Hecke algebra side we fix an injective resolution $\mathrm{ind}_I^G(1)\xrightarrow{\simeq}\mathcal{I}^\bullet$ in $C(\mathrm{Mod}_k(G))$ and introduce the differential graded algebra

$$\mathcal{H}_I^\bullet := \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$$

over k. We recall that

$$\mathcal{H}_I^n = \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n})$$

with differential

$$(da)_q(x) = d(a_q(x)) - (-1)^n a_{q+1}(dx)$$

for $a = (a_q) \in \mathcal{H}_I^n$ and multiplication

$$(ba)_q := (-1)^{mn} a_{q+m} \circ b_q$$

for $a = (a_q) \in \mathcal{H}_I^n$ and $b = (b_q) \in \mathcal{H}_I^m$. The cohomology of $\mathcal{H}_I^\bullet$ is given by

$$h^*(\mathcal{H}_I^\bullet) = \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1));$$

compare [Hartshorne 1966, §I.6]. In particular,

$$h^0(\mathcal{H}_I^\bullet) = \mathcal{H}_I.$$

Remark 7. $h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1))$ and, in particular, $h^i(\mathcal{H}_I^\bullet) = 0$ for $i > d$.

Proof. We compute

$$\begin{aligned} h^*(\mathcal{H}_I^\bullet) &= \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1)) \\ &= h^*(\mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathcal{I}^\bullet)) \\ &= h^*((\mathcal{I}^\bullet)^I) = H^*(I, \mathrm{ind}_I^G(1)). \end{aligned}$$

□

Let $D(\mathcal{H}_I^\bullet)$ be the derived category of differential graded left $\mathcal{H}_I^\bullet$-modules. Note that $\mathcal{H}_I^\bullet$ is a compact generator of $D(\mathcal{H}_I^\bullet)$ [Keller 1998, §2.5]. It is well known that $\mathcal{H}_I^\bullet$ and $D(\mathcal{H}_I^\bullet)$ do not depend, up to quasi-isomorphism and equivalence, respectively, on the choice of the injective resolution $\mathcal{I}^\bullet$. For the convenience of the reader, we briefly recall the argument. Let $\mathrm{ind}_I^G(1) \xrightarrow{\simeq} \mathcal{J}^\bullet$ be a second injective resolution in $C(\mathrm{Mod}_k(G))$, and let $f : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ be a homotopy equivalence inducing the identity on $\mathrm{ind}_I^G(1)$ with homotopy inverse g. We form the differential graded algebra

$$\mathcal{A}^\bullet := \left\{(a, b) \in \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} \times \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} : a \circ f = f \circ a\right\}$$

(with respect to componentwise multiplication) and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^\bullet & \xrightarrow{\mathrm{pr}_2} & \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle b \mapsto f\circ b} \\ \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} & \xrightarrow{a \mapsto a\circ f} & \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet). \end{array}$$

Obviously, the maps pr_i are homomorphisms of differential graded algebras (and the bottom horizontal and right perpendicular arrows are homotopy equivalences of complexes). By direct inspection, one checks that the pr_i, in fact, are quasi-isomorphisms. Hence the differential graded algebras $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$ and $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}}$ are naturally quasi-isomorphic to each other. Moreover, by appealing to [Bernstein and Lunts 1994, Theorem 10.12.5.1], we see that the functors

$$D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}) \xrightarrow[(\mathrm{pr}_2)_*]{\sim} D(\mathcal{A}^\bullet) \xleftarrow[(\mathrm{pr}_1)_*]{\sim} D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}})$$

are equivalences of triangulated categories.

