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ON A QUESTION OF A. BALOG

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We give a partial answer to a conjecture of A. Balog concerning the size of $AA + A$, where A is a finite subset of real numbers. We also prove several new results on the cardinality of $A : A + A$, $AA + AA$ and $A : A + A : A$.

1. Introduction

Let $A \subset \mathbb{R}$ be a finite set. Define the *sumset*, and respectively the *product set*, by

$$A + A := \{a + b : a, b \in A\}$$

and

$$AA := \{ab : a, b \in A\}.$$

The Erdős–Szemerédi conjecture [1983] states that for all $\varepsilon > 0$,

$$\max\{|A + A|, |AA|\} \gg |A|^{2-\varepsilon}.$$

Loosely speaking, the conjecture says that any set of reals (or integers) cannot be highly structured in both a multiplicative and an additive sense. The best result in this direction is due to Solymosi [2009].

Theorem 1. *Let $A \subset \mathbb{R}$ be a set. Then*

$$\max\{|A + A|, |AA|\} \gg |A|^{4/3} \log^{-1/3}|A|.$$

If one considers the set

$$AA + A = \{ab + c : a, b, c \in A\}$$

then the Erdős–Szemerédi conjecture implies that $AA + A$ has size at least $|A|^{2-\varepsilon}$ (we assume for simplicity that $1 \in A$). Balog [2011] formulated the weaker hypothesis that for all $\varepsilon > 0$ one has

$$|AA + A| \gg |A|^{2-\varepsilon}.$$

In that paper he proved the following result, which implies, in particular, that $|AA + A| \gg |A|^{3/2}$ and $|AA + AA| \gg |A||A : A|^{1/2}$.

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Theorem 2. For every finite set of reals $A, B, C, D \subset \mathbb{R}$, we have

$$(1) \quad |AC + A||BC + B| \gg |A||B||C|$$

and

$$(2) \quad |AC + AD||BC + BD| \gg |B : A||C||D|.$$

More precisely (see [Schoen and Shkredov 2013]),

$$|(A \times B) \cdot \Delta(C) + A \times B| \gg |A||B||C|$$

and

$$|(A \times B) \cdot \Delta(C) + (A \times B) \cdot \Delta(D)| \gg |B : A||C||D|,$$

where

$$\Delta(A) := \{(a, a) : a \in A\}.$$

Murphy et al. [2015] obtained a partial answer to a “dual” question on the size of $A(A + A)$. The main result of this paper is the following new bound for $A : A + A$ and $AA + A$, stated more precisely in [Theorem 12](#).

Theorem 3. Let A be a finite subset of positive reals. Then there is $\varepsilon_1 > 0$ such that

$$(3) \quad |A : A + A| \gg |A|^{3/2+\varepsilon_1}.$$

Moreover, if $|A : A| \ll |AA|$ then there exists $\varepsilon_2 > 0$ such that

$$(4) \quad |AA + A| \gg |A|^{3/2+\varepsilon_2}.$$

We also prove several results on the cardinality of $AA + AA$ and $A : A + A : A$; see [Theorem 14](#) and [Proposition 15](#) below.

Roche-Newton and Zhelezov [2015] conjectured there exist absolute constants c and c' such that for any finite $A \subset \mathbb{C}$,

$$\left| \frac{A+A}{A+A} \right| \leq c|A|^2 \implies |A + A| \leq c'|A|.$$

Similar conjectures were made for the sets $(A - A)/(A - A)$, $(A - A)(A - A)$, $A(A + A + A + A)$ and so on. We conclude this paper by giving a partial answer to a variant of Roche-Newton and Zhelezov’s conjecture:

$$|(A + A)(A + A) + (A + A)(A + A)| \ll |A|^2 \implies |A \pm A| \ll |A| \log |A|;$$

see [Corollary 17](#).

