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**AN ORLIK–RAYMOND TYPE CLASSIFICATION OF
SIMPLY CONNECTED 6-DIMENSIONAL TORUS MANIFOLDS
WITH VANISHING ODD-DEGREE COHOMOLOGY**

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Dedicated to Professor Mikiya Masuda on his 60th birthday.

The aim of this paper is to classify simply connected 6-dimensional torus manifolds with vanishing odd-degree cohomology. It is shown that there is a one-to-one correspondence between equivariant diffeomorphism types of these manifolds and 3-valent labelled graphs, called torus graphs, introduced by Maeda, Masuda and Panov. Using this correspondence and combinatorial arguments, we prove that a simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M) = 0$ is equivariantly diffeomorphic to the 6-dimensional sphere S^6 or an equivariant connected sum of copies of 6-dimensional quasitoric manifolds or S^4-bundles over S^2.

1. Introduction

Let M be a $2n$-dimensional closed, connected, oriented manifold with an effective n-dimensional (i.e., half-dimensional) torus T^n-action. We call M, or (M, T), a *torus manifold* if $M^T \neq \varnothing$ (see [Hattori and Masuda 2003]), where M^T is the set of fixed points. A toric manifold (i.e., a nonsingular, complete toric variety viewed as a complex analytic space) with restricted T^n-action is a typical example of a torus manifold. Recall that a toric manifold is a complex $(\mathbb{C}^*)^n$-manifold with a dense orbit (see [Oda 1988; Fulton 1993]), and T^n is the maximal compact subgroup of $(\mathbb{C}^*)^n$. A fundamental result of toric geometry tells us that there is a one-to-one correspondence between toric manifolds and combinatorial objects called fans. Thus, topological (more precisely, geometric) invariants of toric manifolds can be described in terms of combinatorial invariants of fans, such as equivariant cohomology rings, equivariant characteristic classes and other topological invariants.

Hattori and Masuda [2003] introduced a torus manifold as the topological generalization of a toric manifold. They also introduced the combinatorial objects called multifans (see [Masuda 1999; Hattori and Masuda 2003]), and computed topological invariants (such as equivariant characteristic classes or Todd genus for unitary torus manifolds) in terms of multifans. However, unlike the case for toric geometry, a multifan does not contain enough information to determine some topological invariants of torus manifolds (e.g., equivariant cohomology). So, in 2007, Maeda, Masuda and Panov introduced combinatorial objects called torus graphs, which were motivated by the GKM graphs introduced by Guillemin and Zara [2001]. The combinatorial information of torus graphs can completely determine the equivariant cohomology rings of torus manifolds with vanishing odd-degree cohomology, i.e., $H^{\mathrm{odd}}(M; \mathbb{Z}) = 0$ (in this paper, we only consider integer coefficients); see [Masuda and Panov 2006; Maeda et al. 2007], and see also Section 3 in this paper about torus graphs. However, in general, there is no one-to-one correspondence between torus manifolds with $H^{\mathrm{odd}}(M) = 0$ and torus graphs.

So, we are naturally led to ask the following two questions: (1) Which subclasses of torus manifolds are completely determined by combinatorial objects (like multifans or torus graphs)? (2) If we find such a subclass of torus manifolds, how can we classify such torus manifolds? Several mathematicians have answered the first question: for example, Davis and Januszkiewicz [1991] for the subclass called *quasitoric manifolds* (see [Buchstaber and Panov 2002] or Section 4C in this paper), Ishida, Fukukawa and Masuda [2013] for the subclass called *topological toric manifolds*, and Wiemeler [2013] for the class of simply connected 6-dimensional torus manifolds with $H^{\mathrm{odd}}(M) = 0$ (see Theorem 2.7). The aim of this paper is to answer the second question for the class of simply connected 6-dimensional torus manifolds with $H^{\mathrm{odd}}(M) = 0$ using torus graphs.

Let us briefly recall the classification results for torus manifolds with lower dimensions. If T^1 acts on a compact 2-dimensional manifold M, then M is the 2-dimensional sphere S^2, the 2-dimensional real projective space $\mathbb{R}P^2$, the 2-dimensional torus T^2 or the Klein bottle. Because $M^T \neq \varnothing$ and M is oriented, M must be equivariantly diffeomorphic to S^2 with T^1-action (see [Kawakubo 1991]). When dim $M = 4$, by Orlik and Raymond's theorem [1970], we have the following:

Theorem 1.1 (Orlik–Raymond). *Let M be a 4-dimensional simply connected torus manifold. Then, M is equivariantly diffeomorphic to the 4-sphere S^4 or an equivariant connected sum of copies of complex projective spaces $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ (reversed orientation) or a Hirzebruch surface H_k.*

Here a Hirzebruch surface H_k ($k \in \mathbb{Z}$) is a manifold which is defined by the projectivization of the complex 2-dimensional vector bundle $\gamma^{\otimes k} \oplus \epsilon$ over $\mathbb{C}P^1$, where γ and ϵ are the tautological and the trivial complex line bundles over $\mathbb{C}P^1$.

In this paper, we prove an Orlik–Raymond type theorem similar to Theorem 1.1 for simply connected 6-dimensional torus manifolds with $H^{\mathrm{odd}}(M) = 0$. Before we state our main results, we introduce the result for torus manifolds that are not simply connected. One of the consequences of Masuda and Panov's theorem (see Theorem 2.2 in Section 2B) is the following proposition (see also [Wiemeler 2013]).

Proposition 1.2. *Let W be a* 6-*dimensional torus manifold with* $H^{\mathrm{odd}}(W) = 0$ (*it might not be simply connected*). *Then, there are a simply connected* 6-*dimensional torus manifold M with* $H^{\mathrm{odd}}(M) = 0$ *and a homology* 3-*sphere* hS^3 *such that*

$$W \cong M \#_T (hS^3 \times T^3)$$

up to equivariant diffeomorphism.

Here the product manifold $hS^3 \times T^3$ is the product of hS^3 and the 3-dimensional torus T^3 with the free T^3-action on the second factor, and the symbol $\#_T$ represents the equivariant gluing along two free orbits of M and $hS^3 \times T^3$.

In Proposition 1.2, because the fundamental groups $\pi_1(W)$ and $\pi_1(hS^3)$ are isomorphic, W is simply connected if and only if hS^3 is simply connected, i.e., the standard sphere. Our main theorem is a classification of the simply connected torus manifolds that appear in Proposition 1.2.

Theorem 1.3. *Let M be a simply connected* 6-*dimensional torus manifold with* $H^{\mathrm{odd}}(M) = 0$. *Then, M is equivariantly diffeomorphic to the* 6-*sphere* S^6 *or obtained by an equivariant connected sum of copies of* 6-*dimensional quasitoric manifolds or* S^4-*bundles over* S^2 *equipped with the structure of a torus manifold.*

This type of classification, i.e., classification by equivariant connected sum, may be regarded as the 6-dimensional analogue of Orlik and Raymond's classification in Theorem 1.1. So, in this paper, we call this theorem an *Orlik–Raymond type classification* (see [McGavran 1976; Kuroki 2008]).

Remark 1.4. Izmestiev [2001] proved an Orlik–Raymond type classification for a class of 3-dimensional small covers (i.e., the real analogue of quasitoric manifolds; see Section 4B), called a linear model (see also [Lü and Yu 2011; Nishimura 2012]).

The organization of this paper is as follows. In Section 2, we recall the basic facts about torus manifolds. In Section 3, we do the same for torus graphs. In particular, Corollary 3.5 is the key fact used to prove Theorem 1.3. In Section 4, we introduce the torus graphs of S^6, quasitoric manifolds and S^4-bundles over S^2. These torus graphs will be the basic graphs used to classify simply connected 6-dimensional torus manifolds with $H^{\mathrm{odd}}(M) = 0$. In Section 5, we introduce the "oriented" torus graphs and translate the equivariant connected sum around fixed points of torus manifolds to the connected sum around vertices of oriented torus graphs. In Sections 6 and 7, we prove Theorem 1.3. A brief outline of the proof is as follows. By

Corollary 3.5, there is a one-to-one correspondence between 6-dimensional simply connected torus manifolds with $H^{\text{odd}}(M) = 0$ and 3-valent torus graphs. Therefore, to prove Theorem 1.3, it is enough to prove that an (oriented) torus graph can be decomposed into basic torus graphs in Section 4 by the connected sum. We prove this using combinatorial arguments.

2. Orbit spaces of torus manifolds

In this section, we recall some basic facts about torus manifolds (see [Masuda 1999] or [Hattori and Masuda 2003] for details).

2A. *Torus manifolds.* A $2n$-dimensional torus manifold M is said to be *locally standard* if every point in M has a T-invariant open neighborhood U which is weakly equivariantly homeomorphic to an open subset $\Omega_U \subset \mathbb{C}^n$ invariant under the standard T^n-action on $\mathbb{C}^n$, where two group actions (U, T) and (Ω_U, T) are said to be *weakly equivariantly homeomorphic* if there is an equivariant homeomorphism from U to Ω_U up to an automorphism on T^n (see, e.g., [Kuroki 2011, Section 2.1] for details).

Let M_i, $i = 1, \ldots, m$, be a codimension-2 torus submanifold in a $2n$-dimensional torus manifold M which is fixed by some circle subgroup T_i in T. Such an M_i is a $(2n - 2)$-dimensional torus manifold with T/T_i-action, called a *characteristic submanifold*. Because a torus manifold M is compact, the cardinality of all characteristic submanifolds in M is finite. If M is locally standard, each characteristic submanifold is also locally standard.

An *omniorientation* $\mathbb{O}$ of M is a choice of orientation for the torus manifold M as well as for each characteristic submanifold. If there are just m characteristic submanifolds in M, there are exactly 2^{m+1} omniorientations (see [Buchstaber and Panov 2002; Hattori and Masuda 2003]). If M has a T-invariant almost complex structure J (in this case, M is automatically locally standard), then there exists the canonical omniorientation $\mathbb{O}_J$ determined by J. We call the torus manifold M with a fixed omniorientation $\mathbb{O}$ an *omnioriented torus manifold* and denote it by $(M, \mathbb{O})$.