There is the following pair of adjoint functors

$$H : D(G) \longrightarrow D(\mathcal{H}_I^\bullet) \quad \text{and} \quad T : D(\mathcal{H}_I^\bullet) \longrightarrow D(G).$$

For any K-injective complex $V^\bullet$ in $\mathrm{Mod}_k(G)$, the natural chain map

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \xrightarrow{\simeq} \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet)$$

is a quasi-isomorphism. But the left hand term is a differential graded left $\mathcal{H}_I^\bullet$-module in a natural way. In fact, we have the functor

$$\begin{aligned} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) &\longrightarrow K(\mathcal{H}_I^\bullet) \\ V^\bullet &\longmapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \end{aligned}$$

into the homotopy category $K(\mathcal{H}_I^\bullet)$ of differential graded left $\mathcal{H}_I^\bullet$-modules, which allows us to define the composed functor

$$H : D(G) \xrightarrow{i} K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathcal{H}_I^\bullet) \longrightarrow D(\mathcal{H}_I^\bullet).$$

The diagram

(2)
$$\begin{array}{ccc} D(G) & \xrightarrow{H} & D(\mathcal{H}_I^\bullet) \\ & {\scriptstyle RH^0(I,\,\cdot\,)} \searrow & \downarrow{\scriptstyle \text{forget}} \\ & & D(\mathrm{Vec}_k) \end{array}$$

then is commutative up to natural isomorphism.

For the functor T in the opposite direction we first note that $\mathcal{I}^\bullet$ is naturally a differential graded right $\mathcal{H}_I^\bullet$-module so that we can form the graded tensor product $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} M^\bullet$ with any differential graded left $\mathcal{H}_I^\bullet$-module $M^\bullet$. This tensor product is naturally a complex in $C(\mathrm{Mod}_k(G))$. We now define T to be the composite

$$T : D(\mathcal{H}_I^\bullet) \xrightarrow{p} K_{\mathrm{pro},\mathcal{H}_I^\bullet} \xrightarrow{\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet}} K(\mathrm{Mod}_k(G)) \longrightarrow D(G).$$

Here $K_{\mathrm{pro},\mathcal{H}_I^\bullet}$ denotes the full triangulated subcategory of $K(\mathcal{H}_I^\bullet)$ consisting of K-projective modules and $\boldsymbol{p}$ is a quasi-inverse of the equivalence of triangulated categories $K_{\mathrm{pro},\mathcal{H}_I^\bullet} \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet)$; compare [Bernstein and Lunts 1994, Corollary 10.12.2.9].

The usual standard computation shows that T is left adjoint to H.

4. The main theorem

We need one more property of the derived category $D(G)$.

Lemma 8. *The triangulated category $D(G)$ is algebraic.*

Proof. The composite functor

$$D(G) \xrightarrow{\ i\ } K_{\mathrm{inj}}(\mathrm{Mod}_k(G)) \xrightarrow{\subseteq} K(\mathrm{Mod}_k(G))$$

is a fully faithful exact functor between triangulated categories. Hence, the assertion follows from [Krause 2007, Lemma 7.5]. □

In view of Lemmas 4 and 8 and Proposition 6, all assumptions of Keller's theorem [1994, Theorem 4.3; 1998, Theorem 3.3(a)] (compare also [Bondal and van den Bergh 2003, Theorem 3.1.7]) are satisfied and we obtain our main result.

Theorem 9. *The functor H is an equivalence between triangulated categories*

$$D(G) \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet).$$

Of course, it follows formally that the adjoint functor T is a left inverse of H.

Remark 10. The full subcategory $D(G)^{\mathrm{c}}$ of all compact objects in $D(G)$ is the smallest strictly full triangulated subcategory closed under direct summands which contains $\mathrm{ind}_I^G(1)$.

Proof. In view of Lemma 4 and Proposition 6 this follows from [Neeman 1992, Lemma 2.2]. □

The subcategory $D(G)^{\mathrm{c}}$ should be viewed as the analog of the subcategory of perfect complexes in the derived category of a ring; compare [Keller 1998, Lemma 1.4].

