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YUHUA SUN

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UNIQUENESS RESULT ON NONNEGATIVE SOLUTIONS OF A LARGE CLASS OF DIFFERENTIAL INEQUALITIES ON RIEMANNIAN MANIFOLDS

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We consider a large class of differential inequalities on complete connected Riemannian manifolds and provide a sufficient condition in terms of volume growth for the uniqueness of nonnegative solutions to the differential inequalities.

1. Introduction

The purpose of this paper is to give a sufficient condition for the uniqueness of nonnegative solutions to a large class of the differential inequalities

$$Lu + V(x)u^{\sigma} \le 0 \tag{1-1}$$

on a connected geodesically complete noncompact N-dimensional Riemannian manifold M^N with $N \ge 2$. Here the operator L is defined by

$$Lu = \operatorname{div}(A(x, u, \nabla u)), \tag{1-2}$$

where $A(x, \eta, \xi) = (A_i(x, \eta, \xi))$ is a vector field on M^N, and for $i = 1, \dots, N$ the $A_i(x, \eta, \xi)$ are Carathéodorian functions defined on $M^N \times [0, \infty) \times TM^N$, and TM^N is the tangent bundle of M^N. The function V is positive, measurable, and locally integrable on M^N.

Let $m \ge 1$ be an arbitrary given number. We say that the operator L belongs to the class $A(m)$ if there exists a positive constant C such that, for almost all $x \in M^N$, all $\eta \in [0, \infty)$, and all $\xi, \zeta \in T_x M^N$, the following conditions hold:

$$\begin{cases} (A(x, \eta, \xi), \xi) \ge 0, \\ |(A(x, \eta, \xi), \zeta)| \le C(A(x, \eta, \xi), \xi)^{\frac{m-1}{m}} |\zeta|, \end{cases} \tag{1-3}$$

where $(\cdot\,, \cdot)$ is the inner product given by the Riemannian metric, and $|\zeta|$ is the norm of ζ in $T_x M^N$.

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The definition of such a class operator $A(m)$ was first introduced by Mīklyukov [1979; 1980]. Actually, the operators of such a class are quite common. Let us mention some examples:

(1) m-Laplacian operator:

$$L_1 u = \operatorname{div}(|\nabla u|^{m-2}\nabla u), \quad m > 1. \tag{1-4}$$

(2) Mean curvature type operators:

$$L_2 u = \operatorname{div}\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right), \quad m > 1. \tag{1-5}$$

and

$$L_3 u = \operatorname{div}\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \quad m > 1. \tag{1-6}$$

(3) Nonlinear operator:

$$L_4 u = \operatorname{div}(a(x,u,\nabla u)|\nabla u|^{m-2}\nabla u), \quad m > 1. \tag{1-7}$$

The definition of L in (1-3) is less restrictive than the one defined by

$$|A(x,\eta,\xi)| \le C_1|\xi|^{m-1}, \quad |(A(x,\eta,\xi),\xi)| \ge C_2|\xi|^m, \tag{1-8}$$

for some positive constants C_1, C_2. For example, by choosing $a(x,\eta,\xi)$ of (1-7) appropriately, the operator L_4 belongs to $A(m)$ but does not necessarily satisfy (1-8).

Generally speaking, the operator Lu defined by (1-3) may meanwhile belong to several classes denoted by $A(m_1),\dots,A(m_k)$, where $m_1 \le m_2 \le \cdots \le m_k$. For example, the operators of L_2, L_3 belong to both $A(m-1)$ and $A(m)$. Throughout the paper, when we say that L belongs to the class of $A(m)$, we always mean m is the largest value m_k.

The purpose of this paper is to provide a very simple geometric condition of volume growth on M^N to suffice that the only nonnegative solution u of (1-1) is identically zero. Let us emphasize that there is no curvature assumption on manifolds throughout the paper.