The main idea of the proof of [Theorem 3](#) is the following. We need to estimate from below the sumset of two sets A and $A : A$. As in many problems of this type, the usual applications of the Szemerédi–Trotter theorem [Tao and Vu 2006] or Solymosi’s method [Balog 2011] give us a lower bound of the form $|A : A + A| \gg |A|^{3/2}$. In [Schoen and Shkredov 2011] the exponent $3/2$ was improved in the particular

case of sumsets of convex sets. After that the method was developed by several authors; see, e.g., [Konyagin and Rudnev 2013; Li 2011; Li and Roche-Newton 2012; Schoen 2014; Schoen and Shkredov 2013; Shkredov 2013a; 2013b; 2015]. In [Shkredov 2015] it was proved that the bound $|A + B| \gg |A|^{3/2+c}$, $c > 0$, holds for a wide class of *different* sets having roughly comparable sizes. For example, such a bound holds if A and B have small multiplicative doubling. It turns out that if (3) cannot be improved then there is some large set C such that $|AC| \ll |A|$. This allows us to apply results from [Shkredov 2015].

2. Notation

Let \mathbf{G} be an abelian group and $+$ be the group operation. We use the same letter to denote a set $S \subseteq \mathbf{G}$ and its characteristic function $S : \mathbf{G} \rightarrow \{0, 1\}$. By $|S|$ denote the cardinality of S .

Let $f, g : \mathbf{G} \rightarrow \mathbb{C}$ be two functions with finite supports. Put

$$(5) \quad (f * g)(x) := \sum_{y \in \mathbf{G}} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y)g(y + x).$$

Let $A \subseteq \mathbf{G}$ be a set. For any real $\alpha > 0$ let

$$(6) \quad E_{\alpha}^{+}(A) = \sum_{x \in \mathbf{G}} (A \circ A)^{\alpha}(x)$$

be the *higher energy* of A . In the particular case $\alpha = 2$ we write $E^{+}(A) = E_2^{+}(A)$ and $E(A, B)$ for $\sum_{x \in \mathbf{G}} (A \circ A)(x)(B \circ B)(x)$. The quantity $E^{+}(A)$ is called the *additive energy* of a set; see, e.g., [Tao and Vu 2006]. For a sequence $s = (s_1, \dots, s_{k-1})$ put $A_s^{+} = A \cap (A - s_1) \cap \dots \cap (A - s_{k-1})$. Then

$$E_k^{+}(A) = \sum_{s_1, \dots, s_{k-1} \in \mathbf{G}} |A_s^{+}|^2.$$

If we have a group \mathbf{G} with multiplication instead of addition, then we use the symbol $E_{\alpha}^{\times}(A)$ for the corresponding energy of a set A and we write A_s^{\times} for $A \cap (As_1^{-1}) \cap \dots \cap (As_{k-1}^{-1})$. In the case of a unique operation we write just $E_k(A)$, $E(A)$ and A_s .

Let $A, B \subseteq \mathbf{G}$ be two finite sets. The *magnification ratio* $R_B[A]$ of the pair (A, B) (see, e.g., [Tao and Vu 2006]) is defined by

$$(7) \quad R_B[A] = \min_{\emptyset \neq Z \subseteq A} \frac{|B + Z|}{|Z|}.$$

A beautiful result on the magnification ratio was proven by Petridis [2012].

Theorem 4. For any $A, B, C \subseteq G$, we have

$$(8) \quad |B + C + X| \leq R_B[A] \cdot |C + X|,$$

where $X \subseteq A$ and $|B + X| = R_B[A]|X|$.

We conclude the section with Ruzsa’s triangle inequality; see, e.g., [Tao and Vu 2006]. Interestingly, our proof (developing some ideas of [Schoen and Shkredov 2013; Murphy et al. 2015]) describes the situation when the triangle inequality is sharp, namely, when $|B \cap (A - z) - C| \approx |C|$ for many $z \in A - B$.