Another important subcategory of $D(G)$ is the bounded derived category

$$D^{\mathrm{b}}(G) := D^{\mathrm{b}}(\mathrm{Mod}_k(G)).$$

Correspondingly we have the full subcategory $D^{\mathrm{b}}(\mathcal{H}_I^\bullet)$ of all differential graded modules $M^\bullet$ in $D(\mathcal{H}_I^\bullet)$ such that $h^j(M^\bullet) = 0$ for all but finitely many $j \in \mathbb{Z}$. Since I has finite cohomological dimension, the commutative diagram (2) shows that H restricts to a fully faithful functor

$$D^{\mathrm{b}}(G) \longrightarrow D^{\mathrm{b}}(\mathcal{H}_I^\bullet).$$

On the other hand, the behavior of the functor T is controlled by an Eilenberg–Moore spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_{-r}^{h^*(\mathcal{H}_I^\bullet)}(\mathcal{H}_I, h^*(M^\bullet))^s \Longrightarrow h^{r+s}(T(M^\bullet))$$

[May, Theorem 4.1]. This suggests that, except in very special cases, the functor T will not preserve the bounded subcategories.

5. Complements

5.1. ***The top cohomology.*** A first step in the investigation of the DGA $\mathcal{H}_I^\bullet$ might be the computation of its cohomology algebra $h^*(\mathcal{H}_I^\bullet)$. By Remark 7, the latter is concentrated in degrees 0 to d. Of course the usual Hecke algebra $\mathcal{H}_I = h^0(\mathcal{H}_I^\bullet)$ is a subalgebra of $h^*(\mathcal{H}_I^\bullet)$. We determine here the top cohomology $h^d(\mathcal{H}_I^\bullet)$ as a right $\mathcal{H}_I$-module.

Using the I-equivariant linear map

$$\begin{aligned} \pi_I : \mathrm{ind}_I^G(1) &\longrightarrow \mathrm{ind}_I^G(1)^I = \mathcal{H}_I \\ \phi &\longmapsto \Big[h \longmapsto \sum_{g \in I/I \cap hIh^{-1}} \phi(gh) \Big] \end{aligned}$$

we obtain the map

$$\pi_I^* : h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1)) \xrightarrow{H^*(I,\pi_I)} H^*(I, \mathcal{H}_I) = H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I.$$

The last equality in this chain comes from the universal coefficient theorem, which is applicable since I as a Poincaré group [Lazard 1965, Théorème V.2.5.8] has finite cohomology $H^*(I, \mathbb{F}_p)$. Of course, as a ring $\mathcal{H}_I$ is a right module over itself. For our purposes, we have to consider a modification of this module structure which is specific to characteristic p.

As a k-vector space $\mathrm{ind}_I^G(1)^I = \mathcal{H}_I$ has the basis $\{\chi_{IxI}\}_{x \in I \backslash G/I}$ consisting of the characteristic functions of the double cosets IxI. If we denote the multiplication in the algebra $\mathcal{H}_I$, as usual, by the symbol "$*$" for convolution, then in this basis it is given by the formula

$$\chi_{IxI} * \chi_{IhI} = \sum_{y \in I \backslash G/I} c_{x,y;h}\, \chi_{IyI},$$

where the coefficients are

$$c_{x,y;h} = (\chi_{IxI} * \chi_{IhI})(y) = \sum_{y \in G/I} \chi_{IxI}(g)\chi_{IhI}(g^{-1}y) = |IxI \cap yIh^{-1}I/I| \cdot 1_k,$$

with 1_k denoting the unit element in the field k. Of course, for fixed x and h we have $c_{x,y;h} = 0$ for all but finitely many $y \in I\backslash G/I$. But $IxI \cap yIh^{-1}I \neq \varnothing$ implies

$IxI \subseteq IyIh^{-1}I$; by compactness, the latter is a finite union of double cosets. Hence, also for fixed y and h, we have $c_{x,y;h} \neq 0$ for at most finitely many $x \in I\backslash G/I$. It follows that by combining the transpose of these coefficient matrices with the antiautomorphism

$$\begin{aligned} \mathcal{H}_I &\longrightarrow \mathcal{H}_I \\ \chi &\longmapsto \chi^*(g) := \chi(g^{-1}), \end{aligned}$$

we obtain through the formula

$$\chi_{IxI} *_\tau \chi_{IhI} := \sum_{y \in I\backslash G/I} c_{y,x;h^{-1}} \chi_{IyI}$$

a new right action of $\mathcal{H}_I$ on itself. We denote this new module by $\mathcal{H}_I^\tau$.