First, let us give our setting on manifolds. Let M^N be a connected geodesically complete noncompact Riemannian manifold. Denote by μ the Riemannian measure, and by $B(x,r)$ the geodesic ball on M^N of radius r centered at $x \in M^N$. Given that $d(\cdot,\cdot)$ is the geodesic distance and that x_0 is a reference point on M, define $B_r := B(x_0,r)$ for simplicity, where $r = d(x,x_0)$. Assume also throughout the paper that $V(x) \in L^\infty_{\mathrm{loc}}(M^N)$.

The problem of investigating the uniqueness of nonnegative solutions has attracted a lot of attention, especially in the Euclidean space. For example, if $M^N = \mathbb{R}^N$ with $N \ge 2$, in the case of $V(x) \equiv 1$, the problem (1-1) was systematically investigated by

Kurta [1999]. By using the nonlinear capacity arguments, he obtained nonexistence results concerning different differential inequalities. For a specific operator L, let us recommend a series of papers of Mitidieri and Pokhozhaev [1998; 1999; 2001] for a more comprehensive description. Related problems have also been studied in massive literatures; see [Caristi et al. 2008; Caristi and Mitidieri 1997; D'Ambrosio 2009; D'Ambrosio and Mitidieri 2010; Ni and Serrin 1985; 1986] and the references therein.

Let us turn to the results in the Riemannian manifolds setting. The celebrated idea of studying the uniqueness of nonnegative solutions in terms of the volume of the geodesic ball was due to Cheng and Yau [1975]. They obtained the following marvelous result: if the volume estimate

$$\mu(B_r) \le Cr^2$$

holds for all large enough r, then any positive solution to $\Delta u \le 0$ is identically constant.

The amazing point of Cheng and Yau's result is that there is no assumption on either curvature or the behavior of the solution near infinity, only in terms of volume growth.

Very recently, this idea was used and developed in [Grigor'yan and Kondratiev 2010; Grigor'yan and Sun 2014; Sun 2014] to investigate the differential inequality of the form

$$\operatorname{div}(A(x)\nabla u) + V(x)u^\sigma \le 0, \tag{1-9}$$

where $\sigma > 1$. Particularly, when $A(x) = \mathrm{Id}$ and $V(x) = 1$, (1-9) becomes

$$\Delta u + u^\sigma \le 0. \tag{1-10}$$

In [Grigor'yan and Sun 2014] it is proved that if

$$\mu(B_r) \le Cr^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r$$

holds for all large enough r, then the only nonnegative solution of (1-10) is identically zero. Moreover, the exponents $2\sigma/(\sigma-1)$ and $1/(\sigma-1)$ are sharp and cannot be relaxed.

Let us define the weak nonnegative solution of (1-1). For convenience, we introduce the notation

$$A_u = (A(x,u,\nabla u), \nabla u) \tag{1-11}$$

and

$$W^{1,m}_{\mathrm{loc}}(M^N) := \{f \mid f \in L^m_{\mathrm{loc}}(M^N),\ \nabla f \in L^m_{\mathrm{loc}}(M^N)\}, \tag{1-12}$$

and denote by $W_c^{1,m}(M^N)$ the subspace of $W_{\text{loc}}^{1,m}(M^N)$ of functions with compact support.

Definition 1.1. A function u on M^N is called a weak nonnegative solution of (1-1) if $u \in W_{\text{loc}}^{1,m}(M^N)$ and $A_u \in L_{\text{loc}}^1(M^N)$ and if, for any nonnegative function $\psi \in W_c^{1,m}(M^N)$, the following inequality holds:

$$-\int_{M^N} (A(x,u,\nabla u), \nabla\psi)\, d\mu + \int_{M^N} V(x) u^\sigma \psi\, d\mu \le 0, \tag{1-13}$$

where $(\cdot\,,\cdot)$ is the inner product in $T_x(M^N)$ given by a Riemannian metric.