Lemma 5. Let $A, B, C \subseteq G$ be any sets. Then

$$(9) \quad |C||A - B| \leq |A \times B - \Delta(C)| \leq |A - C||B - C|.$$

Proof. We have

$$|A \times B - \Delta(C)| = \sum_{z \in A - B} |B \cap (A - z) - C| \geq |A - B||C|.$$

The inequality above is trivial and the identity follows by the projection of points $(x, y) \in A \times B - \Delta(C)$, $(x, y) = (a - c, b - c)$, $a \in A$, $b \in B$, $c \in C$, onto $z := x - y = a - b \in A - B$. If z is fixed we see that the result of the projection is the intersection of the line $z = x - y$ with our set and moreover the ordinates of the points from the intersection belong to $B \cap (A - z) - C$. It is easy to check that the converse is also true. □

All logarithms are base 2. The signs \ll and \gg are the usual Vinogradov symbols.

3. Preliminaries

As we discussed in the introduction our proof uses some notions from [Shkredov 2015]. So, let us recall the main definition of that paper.

Definition 6. A set $A \subset G$ has *SzT-type* (in other words, A is called a *Szemerédi-Trotter set*) with parameter $\alpha \geq 1$ if for any set $B \subset G$ and an arbitrary $\tau \geq 1$,

$$(10) \quad |\{x \in A + B : (A * B)(x) \geq \tau\}| \ll c(A)|B|^\alpha \cdot \tau^{-3},$$

where $c(A) > 0$ is a constant that depends on the set A only.

Simple calculations (or see [Shkredov 2015, Lemma 7]) give us some connections between various energies of SzT-type sets. Formula (11) below is due to Li [2011].

Lemma 7. Suppose that $A, B, C \subseteq G$ have SzT-type with the same parameter α . Then

$$(11) \quad E^3(A) \ll E_{3/2}^2(A)c(A)|A|^\alpha,$$

$$(12) \quad E(A) \ll c^{1/2}(A)|A|^{1+\alpha/2},$$

and

$$(13) \quad \sum_x (A \circ A)(x)(B \circ B)(x)(C \circ C)(x) \ll (c(A)c(B)c(C))^{1/3}(|A||B||C|)^{\alpha/3} \times \log(\min\{|A|, |B|, |C|\}).$$

Proof. We prove just (11); estimates (12) and (13) can be established by similar arguments.

Let us arrange the convolutions $(A \circ A)(x)$ in decreasing order: $(A \circ A)(x_1) \geq (A \circ A)(x_2) \geq \dots$. By assumption A has SzT-type with parameter α , which implies that $(A \circ A)(x_j) \ll (c(A)|A|^\alpha j^{-1})^{1/3}$. Choosing the parameter $\Delta^{3/2} = c(A)|A|^\alpha E_{3/2}^{-1}(A)$ and applying the obtained bound, we get

$$E(A) = \sum_{j=1}^{|A-A|} (A \circ A)^2(x_j) \leq \Delta^{1/2} E_{3/2}(A) + \sum_{j:(A \circ A)(x_j) \geq \Delta} (c(A)|A|^\alpha j^{-1})^{2/3}.$$

The condition $(A \circ A)(x_j) \geq \Delta$ implies $j^{1/3} \ll (c(A)|A|^\alpha)^{1/3} \Delta^{-1}$. Thus by our choice of Δ , we have

$$E(A) \ll \Delta^{1/2} E_{3/2}(A) + c(A)|A|^\alpha \Delta^{-1} \ll E_{3/2}^{2/3}(A)(c(A)|A|^\alpha)^{1/3}$$

as required. □

We need Lemma 7 from [Raz et al. 2015] (see also Lemma 27 from [Schoen and Shkredov 2013]).

Lemma 8. Any set $A \subset \mathbb{R}$, $\mathbb{R} = (\mathbb{R}, +)$, has SzT-type with $\alpha = 2$ and $c(A) = |A|d(A)$, where

$$(14) \quad d(A) := \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|}.$$

So, any set with small multiplicative doubling or, more precisely, with small quantity (14) has SzT-type, relative to addition, in an effective way. The interested reader can check that the minimum in (14) is actually attained. Careful analysis of our proof shows that we do not need this. Other examples of SzT-type sets can be found in [Shkredov 2015].