2B. *Orbit spaces of locally standard torus manifolds.* The orbit space M/T of a locally standard torus manifold M naturally admits the structure of a "topological" manifold with corners. We next recall the basic facts about a topological manifold with corners (cf. the definition of a smooth manifold with corners in [Lee 2013]) and introduce the structure on M/T.

We will use the notation

$$[n] = \{0, 1, \ldots, n\}$$

and

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}.$$

Let Q^n be an n-dimensional topological manifold with boundary. A *chart with corners* for Q^n is a pair (V, ψ_V), where V is an open subset of Q^n and

$$\psi_V : V \to \mathbb{R}_+^n$$

is homeomorphic from V to a (relatively) open subset $\Omega_V \subset \mathbb{R}_+^n$. Two charts with corners (V, ψ_V) and (W, ψ_W) are said to be (*topologically*) *compatible* if the composition of functions $\psi_V \circ \psi_W^{-1} : \psi_W(V \cap W) \to \psi_V(V \cap W)$ is a strata-preserving homeomorphism. This implies that if $\psi_W(p) \in \mathbb{R}_+^n$ contains exactly k zero-coordinates then $\psi_V(p) \in \mathbb{R}_+^n$ also contains exactly k zero-coordinates for $0 \leq k \leq n$. We call the collection of compatible charts with corners $\{(V, \psi_V)\}$ whose domains cover Q^n an *atlas*. Then, its maximal atlas is called a *structure with corners* of Q^n. A topological manifold with boundary together with a structure with corners is called a (*topological*) *manifold with corners*. Let $p \in Q^n$ be a point of an n-dimensional manifold with corners Q^n. For a chart (V, ψ_V) with corners such that $p \in V$, we define $d(p) \in [n]$ to be the number of zero-coordinates of $\psi_V(p) \in \mathbb{R}_+^n$. By the compatibility of charts, this number is independent of the choice of a chart with corners which contains p. Therefore, the map $d : Q^n \to [n]$ is well defined. The number $d(p)$ is called the *depth* of p. We call the closure of a connected component of $d^{-1}(k)$, $0 \leq k \leq n$, a *codimension-k face*. In particular, the codimension-0 face is Q^n itself. Moreover, codimension-1, codimension-$(n-1)$ and codimension-n faces are called *facets*, *edges* and *vertices*, respectively. The set of all edges and vertices is called a *one-skeleton of Q^n* (or a *graph of Q^n*). By restricting the structure with corners on Q^n to faces, we may regard each codimension-k face as an $(n - k)$-dimensional (sub)manifold with corners.

Definition 2.1 (manifold with faces). An n-dimensional manifold with corners Q is said to be a *manifold with faces* (or a *nice manifold with corners*) if Q satisfies the following conditions:

(1) For every $k \in [n]$, there exists a codimension-k face.

(2) For each codimension-k face H, there are exactly k facets $F_1, \ldots, F_k$ such that H is a connected component of $\bigcap_{i=1}^{k} F_i$; moreover, $H \cap F \neq H$ for any facet $F \neq F_i$ ($i = 1, \ldots, k$).

Let (M, T) be a torus manifold. When (M, T) is locally standard, by the differentiable slice theorem, the orbit space M/T has the structure of an n-dimensional manifold with faces. On the other hand, when M satisfies $H^{\mathrm{odd}}(M) = 0$, its orbit space M/T satisfies a stronger condition by the following theorem (see [Masuda and Panov 2006, Lemma 2.1 and Theorem 2]).

Theorem 2.2 (Masuda–Panov). *Let M be a $2n$-dimensional torus manifold. Then, the following conditions are equivalent*:

(1) $H^{\mathrm{odd}}(M) = 0$.

(2) *The T-action on M is locally standard and its orbit space M/T has the structure of an n-dimensional face acyclic manifold with corners.*

Here, an n-dimensional *face acyclic* manifold with corners Q is an n-dimensional manifold with faces such that all faces F of Q (including Q) are acyclic, i.e., $H_*(F) \simeq H_0(F) \simeq \mathbb{Z}$. For example, if Q is a simply connected 3-dimensional face acyclic manifold with corners, then it is easy to check that the boundary of Q is homeomorphic to the 2-sphere S^2. Moreover, in this case, we can also check that Q itself is homeomorphic to the 3-dimensional disk D^3. Therefore, as one of the consequences of Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let M be a simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M) = 0$. Then, its orbit space M/T is homeomorphic to the 3-dimensional disk.*

By the definition of a manifold with faces Q, we can define a simplicial poset (partially ordered set) $\mathcal{P}(Q)$, called a *face poset* of Q (see [Masuda 2005]), to be the set of faces in Q with the empty set $\varnothing$ ordered by inclusion, where $\varnothing$ is the smallest element under this ordering, say $\preceq$. We often denote the face poset structure of Q by $(\mathcal{P}(Q), \preceq)$. Let Q_1 and Q_2 be n-dimensional manifolds with faces. We say Q_1 and Q_2 are *combinatorially equivalent* if their face posets $(\mathcal{P}(Q_1), \preceq_1)$ and $(\mathcal{P}(Q_2), \preceq_2)$ are isomorphic as posets (i.e., there is an order-preserving bijection between them). We denote the equivalence by $Q_1 \approx_c Q_2$. By the definition of weakly equivariant homeomorphism, if two locally standard torus manifolds M_1 and M_2 are weakly equivariantly homeomorphic then $M_1/T \approx_c M_2/T$.

2C. *Characteristic functions.* Let M be a $2n$-dimensional locally standard torus manifold. By the argument demonstrated in Section 2B, the orbit map $\pi : M \to M/T = Q$ may be regarded as the projection onto some manifold with faces Q. Let $\mathcal{F}(Q) = \{F_1, \ldots, F_m\} \subset \mathcal{P}(Q)$ be the set of all facets in Q. By the definition of facet $F_i \in \mathcal{F}(Q)$, its preimage $\pi^{-1}(F_i)$ is a characteristic submanifold M_i. Then, there exists the circle subgroup T_i ($\subset T$) fixing $M_i = \pi^{-1}(F_i)$ (recall that $\dim M_i = 2n - 2$). Recall that T_i is determined by a primitive element in $\mathfrak{t}_{\mathbb{Z}} \simeq \mathbb{Z}^n$ (the lattice of the Lie algebra of T). Therefore, using this primitive element (up to sign) in $\mathfrak{t}_{\mathbb{Z}}$, we can define the map

$$\lambda : \mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}/\{\pm 1\},$$

where $\mathfrak{t}_{\mathbb{Z}}/\{\pm 1\}$ represents the quotient of $\mathfrak{t}_{\mathbb{Z}}$ by signs. We call λ a *characteristic function*.

Now the choice of omniorientation $\mathbb{O}$ of M determines the sign of λ as follows. Fix an omniorientation $\mathbb{O}$ of M. Namely, we fix the orientation of the tangent bundle

of M (resp. M_i), say τ (resp. τ_i). Restricting τ to the submanifold M_i, say $\tau|_{M_i}$, we obtain the T^n-equivariant decomposition $\tau|_{M_i} \simeq \tau_i \oplus \nu_i$, where ν_i is the T_i-equivariant normal bundle of M_i. Therefore, because we fix the orientation of $\tau|_{M_i}$ (induced from the orientation of τ) and that of τ_i, we may choose an orientation of ν_i such that the orientation of $\tau|_{M_i}$ coincides with that of $\tau_i \oplus \nu_i$ (thus, we may regard ν_i as the complex line bundle over M_i). Because T_i acts on ν_i, we may choose an orientation of T_i such that the T_i-action preserves the orientation of ν_i. This orientation of T_i determines the sign of $\lambda(F_i)$ for $i = 1, \dots, m$. In this way, we have the function

$$\lambda_{\mathbb{O}} : \mathscr{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}.$$

In this paper, this is called an *omnioriented characteristic function* (of $(M, \mathbb{O})$).

Remark 2.4. The characteristic function defined in [Wiemeler 2013] may be regarded as the characteristic function λ above. On the other hand, the characteristic function defined in [Davis and Januszkiewicz 1991] may be regarded as the characteristic function $\lambda_{\mathbb{O}}$ above by taking an appropriate omniorientation (see also [Buchstaber and Panov 2002, Section 5.2]).

Let $p \in M^T$. We define the subset $I_p \subset [m]$ by

$$I_p = \{i \in [m] \mid p \in M_i\}.$$

By the differentiable slice theorem around $p \in M^T$, we have that its cardinality $|I_p|$ equals n for every $p \in M^T$. Put $I_p = \{i_1, \dots, i_n\}$. Because the T-action on M is effective, $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ spans $\mathfrak{t}_{\mathbb{Z}}^*/\{\pm 1\}$, i.e., the determinant of the induced $(n \times n)$-matrix

$$(\lambda(F_{i_1}) \cdots \lambda(F_{i_n}))$$

satisfies

(2-1) $$\det(\lambda(F_{i_1}) \cdots \lambda(F_{i_n})) = \pm 1.$$

Similarly, we have

(2-2) $$\det(\lambda_{\mathbb{O}}(F_{i_1}) \cdots \lambda_{\mathbb{O}}(F_{i_n})) = \pm 1$$

for each set of n facets such that $\bigcap_{j=1}^{n} F_{i_j} = \{p\}$ for some vertex $p \in Q$ (called the *facets around a vertex*).

Motivated by the above observations, we may abstractly define the characteristic function on a manifold with faces as follows (see [Buchstaber and Panov 2002; Davis and Januszkiewicz 1991] for simple polytopes and [Masuda and Panov 2006; Wiemeler 2013] for manifolds with faces).

Definition 2.5. Let Q be an n-dimensional manifold with faces and $\mathscr{F}(Q)$ be the set of its facets. Let $\mathfrak{t}_{\mathbb{Z}}$ be the lattice of the Lie algebra of T^n and $\mathfrak{t}_{\mathbb{Z}}/\{\pm 1\}$ be its quotient

by $\{\pm 1\}$. A function $\lambda : \mathscr{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}/\{\pm 1\}$ is said to be a *characteristic function* if λ satisfies (2-1) for the facets around every vertex, and a function $\lambda_{\mathbb{O}} : \mathscr{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}$ is said to be an *omnioriented characteristic function* if $\lambda_{\mathbb{O}}$ satisfies (2-2) for the facets around every vertex.