Remark 1.2. If u is a weak nonnegative solution of (1-1), and the operator L belongs to the class $A(m)$, we know

$$\begin{aligned}\int_{M^N} (A(x,u,\nabla u), \nabla\psi)\, d\mu &\le C \int_{M^N} |\nabla\psi| A_u^{\frac{m-1}{m}}\, d\mu \\ &\le C\Big(\int_{M^N} |\nabla\psi|^m\, d\mu\Big)^{\frac{1}{m}} \Big(\int_{\operatorname{supp}(\psi)} A_u\, d\mu\Big)^{\frac{m-1}{m}} < \infty.\end{aligned}$$

Hence, by the definition of the solution, we know the second integral in (1-13) is bounded.

Define

$$p = \frac{m\sigma}{\sigma - m + 1}, \quad q = \frac{m-1}{\sigma - m + 1}, \tag{1-14}$$

and introduce a new measure ν defined by

$$d\nu = V^{-\frac{m-1}{\sigma - m + 1}}\, d\mu. \tag{1-15}$$

Assume that V satisfies the following condition: for some nonnegative constants δ_1, δ_2, the estimate

$$cr^{-\delta_1} \le V(x) \le Cr^{\delta_2} \tag{V}$$

holds for all large enough r.

Theorem 1.3. *Assume that operator L in (1-1) belongs to the class of $A(m)$ with $1 < m < \sigma + 1$. Assume also that (V) holds with $\delta_1, \delta_2 \ge 0$. If the inequality*

$$\nu(B_r \setminus \overline{B_1}) \le Cr^p \ln^q r \tag{1-16}$$

holds for all large enough r, then the only nonnegative solution of (1-1) is identically zero.

Remark 1.4. It is not clear that the sharpness of exponents p and q in (1-16) holds for all the operators of the class $A(m)$. However, in many specific cases, the exponents p, q are sharp; one can refer to [Grigor'yan and Sun 2014; Sun 2014; 2015].

Notation. The letters C, C', C_0, C_1, ... denote positive constants whose values are unimportant and may vary at different occurrences.

In Section 2, we show the proof of Theorem 1.3. In Section 3, we present two examples to show that our result is very inclusive.

2. Proof of Theorem 1.3

Let u be a nonnegative solution of (1-1). Fix some ball B_R, where $R > 0$ is to be chosen later. Take a Lipschitz function φ on M^N with compact support, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of $\overline{B_R}$. Particularly, $\varphi \in W_c^{1,m}(M^N)$. We use the following test function for (1-13):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^{-t}, \tag{2-1}$$

where $\rho > 0$ is a parameter near zero, and s will be chosen to be a large enough fixed constant, and t will take arbitrarily small positive values near zero.

Since $1/(u+\rho)$ is bounded, ψ_ρ has compact support and is bounded. The identity

$$\nabla\psi_\rho = -t\varphi^s(u+\rho)^{-t-1}\nabla u + s\varphi^{s-1}(u+\rho)^{-t}\nabla\varphi$$

implies that $\nabla\psi_\rho \in L^m(M^N)$, hence, $\psi_\rho \in W_c^{1,m}(M^N)$. We obtain from (1-13) that

$$\begin{aligned} t\int_{M^N}\varphi^s(u+\rho)^{-t-1}A_u\,d\mu + \int_{M^N}\varphi^s V u^\sigma (u+\rho)^{-t}\,d\mu \\ \le s\int_{M^N}\varphi^{s-1}(u+\rho)^{-t}(A(x,u,\nabla u),\nabla\varphi)\,d\mu. \end{aligned} \tag{2-2}$$

Estimate the right-hand side of (2-2) by the Young inequality

$$\int_{M^N} fg\,d\mu \le \epsilon\int_{M^N}|f|^{p_0}\,d\mu + C_\epsilon\int_{M^N}|g|^{p_0'}\,d\mu, \tag{2-3}$$