Remark. We compute

$$\begin{aligned} |IyI/I| \cdot c_{x,y;h} &= |IyI/I| \cdot (\chi_{IxI} * \chi_{IhI})(y) \\ &= \sum_{z \in G/I} \chi_{IyI}(z)\big(\chi_{IxI} * \chi^*_{Ih^{-1}I}\big)(z) \\ &= \big(\chi_{IyI} * (\chi_{IxI} * \chi^*_{Ih^{-1}I})^*\big)(1) \\ &= \big((\chi_{IyI} * \chi_{Ih^{-1}I}) * \chi^*_{IxI}\big)(1) \\ &= \sum_{z \in G/I} (\chi_{IyI} * \chi_{Ih^{-1}I})(z)\chi_{IxI}(z) \\ &= |IxI/I| \cdot (\chi_{IyI} * \chi_{Ih^{-1}I})(x) \\ &= |IxI/I| \cdot c_{y,x;h^{-1}}. \end{aligned}$$

This, of course, is valid with integral coefficients (instead of k). Moreover, $|IxI/I|$ is always a power of p. It follows that over any field of characteristic different from p one has $\mathcal{H}_I^\tau \cong \mathcal{H}_I$. It also follows that $c_{x,y;h} = c_{y,x;h^{-1}}$ whenever both are nonzero.

It is straightforward to check that

$$\pi_I(\phi) *_\tau \chi_{IhI} = \pi_I(\phi * \chi_{IhI})$$

holds true for any $\phi \in \mathrm{ind}_I^G(1)$ and any $h \in G$. Hence,

$$\pi_I : \mathrm{ind}_I^G(1) \longrightarrow \mathcal{H}_I^\tau \quad \text{and} \quad \pi_I^* : h^*(\mathcal{H}_I^\bullet) \longrightarrow H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

are maps of right $\mathcal{H}_I$-modules.

Proposition 11. *The map π_I^d is an isomorphism*

$$h^d(\mathcal{H}_I^\bullet) \xrightarrow{\cong} H^d(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

of right $\mathcal{H}_I$-modules. By fixing a basis of the one dimensional $\mathbb{F}_p$-vector space $H^d(I, \mathbb{F}_p)$, we therefore obtain $h^d(\mathcal{H}_I^\bullet) \cong \mathcal{H}_I^\tau$ as right $\mathcal{H}_I$-modules.

Proof. It remains to show that π_I^d is bijective. We have the I-equivariant decomposition

$$\mathrm{ind}_I^G(1) = \bigoplus_{x \in I \backslash G/I} \mathrm{ind}_{I \cap xIx^{-1}}^I(1).$$

The map π_I restricts to

$$\pi_I : \mathrm{ind}_{I \cap xIx^{-1}}^I(1) \longrightarrow k \cdot \chi_{IxI} \subseteq \mathcal{H}_I$$
$$\phi \longmapsto \Big(\sum_{y \in I/I \cap xIx^{-1}} \phi(y) \Big) \cdot \chi_{IxI}.$$

Since $H^*(I, \cdot)$ commutes with arbitrary direct sums it therefore suffices to show that the map

$$H^d\Big(I, \phi \longmapsto \sum_{y \in I/I \cap xIx^{-1}} \phi(y)\Big) : H^d(I, \mathrm{ind}_{I \cap xIx^{-1}}^I(1_{\mathbb{F}_p})) \longrightarrow H^d(I, \mathbb{F}_p)$$

is bijective. Using Shapiro's lemma this latter map identifies (compare [Serre 1994, §I.2.5]) with the corestriction map

$$\mathrm{Cor} : H^d(I \cap xIx^{-1}, \mathbb{F}_p) \longrightarrow H^d(I, \mathbb{F}_p),$$

which for Poincaré groups of dimension d is an isomorphism of one dimensional vector spaces [op. cit., (4) on p. 37]. □