Now let us prove a simple result on $d(A)$ that follows from Petridis’s Theorem 4.

Lemma 9. Let $A \subseteq \mathbb{R}^+$ be a set. Then $d(A) = d(A^{-1})$, and for any nonempty C we have

$$(15) \quad d(AA) \leq \frac{|AC|^4}{|AA||C|^3}, \quad d(A:A) \leq \frac{|AC|^4}{|A:A||C|^3}.$$

In particular,

$$(16) \quad d(AA) \leq \frac{|A|^2 d^2(A)}{|AA||C|}, \quad d(A : A) \leq \frac{|A|^2 d^2(A)}{|A : A||C|},$$

where C is a set where the minimum in (14) is attained.

Proof. The identity $d(A) = d(A^{-1})$ is obvious. Let us prove (16). By Theorem 4 there is $X \subseteq C$ such that $|AAX| \leq R|AX|$, where $R = R_A[C]$ is defined by formula (7). We have

$$(17) \quad d(AA) \leq \frac{|AAX|^2}{|AA||X|} \leq R^2 \frac{|AX|^2}{|AA||X|} = \frac{|AX|^4}{|AA||X|^3} \leq \frac{|AC|^4}{|AA||C|^3}$$

and the first bound of (15) is obtained. Similarly, let $Y \subseteq C$ be as given by Theorem 4 and put $R = R_A[C^{-1}]$. Then $|(A : A)Y| \leq R|A^{-1}Y| \leq R|A^{-1}C|$, $R = |AY^{-1}|/|Y| \leq |AC^{-1}|/|C|$, and arguments similar to (17) can be applied. \square

Finally, we formulate a full version of Theorem 1.

Theorem 10. *Let $A, B \subseteq \mathbb{R}$ be sets, and let $\tau > 0$ be a real number. Then*

$$(18) \quad \{|x : |A \cap xB| \geq \tau\} \ll \frac{|A + A||B + B|}{\tau^2}.$$

In particular,

$$(19) \quad E^\times(A, B) \ll |A + A||B + B| \cdot \log(\min\{|A|, |B|\}).$$

4. Proof of the main results

Our proof relies on a partial case of Theorem 14 from [Shkredov 2015].

Theorem 11. *Suppose $A, A_* \subset \mathbb{R}$ have SzT-type with the same parameter $\alpha = 2$. Then*

$$(20) \quad |A \pm A_*| \gg \max \left\{ d(A_*)^{-1/3} d(A)^{-2/9} |A_*|^{8/9} |A|^{2/3}, d(A)^{-1/3} d(A_*)^{-2/9} |A|^{8/9} |A_*|^{2/3}, \right. \\ \left. \min \{ d(A_*)^{-2/27} d(A)^{-13/27} |A_*|^{14/9}, d(A)^{-2/27} d(A_*)^{-13/27} |A|^{14/9} \} \right\} \\ \times (\log(|A||A_*|))^{-2/9}.$$

Now we can prove the main result of the paper.

Theorem 12. *Let A be a finite subset of positive reals. Then*

$$(21) \quad |A : A + A| \gg |A|^{3/2+1/82} \cdot (\log|A|)^{-2/41},$$

and

$$(22) \quad |AA + A| \gg |AA|^{11/41} |A : A|^{-11/41} |A|^{3/2+1/82} (\log|A|)^{-2/41}.$$

Proof. Put $l = \log|A|$. We will assume that $|A : A + A| \ll M|A|^{3/2}$ and that $|AA + A| \ll M|A|^{3/2}$, where M is a small power of $|A|$, that is, $M = |A|^\epsilon$, and obtain a contradiction. Let us begin with (21) because the proof of the second inequality requires some additional steps.