where $1/p_0 + 1/p_0' = 1$. Letting $p_0 = m/(m-1)$, and using (1-3), we obtain

$$\begin{aligned} s&\int_{M^N}\varphi^{s-1}(u+\rho)^{-t}(A(x,u,\nabla u),\nabla\varphi)\,d\mu \\ &\le Cs\int_{M^N}\varphi^{s-1}(u+\rho)^{-t}A_u^{\frac{m-1}{m}}|\nabla\varphi|\,d\mu \\ &= C\int_{M^N}\Big[t^{\frac{1}{p_0}}\varphi^{\frac{s}{p_0}}(u+\rho)^{-\frac{t+1}{p_0}}A_u^{\frac{m-1}{m}}\Big]\Big[\frac{s}{t^{p_0}}\varphi^{\frac{s}{p_0'}-1}(u+\rho)^{1-\frac{t+1}{p_0'}}|\nabla\varphi|\Big]\,d\mu \\ &\le \frac{t}{2}\int_{M^N}\varphi^s(u+\rho)^{-t-1}A_u\,d\mu + C\frac{s^m}{t^{m-1}}\int_{M^N}\varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m\,d\mu. \end{aligned}$$

Substituting the above into (2-2), and canceling out half of the first term in (2-2), we obtain

$$\text{(2-4)}\quad \frac{t}{2}\int_{M^N}\varphi^s(u+\rho)^{-t-1}A_u\,d\mu+\int_{M^N}\varphi^s V u^\sigma(u+\rho)^{-t}\,d\mu \le C\frac{s^m}{t^{m-1}}\int_{M^N}\varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m\,d\mu.$$

Using the Young inequality again in the right-hand side of (2-4) with

$$p_1=\frac{\sigma-t}{m-t-1},\qquad p_1'=\frac{\sigma-t}{\sigma-m+1},$$

we obtain

$$\begin{aligned}\text{(2-5)}\quad &\frac{s^m}{t^{m-1}}\int_{M^N}\varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m\,d\mu\\ &=\int_{M^N}\Big[\varphi^{\frac{s}{p_1}}V^{\frac{1}{p_1}}(u+\rho)^{\frac{\sigma-t}{p_1}}\Big]\Big[\frac{s^m}{t^{m-1}}\varphi^{\frac{s}{p_1'}-m}V^{-\frac{1}{p_1}}|\nabla\varphi|^m\Big]\,d\mu\\ &\le\frac{1}{2}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}\,d\mu\\ &\quad+C\Big(\frac{s^m}{t^{m-1}}\Big)^{\frac{\sigma-t}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu.\end{aligned}$$

Using in the right-hand side of (2-5) the simple inequality

$$\Big(\frac{s^m}{t^{m-1}}\Big)^{\frac{\sigma-t}{\sigma-m+1}}\le\Big(\frac{s^m}{t^{m-1}}\Big)^{\frac{\sigma}{\sigma-m+1}}$$

and combining (2-5) with (2-4), we obtain that

$$\begin{aligned}\text{(2-6)}\quad &\frac{t}{2}\int_{M^N}\varphi^s(u+\rho)^{-t-1}A_u\,d\mu+\int_{M^N}\varphi^s V u^\sigma(u+\rho)^{-t}\,d\mu\\ &\le\frac{1}{2}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}\,d\mu\\ &\quad+Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu,\end{aligned}$$

where the value of s is absorbed into the constant C.

It is easy to obtain from the definition of the solution the boundedness of the term

$$\int_{M^N}\varphi^s V u^\sigma(u+\rho)^{-t}\,d\mu.$$

Then the boundedness of $\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}\,d\mu$ follows by the boundedness of

$$\int_{M^N}\varphi^s V u^\sigma(u+\rho)^{-t}\,d\mu,$$

and by the fact that $V \in L^1_{\mathrm{loc}}(M^N)$.