We denote an n-dimensional manifold with faces Q with its characteristic function λ (resp. omnioriented characteristic function $\lambda_{\mathbb{O}}$) by (Q, λ) (resp. $(Q, \lambda_{\mathbb{O}})$).

Let Q_1 and Q_2 be manifolds with faces, and let λ_1 and λ_2 be their characteristic functions and $\lambda_{\mathbb{O}_1}$ and $\lambda_{\mathbb{O}_2}$ be their omnioriented characteristic functions, respectively. Assume that $Q_1 \approx_c Q_2$, induced by the bijective map $\tilde{f} : \mathscr{P}(Q_1) \to \mathscr{P}(Q_2)$. Denote its restriction onto the set of facets by

$$f = \tilde{f}|_{\mathscr{F}(Q_1)} : \mathscr{F}(Q_1) \to \mathscr{F}(Q_2).$$

We say that (Q_1, λ_1) and (Q_2, λ_2) are *combinatorially equivalent* if the following diagram commutes:

$$
\begin{array}{ccc}
\mathscr{F}(Q_1) & \xrightarrow{\ \lambda_1\ } & \mathfrak{t}_{\mathbb{Z}}/\{\pm 1\} \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{Id}} \\
\mathscr{F}(Q_2) & \xrightarrow{\ \lambda_2\ } & \mathfrak{t}_{\mathbb{Z}}/\{\pm 1\}
\end{array}
$$

Similarly, $(Q_1, \lambda_{\mathbb{O}_1})$ and $(Q_2, \lambda_{\mathbb{O}_2})$ are combinatorially equivalent if the following diagram commutes:

$$
\begin{array}{ccc}
\mathscr{F}(Q_1) & \xrightarrow{\ \lambda_{\mathbb{O}_1}\ } & \mathfrak{t}_{\mathbb{Z}} \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{Id}} \\
\mathscr{F}(Q_2) & \xrightarrow{\ \lambda_{\mathbb{O}_2}\ } & \mathfrak{t}_{\mathbb{Z}}
\end{array}
$$

Note that the characteristic function λ can be obtained by ignoring signs from the omnioriented characteristic function $\lambda_{\mathbb{O}}$; we call such a λ an *induced characteristic function* from $\lambda_{\mathbb{O}}$. On the other hand, by choosing a sign for each facet, we can obtain an omnioriented characteristic function $\lambda_{\mathbb{O}}$ from the characteristic function λ; we call such a $\lambda_{\mathbb{O}}$ an *induced oriented characteristic function* from λ.

Lemma 2.6. *If $(Q_1, \lambda_{\mathbb{O}_1})$ and $(Q_2, \lambda_{\mathbb{O}_2})$ are combinatorially equivalent, then their induced (Q_1, λ_1) and (Q_2, λ_2) are also combinatorially equivalent.*

If (Q_1, λ_1) and (Q_2, λ_2) are combinatorially equivalent, then there are induced omnioriented characteristic functions $\lambda_{\mathbb{O}_1}$ and $\lambda_{\mathbb{O}_2}$ such that $(Q_1, \lambda_{\mathbb{O}_1})$ and $(Q_2, \lambda_{\mathbb{O}_2})$ are combinatorially equivalent.

We now introduce one of the key facts used to prove our main theorem (see [Wiemeler 2013, Theorems 1.3 and 6.1]).

Theorem 2.7 (Wiemeler). *Let M_1 and M_2 be 6-dimensional simply connected torus manifolds with $H^{\mathrm{odd}}(M_1) = H^{\mathrm{odd}}(M_2) = 0$, and let (Q_1, λ_1) and (Q_2, λ_2) be*

their orbit spaces with characteristic functions. Then, the following statements are equivalent:

(1) (Q_1, λ_1) *and* (Q_2, λ_2) *are combinatorially equivalent.*

(2) M_1 *and* M_2 *are equivariantly homeomorphic.*

(3) M_1 *and* M_2 *are equivariantly diffeomorphic.*

Therefore, by Corollary 2.3 and Theorem 2.7, to classify all 6-dimensional simply connected torus manifolds with $H^{\mathrm{odd}}(M) = 0$, it is enough to classify all (Q, λ)'s up to combinatorial equivalence, where Q is a 3-dimensional disk equipped with the structure of a manifold with faces.

3. Torus graph induced from manifold with faces

Let $(M, \mathbb{O})$ be an omnioriented locally standard $2n$-dimensional torus manifold and $(Q, \lambda_{\mathbb{O}})$ be its orbit space with an omnioriented characteristic function. From the one-skeleton of $(Q, \lambda_{\mathbb{O}})$, we can define a labelled graph called a *torus graph*. One of the key steps in proving the main theorem is to classify all possible torus graphs (see Section 7). We first recall the definition of torus graph given by Maeda et al. [2007].

Let Γ be the graph of Q. Let $V(\Gamma)$ be its vertices and $E(\Gamma)$ be its oriented edges, i.e., we distinguish two edges pq and qp. For $p \in V(\Gamma)$, we denote the set of outgoing edges from p by $E_p(\Gamma)$. Because Q is an n-dimensional manifold with faces, $|E_p(\Gamma)| = n$ and each edge $e \in E(\Gamma)$ is a connected component of $\bigcap_{i=1}^{n-1} F_i$ for some $F_1, F_2, \ldots, F_{n-1} \in \mathscr{F}(Q)$. Moreover, for a $p \in V(\Gamma)$ which is one of two vertices on e, there is another facet $F_n \in \mathscr{F}(Q)$ such that $\{p\}$ is a connected component of $\bigcap_{i=1}^{n} F_i$. In other words, F_n may be regarded as a normal facet of $e \in E(\Gamma)$ on $p \in V(\Gamma)$. Put $\lambda_{\mathbb{O}}(F_i) = a_i \in \mathfrak{t}_{\mathbb{Z}} \simeq \mathbb{Z}^n$. Then, there exists a unique $\alpha \in \mathfrak{t}_{\mathbb{Z}}^*$ such that

$$(3\text{-}1) \qquad \langle \alpha, a_i \rangle = 0 \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad \langle \alpha, a_n \rangle = +1,$$

where $\langle \, \cdot \, , \, \cdot \, \rangle$ represents the pairing of $\mathfrak{t}^*$ and $\mathfrak{t}$. Therefore, in this way, we can define a map $\mathscr{A}: E(\Gamma) \to \mathfrak{t}_{\mathbb{Z}}^*$ from the omnioriented characteristic function $\lambda_{\mathbb{O}}$. This map $\mathscr{A}$ is called an *axial function* on Γ. We call the labelled graph $(\Gamma, \mathscr{A})$ a *torus graph* induced from $(Q, \lambda_{\mathbb{O}})$ (or equivalently $(M, \mathbb{O})$). We denote such a torus graph by $\Gamma(Q, \lambda_{\mathbb{O}})$ (or $(\Gamma_M, \mathscr{A}_M)$). We can easily check the following proposition using the definition of torus graph (see also [Maeda et al. 2007]).

Proposition 3.1. *Let* $(\Gamma, \mathscr{A})$ *be a torus graph induced from* $(Q, \lambda_{\mathbb{O}})$. *Then,* Γ *is an n-valent regular graph, i.e.,* $|E_p(\Gamma)| = n$ *for all* $p \in V(\Gamma)$, *and* $(\Gamma, \mathscr{A})$ *satisfies the following conditions*:

(1) $\mathscr{A}(e) = \pm \mathscr{A}(\bar{e})$, *where* $\bar{e}$ *is the orientation-reversed edge of e.*

(2) $\{\mathscr{A}(e) \mid e \in E_p(\Gamma)\}$ *spans* $\mathfrak{t}_{\mathbb{Z}}^*$ *for all vertices* $p \in V(\Gamma)$.

(3) *There is a bijection* $\nabla_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ *for all edges whose initial vertex is* p *and terminal vertex is* q *such that*

    (a) $\nabla_{\bar{e}} = \nabla_e^{-1}$,
    (b) $\nabla_e(e) = \bar{e}$,
    (c) $\mathscr{A}(\nabla_{pq}(e)) - \mathscr{A}(e) \equiv 0 \bmod \mathscr{A}(pq)$ *for all* $e \in E_p(\Gamma)$.

We call $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ a *connection* on $(\Gamma, \mathscr{A})$.

Remark 3.2. The original definition of torus graph (induced from an omnioriented torus manifold) uses *tangential representations*; see [Masuda and Panov 2006; Maeda et al. 2007]. The definition of torus graph given above is essentially the same as the original definition.

In [Maeda et al. 2007], motivated by the GKM graph introduced by Guillemin and Zara [2001], an n-valent graph Γ with a label $\mathscr{A} : E(\Gamma) \to \mathfrak{t}_{\mathbb{Z}}^*$ which satisfies the three conditions in Proposition 3.1 is called an (*abstract*) *torus graph* (i.e., there might be no geometric objects which define $(\Gamma, \mathscr{A})$).

We next define the equivalence relation between two torus graphs. We call the map $f : \Gamma_1 = (V(\Gamma_1), E(\Gamma_1)) \to \Gamma_2 = (V(\Gamma_2), E(\Gamma_2))$ a *graph isomorphism* if the restricted maps $f|_V : V(\Gamma_1) \to V(\Gamma_2)$ and $f|_E : E(\Gamma_1) \to E(\Gamma_2)$ are bijective and the following diagram commutes:

$$
\begin{array}{ccc}
E(\Gamma_1) & \xrightarrow{\ f|_E\ } & E(\Gamma_2) \\
\downarrow{\scriptstyle \pi_{V_1}} & & \downarrow{\scriptstyle \pi_{V_2}} \\
V(\Gamma_1) & \xrightarrow{\ f|_V\ } & V(\Gamma_2)
\end{array}
$$

Here the map $\pi_V : E(\Gamma) \to V(\Gamma)$ is the projection onto the initial vertex, i.e., $\pi_V(pq) = p$. In other words, the bijection $f|_V$ preserves the edges. Now we may define the equivalence relation.