5.2. ***The easiest example.*** As an example, we will make explicit the case where $G = I = \mathbb{Z}_p$ is the additive group of p-adic integers, which we nevertheless write multiplicatively with unit element e. In order to distinguish it from the unit element $1 \in k$ we will denote the multiplicative unit in $\mathbb{Z}_p$ by γ. Let Ω denote the completed group ring of $\mathbb{Z}_p$ over k. We have:

(a) The category $\mathrm{Mod}_k(G)$ coincides with the category of torsion Ω-modules.

(b) Sending $\gamma - 1$ to t defines an isomorphism of k-algebras $\Omega \cong k[[t]]$ between Ω and the formal power series ring in one variable t over k.

For any V in $\mathrm{Mod}_k(G)$ we have the smooth G-representation $C^\infty(G, V)$ of all V-valued locally constant functions on G, where $g \in G$ acts on $f \in C^\infty(G, V)$ by ${}^g f(h) := g(f(g^{-1}h))$. One easily checks:

(c) $C^\infty(G, V) = C^\infty(G, k) \otimes_k V$ with the diagonal G-action on the right hand side.

(d) The map $\mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, C^\infty(G, V)) \xrightarrow{\cong} \mathrm{Hom}_k(W, V)$ sending F to $[w \mapsto F(w)(e)]$ is an isomorphism for any W in $\mathrm{Mod}_k(G)$. It follows that $C^\infty(G, V)$ is an injective object in $\mathrm{Mod}_k(G)$.

(e) The short exact sequence

$$0 \longrightarrow V \longrightarrow C^\infty(G,k) \otimes_k V \xrightarrow{\gamma_* - 1 \otimes \mathrm{id}} C^\infty(G,k) \otimes_k V \longrightarrow 0, \tag{3}$$

where $\gamma_*(\phi)(h) = \phi(h\gamma)$ is an injective resolution of V in $\mathrm{Mod}_k(G)$.

(f) For any $g \in G$ define the map $F_g : C^\infty(G,k) \to C^\infty(G,k)$ by $F_g(\phi)(h) := \phi(hg)$. In particular, $F_\gamma = \gamma_*$. Sending g to F_g defines an isomorphism of k-algebras

$$\Omega \xrightarrow{\cong} \mathrm{End}_{\mathrm{Mod}_k(G)}(C^\infty(G,k)).$$

Obviously $\mathrm{ind}_I^G(1) = k$ is the trivial G-representation. By (3) we may take for $\mathcal{I}^\bullet$ the injective resolution

$$C^\infty(G,k) \xrightarrow{\gamma_* - 1} C^\infty(G,k) \longrightarrow 0 \longrightarrow \cdots$$

Using (f) we deduce that $\mathcal{H}_I^\bullet$ is

$$\cdots \longrightarrow \mathcal{H}_I^{-1} = \Omega \xrightarrow{d^{-1}} \mathcal{H}_I^0 = \Omega \times \Omega \xrightarrow{d^0} \mathcal{H}_I^1 = \Omega \longrightarrow \cdots$$

with

$$d^{-1}a = ((\gamma - 1)a, (\gamma - 1)a) \quad \text{and} \quad d^0(a,b) = (\gamma - 1)(a - b)$$

and multiplication

$$\begin{aligned}(a_{-1}, (a_0, b_0), a_1) \cdot (a'_{-1}, (a'_0, b'_0), a'_1) \\ = (a'_0 a_{-1} + a'_{-1} b_0, (a'_0 a_0 - a'_{-1} a_1, b'_0 b_0 - a'_1 a_{-1}), a'_1 a_0 + b'_0 a_1).\end{aligned}$$