By the dominated convergence theorem, we know

$$\lim_{\rho\downarrow 0}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}\,d\mu=\int_{M^N}\varphi^s V u^{\sigma-t}\,d\mu.$$

Letting $\rho\downarrow 0$ in (2-6) and applying the monotone convergence theorem, we have

$$\frac{t}{2}\int_{M^N}\varphi^s u^{-t-1}A_u\,d\mu+\int_{M^N}\varphi^s V u^{\sigma-t}\,d\mu \le \frac{1}{2}\int_{M^N}\varphi^s V u^{\sigma-t}\,d\mu+Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu,$$

which is

$$\frac{t}{2}\int_{M^N}\varphi^s u^{-t-1}A_u\,d\mu+\frac{1}{2}\int_{M^N}\varphi^s V u^{\sigma-t}\,d\mu \le Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu. \tag{2-7}$$

Applying (1-13) once more, using another test function $\psi=\varphi^s$, we obtain

$$\begin{aligned}\int_{M^N}\varphi^s V u^{\sigma}\,d\mu &\le s\int_{M^N}\varphi^{s-1}(A(x,u,\nabla u),\nabla\varphi)\,d\mu\\ &\le Cs\int_{M^N}\varphi^{s-1}A_u^{\frac{m-1}{m}}|\nabla\varphi|\,d\mu\\ &\le Cs\Big(\int_{M^N}\varphi^s u^{-t-1}A_u\,d\mu\Big)^{\frac{m-1}{m}}\Big(\int_{M^N}\varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m\,d\mu\Big)^{\frac{1}{m}}.\end{aligned} \tag{2-8}$$

From (2-7), we obtain

$$\int_{M^N}\varphi^s u^{-t-1}A_u\,d\mu\le Ct^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu.$$

Substituting into (2-8) yields

$$\int_{M^N}\varphi^s V u^{\sigma}\,d\mu\le C\Big[t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}\,d\mu\Big]^{\frac{m-1}{m}} \times\Big[\int_{M^N}\varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m\,d\mu\Big]^{\frac{1}{m}}. \tag{2-9}$$

Recalling that $\nabla\varphi=0$ on B_R and applying the Hölder inequality to the last term of (2-9) with the Hölder couple

$$p_2=\frac{\sigma}{(t+1)(m-1)},\qquad p_2'=\frac{\sigma}{\sigma-(t+1)(m-1)},$$

we obtain

$$\begin{aligned}\text{(2-10)}\quad &\int_{M^N} \varphi^{s-m} u^{(t+1)(m-1)} |\nabla\varphi|^m \, d\mu \\ &= \int_{M^N\setminus B_R} \bigl(\varphi^{\frac{s}{p_2}} V^{\frac{1}{p_2}} u^{(t+1)(m-1)}\bigr)\bigl(\varphi^{\frac{s}{p_2'}-m} V^{-\frac{1}{p_2}} |\nabla\varphi|^m\bigr)\, d\mu \\ &\le \biggl(\int_{M^N\setminus B_R} \varphi^s V u^\sigma \, d\mu\biggr)^{\frac{(t+1)(m-1)}{\sigma}} \\ &\quad \times \biggl(\int_{M^N\setminus B_R} \varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}} V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\biggr)^{\frac{\sigma-(t+1)(m-1)}{\sigma}}.\end{aligned}$$

Substituting (2-10) into (2-9), choosing s large enough, and noting that $\varphi \le 1$, we obtain

$$\begin{aligned}\text{(2-11)}\quad \int_{M^N} \varphi^s V u^\sigma \, d\mu \le C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} &\biggl(\int_{M^N} V^{-\frac{m-1-t}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu\biggr)^{\frac{m-1}{m}} \\ &\times \biggl(\int_{M^N} V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\biggr)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}} \\ &\times \biggl(\int_{M^N\setminus B_R} \varphi^s V u^\sigma \, d\mu\biggr)^{\frac{(t+1)(m-1)}{m\sigma}}.\end{aligned}$$

From the definition of the solution, we know $\int_{M^N} \varphi^s V u^\sigma \, d\mu$ is finite. It follows from (2-11) that

$$\begin{aligned}\text{(2-12)}\quad &\biggl(\int_{M^N} \varphi^s V u^\sigma \, d\mu\biggr)^{1-\frac{(t+1)(m-1)}{m\sigma}} \\ &\le C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \biggl(\int_{M^N} V^{-\frac{m-1-t}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu\biggr)^{\frac{m-1}{m}} \\ &\quad \times \biggl(\int_{M^N} V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\biggr)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}.\end{aligned}$$

