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COMPACTNESS AND THE PALAIS–SMALE PROPERTY FOR CRITICAL KIRCHHOFF EQUATIONS IN CLOSED MANIFOLDS

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We prove the Palais–Smale property and the compactness of solutions for critical Kirchhoff equations using solely energy arguments in the situation where no sign assumption is made on the solutions. We then prove the existence of a mountain-pass solution to the equation, discuss its ground-states structure, and, in extreme cases, prove uniqueness of this solution.

The Kirchhoff equation [1883] was proposed as an extension of the classical wave equation of D'Alembert for the vibration of elastic strings. The model takes into account the small vertical vibrations of a stretched elastic string when the tension is variable but the ends of the string are fixed. The equation in [loc. cit.] was written as

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where L is the length of the string, h is the area of the cross-section, E is the young modulus of the material (also referred to as the elastic modulus — it measures the string's resistance to being deformed elastically), ρ is the mass density, and P_0 is the initial tension. Almost one century later, Jacques-Louis Lions [1978] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as

$$\frac{\partial^2 u}{\partial t^2} + \left(a + b\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x, u),$$

where

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$$

is the Laplace–Beltrami Euclidean Laplacian. We investigate in this paper the stationary version of this equation, in the case of closed manifolds, and when f is the critical pure power nonlinearity. We prove the surprising result that the equation satisfies the Palais–Smale property when a and b are large (in a sense to be made

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precised in Theorem 1 below). As usual, solutions of the stationary equation (with the square of the phase added as a potential) correspond to standing wave solutions of the evolution equation.

In what follows, we let (M, g) be a closed n-dimensional Riemannian manifold of dimension $n \geq 4$, $a, b > 0$ be positive real numbers, and $h \in C^1(M, \mathbb{R})$. The Kirchhoff equation we investigate is written as

$$\left(a + b\int_M |\nabla u|^2 \, dv_g\right)\Delta_g u + hu = |u|^{2^\star - 2}u, \tag{1}$$

where $2^\star = 2n/(n-2)$ is the critical Sobolev exponent. It is an appealing mathematical model because of its nonlocal nature and its integrodifferential structure. It has been paid much attention over the past years. Among other possible references (the following list is far from being exhaustive), we mention Figueiredo [2013], Figueiredo, Ikoma, and Santos [Figueiredo et al. 2014], Figueiredo and Santos [2012], He and Zou [2012], and the references in these papers. The case of positive solutions in the curved setting of closed manifolds has been investigated in Hebey and Thizy [2015a; 2015b]. We treat here the case where absolutely no sign assumption is made on the solutions. As a remark, the equation always has a pair of constant solutions if $h > 0$ is constant.

In what follows, we let H^1 be the Sobolev space of functions in L^2 with one derivative in L^2. We let also $I : H^1 \to \mathbb{R}$ be the functional

$$I(u) = \frac{a}{2}\int_M |\nabla u|^2 \, dv_g + \frac{b}{4}\left(\int_M |\nabla u|^2 \, dv_g\right)^2 + \frac{1}{2}\int_M hu^2 \, dv_g - \frac{1}{2^\star}\int_M |u|^{2^\star} dv_g. \tag{2}$$

As is easily checked, critical points of I are solutions of (1). In particular, (1) has a variational structure. A sequence $(u_\alpha)_\alpha$ in H^1 is said to be a *Palais–Smale sequence* for I if the sequence $(I(u_\alpha))_\alpha$ is bounded with respect to α, and $I'(u_\alpha) \to 0$ in $(H^1)'$ as $\alpha \to +\infty$. Following standard terminology, we say that I satisfies the *Palais–Smale property* if Palais–Smale sequences for I converge, up to a subsequence, in H^1. Let S_n be the sharp Euclidean Sobolev constant given by $S_n = \frac{1}{4}n(n-2)\omega_n^{2/n}$, where ω_n is the volume of the unity n-sphere. We define the dimensional constant $C(n)$ by

$$C(n) = \frac{2(n-4)^{(n-4)/2}}{(n-2)^{(n-2)/2} S_n^{n/2}}. \tag{3}$$

The main result of this paper provides very simple criteria on a and b for the equation to be compact and I to satisfy the Palais–Smale property. Our main result is stated as follows.