Using (b) we then identify $\mathcal{H}_I^\bullet$ with the upper row in the commutative diagram

$$\begin{array}{ccccc} k[\![t]\!] & \xrightarrow{a \mapsto (ta, ta)} & k[\![t]\!] \times k[\![t]\!] & \xrightarrow{(a,b) \mapsto t(a-b)} & k[\![t]\!] \\ \uparrow & & \uparrow{\scriptstyle a \mapsto (a,a)} & & \uparrow{\scriptstyle \subseteq} \\ 0 & \longrightarrow & k & \xrightarrow{0} & k \end{array}$$

We view the bottom row as the differential graded algebra of dual numbers $k[\epsilon]/(\epsilon^2)$ in degrees 0 and 1 with the zero differential. It is easy to check that the vertical arrows in the above diagram constitute a quasi-isomorphism of differential graded algebras. In particular, this says that $\mathcal{H}_I^\bullet$ is quasi-isomorphic to its cohomology algebra with zero differential (ϵ corresponds to the projection map $G = \mathbb{Z}_p \to \mathbb{F}_p \subseteq k$, as a generator of $H^1(G,k) = \mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p, k)$). According to our Theorem 9, we therefore obtain that H composed with the pullback along the above quasi-isomorphism is an equivalence of triangulated categories

$$D(\mathbb{Z}_p) \xrightarrow{\simeq} D(k[\epsilon]/(\epsilon^2)). \tag{4}$$

We finish by determining this functor explicitly. Let V be an object in $\mathrm{Mod}_k(G)$. Using the injective resolution (3) we can represent $H(V)$ by the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}\big([C^\infty(G,k) \xrightarrow{\gamma_*-1} C^\infty(G,k)],\ [C^\infty(G,k)\otimes_k V \xrightarrow{\gamma_*-1\otimes \mathrm{id}} C^\infty(G,k)\otimes_k V]\big).$$

Furthermore, using the identifications in (c) and (d), this latter complex can be computed to be the complex

$$\mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{d^{-1}} \mathrm{Hom}_k(C^\infty(G,k),V)\times \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{d^0} \mathrm{Hom}_k(C^\infty(G,k),V)$$

in degrees -1, 0, and 1 with the differentials

$$d^{-1}f = (f\circ(\gamma_*-1),\ f\circ(\gamma_*-1)+(\gamma-1)\circ f\circ \gamma_*) \quad \text{and}$$
$$d^0(f_0,f_1) = (\gamma-1)\circ f_0\circ\gamma_* + (f_0-f_1)\circ(\gamma_*-1).$$

Let $\delta_e \in \mathrm{Hom}_k(C^\infty(G,k),k)$ denote the "Dirac distribution" $\delta_e(\phi) := \phi(e)$ in the unit element. The diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \mathrm{Hom}_k(C^\infty(G,k),V) \\
\downarrow & & \downarrow d^{-1} \\
V & \xrightarrow{v\mapsto(\delta_e(\,\cdot\,)v,\,\delta_e(\,\cdot\,)\gamma(v))} & \mathrm{Hom}_k(C^\infty(G,k),V)\times\mathrm{Hom}_k(C^\infty(G,k),V) \\
\gamma-1\downarrow & & \downarrow d^0 \\
V & \xrightarrow{v\mapsto\delta_e(\,\cdot\,)v} & \mathrm{Hom}_k(C^\infty(G,k),V)
\end{array}$$