Note that the first integral in the right-hand side of (2-12) has the estimate

$$\text{(2-13)}\quad \int_{M^N} V^{-\frac{m-1-t}{\sigma-m+1}} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu \le \int_{M^N} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} V^{\frac{t}{\sigma-m+1}} \, d\nu,$$

where we have used that $d\nu = V^{-\frac{m-1}{\sigma-m+1}} \, d\mu$. Similarly, the second integral in the right-hand side of (2-12) can be estimated as follows:

$$\begin{aligned}\text{(2-14)}\quad &\int_{M^N} V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu \\ &\le \int_{M^N} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}} \, d\nu.\end{aligned}$$

Substituting (2-13) and (2-14) into (2-11), we have

$$
\begin{aligned}
\text{(2-15)}\quad \int_{M^N} \varphi^s V u^\sigma \, d\mu
&\le C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_{M^N} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} V^{\frac{t}{\sigma-m+1}} \, d\nu \right)^{\frac{m-1}{m}} \\
&\quad \times \left(\int_{M^N} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}} \, d\nu \right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}} \\
&\quad \times \left(\int_{M^N \setminus B_R} \varphi^s V u^\sigma \, d\mu \right)^{\frac{(t+1)(m-1)}{m\sigma}}.
\end{aligned}
$$

Substituting (2-13) and (2-14) into (2-12), we obtain

$$
\begin{aligned}
\text{(2-16)}\quad \left(\int_{M^N} \varphi^s V u^\sigma \, d\mu \right)^{1-\frac{(t+1)(m-1)}{m\sigma}}
&\le C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_{M^N} |\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}} V^{\frac{t}{\sigma-m+1}} \, d\nu \right)^{\frac{m-1}{m}} \\
&\quad \times \left(\int_{M^N} |\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}} \, d\nu \right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}.
\end{aligned}
$$

Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ be a sequence for which each $\tilde{\varphi}_k$ is a Lipschitz function such that $\operatorname{supp}(\tilde{\varphi}_k) \subset B_{2^k}$, and $\tilde{\varphi}_k = 1$ in a neighborhood of $B_{2^{k-1}}$, and

$$
\text{(2-17)}\qquad |\nabla\tilde{\varphi}_k| \begin{cases} \le \frac{C}{2^{k-1}} & \text{for } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ = 0 & \text{otherwise,} \end{cases}
$$

where C does not depend on k.

Fix some $n \in \mathbb{N}$ and set

$$
\text{(2-18)}\qquad t = \frac{1}{n}
$$

and

$$
\text{(2-19)}\qquad \varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}.
$$

Note that $\varphi_n = 1$ on B_{2^n}, and $\varphi_n = 0$ outside $B_{2^{2n}}$, and $0 \le \varphi_n \le 1$ on M^N. Note that, for any $a \ge 1$, using that $\operatorname{supp}(\nabla\tilde{\varphi}_k)$ are disjoint, we have

$$
\text{(2-20)}\qquad |\nabla\varphi_n|^a = \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a}.
$$

It is easy to see that

$$
\varphi_n \in W^{1,m}_{\mathrm{loc}}(M^N).
$$

Consider the integral

$$J_n(a,b)=\int_{M^N}|\nabla\varphi_n|^a V^b\,d\mu, \tag{2-21}$$

where a, b are taking values from

$$(a,b)=\begin{cases}\left(\frac{m(\sigma-t)}{\sigma-m+1},\ \frac{t}{\sigma-m+1}\right),\\ \left(\frac{m\sigma}{\sigma-(t+1)(m-1)},\ -\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}\right).\end{cases} \tag{2-22}$$

We write a in the form

$$a=p+lt, \tag{2-23}$$

with the corresponding two values of l,

$$l_1=-\frac{m}{\sigma-m+1},\quad l_2=\frac{m\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}, \tag{2-24}$$

where $p=m\sigma/(\sigma-m+1)$.