Recall the arguments from [Balog 2011] or see the proof of Theorem 31 from [Schoen and Shkredov 2013]. Let l_i be the line $y = q_i x$. Thus, $(x, y) \in l_i \cap A^2$ if and only if $x \in A_{q_i}^\times$. Let $q_1, \dots, q_n \in \Pi \subseteq A : A$ be such that $q_1 < q_2 < \dots < q_n$. Here Π is a set which can vary, in principle, and at the moment we choose Π such that $|A_{q_i}^\times| \geq 2^{-1}|A|^2/|A : A|$ for all $q_i \in \Pi$. Thus, $\sum_{q_i \in \Pi} |A_{q_i}^\times| \geq \frac{1}{2}|A|^2$. We multiply all points of A^2 lying on the line l_i by $\Delta(A^{-1})$, so we obtain $|A_{q_i}^\times : A|$ points still belonging to the line l_i , and then we consider the sumset of the resulting set with $l_{i+1} \cap A^2$. Clearly, we get $|A_{q_i}^\times : A| |A_{q_{i+1}}^\times|$ points from the set $(A : A + A)^2$ lying between the lines l_i and l_{i+1} . Put

$$(23) \quad d(A) \leq \tilde{d}(A) := \min_{i=2, \dots, n} \frac{|AA_{q_i}^\times|^2}{|A| |A_{q_i}^\times|}.$$

Therefore, using the definition of $\tilde{d}(A)$, we have

$$(24) \quad \begin{aligned} M^2|A|^3 &\gg |A : A + A|^2 \\ &\geq \sum_{i=1}^{n-1} |A_{q_i}^\times| |A_{q_{i+1}}^\times : A| \\ &\geq |A|^{1/2} \tilde{d}^{1/2}(A) \sum_{i=1}^{n-1} |A_{q_i}^\times| |A_{q_{i+1}}^\times|^{1/2} \\ &\gg |A|^{3/2} \tilde{d}^{1/2}(A) |A : A|^{-1/2} \sum_{i=1}^{n-1} |A_{q_i}^\times| \\ &\gg |A|^{7/2} \tilde{d}^{1/2}(A) |A : A|^{-1/2}. \end{aligned}$$

Thus,

$$(25) \quad d(A) \leq \tilde{d}(A) = \min_{i=2, \dots, n} \frac{|AA_{q_i}^\times|^2}{|A| |A_{q_i}^\times|} \ll \frac{M^4 |A : A|}{|A|}.$$

To estimate $d(A : A)$ and $d(AA)$ we use Lemma 9. In other words, taking our $C = A_{q_i}^\times$ to minimize (25), we get

$$(26) \quad d(AA) \leq \frac{|AA_{q_i}^\times|^4}{|AA| |A_{q_i}^\times|^3}, \quad d(A : A) \leq \frac{|AA_{q_i}^\times|^4}{|A : A| |A_{q_i}^\times|^3} \ll \frac{M^8 |A : A|}{|C|}.$$

Applying the first inequality of [Theorem 11](#) with $A = A$ and $A_* = A : A$, we obtain

$$\begin{aligned} M|A|^{3/2} &\geq |A : A + A| \\ &\gg |A : A|^{8/9} |A|^{2/3} d^{-2/9}(A) \left(\frac{M^8 |A : A|}{|C|} \right)^{-1/3} l^{-2/9} \\ &= |A : A|^{5/9} |A|^{2/3} d^{-2/9}(A) |C|^{1/3} M^{-8/3} l^{-2/9} \\ &\gg |A|^{14/9} M^{-32/9} l^{-2/9}, \end{aligned}$$

and hence $M \gg l^{-2/41} |A|^{1/82}$. This implies [\(21\)](#).