is commutative. We claim that the horizontal arrows form a quasi-isomorphism $\alpha^\bullet$. In order to define a map in the opposite direction we let $\phi_1 \in C^\infty(G,k)$ denote the constant function with value 1. Using that $\gamma_*(\phi_1)=\phi_1$, one checks that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_k(C^\infty(G,k),V) & \longrightarrow & 0 \\
d^{-1}\downarrow & & \downarrow \\
\mathrm{Hom}_k(C^\infty(G,k),V)\times\mathrm{Hom}_k(C^\infty(G,k),V) & \xrightarrow{(f_0,f_1)\mapsto f_0(\phi_1)} & V \\
d^0\downarrow & & \downarrow \gamma-1 \\
\mathrm{Hom}_k(C^\infty(G,k),V) & \xrightarrow{f\mapsto f(\phi_1)} & V
\end{array}$$

is commutative. Hence the horizontal arrows define a homomorphism of complexes $\beta^\bullet$ such that $\beta^\bullet\circ\alpha^\bullet = \mathrm{id}$. Applying $\mathrm{Hom}_k(\,\cdot\,,V)$ to our injective resolution of k, we obtain the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{f\mapsto f\circ(\gamma_*-1)} \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{\beta^1} V \longrightarrow 0.$$

This implies that d^{-1} is injective and that $\mathrm{im}(d^0) \supseteq \ker(\beta^1)$. The former says that the cohomology in degree -1 is zero. Because of

$$\mathrm{Hom}_k(C^\infty(G,k), V) = \ker(\beta^1) \oplus \mathrm{im}(\alpha^1), \tag{5}$$

the latter shows the surjectivity of $h^1(\alpha^\bullet)$. Hence $h^1(\alpha^\bullet)$ is bijective. A pair (f_0, f_1) represents a class in $\ker(h^0(\beta^\bullet))$ if and only if $d^0(f_0, f_1) = 0$ and $\beta^0(f_0, f_1) = 0$. The first condition implies that

$$f_1 \circ (\gamma_* - 1) = (\gamma - 1) \circ f_0 \circ \gamma_* + f_0 \circ (\gamma_* - 1).$$

By (5) the second condition says that we may write $f_0 = \delta_e(\,\cdot\,)v + f \circ (\gamma_* - 1)$ for $v := f_0(\phi_1) \in V$ and some $f \in \mathrm{Hom}_k(C^\infty(G,k), V)$. Inserting this into the above equation we obtain

$$f_1 \circ (\gamma_* - 1) = \delta_e(\,\cdot\,)(\gamma(v) - v) + (\gamma \circ f \circ \gamma_* - f) \circ (\gamma_* - 1).$$

It follows that

$$\gamma(v) = v \quad \text{and} \quad f_1 = (\gamma \circ f \circ \gamma_* - f).$$

Using this last identity one checks that $(f_0, f_1) = d^{-1}f + (\delta_e(\,\cdot\,)v, 0)$. But we have $0 = d^0(\delta_e(\,\cdot\,)v, 0) = \delta_e(\gamma_* \cdot)(\gamma - 1)(v) + \delta_e((\gamma_* - 1)\cdot)v = \delta_e((\gamma_* - 1)\cdot)v$, which implies that $v = 0$. We conclude that $h^0(\beta^\bullet)$ is injective and hence bijective and that therefore $h^0(\alpha^\bullet)$ is bijective.

A differential graded $k[\epsilon]/(\epsilon^2)$-module is the same as a graded k-vector space with two anticommuting differentials ϵ and d of degree 1. Given the smooth G-representation V, we form the graded $k[\epsilon]/(\epsilon^2)$-module $k[\epsilon]/(\epsilon^2) \otimes_k V$ (sitting in degrees 0 and 1) and equip it with the differential $d_V(v_0 + v_1\epsilon) := (\gamma - 1)(v_0)\epsilon$. The above computations together with the fact that ϵ corresponds to the identity in $\mathcal{H}_I^1 = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^0, \mathcal{I}^1) = \mathrm{End}_{\mathrm{Mod}_k(G)}(C^\infty(G,k))$ proves the following:

Proposition 12. *The equivalence (4) sends V in $\mathrm{Mod}_k(G)$ to the differential graded module $(k[\epsilon]/(\epsilon^2) \otimes_k V, d_V)$.*

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In memoriam: Robert Steinberg