Definition 3.3. Let $(\Gamma_1, \mathscr{A}_1)$ and $(\Gamma_2, \mathscr{A}_2)$ be torus graphs. We say $(\Gamma_1, \mathscr{A}_1)$ and $(\Gamma_2, \mathscr{A}_2)$ are *equivalent* if there is a graph isomorphism $f : \Gamma_1 \to \Gamma_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E(\Gamma_1) & \xrightarrow{\ \mathscr{A}_1\ } & \mathfrak{t}_{\mathbb{Z}}^* \\
\downarrow{\scriptstyle f|_E} & & \downarrow{\scriptstyle \mathrm{Id}} \\
E(\Gamma_2) & \xrightarrow{\ \mathscr{A}_2\ } & \mathfrak{t}_{\mathbb{Z}}^*
\end{array}
$$

Assume $(\Gamma, \mathscr{A}) = \Gamma(Q, \lambda_{\mathbb{O}})$. Let $\mathscr{P}_k(\Gamma, \mathscr{A})$ be the set of k-valent torus subgraphs in $(\Gamma, \mathscr{A})$, i.e., k-valent subgraphs in Γ closed under the connection ∇, where $-1 \leq k \leq n$ and we define $\mathscr{P}_{-1}(\Gamma, \mathscr{A}) = \{\varnothing\}$. Then, the set

$$\mathscr{P}(\Gamma, \mathscr{A}) = \bigcup_{k=-1}^{n} \mathscr{P}_k(\Gamma, \mathscr{A})$$

admits the structure of a simplicial poset by inclusion (see [Maeda et al. 2007]). We denote this structure by $(\mathscr{P}(\Gamma, \mathscr{A}), \preceq)$. Let $\mathscr{P}(Q)$ be the face poset of Q (see Section 2B) and $\mathscr{P}_k(Q)$ be the set of all k-dimensional faces, where $-1 \leq k \leq n$ and $\mathscr{P}_{-1}(Q) = \{\varnothing\}$. Then, each element of $\mathscr{P}_k(\Gamma, \mathscr{A})$ is nothing but the graph of an element in $\mathscr{P}_k(Q)$. This implies that the poset $(\mathscr{P}(\Gamma, \mathscr{A}), \preceq)$ is equivalent to the poset $(\mathscr{P}(Q), \preceq)$. Therefore, we have the following lemma.

Lemma 3.4. *The following two statements are equivalent*:

(1) *Two manifolds with faces with omnioriented characteristic functions $(Q_1, \lambda_{\mathbb{O}_1})$ and $(Q_2, \lambda_{\mathbb{O}_2})$ are combinatorially equivalent.*

(2) *Their induced torus graphs $\Gamma(Q_1, \lambda_{\mathbb{O}_1})$ and $\Gamma(Q_2, \lambda_{\mathbb{O}_2})$ are equivalent.*

By Lemma 2.6, Theorem 2.7 and Lemma 3.4, we have the following corollary.

Corollary 3.5. *Let (M_1, T) and (M_2, T) be 6-dimensional simply connected torus manifolds with vanishing odd-degree cohomology. Then, the following statements are equivalent*:

(1) *(M_1, T) and (M_2, T) are equivariantly diffeomorphic.*

(2) *Their orbit spaces, i.e., 3-dimensional disks with the structures of manifolds with faces, with characteristic functions $(M_1/T, \lambda_1)$ and $(M_2/T, \lambda_2)$ are combinatorially equivalent.*

(3) *There are omnioriented characteristic functions $\lambda_{\mathbb{O}_1}$ and $\lambda_{\mathbb{O}_2}$ such that their induced 3-valent torus graphs $\Gamma(M_1/T, \lambda_{\mathbb{O}_1})$ and $\Gamma(M_2/T, \lambda_{\mathbb{O}_2})$ are equivalent.*

Therefore, to prove our main theorem (Theorem 7.1), it is enough to classify all 3-valent torus graphs $(\Gamma, \mathscr{A})$, induced from $(M, \mathbb{O})$, up to equivalence.

4. Basic 6-dimensional torus manifolds

Let (M, T) be a simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M) = 0$, and let $(\Gamma_M, \mathscr{A}_M)$ $(= (\Gamma, \mathscr{A}))$ be its torus graph induced by some omniorientation. As a preliminary to proving the main theorem (Theorem 7.1), in this section we introduce some of the basic torus graphs $(\Gamma, \mathscr{A})$ and their corresponding 6-dimensional torus manifolds (M, T).
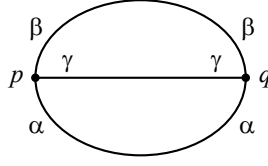
Figure 1. The torus graph $(\Gamma_{\mathrm{sp}}, \mathcal{A}_{\alpha,\beta,\gamma})$, where α, β, $\gamma \in \mathfrak{t}_{\mathbb{Z}}^* \simeq \mathbb{Z}^3$ are a $\mathbb{Z}$-basis.

4A. 6-*sphere*. Because the induced torus graphs from (M, T) are 3-valent, if there is a 3-multiple edge, i.e., three edges that are incident to the same two vertices, then it follows from Proposition 3.1 that such a torus graph must be the torus graph in Figure 1, denoted $(\Gamma_{\mathrm{sp}}, \mathcal{A}_{\alpha,\beta,\gamma})$.

Put $\alpha = k_{11}e_1 + k_{12}e_2 + k_{13}e_3$, $\beta = k_{21}e_1 + k_{22}e_2 + k_{23}e_3$ and $\gamma = k_{31}e_1 + k_{32}e_2 + k_{33}e_3$, using the standard basis e_1, e_2, e_3 in $\mathfrak{t}_{\mathbb{Z}}^* \simeq \mathbb{Z}^3$. Then, we have

$$(4\text{-}1) \qquad \det \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} = \pm 1.$$

Let $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ be the unit sphere, i.e., the set $(z_1, z_2, z_3, r) \in \mathbb{C}^3 \oplus \mathbb{R}$ such that $|z_1|^2 + |z_2|^2 + |z_3|^2 + r^2 = 1$. Define the T^3-action on the first three complex coordinates in S^6 by

$$(4\text{-}2) \qquad (t_1, t_2, t_3)(z_1, z_2, z_3, r) \mapsto (\rho_1(t)z_1, \rho_2(t)z_2, \rho_3(t)z_3, r),$$

where $t = (t_1, t_2, t_3) \in T$ and $\rho_i : T \to S^1$, $i = 1, 2, 3$, is a 1-dimensional complex representation defined by

$$\rho_i(t_1, t_2, t_3) = t_1^{k_{i1}} t_2^{k_{i2}} t_3^{k_{i3}}.$$

Then, by choosing an appropriate omniorientation on S^6, we have that its induced torus graph is equivalent to $(\Gamma_{\mathrm{sp}}, \mathcal{A}_{\alpha,\beta,\gamma})$. Therefore, using Corollary 3.5, we have the following lemma.

Lemma 4.1. *Let $(M, \mathbb{O})$ be an omnioriented 6-dimensional simply connected torus manifold with $H^{\mathrm{odd}}(M) = 0$. If its induced torus graph is $(\Gamma_{\mathrm{sp}}, \mathcal{A}_{\alpha,\beta,\gamma})$, then (M, T) is equivariantly diffeomorphic to one of (S^6, T) defined by (4-2).*

4B. S^4-*bundles over* S^2. Assume that a 3-valent torus graph $(\Gamma, \mathcal{A})$ does not have 3-multiple edges but does have multiple edges, i.e., two edges that are incident to the same two vertices. In this section, we classify the easiest case of such torus graphs.

Because Γ is a one-skeleton of a 3-dimensional manifold with faces Q, we have $|V(\Gamma)| \geq 4$. Assume that $|V(\Gamma)| = 4$. Then, we can easily check that such a torus
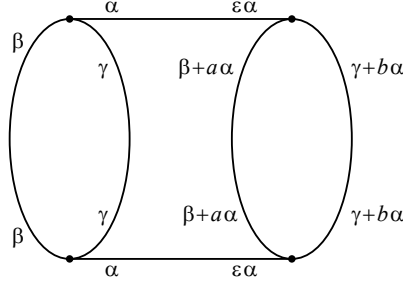
Figure 2. The torus graph $(\Gamma_S, \mathscr{A}_S) = (\Gamma_S, \mathscr{A}^{\epsilon,a,b}_{\alpha,\beta,\gamma})$, where $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$, and $\alpha, \beta, \gamma \in \mathfrak{t}^*_{\mathbb{Z}}$ are a $\mathbb{Z}$-basis of $\mathfrak{t}^*_{\mathbb{Z}}$.

manifold is the one-skeleton of the 3-simplex (see Figure 3 in Section 4C) or the graph drawn in Figure 2, say Γ_S. It is well known that the torus manifold whose torus graph is the one-skeleton of the 3-simplex is equivariantly diffeomorphic to the complex projective space with some T-action (see, e.g., [Davis and Januszkiewicz 1991], and see also Figure 3 in Section 4C). So, we only study the torus manifold which induces the graph Γ_S. Because Q is homeomorphic to D^3, we may regard a Q whose one-skeleton is Γ_S as the product $D^2 \times I$, where D^2 is the 2-dimensional disk and I is the interval. By considering all functions on facets of Q which satisfy (2-2), we can classify all omnioriented characteristic functions $\lambda_{\mathbb{O}}$ on Q. Then, in the same way we induced the axial function $\mathscr{A}_S$ from $(Q, \lambda_{\mathbb{O}})$ in Section 3, we can obtain all possible axial functions on Γ_S, as shown in Figure 2.