It remains to prove [\(22\)](#). In this case we multiply all points of A^2 lying on the line l_i by $\Delta(A)$, so we obtain $|AA_{q_i}^\times|$ points still belonging to the line l_i , and then we consider the sumset of the resulting set with $l_{i+1} \cap A^2$. Clearly, we obtain $|AA_{q_i}^\times| |A_{q_{i+1}}^\times|$ points from the set $(AA + A)^2$. Thus,

$$(27) \quad M^2 |A|^3 \gg |AA + A|^2 \geq \sum_{i=1}^{n-1} |A_{q_i}^\times| |A_{q_{i+1}}^\times|,$$

and we repeat the arguments above. The proof gives us

$$(28) \quad |AA + A| \gg |AA|^{11/41} |A|^{-4/41} (E_{3/2}^\times(A))^{22/41} l^{-2/41}.$$

Here we have chosen the set Π as $\sum_{q \in \Pi} |A_q^\times|^{3/2} \gg E_{3/2}^\times(A)$ or, in other words, $|A_q^\times| \gg (E_{3/2}^\times(A))^2 |A|^{-4}$. Using the Hölder inequality, combined with [\(28\)](#), we get

$$|AA + A| \gg |AA|^{11/41} |A : A|^{-11/41} |A|^{62/41} l^{-2/41}. \quad \square$$

Remark 13. Using the full power of [Theorem 14](#) from [\[Shkredov 2015\]](#), one can obtain further results connecting $|AA : A|$ and $|A : AA|$ with $|AA + A|$ and $|A : A + A|$ and so on. We do not make such calculations.

The same method allows us to improve the result of Balog concerning the size of $AA + AA$ and $A : A + A : A$.

Theorem 14. *Let $A \subset \mathbb{R}$ be a set. Then*

$$(29) \quad |A : A + A : A| \gg |A : A|^{14/29} |A|^{30/29} (\log |A|)^{-2/29},$$

and

$$(30) \quad |AA + AA| \gg |AA|^{19/29} |A : A|^{-5/29} |A|^{30/29} (\log |A|)^{-2/29}.$$

Proof. As in the proof of [Theorem 12](#), we define l_i to be the line $y = q_i x$ and let $q_1, \dots, q_n \in \Pi \subseteq A : A$ be such that $q_1 < q_2 < \dots < q_n$ and $|A_{q_i}^\times| \geq 2^{-1} |A|^2 / |A : A|$ for any $q_i \in \Pi$. Thus, $\sum_i |A_{q_i}^\times| \geq \frac{1}{2} |A|^2$. We multiply all points of A^2 lying on the line l_i by $\Delta(A^{-1})$, so we obtain $|A_{q_i}^\times : A|$ points still belonging to the line l_i , and then we consider the sumset of the resulting set with itself. Clearly, we get

$|A_{q_i}^\times : A| |A_{q_{i+1}}^\times : A|$ points from the set $(A : A + A : A)^2$ lying between the lines l_i and l_{i+1} . Therefore, we have

$$\begin{aligned}
 (31) \quad \sigma^2 &:= |A : A + A : A|^2 \\
 &\geq \sum_{i=1}^{n-1} |A_{q_i}^\times : A| |A_{q_{i+1}}^\times : A| \\
 &\geq \tilde{d}(A) |A| \sum_{i=1}^{n-1} |A_{q_i}^\times|^{1/2} |A_{q_{i+1}}^\times|^{1/2} \\
 &\gg |A|^3 \tilde{d}(A),
 \end{aligned}$$

where

$$(32) \quad \tilde{d}(A) := \min_{i=1, \dots, n} \frac{|A_{q_i}^\times : A|^2}{|A| |A_{q_i}^\times|}.$$

This gives us $d(A) \leq \tilde{d}(A) \ll \sigma^2 |A|^{-3}$. Using [Theorem 11](#) with $A = A_* = A : A$, we obtain

$$\begin{aligned}
 (33) \quad \sigma &\gg |A : A|^{14/9} \left(\frac{\sigma^4}{|A|^4 |A : A| |C|} \right)^{-5/9} l^{-2/9} \\
 &\gg |A : A|^{19/9} |A|^{20/9} |C|^{5/9} \sigma^{-20/9} l^{-2/9} \\
 &\gg |A : A|^{14/9} \sigma^{-20/9} |A|^{10/3} l^{-2/9}.
 \end{aligned}$$

After some calculations, we get $\sigma \gg |A : A|^{14/29} |A|^{30/29} l^{-2/29}$.