For $b\geq 0$, we know

$$\begin{aligned} J_n(a,b)&=\int_{M^N}|\nabla\varphi_n|^a V^b\,d\nu\\ &=\int_{M^N}\frac{\sum_{k=n+1}^{2n}|\nabla\tilde{\varphi}_k|^a}{n^a}V^b\,d\nu\\ &\leq\sum_{k=n+1}^{2n}\int_{B_{2^k}\setminus B_{2^{k-1}}}\frac{|\nabla\tilde{\varphi}_k|^a}{n^a}V^b\,d\nu\\ &\leq C\sum_{k=n+1}^{2n}\int_{B_{2^k}\setminus B_{2^{k-1}}}\left(\frac{2^{1-k}}{n}\right)^a r^{\delta_2 b}\,d\nu\\ &\leq C\sum_{k=n+1}^{2n}\left(\frac{2^{1-k}}{n}\right)^a(2^k)^{\delta_2 b}\nu(B_{2^k}\setminus B_1)\end{aligned} \tag{2-25}$$

Note that $a=p+lt$, and $n+1\leq k\leq 2n$, and

$$\begin{aligned}\left(\frac{2^{1-k}}{n}\right)^a(2^k)^{\delta_2 b}&=\left(\frac{2^{-k}}{n}\right)^p\left(\frac{2^{-k}}{n}\right)^{lt}(2^k)^{\delta_2 b}\\ &\leq\left(\frac{2^{-k}}{n}\right)^p(2^k)^{\delta_2 b}\sup_{n+1\leq k\leq 2n}\left(\frac{2^{-k}}{n}\right)^{lt}\\ &\leq C\left(\frac{2^{-k}}{n}\right)^p(2^k)^{\delta_2 b}.\end{aligned} \tag{2-26}$$

Substituting (2-26) into (2-25), and using the volume growth (1-16), we obtain

$$J_n(a,b) \le C\sum_{k=n+1}^{2n}\Big(\frac{2^{-k}}{n}\Big)^p (2^k)^{\delta_2 b}\nu(B_{2^k}\setminus B_1) \tag{2-27}$$
$$\le C\sum_{k=n+1}^{2n}\Big(\frac{2^{-k}}{n}\Big)^p (2^k)^{\delta_2 b}(2^k)^p \ln^q(2^k)$$
$$\le C\frac{1}{n^p}\sum_{k=n+1}^{2n} k^q 2^{k\delta_2 b} \le Cn^{q+1-p}2^{2n\delta_2 b} \le Cn^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_2 b}.$$

Similarly, for the case of $b \le 0$, we obtain

$$J_n(a,b) \le Cn^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{-2n\delta_1 b}. \tag{2-28}$$

Taking the sequence $\{\varphi_n\}$ in (2-16), we obtain

(2-29)
$$\Big(\int_{M^N}\varphi_n^s V u^\sigma\,d\mu\Big)^{1-\frac{(t+1)(m-1)}{m\sigma}}$$
$$\le Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\Big(J_n\Big(\frac{m(\sigma-t)}{\sigma-m+1},\frac{t}{\sigma-m+1}\Big)\Big)^{\frac{m-1}{m}}$$
$$\times\Big(J_n\Big(\frac{m\sigma}{\sigma-(t+1)(m-1)},\frac{-t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}\Big)\Big)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}.$$

Substituting (2-27) and (2-28) and noting that $t = 1/n$, we obtain

$$\Big(\int_{M^N}\varphi_n^s V u^\sigma\,d\mu\Big)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}} \tag{2-30}$$
$$\le Cn^{\frac{m-1}{m}+\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\Big(n^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_2\frac{\frac{1}{n}}{\sigma-m+1}}\Big)^{\frac{m-1}{m}}$$
$$\times\Big(n^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_1\frac{\frac{1}{n}\sigma(m-1)}{[\sigma-(\frac{1}{n}+1)(m-1)](\sigma-m+1)}}\Big)^{\frac{\sigma-(\frac{1}{n}+1)(m-1)}{m\sigma}}$$
$$\le Cn^{\frac{(m-1)^2}{n(\sigma-m+1)}}2^{\frac{2(\delta_1+\delta_2)(m-1)}{m(\sigma-m+1)}}.$$