The torus graph $(\Gamma_S, \mathscr{A}_S)$ in Figure 2 can be induced from an S^4-bundle over S^2 as follows. First, by choosing $\epsilon = \pm 1$, we may define two free T^1-actions on $S^3 \subset \mathbb{C}^2$:

$$(w, z) \mapsto (t^{-1}w, t^{\epsilon}z).$$

We denote S^3 with the above T^1-action by S^3_ϵ. Note that S^3_ϵ/T^1 is diffeomorphic to the 2-sphere S^2, and a complex line bundle over S^2 can be denoted by

$$S^3_\epsilon \times_{T^1} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the complex 1-dimensional T^1-representation space by k-times rotation for some $k \in \mathbb{Z}$. Let $S^3_\epsilon \times_{T^1} \mathbb{R}$ be the trivial real line bundle over S^2. Take the unit sphere bundle of the following Whitney sum of three vector bundles for $a, b \in \mathbb{Z}$:

$$S^3_\epsilon \times_{T^1} (\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R}).$$

Then, we obtain the S^4-bundle over S^2 denoted by

$$M(\epsilon, a, b) = S^3_\epsilon \times_{S^1} S(\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R}),$$

for $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$. Namely, we can identify elements in $M(\epsilon, a, b)$ by

$$[(w, z), (x, y, r)] = [(t^{-1}w, t^\epsilon z), (t^a x, t^b y, r)]$$

for any $t \in T^1$ such that $|w|^2 + |z|^2 = 1$ and $|x|^2 + |y|^2 + r^2 = 1$. Define a T^3-action on $M(\epsilon, a, b)$ by

$$[(w, z), (x, y, r)] \mapsto [(t_1 w, z), (t_2 x, t_3 y, r)],$$

where $(t_1, t_2, t_3) \in T^3$. Fix an omniorientation on $M(\epsilon, a, b)$ by the induced orientations from $S_\epsilon^3 \times S^4 \subset \mathbb{C}^2 \times (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R})$. Then, considering the tangential representations around each fixed point, it is easy to check that the induced torus graph is $(\Gamma_S, \mathscr{A}_{e_1, e_2, e_3}^{\epsilon, a, b})$, where e_1, e_2, e_3 are the standard basis of $\mathfrak{t}_{\mathbb{Z}} \simeq \mathbb{Z}^3$. Therefore, by taking the appropriate automorphism of T^3, we can construct each torus graph $(\Gamma_S, \mathscr{A}_S)$ in Figure 2 from $M(\epsilon, a, b)$. Note that if $\epsilon = -1$ and $a = b$, then this is nothing but one of the torus manifolds which appeared in the classifications of torus manifolds with codimension-1 extended actions in [Kuroki 2011].

By the argument above and Corollary 3.5, we have the following lemma.

Lemma 4.2. *Let* $(M, \mathbb{O})$ *be an omnioriented* 6-*dimensional simply connected torus manifold with* $H^{\mathrm{odd}}(M) = 0$. *If its induced torus graph has four vertices, then* (M, T) *is equivariantly diffeomorphic to one of the following*:

(1) $\mathbb{C}P^3$ *with the standard* T^3-*action up to automorphism of* T^3;

(2) $M(\epsilon, a, b)$ *for some* $\epsilon = \pm 1$ *and* $a, b \in \mathbb{Z}$.

4C. 6-*dimensional quasitoric manifolds*. Assume that there are no multiple edges in a 3-valent torus graph $(\Gamma, \mathscr{A})$, i.e., there are no two edges that are incident to the same two vertices. A graph Γ is called *simple* if Γ does not have both multiple edges and loops. In this section and in Section 5, we study simple torus graphs which can be realized as the one-skeleton of a manifold with faces homeomorphic to D^3.

The typical example of such torus manifolds whose torus graphs are simple is a quasitoric manifold (introduced by Davis and Januszkiewicz [1991]; see also [Buchstaber and Panov 2002]). A *quasitoric manifold* is defined by a torus manifold whose orbit space is a *simple convex polytope*, i.e., a convex polytope admitting the structure of a manifold with faces. For example, the complex projective space $\mathbb{C}P^n$ with the standard T^n-action is the quasitoric manifold whose orbit space is the n-dimensional simplex. Figure 3 shows the torus graph induced from $(\mathbb{C}P^3, \mathbb{O}_\mathbb{C})$, i.e., the omniorientation $\mathbb{O}_\mathbb{C}$ induced from the standard complex structure on $\mathbb{C}P^3$ and the standard T-action on $\mathbb{C}P^3$.

We next characterize when torus graphs are induced from simple convex polytopes, i.e., induced from quasitoric manifolds. The Steinitz theorem (see [Ziegler
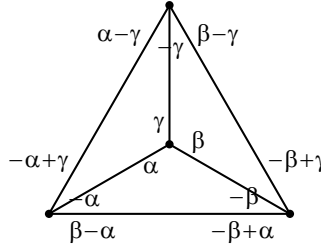
Figure 3. The torus graph induced from $(\mathbb{C}P^3, \mathbb{O}_{\mathbb{C}})$.

1995, Chapter 4]) tells us that a graph Γ is the one-skeleton of a 3-dimensional convex polytope if and only if Γ is a simple, planar and 3-connected graph, where Γ is called a 3-*connected* graph if it remains connected whenever fewer than three vertices are removed. It easily follows from the Steinitz theorem that we have the following lemma.

Lemma 4.3. *Let Q be a manifold with faces and Γ be its graph. Assume that Q is homeomorphic to the 3-disk D^3 and there are no multiple edges. Then, the following statements are equivalent*:

(1) *Q is combinatorially equivalent to a 3-dimensional simple convex polytope P.*

(2) *Γ is a 3-connected graph.*

Combining this result with Corollary 3.5, we have the following fact.

Lemma 4.4. *Let $(M, \mathbb{O})$ be an omnioriented 6-dimensional simply connected torus manifold with $H^{\mathrm{odd}}(M) = 0$. Then, the following statements are equivalent*:

(1) *(M, T) is equivariantly diffeomorphic to a quasitoric manifold.*

(2) *Its induced torus graph Γ is a 3-connected graph with no multiple edges.*

5. Connected sum of torus graphs and other 6-dimensional torus manifolds

By the arguments in Section 4, only the following case remains: the simply connected 6-dimensional torus manifolds with $H^{\mathrm{odd}}(M) = 0$ whose induced torus graphs are simple but not 3-connected. Such torus manifolds can be constructed using the connected sum of "oriented" torus graphs. The purpose of this section is to introduce oriented torus graphs and their connected sum (see also [Darby 2015]).

We first recall the equivariant connected sum of torus manifolds. Let M_1, M_2 be $2n$-dimensional torus manifolds and $p \in M_1^T$, $q \in M_2^T$ be fixed points. Using the slice theorem, we may take T-invariant open neighborhoods $U_1 \subset M_1$ of p and $U_2 \subset M_2$ of q. Assume that U_1 and U_2 are equivariantly diffeomorphic. Then, $U_1 \backslash \{p\}$ and $U_2 \backslash \{q\}$ are equivariantly diffeomorphic to $S^{2n-1} \times I$, where $S^{2n-1} \subset \mathbb{C}^n$ with some effective T^n-action and $I = (-\epsilon, \epsilon)$ with the trivial T^n-action for some

$\epsilon > 0$. We glue these two neighborhoods by φ defined by the identity on S^{2n-1} and the map $r \mapsto -r$ on I for $r \in I$. Namely, we can glue $M_1 \setminus \{p\}$ and $M_2 \setminus \{q\}$ by the identification

$$(5\text{-}1) \quad M_1 \setminus \{p\} \supset U_1 \setminus \{p\} \xrightarrow{\simeq} S^{2n-1} \times I \xrightarrow{\varphi} S^{2n-1} \times I \xrightarrow{\simeq} U_2 \setminus \{q\} \subset M_2 \setminus \{q\}.$$

The T^n-manifold obtained in this way is denoted by $M_1 \# M_2$ or $M_1 \#_{(p,q)} M_2$ (if we emphasize fixed points $p \in M_1^T$ and $q \in M_2^T$). Because each torus manifold has more than two fixed points, $M_1 \# M_2$ is again a torus manifold. We call this operation the *equivariant connected sum*.

Lemma 5.1. *If two torus manifolds M_1 and M_2 are simply connected and $H^{\mathrm{odd}}(M_1) = H^{\mathrm{odd}}(M_2) = 0$, then $M_1 \# M_2$ is also simply connected and $H^{\mathrm{odd}}(M_1 \# M_2) = 0$.*

Proof. It is easy to check the statement using van Kampen's theorem and the Mayer–Vietoris exact sequence. □

Assume that $(M_1, \mathbb{O}_1)$ and $(M_2, \mathbb{O}_2)$ are 6-dimensional omnioriented simply connected torus manifolds with $H^{\mathrm{odd}}(M_1) = H^{\mathrm{odd}}(M_2) = 0$. Let $(\Gamma_1, \mathscr{A}_1)$ and $(\Gamma_2, \mathscr{A}_2)$ be their induced 3-valent torus graphs. Assume that we can glue $p \in M_1^T$ and $q \in M_2^T$ by the connected sum. Then, by considering the restriction of φ in (5-1) onto $S^{2n-1} \subset \mathbb{C}^n$, i.e., the identity map, the axial functions around $p \in V(\Gamma_1)$ and $q \in V(\Gamma_2)$ must satisfy

$$(5\text{-}2) \qquad \{\mathscr{A}_1(e) \mid e \in E_p(\Gamma_1)\} = \{\mathscr{A}_2(e) \mid e \in E_q(\Gamma_2)\}.$$

However, at this stage, the torus graphs $(\Gamma_1, \mathscr{A}_1)$ and $(\Gamma_2, \mathscr{A}_2)$ do not contain information about the orientations of M_1 and M_2. To do the connected sum, we need the orientations around $p \in M_1^T$ and $q \in M_2^T$. To encode the orientations around fixed points, we need the following notion.

Definition 5.2. Let $(\Gamma, \mathscr{A})$ be a torus graph. We call a triple $(\Gamma, \mathscr{A}, \sigma)$ with a map $\sigma : V(\Gamma) \to \{+1, -1\}$ an *oriented torus graph* if σ satisfies the following condition for all $e \in E(\Gamma)$:

$$\sigma(\pi_V(e))\mathscr{A}(e) = -\sigma(\pi_V(\bar{e}))\mathscr{A}(\bar{e}),$$

where $\pi_V(e) \in V(\Gamma)$ is the initial vertex of $e \in E(\Gamma)$, i.e., for $e = pq$, $\pi_V(e) = p$ and $\pi_V(\bar{e}) = q$. We call such a map σ an *orientation* of $(\Gamma, \mathscr{A})$.