To obtain (30) we use the previous arguments. We have

$$\begin{aligned}
 (34) \quad \sigma^2 &:= |AA + AA|^2 \\
 &\geq \sum_{i \in \Pi} |AA_{q_i}^\times| |AA_{q_{i+1}}^\times| \\
 &\geq d(A) |A| \sum_{i \in \Pi} |A_{q_i}^\times|^{1/2} |A_{q_{i+1}}^\times|^{1/2} \\
 &\gg d(A) |A| |\Pi| \Delta,
 \end{aligned}$$

choosing $\Pi \subseteq A : A$ such that for any $q \in \Pi$ one has $|A|^2 / |A : A| \ll \Delta \leq |A_q^\times|$. Clearly, such a set Π exists by simple average arguments. Calculations like those in (33) give us

$$\sigma \gg |AA|^{14/9} \left(\frac{\sigma^4}{|AA| |\Pi|^2 \Delta^3} \right)^{-5/9} l^{-2/9} \gg |AA|^{19/9} (|\Pi| \Delta^{3/2})^{10/9} \sigma^{-20/9} l^{-2/9}.$$

After some computations, we obtain

$$\sigma \gg |AA|^{19/29} |A : A|^{-5/29} |A|^{30/29} l^{-2/29}.$$

□

Finally, let us obtain a result on $AA + A$ and $AA + AA$ of another type.

Proposition 15. *Let $A \subset \mathbb{R}$ be a set. Then*

$$(35) \quad |AA + A|^4, |A : A + A|^4 \gg |A|^{-2} (E_{3/2}^\times(A))^2 E_3^+(A) \log^{-3} |A|,$$

and

$$(36) \quad |AA + AA|^2, |A : A + A : A|^2 \gg E_3^+(A) \log^{-3} |A|.$$

Moreover,

$$(37) \quad |AA + A|^4, |A : A + A|^4 \gg \frac{|A|^{10}}{|A : A| |A - A|^2},$$

and

$$(38) \quad |AA + AA|^2, |A : A + A : A|^2 \gg \frac{|A|^6}{|A - A|^2}.$$

Proof. Put $l = \log |A|$. Using [Lemma 7](#), we obtain that for any A, B and C

$$(39) \quad \sum_x (A \circ A)(x)(B \circ B)(x)(C \circ C)(x) \ll |A| |B| |C| (d(A)d(B)d(C))^{1/3} \log(|A| |B| |C|).$$

In the particular case $A = B = C$, the definition of $d(A)$ gives us

$$(40) \quad |AA_s^\times|^2, |A : A_s^\times|^2 \gg |A|^{-2} |A_s^\times| E_3^+(A) l^{-1}$$

for any $s \in A : A$. Using pigeonholing, choose $\Pi \subseteq A : A$ such that $|A_q^\times|$ differs at most twice from $|\Pi|$ and such that $\sum_{q \in \Pi} |A_q^\times|^{3/2} \gg E_{3/2}^\times(A) l^{-1}$. Applying [\(24\)](#), [\(27\)](#), [\(40\)](#) and the last bound, we obtain [\(35\)](#). Using [\(40\)](#) one more time and Katz–Koester inclusion [\[2010\]](#), namely,

$$(41) \quad AA_s^\times \subseteq AA \cap_s AA, \quad A : A_s^\times \subseteq (A : A) \cap_s^{-1} (A : A),$$

as well as formula [\(18\)](#) of Solymosi’s result, we get [\(36\)](#). Another way to prove [\(36\)](#) is just to use formulas [\(31\)](#) and [\(34\)](#), combined with [\(40\)](#).