Noting that $\varphi_n = 1$ on B_{2^n} and taking the lim sup of both sides in (2-30) as $n \to \infty$, we obtain

$$\int_{M^N} V u^\sigma\,d\mu \le C < \infty. \tag{2-31}$$

Applying similar arguments to (2-15), we obtain that

$$\int_{M^N} \varphi_n^s V u^\sigma \, d\mu \le C\Big(\int_{M^N \setminus B_{2^n}} \varphi_n^s V u^\sigma \, d\mu\Big)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}. \tag{2-32}$$

Since $\varphi_n = 1$ on B_{2^n}, we have

$$\int_{B_{2^n}} V u^\sigma \, d\mu \le C\Big(\int_{M^N \setminus B_{2^n}} \varphi_n^s V u^\sigma \, d\mu\Big)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}. \tag{2-33}$$

Combining this with (2-31) and letting $n \to \infty$, we obtain that

$$\int_{M^N} V u^\sigma \, d\mu = 0,$$

since $V > 0$ for almost all $x \in M^N$. Thus $u \equiv 0$. □

3. Examples

Our result can cover many known results in the case of $M^N = R^N$. Let us mention two of these examples.

Example 1. Let us investigate the inequality

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + V(x)u^\sigma \le 0, \quad \text{in } \mathbb{R}^N, \tag{3-1}$$

where $V(x) = 1/|x|^\gamma$ for $|x| \ge 1$, and $N > m > \max\{1, \gamma\}$, and $\sigma > m-1$.

By [Filippucci 2009, Corollary 1.5], we know if

$$\sigma \le \frac{(N-\gamma)(m-1)}{N-m}, \tag{3-2}$$

then (3-1) has no positive solutions in some natural class. Compared to our result of Theorem 1.3, we know for large r

$$\nu(B_r \setminus \overline{B_1}) = \int_{B_r \setminus \overline{B_1}} V^{-\frac{m-1}{\sigma-m+1}} \, d\mu = \omega_N \int_1^r s^{\frac{\gamma(m-1)}{\sigma-m+1}} s^{N-1} \, ds \approx C r^{N+\frac{\gamma(m-1)}{\sigma-m+1}}, \tag{3-3}$$

where ω_N is the surface area of the unit ball in $\mathbb{R}^N$, and μ is the Lebesgue measure, and the sign $\approx$ means that both the inequalities $\le$ and $\ge$ are satisfied but with different values of different constants c, C.

By (3-3), it follows that the condition (1-16) is equivalent to

$$N + \frac{\gamma(m-1)}{\sigma-m+1} \le p = \frac{m\sigma}{\sigma-m+1}, \tag{3-4}$$

which in turn is equivalent to (3-2).

Example 2. Consider the differential inequality

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+u^\sigma \le 0, \quad \text{in } \mathbb{R}^N, \tag{3-5}$$

where $N > 2$, $\sigma > 1$. This problem was investigated in [Mitidieri and Pokhozhaev 1999]. They obtained that if

$$\sigma \le \frac{N}{N-2}, \tag{3-6}$$

then (3-5) has no positive solutions. Note that the operator in (3-5) belongs to the class of $A(2)$, and that $\nu(B_r \setminus \overline{B_1}) = \mu(B_r \setminus \overline{B_1}) \approx Cr^N$. By Theorem 1.3, we know if

$$N \le \frac{2\sigma}{\sigma-1}, \tag{3-7}$$

then (3-5) has no positive solution. It is easy to check that (3-6) and (3-7) are equivalent.

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YUHUA SUN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BIELEFELD
D-33501 BIELEFELD
GERMANY
Current address:
SCHOOL OF MATHEMATICAL SCIENCES AND LPMC
NANKAI UNIVERSITY
94 WEIJIN ROAD
TIANJIN 300071
CHINA
sunyuhua@nankai.edu.cn

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