Remark 5.3. Let $(M, \mathbb{O})$ be an omnioriented torus manifold. The oriented torus graph $(\Gamma, \mathscr{A}, \sigma)$ of $(M, \mathbb{O})$ is defined as follows. Let $p \in M^T$. Then, there exist exactly n characteristic submanifolds $M_1, \dots, M_n$ such that p is a connected component of $\bigcap_{i=1}^n M_i$. Now the fixed orientations of $M_1, \dots, M_n$ determine the decomposition of the tangential representation; i.e., $\psi_p : T_p M \xrightarrow{\simeq} V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$ is determined by fixing the orientations of $M_1, \dots, M_n$. On the other hand, the

orientation of M determines the orientation of $T_p M$. So, we define the map $\sigma : V(\Gamma) = M^T \to \{+1, -1\}$ by

$$\sigma(p) = \begin{cases} +1 & \text{if } \psi_p \text{ preserves the orientations,} \\ -1 & \text{if } \psi_p \text{ reverses the orientations.} \end{cases}$$

Let $(\Gamma_1, \mathcal{A}_1, \sigma_1)$ and $(\Gamma_2, \mathcal{A}_2, \sigma_2)$ be the induced oriented torus graphs from $(M_1, \mathbb{O}_1)$ and $(M_2, \mathbb{O}_2)$. If we can glue $p \in M_1^T$ and $q \in M_2^T$ by the connected sum, then both (5-2) and

$$(5\text{-}3) \qquad\qquad\qquad \sigma_1(p) \neq \sigma_2(q)$$

hold ((5-3) corresponds to the fact that the orientations on $T_p M_1$ and $T_q M_2$ are different). The induced (oriented) torus graph by $M_1 \#_{(p,q)} M_2$ is nothing but the one-skeleton of the connected sum $Q_1 \#_{(p,q)} Q_2$ of manifolds with faces, where Q_i is the orbit space of M_i, $i = 1, 2$ (see [Izmest'ev 2001, Definition 3; Kuroki 2010, Section 3.1] for details about the connected sum of polytopes). Therefore, conversely, if $p \in V(\Gamma_1)$ and $q \in V(\Gamma_2)$ satisfy (5-2) and (5-3), then we can do the *connected sum of* (*oriented*) *torus graphs* between $(\Gamma_1, \mathcal{A}_1, \sigma_1)$ and $(\Gamma_2, \mathcal{A}_2, \sigma_2)$, say $(\Gamma, \mathcal{A}, \sigma) = (\Gamma_1, \mathcal{A}_1, \sigma_1) \# (\Gamma_2, \mathcal{A}_2, \sigma_2)$ or $(\Gamma_1, \mathcal{A}_1, \sigma_1) \#_{(p,q)} (\Gamma_2, \mathcal{A}_2, \sigma_2)$ (if we emphasize the vertices $p \in V(\Gamma_1)$ and $q \in V(\Gamma_2)$). More precisely, $(\Gamma, \mathcal{A}, \sigma) = (\Gamma_1, \mathcal{A}_1, \sigma_1) \# (\Gamma_2, \mathcal{A}_2, \sigma_2)$ is defined as follows (see Figure 4).

(1) $V(\Gamma) = V(\Gamma_1) \setminus \{p\} \sqcup V(\Gamma_2) \setminus \{q\}$.

(2) $E(\Gamma)$ is given by

$(E(\Gamma_1) \setminus \{pp_1, pp_2, pp_3\}) \sqcup (E(\Gamma_2) \setminus \{qq_1, qq_2, qq_3\}) \sqcup \{p_1q_1, p_2q_2, p_3q_3\},$

where $\mathcal{A}_1(pp_i) = \mathcal{A}_2(qq_i)$ for $i = 1, 2, 3$.

(3) $\mathcal{A} : E(\Gamma) \to (\mathfrak{t}_{\mathbb{Z}}^3)^*$ is defined by $\mathcal{A}(e) = \mathcal{A}_1(e)$ and $\mathcal{A}(f) = \mathcal{A}_2(f)$ for e in $E(\Gamma_1) \setminus \{pp_1, pp_2, pp_3\}$ and f in $E(\Gamma_2) \setminus \{qq_1, qq_2, qq_3\}$, and $\mathcal{A}(p_iq_i) = \mathcal{A}_1(p_i p)$ and $\mathcal{A}(q_i p_i) = \mathcal{A}_2(q_i q)$.

(4) $\sigma : V(\Gamma) \to \{+1, -1\}$ is defined by $\sigma(r) = \sigma_1(r)$ for $r \in V(\Gamma_1) \setminus \{p\}$ and $\sigma(r') = \sigma_2(r')$ for $r' \in V(\Gamma_2) \setminus \{q\}$.
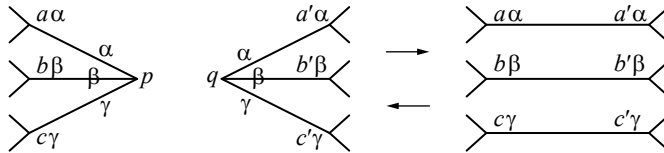
Figure 4. The local figure of the connected sum $\#_{(p,q)}$ of a torus manifold (left to right) and its inverse $\#_{(p,q)}^{-1}$ (right to left), where $\sigma_1(p) \neq \sigma_2(q)$. Here, α, β, γ are a $\mathbb{Z}$-basis of $(\mathfrak{t}_{\mathbb{Z}}^3)^*$ and $a, a', b, b', c, c' = \pm 1$.
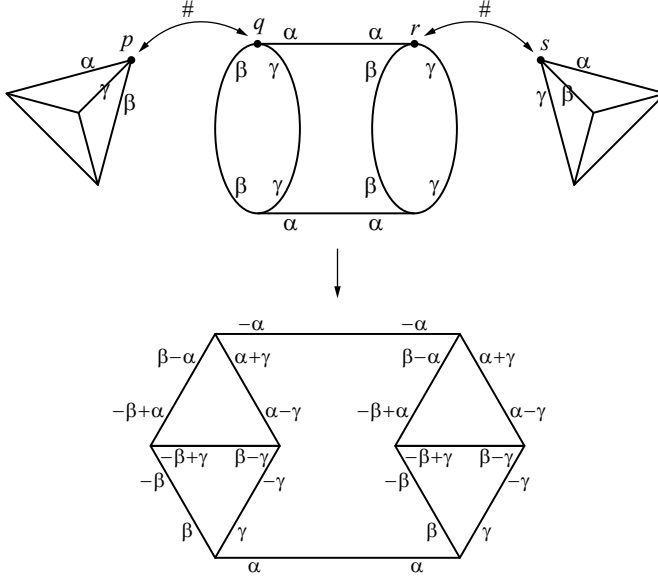
Figure 5. The torus graph (with appropriate orientations, e.g, $\sigma(p) = +1$, $\sigma(q) = -1$, $\sigma(r) = +1$, $\sigma(s) = -1$) induced from $\mathbb{C}P^3 \# (S^2 \times S^4) \# \overline{\mathbb{C}P^3}$.

Then, we can easily check that $(\Gamma, \mathscr{A}, \sigma)$ is an oriented torus graph. Using Corollary 3.5 and the arguments above, we have the following lemma.

Lemma 5.4. *Let M_1 and M_2 be 6-dimensional simply connected torus manifolds with $H^{\mathrm{odd}}(M_1) = H^{\mathrm{odd}}(M_2) = 0$, and let $(\Gamma_1, \mathscr{A}_1, \sigma_1)$ and $(\Gamma_2, \mathscr{A}_2, \sigma_2)$ be their respective induced oriented torus graphs from some omniorientations. If $(\Gamma, \mathscr{A}, \sigma) = (\Gamma_1, \mathscr{A}_1, \sigma_1) \#_{(p,q)} (\Gamma_2, \mathscr{A}_2, \sigma_2)$, then $(\Gamma, \mathscr{A}, \sigma)$ is the oriented torus graph induced from $M = M_1 \#_{(p,q)} M_2$ with some omniorientation.*

Using the connected sum, we can construct the torus manifolds which do not appear in Section 4. One such example is

$$\mathbb{C}P^3 \# (S^2 \times S^4) \# \overline{\mathbb{C}P^3},$$

where $\overline{\mathbb{C}P^3}$ is the reversed orientation of $\mathbb{C}P^3$. Figure 5 shows the torus graph induced from $\mathbb{C}P^3 \# (S^2 \times S^4) \# \overline{\mathbb{C}P^3}$ (see the axial functions in Figures 2 and 3 for details). We can easily check that this graph is 3-valent, simple and planar but not 3-connected; therefore, by Lemma 4.4, this manifold is not a quasitoric manifold.

6. Some combinatorial lemmas

To prove the main theorem (Theorem 7.1), we need the following two lemmas.
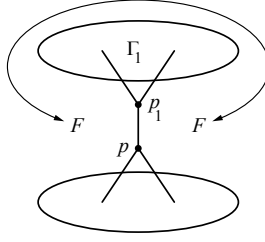
Figure 6. The figure explained in the proof of Lemma 6.1. The facet F has a self-intersection on the edge $p_1 p$.

Lemma 6.1. *Let Q be a 3-dimensional manifold with faces which is homeomorphic to D^3 and Γ be its graph. Then, $\Gamma \setminus \{p\}$ is connected for all vertices $p \in V(\Gamma)$.*

Proof. Because Q is homeomorphic to the 3-disk D^3, Γ may be regarded as a planar graph by the stereographic projection of $\partial Q = S^2$. Assume $\Gamma \setminus \{p\}$ is not connected. Because Q is a 3-dimensional manifold with faces, there are exactly three outgoing edges from p, say pp_1, pp_2 and pp_3. Therefore, we may assume that there exists a connected component Γ_1 in $\Gamma \setminus \{p\}$ such that $p_1 \in V(\Gamma_1)$ but $p_2, p_3 \notin V(\Gamma_1)$ (see Figure 6). Since Γ_1 is also a planar 3-valent graph except on the vertex p_1 (because $p \notin V(\Gamma_1)$), there is a 2-valent subgraph in Γ_1, say $\partial \Gamma_1$, such that $\partial \Gamma_1$ splits $\partial Q = S^2$ into two connected components H_+ and H_-, where $\Gamma_1 \setminus \partial \Gamma_1 \subset H_+ \setminus \partial \Gamma_1$ but $\Gamma_1 \not\subset H_-$. This implies that there is a facet F in Q such that ∂F contains $\partial \Gamma_1$ and $p_1 p$. However, in this case, $p_1 p$ must be a self-intersection edge of F (see Figure 6). This contradicts that Q is a manifold with faces. $\square$

By Lemma 6.1, if Γ is not 3-connected, then there are two vertices $p, q \in V(\Gamma)$ such that $\Gamma \setminus \{p, q\}$ is not connected but both $\Gamma \setminus \{p\}$ and $\Gamma \setminus \{q\}$ are connected. More precisely, we have the following lemma.