Theorem 1. *Suppose that (M, g) is a closed n-dimensional Riemannian manifold of dimension $n \geq 4$, that $a, b > 0$ are positive real numbers, and that $h \in C^1(M, \mathbb{R})$ makes $\Delta_g + h/a$ positive. Assume that $b \gg 1$ when $n = 4$, and that $a^{(n-4)/2} b > C(n)$*

when $n \geq 5$, where $C(n)$ is as in (3). Then, I satisfies the Palais–Smale property and the set of solutions of (1) is compact in the C^2-topology.

It is very surprising that such a compactness result, in strong topologies, for an equation with critical nonlinearity, can be obtained without the whole machinery of strong pointwise estimates (see Hebey [2014] for a reference in book form on this machinery). Moreover, no assumption of positiveness is made on the solutions in Theorem 1.

Proof of Theorem 1. (i) We prove that Palais–Smale sequences for I are bounded in H^1, assuming that $b \gg 1$ when $n = 4$. Let $(u_\alpha)_\alpha$ be a Palais–Smale sequence for I. Then, we get that $I(u_\alpha) = O(1)$ and $I'(u_\alpha) . (u_\alpha) = o(\|u_\alpha\|_{H^1})$, where $\|\cdot\|_{H^1}$ is the H^1-norm given for $u \in H^1$ by

$$\|u\|_{H^1}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

In particular,

$$a\int_M\Big(|\nabla u_\alpha|^2 + \frac{h}{a}u_\alpha^2\Big)\,dv_g + b\Big(\int_M |\nabla u_\alpha|^2\,dv_g\Big)^2 = \int_M |u_\alpha|^{2^\star}dv_g + o(\|u_\alpha\|_{H^1}) \tag{4}$$

and that

$$\frac{a}{2}\int_M\Big(|\nabla u_\alpha|^2 + \frac{h}{a}u_\alpha^2\Big)dv_g + \frac{b}{4}\Big(\int_M |\nabla u_\alpha|^2 dv_g\Big)^2 = \frac{1}{2^\star}\int_M |u_\alpha|^{2^\star}dv_g + O(1). \tag{5}$$

By the Sobolev–Poincaré inequality, there exist $C_1, C_2 > 0$ such that

$$\|u_\alpha\|_{L^{2^\star}}^{2^\star} \leq C_1\|\nabla u_\alpha\|_{L^2}^{2^\star} + C_2|\bar{u}_\alpha|^{2^\star} \tag{6}$$

for all α, where

$$\bar{u}_\alpha = \frac{1}{V_g}\int_M u_\alpha\,dv_g$$

is the average of u_α, and by the Poincaré inequality,

$$\|u_\alpha - \bar{u}_\alpha\|_{L^2}^2 \leq \frac{1}{\lambda_1}\|\nabla u_\alpha\|_{L^2}^2 \tag{7}$$

for all α, where $\lambda_1 = \lambda_1(M, g) > 0$ is the first nonzero eigenvalue of Δ_g. It clearly follows from the positivity of $\Delta_g + h/a$, (5), and (6) that if either $n = 4$ and $b > C_1$ or $n \geq 5$ and if $\bar{u}_\alpha = O(1)$, then $\|u_\alpha\|_{H^1} = O(1)$. We may therefore assume that $\bar{u}_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, still by the positivity of $\Delta_g + h/a$, (5), and (6),

$$\int_M |\nabla u_\alpha|^2\,dv_g = \begin{cases} \frac{1}{b}O(\bar{u}_\alpha^2) & \text{if } n = 4,\\ o(\bar{u}_\alpha^2) & \text{if } n \geq 5,\end{cases} \tag{8}$$

where we assume that $b > C_1$ when $n = 4$. Now, we write that

$$u_\alpha = \bar{u}_\alpha(1 + \varphi_\alpha). \tag{9}$$

Then, $\int_M \varphi_\alpha \, dv_g = 0$ and

$$\bar{u}_\alpha^2 \int_M |\nabla \varphi_\alpha|^2 \, dv_g = \int_M |\nabla u_\alpha|^2 \, dv_g. \tag{10}$$

It follows from (8), (10), the Poincaré inequality, (7), and (10) that

$$\|\varphi_\alpha\|_{H^1}^2 = \begin{cases} O\left(\frac{1}{b}\right) & \text{if } n = 4, \\ o(1) & \text{if } n \geq 5. \end{cases} \tag{11}$$