Inequalities [\(37\)](#) and [\(38\)](#) follow similarly to [\(35\)](#) and [\(36\)](#) from a direct application of [Definition 6](#) and the Hölder inequality. For example, let us show how to get

the first estimate of (37). Taking $B = -A$ and the parameter $\tau = |A|^2/(2|A - A|)$ in Definition 6, we obtain

$$d(A) \gg \frac{|A|^3}{|A - A|^2}.$$

Applying (24) and the lower bound for $d(A)$, we get

$$\begin{aligned} |A : A + A|^2 &\geq \sum_{i=1}^{n-1} |A_{q_i}^\times| |A_{q_{i+1}}^\times : A| \\ &\geq |A|^{1/2} d^{1/2}(A) \sum_{i=1}^{n-1} |A_{q_i}^\times| |A_{q_{i+1}}^\times| \\ &\gg |A|^{1/2} (|A|^3 |A - A|^{-2})^{1/2} (|A|^2 / |A : A|)^{1/2} \sum_{i=1}^{n-1} |A_{q_i}^\times| \\ &\gg |A|^{5/2} (|A|^3 |A - A|^{-2})^{1/2} (|A|^2 / |A : A|)^{1/2} \end{aligned}$$

as required. □

Remark 16. Applying arguments in the proof of (36) as well as formula (12) of Lemma 7, we obtain a similar bound, namely,

$$E^+(A) \ll |A| |AA + AA|$$

(actually, using methods from [Shkredov 2013a] one can improve the inequality). It is interesting to compare this estimate with Solymosi’s upper bound for the multiplicative energy (19). Using formula (11) of Lemma 7, we also have

$$(E^+(A))^{3/2} E_{3/2}^\times(A) \ll E_{3/2}^+(A) |A| |AA + A|^2.$$

Combining inequality (36) with some estimates from [Shkredov 2014], we obtain a result in the spirit of [Roche-Newton and Zhelezov 2015].

Corollary 17. *Let $A \subset \mathbb{R}$ be a set. Suppose that*

$$(42) \quad |(A + A)(A + A) + (A + A)(A + A)| \ll |A|^2 \quad \text{and} \quad E^+(A) |A - A| \ll |A|^4.$$

Then

$$(43) \quad |A - A| \ll |A| \log^{12/7} |A|.$$

The same holds if one replaces addition with subtraction and multiplication with division in the first condition of (42).

If just the first condition of (42) holds (with plus) then

$$(44) \quad |A \pm A| \ll |A| \log^3 |A|,$$

and if it holds with minus then

$$(45) \quad |A - A| \ll |A| \log^3 |A|.$$

Again, one can replace multiplication with division in the first condition of (42).

Proof. Let us deal with the situation of the sum and the product. Other cases can be considered similarly. By Theorem 30 from [Shkredov 2014] and our second condition, one has

$$E_3^+(A \pm A) \geq |A|^{45/4} |A - A|^{-1/2} (E^+(A))^{-9/4} \gg |A|^{9/4} |A - A|^{7/4}.$$

On the other hand, using formula (36) from Proposition 15 and our first condition, we get

$$|A|^4 \log^3 |A| \gg E_3^+(A \pm A) \gg |A|^{9/4} |A - A|^{7/4}$$

as required.

Finally, using the additive variant of Katz–Koester inclusion (41) (or see Proposition 29 from [Shkredov 2014]), we obtain

$$|A|^3 |A \pm A| \leq E_3^+(A + A) \ll |A|^4 \log^3 |A|,$$

and

$$|A|^3 |A - A| \leq E_3^+(A - A) \ll |A|^4 \log^3 |A|. \quad \square$$

A simpler proof of a stronger result was kindly pointed out to the author by Oliver Roche-Newton. Indeed applying estimate (2) with $A = B = A + A$, $C = A$ and $D = A + A$, we obtain

$$\begin{aligned} |A|^4 &\gg |(A + A)A + (A + A)(A + A)|^2 \\ &\gg |(A + A) : (A + A)| |A| |A + A| \\ &\gg |A|^3 |A + A|, \end{aligned}$$

and the result follows. Here we have used the estimate $|(A + A) : (A + A)| \geq |A|^2$ from [Balog and Roche-Newton 2015]. Applying the well-known Ungar bound $|(A - A) : (A - A)| \geq |A|^2$ and taking $C = A^{-1}$ and $D = (A + A)^{-1}$, one can replace division with multiplication.

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