Lemma 6.2. *Let Q be a 3-dimensional manifold with faces which is homeomorphic to D^3 and Γ be its graph. Assume that there are two vertices $p, q \in V(\Gamma)$ such that $\{p, q\} \not\subset V(F)$ for any facets F, i.e, p and q are not on the same facet F. Then, $\Gamma \setminus \{p, q\}$ is connected.*

Proof. Assume that p and q are not on the same facet of Q. Because Q is a manifold with faces, there are mutually distinct facets $F_1, \ldots, F_6$ such that $\{p\}$ is a component of $F_1 \cap F_2 \cap F_3$ and $\{q\}$ is a component of $F_4 \cap F_5 \cap F_6$, and we can take vertices p_1, p_2, p_3 and q_1, q_2, q_3 such that pp_i and qq_i are all outgoing edges from p and q for $i = 1, 2, 3$. Take two vertices r and s from $\Gamma \setminus \{p, q\}$. By Lemma 6.1, $\Gamma \setminus \{q\}$ is connected. So there is a path γ from r to s in $\Gamma \setminus \{q\}$. If γ does not go through p, then r and s are connected in $\Gamma \setminus \{p, q\}$. Assume that this path γ goes through p. Then γ goes through exactly two vertices p_i, p_j (we may assume p_1 and p_2). Moreover, one of the facets F_1, F_2, F_3, say F_1, contains both p_1 and p_2.

Note that F_1 corresponds to the 2-valent subgraph in Γ. Therefore, we can take the path γ_p connecting p_1 and p_2 on F_1 which is not the path $p_1 p p_2$. Because p and q are not on the same facet, in particular $q \notin V(F_1)$, the path γ_p does not contain q. Hence, the connected subgraph $\gamma \cup \gamma_p$ contains both r and s but does not contain both p and q. Thus, we can take the path γ' from r to s in $\gamma \cup \gamma_p \subset \Gamma \setminus \{p, q\}$. This establishes that $\Gamma \setminus \{p, q\}$ is connected. $\qquad\square$

In summary, by Lemmas 6.1 and 6.2, we have the following fact.

Corollary 6.3. *Let Γ be a one-skeleton of a 3-dimensional manifold with faces Q. Then, for all $p \in V(\Gamma)$, $\Gamma \setminus \{p\}$ is connected. Furthermore, if $\Gamma \setminus \{p, q\}$ is not connected, then p and q are on the same facet.*

7. Proof of main theorem

The main theorem of this paper can be stated as follows:

Theorem 7.1. *Let M be a simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M) = 0$. Then, either M is equivariantly diffeomorphic to*

(1) *$S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ with a torus action induced from a (faithful) representation of T^3 on $\mathbb{C}^3$,*

(2) *a 6-dimensional quasitoric manifold X, or*

(3) *an S^4-bundle over S^2 which is equivariantly diffeomorphic to $M(\epsilon, a, b)$ for some $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$;*

or else there are some 6-dimensional quasitoric manifolds X_h for some $h = 1, \ldots, k$, and some S^4-bundles over S^2, say $S_i = M(\epsilon_i, a_i, b_i)$ (for some $\epsilon_i = \pm 1$, $a_i, b_i \in \mathbb{Z}$ and $i = 1, \ldots, \ell$), such that M is equivariantly diffeomorphic to

$$\left(\overset{k}{\underset{h=1}{\#}} X_h \right) \# \left(\overset{\ell}{\underset{i=1}{\#}} S_i \right),$$

where $\#$ represents the equivariant connected sum around fixed points, $k + \ell \geq 2$ for $k \geq 0$, $\ell \geq 1$, and the case $k = 0$ means that there is no X_h factor.

In this final section, we prove Theorem 7.1.

Let M be a simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M) = 0$, Q be its orbit space which is homeomorphic to D^3 and $(\Gamma_M, \mathscr{A}_M)$ be its induced oriented torus graph (we omit the orientation).

Because Γ_M is a one-skeleton of a manifold with faces which is homeomorphic to D^3, it is easy to check that $|V(\Gamma_M)| \neq 1, 3$. If $|V(\Gamma_M)| = 2$, by Lemma 4.1, we have that M is equivariantly diffeomorphic to S^6, i.e., statement (1). If $|V(\Gamma_M)| = 4$, it follows from Lemma 4.2 that M is equivariantly diffeomorphic to a quasitoric manifold $\mathbb{C}P^3$ or $M(\epsilon, a, b)$ for some $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$, i.e., statement (2) or (3)

occurs. So we need only prove the case when $|V(\Gamma_M)| \geq 5$.

We first establish the following lemma.

Lemma 7.2. *Assume that $|V(\Gamma_M)| \geq 5$ and there is a multiple edge in Γ_M. Then, $(\Gamma_M, \mathcal{A}_M)$ can be decomposed as*

$$(\Gamma_M, \mathcal{A}_M) = (\Gamma_X, \mathcal{A}_X) \# (\Gamma_{S_1}, \mathcal{A}_{S_1}) \# \cdots \# (\Gamma_{S_{\ell'}}, \mathcal{A}_{S_{\ell'}})$$

or

$$(\Gamma_M, \mathcal{A}_M) = (\Gamma_{S_1}, \mathcal{A}_{S_1}) \# \cdots \# (\Gamma_{S_{\ell'}}, \mathcal{A}_{S_{\ell'}}),$$

where $(\Gamma_X, \mathcal{A}_X)$ is a torus graph without multiple edges and $S_i = M(\epsilon_i, a_i, b_i)$ for $i = 1, \ldots, \ell'$.

Proof. Assume two vertices p and q are connected by a multiple edge, i.e., two edges (see the bottom graph in Figure 7). Then, by the connection of the torus graph (see Proposition 3.1), it is easy to check that the axial functions around the
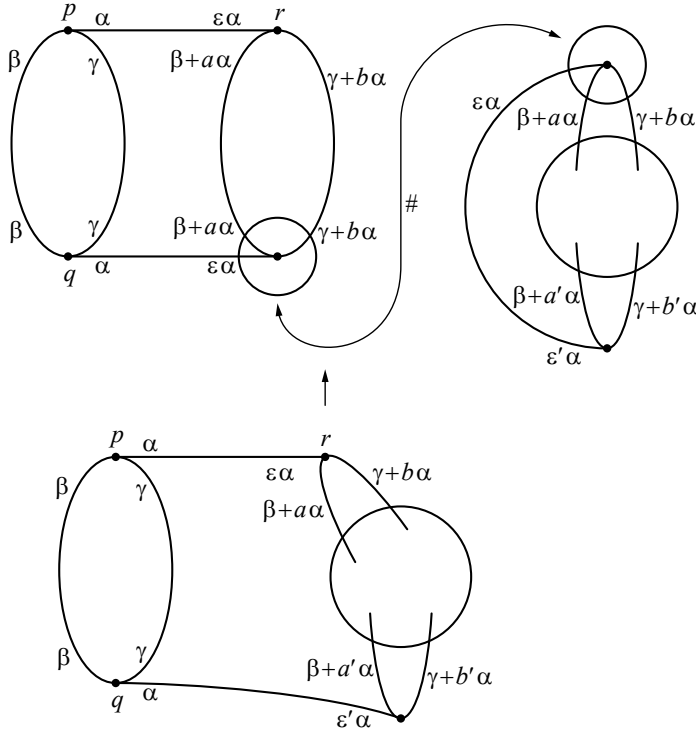
Figure 7. We may regard α, β, γ as any generators in $(\mathfrak{t}_{\mathbb{Z}}^3)^*$ and $a, a', b, b' \in \mathbb{Z}$ and $\epsilon, \epsilon' = \pm 1$. The bottom graph is $(\Gamma_M, \mathcal{A}_M)$, the upper-left graph is $(\Gamma_{S_1}, \mathcal{A}_{S_1})$ and the upper-right graph is $(\Gamma_{M'}, \mathcal{A}_{M'})$. If we fix the orientation of $(\Gamma_M, \mathcal{A}_M)$ then the orientations of $(\Gamma_{S_1}, \mathcal{A}_{S_1})$ and $(\Gamma_{M'}, \mathcal{A}_{M'})$ are automatically determined.

vertex r of the bottom graph in Figure 7 satisfy the axial functions expressed in that figure, where we can take α, β, γ as a $\mathbb{Z}$-basis of $(\mathfrak{t}_\mathbb{Z}^3)^*$. In this case, we can do an (inverse) connected sum such as the one expressed in Figure 7 (from the bottom to the top). Then, the induced torus graph $(\Gamma_M, \mathscr{A}_M)$ is decomposed into two induced torus graphs $(\Gamma_{S_1}, \mathscr{A}_{S_1})$ and $(\Gamma_{M'}, \mathscr{A}_{M'})$, where M' is some simply connected 6-dimensional torus manifold with $H^{\mathrm{odd}}(M') = 0$ by Lemma 5.1. Namely, we have

$$(\Gamma_M, \mathscr{A}_M) = (\Gamma_{M'}, \mathscr{A}_{M'}) \,\#\, (\Gamma_{S_1}, \mathscr{A}_{S_1}).$$

If there are no multiple edges in $\Gamma_{M'}$, then we may put $\Gamma_{M'} = \Gamma_X$. Assume that there is a multiple edge in $\Gamma_{M'}$. If there are only four vertices in $\Gamma_{M'}$, then we may put M' as $S_2 = M(\epsilon_2, a_2, b_2)$ by Lemma 4.2. When there are more than four vertices in $\Gamma_{M'}$, we iterate the above argument, establishing the lemma.   □

Therefore, to prove Theorem 7.1, it is enough to prove the following lemma.