In particular, by (9) and (11),

$$\int_M \left(|\nabla u_\alpha|^2 + \frac{h}{a} u_\alpha^2\right) dv_g = \bar{u}_\alpha^2 (1 + A_\alpha) \quad \text{and} \quad \int_M |u_\alpha|^{2^\star} \, dv_g = \bar{u}_\alpha^{2^\star} (1 + B_\alpha), \tag{12}$$

where $A_\alpha = O\left(\frac{1}{b}\right)$ and $B_\alpha = O\left(\frac{1}{b}\right)$ if $n = 4$, and $A_\alpha = o(1)$ and $B_\alpha = o(1)$ if $n \geq 5$. Subtracting $\frac{1}{4}$ of (4) from (5) yields

$$\frac{a}{4} \int_M \left(|\nabla u_\alpha|^2 + \frac{h}{a} u_\alpha^2\right) dv_g = \left(\frac{1}{2^\star} - \frac{1}{4}\right) \int_M |u_\alpha|^{2^\star} dv_g + O(1) + O(\|u_\alpha\|_{H^1}). \tag{13}$$

Picking $b \gg 1$ when $n = 4$, the contradiction follows by combining (12) and (13). This proves that $(u_\alpha)_\alpha$ is bounded in H^1.

(ii) We prove that I satisfies the Palais–Smale property assuming that $b \gg 1$ when $n = 4$, and that $a^{(n-4)/2} b > C(n)$ when $n \geq 5$. We let

$$K_\alpha = a + b \int_M |\nabla u_\alpha|^2 \, dv_g, \tag{14}$$

$h_\alpha = K_\alpha^{-1} h$, and

$$v_\alpha = \left(\frac{1}{K_\alpha}\right)^{\frac{1}{2^\star - 2}} u_\alpha. \tag{15}$$

We define $I_\alpha : H^1 \to \mathbb{R}$ by

$$I_\alpha(u) = \frac{1}{2} \int_M \left(|\nabla u|^2 + h_\alpha u^2\right) dv_g - \frac{1}{2^\star} \int_M |u|^{2^\star} dv_g. \tag{16}$$

According to (i), and up to passing to a subsequence, $K_\alpha \to K_\infty$ as $\alpha \to +\infty$ for some $K_\infty \geq a$. In particular, $(h_\alpha)_\alpha$ converges in C^k for all k, and $(v_\alpha)_\alpha$ is bounded in H^1. This implies that $I_\alpha(v_\alpha) = O(1)$, and, as one can check,

$$I'_\alpha(v_\alpha) \, . \, (\varphi) = \left(\frac{1}{K_\alpha}\right)^{\frac{2^\star - 1}{2^\star - 2}} I'(u_\alpha) \, . \, (\varphi)$$

for all $\varphi \in H^1$. Then $(v_\alpha)_\alpha$ is a Palais–Smale sequence for the family $(I_\alpha)_\alpha$ (in the sense of Hebey [2014]). In particular the H^1-decomposition as in Struwe [1984] applies (see Druet, Hebey, and Robert [Druet et al. 2004], Hebey [2014], and Vétois

[2007] for the closed setting with varying potentials), and we get that there exists $v_\infty \in H^1$, $k \in \mathbb{N}$, and $k+1$ sequences $(B_{1,\alpha})_\alpha, \dots, (B_{k,\alpha})_\alpha$, $(R_\alpha)_\alpha$ in H^1 such that

$$v_\alpha = v_\infty + \sum_{i=1}^{k} B_{i,\alpha} + R_\alpha \text{in } M \tag{17}$$

and

$$\int_M |\nabla v_\alpha|^2 \, dv_g = \int_M |\nabla v_\infty|^2 \, dv_g + \sum_{i=1}^{k} \int_M |\nabla B_{i,\alpha}|^2 \, dv_g + o(1) \tag{18}$$

for all α, $R_\alpha \to 0$ in H^1 as $\alpha \to +\infty$ and the "bubbles" $(B_{i,\alpha})_\alpha$ satisfy the following properties for any $i = 1, \dots, k$:

(a) $B_{i,\alpha} \to 0$ in L^2 as $\alpha \to +\infty$,

(b) $\|B_{i,\alpha}\| = O(1)$, and

(c) $\int_M |\nabla B_{i,\alpha}|^2 \, dv_g \geq S_n^{n/2} + o(1)$ for all α,

where S_n is the sharp Euclidean constant as in (3). In (c), there is equality if each $B_{i,\alpha}$ is positive. Then, since $(u_\alpha)_\alpha$ is bounded in H^1, and by (17)–(18), we get that, up to passing to a subsequence,

$$\begin{aligned} K_\alpha &= a + b \int_M |\nabla u_\alpha|^2 \, dv_g \\ &= a + b K_\alpha^{2/(2^\star-2)} \int_M |\nabla v_\alpha|^2 \, dv_g \\ &= a + b K_\alpha^{2/(2^\star-2)} \int_M |\nabla v_\infty|^2 \, dv_g + b K_\alpha^{2/(2^\star-2)} \sum_{i=1}^{k} \int_M |\nabla B_{i,\alpha}|^2 \, dv_g + o(1) \\ &= a + b K_\infty^{2/(2^\star-2)} \int_M |\nabla v_\infty|^2 \, dv_g + b C K_\infty^{2/(2^\star-2)} + o(1), \end{aligned} \tag{19}$$

where $C \geq k S_n^{n/2}$. In particular, by (19),

$$K_\infty = a + b K_\infty^{2/(2^\star-2)} \int_M |\nabla v_\infty|^2 \, dv_g + b C K_\infty^{2/(2^\star-2)}. \tag{20}$$

When $n = 4$, we have $2/(2^\star - 2) = 1$, and (20) implies that $k = 0$ in (17) as soon as $b \gg 1$. In particular, the sequence $(u_\alpha)_\alpha$ converges strongly in H^1, and I satisfies the Palais–Smale property. When $n \geq 5$, we define

$$f(x) = bk S_n^{n/2} x^{(n-2)/2} - x + a.$$

By (20), and since $C \geq kS_n^{n/2}$, we have that $f(K_\infty) \leq 0$. Assuming that $k \geq 1$, noting that f is minimum at x_0, where

$$x_0 = \left(\frac{2}{(n-2)bkS_n^{n/2}}\right)^{2/(n-4)},$$

we compute that

$$f(x_0) = -\frac{n-4}{n-2}\left(bkS_n^{n/2}\right)^{-2/(n-4)}\left(\frac{2}{n-2}\right)^{2/(n-4)} + a. \tag{21}$$

If $f(K_\infty) \leq 0$, then $f(x_0) \leq 0$, and by (21), $bka^{(n-4)/2} \leq C(n)$. Since by assumption $a^{(n-4)/2}b > C(n)$, it must be the case that $k = 0$ in (17). In particular, the sequence $(u_\alpha)_\alpha$ converges strongly in H^1, and I satisfies the Palais–Smale property also when $n \geq 5$.

(iii) We prove the compactness of (1) assuming that $b \gg 1$ when $n = 4$ and that $a^{(n-4)/2}b > C(n)$ when $n \geq 5$. Noting that a bounded sequence in H^1 of solutions of (1) is a Palais–Smale sequence for I, according to what we proved above, it suffices to prove that if $(u_\alpha)_\alpha$ is a sequence of solutions of (1), then $(u_\alpha)_\alpha$ is bounded in H^1 when $n \geq 5$ and when $n = 4$ and $b \gg 1$. By the Palais–Smale property we would indeed get that, up to passing to a subsequence, $(u_\alpha)_\alpha$ converges in H^1, and by standard elliptic theory, this actually implies that the sequence converges in C^2. Now, we multiply the equation by u_α and integrate over M, yielding

$$a\int_M \left(|\nabla u_\alpha|^2 + \frac{h}{a}u_\alpha^2\right) dv_g + b\left(\int_M |\nabla u_\alpha|^2\, dv_g\right)^2 = \int_M |u_\alpha|^{2^\star} dv_g \tag{22}$$

for all α. We clearly get from (6) and (22) that $(u_\alpha)_\alpha$ is bounded in H^1 if the sequence $(\bar{u}_\alpha)_\alpha$ is bounded (and $b \gg 1$ when $n = 4$). We may thus assume that $\bar{u}_\alpha \to +\infty$ as $\alpha \to +\infty$. By (6) and (22), we get that (8) holds. Writing (9), we then get that (12) holds and also that

$$\int_M |u_\alpha|\, dv_g = |\bar{u}_\alpha|(1 + C_\alpha), \tag{23}$$

where $C_\alpha = O\left(\frac{1}{b}\right)$ if $n = 4$, and $C_\alpha = o(1)$ if $n \geq 5$. Integrating the equation,

$$\int_M hu_\alpha\, dv_g = \int_M |u_\alpha|^{2^\star - 2} u_\alpha\, dv_g. \tag{24}$$

The contradiction follows from (12), (23), and (24). This proves the above claim that $(u_\alpha)_\alpha$ is bounded in H^1. This also proves that the set of solutions of (1) is compact in the C^2-topology. □

At this point we define a mountain-pass solution of (1) as a solution which we obtain from I by the use of the mountain-pass lemma. We easily get from Theorem 1 that the following existence result holds true.

Proposition 2. *Suppose that (M, g) is a closed Riemannian manifold of dimension $n \geq 4$, that a and b are positive real numbers, and that $h \in C^1(M, \mathbb{R})$ is such that $\Delta_g + h/a$ is positive. Assume that $b \gg 1$ when $n = 4$, and that $a^{(n-4)/2} b > C(n)$ when $n \geq 5$, where $C(n)$ is as in (3). Then, (1) possesses a nontrivial mountain-pass solution.*

Proof of Proposition 2. Let $u_0 \equiv 1$. Then, I is C^1, $I(0) = 0$, $I(T u_0) < 0$ for $T \gg 1$, and by the coercivity of $\Delta_g + h/a$, there exist $C_1, C_2 > 0$ such that $I(u) \geq C_1 \|u\|_{H^1}^2 - C_2 \|u\|_{H^1}^{2^\star}$ for all u. Then, we can apply the mountain-pass lemma of Ambrosetti and Rabinowitz [1973] and we get that there exists a sequence $(u_\alpha)_\alpha$ in H^1 such that $I(u_\alpha) = c + o(1)$ and $I'(u_\alpha).(\psi) = o(\|\psi\|_{H^1})$ for all $\psi \in H^1$, where

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I(u),$$

and Γ is the set of continuous paths from 0 to $T u_0$. Obviously, $c > 0$. By Theorem 1, up to passing to a subsequence, $(u_\alpha)_\alpha$ converges in H^1. Let u_∞ be the limit in H^1 of the sequence u_α. Then $I(u_\infty) = c$, $u_\infty \not\equiv 0$, and by passing to the limit in the equation $I'(u_\alpha) . (\varphi) = o(1)$ for all $\varphi \in H^1$, we get that u_∞ solves (1). □

It is easily seen that the mountain-pass solution u_∞ obtained in Proposition 2 has a nice ground-state structure when $n = 4$. We define the Nehari manifold $\mathcal{N}$ attached to I by

$$\mathcal{N} = \{u \in H^1 \setminus \{0\} \mid I'(u) . (u) = 0\}. \tag{25}$$

The following 4-dimensional ground-state characterization of the solution obtained in Proposition 2 holds true.

Proposition 3. *Suppose that (M, g) is a closed 4-dimensional Riemannian manifold, that a and b are positive real numbers, and that $h \in C^1(M, \mathbb{R})$ is such that $\Delta_g + h/a$ is positive. Assume that $b \gg 1$. Then, the mountain-pass solution u_∞ obtained in Proposition 2 has a ground-state structure given by*

$$I(u_\infty) = \inf_{u \in \mathcal{N}} I(u), \tag{26}$$

where $\mathcal{N}$ is the Nehari manifold attached to I given by (25).

Proof of Proposition 3. We obviously have that $u_\infty \in \mathcal{N}$, and thus there holds that $I(u_\infty) \geq \inf_{u \in \mathcal{N}} I(u)$. Given $\tilde{u} \in H^1 \setminus \{0\}$, we define the mountain-pass energy level $c_{\tilde{u}}$ by

$$c_{\tilde{u}} = \inf_{\gamma \in \Gamma_{\tilde{u}}} \sup_{u \in \gamma} I(u),$$

where $\Gamma_{\tilde{u}}$ is the set of continuous paths from 0 to $\tilde{u}$. Let $u_0 \equiv 1$ be as in the proof of Proposition 2. Let $T_0 \gg 1$ be such that $I(T_0 u_0) < 0$. By construction (see the proof of Proposition 2), it holds that $I(u_\infty) = c_{T_0 u_0}$. Let $u \in \mathcal{N}$. Then $I(u) = I(|u|)$,

$$a\int_M \Big(|\nabla u|^2 + \frac{h}{a}u^2\Big)\,dv_g + b\Big(\int_M |\nabla u|^2\,dv_g\Big)^2 = \int_M u^4\,dv_g$$

and for $t \geq 0$,

$$\begin{aligned} I(t|u|) &= \frac{at^2}{2}\int_M \Big(|\nabla u|^2 + \frac{h}{a}u^2\Big)\,dv_g + \frac{bt^4}{4}\Big(\int_M |\nabla u|^2\,dv_g\Big)^2 - \frac{t^4}{4}\int_M u^4\,dv_g \\ &= \frac{at^2(2-t^2)}{4}\int_M \Big(|\nabla u|^2 + \frac{h}{a}u^2\Big)\,dv_g. \end{aligned} \tag{27}$$