Lemma 7.3. *Assume that $|V(\Gamma_M)| \geq 5$ and there are no multiple edges in Γ_M. Then, $(\Gamma_M, \mathscr{A}_M)$ can be decomposed as*

$$(\Gamma_M, \mathscr{A}_M) = (\Gamma_{X_1}, \mathscr{A}_{X_1}) \,\#\, \cdots \,\#\, (\Gamma_{X_k}, \mathscr{A}_{X_k}) \,\#\, (\Gamma_{S_1}, \mathscr{A}_{S_1}) \,\#\, \cdots \,\#\, (\Gamma_{S_{\ell''}}, \mathscr{A}_{S_{\ell''}}),$$

where $(\Gamma_{X_h}, \mathscr{A}_{X_h})$ for $h = 1, \ldots, k$ is the torus graph induced from a quasitoric manifold X_h, and $S_i = M(\epsilon_i, a_i, b_i)$ for $i = 1, \ldots, \ell''$.

Proof. If $\Gamma_M (= \Gamma)$ is 3-connected, then it follows from Lemma 4.4 that the statement holds, i.e., $k = 1$, $\ell'' = 0$. Therefore, we may assume Γ is not 3-connected. In this case, by Corollary 6.3, there is a 2-valent torus subgraph $F \subset \Gamma$ such that $\Gamma \setminus \{p, q\}$ is not connected for some $p, q \in V(F)$.

If F is a triangle (i.e., $|V(F)| = 3$), using a method similar to that demonstrated in the proof of Lemma 6.1, we have that there is a face in Q which has a self-intersection edge. This contradicts that Q is a manifold with faces. Therefore, we may assume $|V(F)| \geq 4$. We first assume that pq is an edge of F. Then, there are two graphs Γ_1 and Γ_2 which are the connected components of $\Gamma \setminus \{p, q\}$ expressed in Figure 8. If we remove the two vertices r and q from Γ instead of p and q, where $r \in V(\Gamma_2)$ such that pr is an edge, then $\Gamma \setminus \{r, q\}$ is also not connected (see Figure 8). Therefore, we may assume that
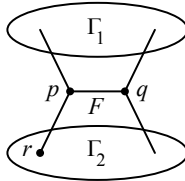
Figure 8. If we remove r and q from Γ instead of p and q, the graph is also disconnected.

(1) $p, q \in V(\Gamma)$ are such that $\Gamma \setminus \{p, q\}$ is not connected,

(2) $pq \notin E(\Gamma)$,

(3) there is a 2-valent torus subgraph (facet) F with $|V(F)| \geq 4$ in Γ such that $p, q \in V(F)$.

We call such a facet F a *singular facet*.

Let F be a singular facet. Assume $|V(F)| \geq 6$. In this case, by an argument similar to the one just before, we may take p and q to be in the position of Figure 9, i.e., p and q are on two separated edges rp and sq which are common edges of two facets F and F' in Figure 9 (note that r and s might be connected by an edge). Moreover, by considering the omnioriented characteristic functions of the facets F and F', we may take the axial functions around the facet F to be as in Figure 9.

By taking an appropriate orientation, we can do the connected sum as in Figure 10; here we denote the (oriented) torus graph containing Γ_1 by $(\widetilde{\Gamma}_1, \widetilde{\mathscr{A}}_1)$ and that containing Γ_2 by $(\widetilde{\Gamma}'_2, \widetilde{\mathscr{A}}'_2)$. The torus graph obtained by this connected sum is nothing but the torus graph $(\Gamma, \mathscr{A})$ in Figure 9. Note that $\widetilde{\Gamma}_1$ is simple and planar, while $\widetilde{\Gamma}'_2$ is just planar. With a method similar to that demonstrated in Figure 7, $(\widetilde{\Gamma}'_2, \widetilde{\mathscr{A}}'_2)$ can be obtained from the connected sum of $(\Gamma_S, \mathscr{A}_S)$ and the simple, planar graph $(\widetilde{\Gamma}_2, \widetilde{\mathscr{A}}_2)$ (containing Γ_2), where $(\Gamma_S, \mathscr{A}_S)$ is one of the torus graphs (by taking the appropriate axial functions) in Figure 2. Namely, the torus graph in Figure 9 can be obtained from the connected sum

$$(\Gamma, \mathscr{A}) = (\widetilde{\Gamma}_1, \widetilde{\mathscr{A}}_1) \,\#\, (\Gamma_S, \mathscr{A}_S) \,\#\, (\widetilde{\Gamma}_2, \widetilde{\mathscr{A}}_2).$$

Here, it is easy to check that $\widetilde{\Gamma}_i$ consists of Γ_i and the other two facets, say $\widetilde{F}(i)$ and $\widetilde{F}'(i)$ (induced from F and F' in Γ). Because of Figure 10, the number of
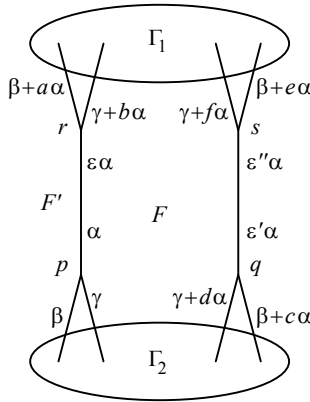
Figure 9. The axial functions around F when $|V(F)| \geq 6$, where $\epsilon, \epsilon', \epsilon'' = \pm 1$ and $a, b, c, d, e, f \in \mathbb{Z}$. Here, F' is a facet which intersects F on pr and qs.
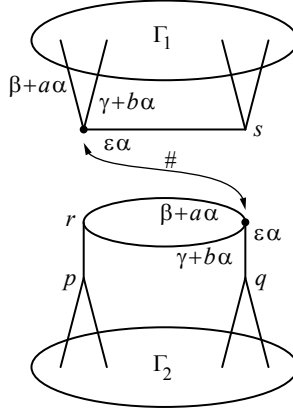
Figure 10. The torus graph $(\Gamma, \mathcal{A})$ in Figure 9 splits into two torus graphs $(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1)$ (upper) and $(\widetilde{\Gamma}'_2, \widetilde{\mathcal{A}}'_2)$ (lower). Here, we omit the axial functions around the vertices p, q, r, s because they are exactly the same as those in Figure 9.

vertices of $\widetilde{F}(i)$ and $\widetilde{F}'(i)$ is reduced; in particular, the number of vertices of the facet $\widetilde{F}(i)$ induced from the singular facet F is strictly less than 6. If both $(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1)$ and $(\widetilde{\Gamma}_2, \widetilde{\mathcal{A}}_2)$ are 3-connected, then these torus graphs are induced from quasitoric manifolds, i.e, the statements of Lemma 7.3 hold. Assume that $(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1)$ is not 3-connected. Then, by the above arguments, there is a singular facet F in $(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1)$. If $|V(F)| \geq 6$, then $(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1)$ also decomposes as

$$(\widetilde{\Gamma}_1, \widetilde{\mathcal{A}}_1) = (\widetilde{\Gamma}_3, \widetilde{\mathcal{A}}_3) \,\#\, (\Gamma_{S'}, \mathcal{A}_{S'}) \,\#\, (\widetilde{\Gamma}_4, \widetilde{\mathcal{A}}_4),$$

using arguments similar to those Figure 10. Iterating, we may reduce all singular facets with $|V(F)| \geq 6$. More precisely, we may decompose $(\Gamma, \mathcal{A})$ in Figure 9 as

$$(\Gamma, \mathcal{A}) = \underset{i=1}{\overset{\ell}{\#}} \{(\Gamma_i, \mathcal{A}_i) \,\#\, (\Gamma_{S_i}, \mathcal{A}_{S_i}) \,\#\, (\Gamma_{i+\ell}, \mathcal{A}_{i+\ell})\},$$

where $(\Gamma_{S_i}, \mathcal{A}_{S_i})$ for $i = 1, \ldots, \ell$ is a torus graph in Figure 2 and $(\Gamma_h, \mathcal{A}_h)$ for $h = 1, \ldots, 2\ell$ is a 3-valent simple and planar torus graph such that either

- $(\Gamma_h, \mathcal{A}_h)$ is 3-connected (in this case, induced from a quasitoric manifold), or
- all singular facets F satisfy $|V(F)| = 4$ or 5.

Assume that the number of vertices in every singular facet of the torus graph $(\Gamma, \mathcal{A})$ is less than or equal to 5. Then, such a torus graph is one of the torus graphs expressed in Figure 11. However, because Γ is the one-skeleton of a manifold with faces and is not 3-connected, it is easy to check that there exists a singular facet F' such that $F' \cap F = \{pr, qs\}$ and $|V(F')| \geq 6$. This gives a contradiction. Hence, this case does not occur. This establishes Lemma 7.3. $\qquad\square$
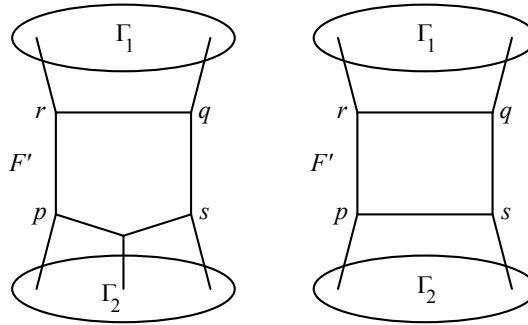
Figure 11. The singular facets F with $|V(F)| = 5$ or 4. Here, F' is a facet which intersects F on pr and qs.

Consequently, by Lemmas 5.4, 7.2 and 7.3, we have the statement of Theorem 7.1. Finally, by Theorem 7.1 and the Mayer–Vietoris exact sequence, we also have the following well-known result.

Corollary 7.4. *Let M be a simply connected 6-dimensional torus manifold whose cohomology ring is generated by the second-degree cohomology. Then, M is a quasitoric manifold.*

Acknowledgments

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