In particular, $I(T_1|u|) < 0$ for $T_1 > \sqrt{2}$. Let $u_1 = |u|$ and $T_1 \gg 1$. It is easily checked (since u_0 is constant) that

$$I(tT_1u_1 + (1-t)T_0u_0) \leq t^2 I(T_1u_1) - \frac{(1-t)^2 T_0^2 u_0^2 V_g}{4} < 0$$

for all $0 \leq t \leq 1$, where V_g is the volume of (M, g). In particular, $c_{T_0u_0} = c_{T_1u_1}$ since T_0u_0 and T_1u_1 can be connected by a continuous path along which I is everywhere negative. So,

$$c_{T_0u_0} \leq \sup_{0\leq t\leq T_1} I(tu_1). \tag{28}$$

By (27) we see that $t \to I(tu_1)$ is maximal at $t = 1$, and thus $c_{T_0u_0} \leq I(u)$ by (28). This proves that $I(u_\infty) \leq I(u)$ for all $u \in \mathcal{N}$, and thus that (26) holds. □

Balancing Proposition 2 we prove that the following uniqueness result, in the sense of Brézis and Li [2006], holds.

Proposition 4. *Suppose that (M, g) is a closed Riemannian manifold of dimension $n \geq 4$ and that h is a positive constant. Let $\varepsilon_0 > 0$ arbitrary. For $a, b \gg 1$ when $n = 4$, and $a \gg 1$, $b \geq \varepsilon_0$ when $n \geq 5$, the sole nontrivial pair of solutions of (1) is the pair $(-u, u)$ of constant solutions, where $u = h^{(n-2)/4}$.*

Proof of Proposition 4. Let $\varepsilon_0 > 0$ be given arbitrarily small. We prove the result by contradiction. We assume that there exist sequences $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$ of positive real numbers, and a sequence $(u_\alpha)_\alpha$ of nonconstant solutions of

$$\Big(a_\alpha + b_\alpha\int_M |\nabla u_\alpha|^2\,dv_g\Big)\Delta_g u_\alpha + hu_\alpha = |u_\alpha|^{2^\star-2}u_\alpha \tag{29}$$

for all α such that $a_\alpha \to +\infty$ and $b_\alpha \to +\infty$ as $\alpha \to +\infty$ when $n = 4$, and such that $a_\alpha \to +\infty$ as $\alpha \to +\infty$ and $b_\alpha \geq \varepsilon_0$ for all α when $n \geq 5$. As in the proof of Theorem 1, this implies that $\|u_\alpha\|_{H^1} = O(1)$. Suppose that K_α is as in (14), $h_\alpha = K_\alpha^{-1} h$, and v_α is as in (15). Then,

$$\Delta_g v_\alpha + h_\alpha v_\alpha = |v_\alpha|^{2^\star - 2} v_\alpha, \tag{30}$$

and $K_\alpha \to +\infty$ since $a_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, by elliptic regularity, $v_\alpha \to 0$ in C^0. Multiplying (30) by $v_\alpha - \bar{v}_\alpha$, and integrating over M,

$$\begin{aligned}\lambda_1 \int_M (v_\alpha - \bar{v}_\alpha)^2 \, dv_g &\le \int_M (v_\alpha - \bar{v}_\alpha)\big(|v_\alpha|^{2^\star - 2} v_\alpha - |\bar{v}_\alpha|^{2^\star - 2} \bar{v}_\alpha\big) \, dv_g \\ &\le C \|v_\alpha\|_{L^\infty}^{2^\star - 2} \int_M (v_\alpha - \bar{v}_\alpha)^2 \, dv_g\end{aligned} \tag{31}$$

for all α, where $C > 0$ is independent of α, and $\lambda_1 > 0$ is the first nontrivial eigenvalue of Δ_g. Since $v_\alpha \to 0$ in C^0, (31) implies that $v_\alpha = \bar{v}_\alpha$, and we get a contradiction. □

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