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EQUIVARIANT PRINCIPAL BUNDLES AND LOGARITHMIC CONNECTIONS ON TORIC VARIETIES

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Let M be a smooth complex projective toric variety equipped with an action of a torus T, such that the complement D of the open T-orbit in M is a simple normal crossing divisor. Let G be a complex reductive affine algebraic group. We prove that an algebraic principal G-bundle $E_G \to M$ admits a T-equivariant structure if and only if E_G admits a logarithmic connection singular over D. If $E_H \to M$ is a T-equivariant algebraic principal H-bundle, where H is any complex affine algebraic group, then E_H in fact has a canonical integrable logarithmic connection singular over D.

1. Introduction

Our aim is to give characterizations of the equivariant principal bundles on smooth complex projective toric varieties.

Let M be a smooth complex projective toric variety equipped with an action

$$\rho : T \times M \to M$$

of a torus T. For any point $t \in T$, define the automorphism

$$\rho_t : M \to M, \quad x \mapsto \rho(t, x).$$

We assume that the complement D of the open T-orbit in M is a simple normal crossing divisor.

Let G be a complex reductive affine algebraic group, and let E_G be an algebraic principal G-bundle on M. In Proposition 4.1 we prove the following:

The principal G-bundle E_G admits a T-equivariant structure if and only if the pulled-back principal G-bundle $\rho_t^* E_G$ is isomorphic to E_G for every $t \in T$.

When $G = \mathrm{GL}(n, \mathbb{C})$, this result was proved by Klyachko [1989, p. 342, Proposition 1.2.1].

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Using the above characterization of T-equivariant principal G-bundles on M, we prove the following (see Theorem 4.2):

The principal G-bundle E_G admits a logarithmic connection singular over D if and only if E_G admits a T-equivariant structure.

The "if" part of Theorem 4.2 does not require G to be reductive. More precisely, any T-equivariant principal H-bundle $E_H \to M$, where H is any complex affine algebraic group, admits a canonical integrable logarithmic connection singular over D (see Proposition 3.2).

2. Equivariant bundles

Let $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ be the multiplicative group. Take a complex algebraic group T which is isomorphic to a product of copies of $\mathbb{G}_m$. Let M be a smooth irreducible complex projective variety equipped with an algebraic action of T

$$\rho : T \times M \to M \tag{2-1}$$

such that

- there is a Zariski open dense subset $M^0 \subset M$ with $\rho(T, M^0) = M^0$,
- the action of T on M^0 is free and transitive, and
- the complement $M \setminus M^0$ is a simple normal crossing divisor of M.

In particular, M is a smooth projective toric variety. Note that M^0 is the unique T-orbit in M with trivial isotropy.

Let G be a connected complex affine algebraic group. A *T-equivariant* principal G-bundle on M is a pair $(E_G, \tilde{\rho})$, where

$$p : E_G \to M$$

is an algebraic principal G-bundle, and

$$\tilde{\rho} : T \times E_G \to E_G$$

is an algebraic action of T on the total space of E_G, such that

- $p \circ \tilde{\rho} = \rho \circ (\mathrm{Id}_T \times p)$, where ρ is the action in (2-1), and
- the actions of T and G on E_G commute.

Fix a point $x_0 \in M^0 \subset M$. Let

$$\iota : \rho(T, x_0) = M^0 \hookrightarrow M \tag{2-2}$$

be the inclusion map. Let $M^0 \times G$ be the trivial principal G-bundle on M^0. It has a tautological integrable algebraic connection given by its trivialization.

Let $(E_G, \tilde{\rho})$ be a T-equivariant principal G-bundle on M. Fix a point $z_0 \in (E_G)_{x_0}$. Using z_0, the action $\tilde{\rho}$ produces an isomorphism of principal G-bundles between $M^0 \times G$ and the restriction $E_G|_{M^0}$. This isomorphism of principal G-bundles is uniquely determined by the following two conditions:

- this isomorphism is T-equivariant (the action of T on $M^0 \times G$ is given by the action of T on M^0), and
- it takes the point $z_0 \in E_G$ to $(x_0, e) \in M^0 \times G$.

Using this trivialization of $E_G|_{M^0}$, the tautological integrable algebraic connection on $M^0 \times G$ produces an integrable algebraic connection $\mathcal{D}^0$ on $E_G|_{M^0}$. We note that the connection $\mathcal{D}^0$ is independent of the choice of the points x_0 and z_0. Indeed, the flat sections for $\mathcal{D}^0$ are precisely the orbits of T in $E_G|_{M^0}$. Note that this description of $\mathcal{D}^0$ does not require choosing base points in M^0 and $E_G|_{M^0}$.

In Proposition 3.2 it will be shown that $\mathcal{D}^0$ extends to a logarithmic connection on E_G over M singular over the simple normal crossing divisor $M \setminus M^0$.

3. Logarithmic connections

A canonical trivialization. The Lie algebra of T will be denoted by $\mathfrak{t}$. Let

$$\mathcal{V} := M \times \mathfrak{t} \to M \tag{3-1}$$

be the trivial vector bundle with fiber $\mathfrak{t}$. The holomorphic tangent bundle of M will be denoted by TM. Consider the action of T on M in (2-1). It produces a homomorphism of $\mathcal{O}_M$-coherent sheaves

$$\beta : \mathcal{V} \to TM. \tag{3-2}$$

Let

$$D := M \setminus M^0$$

be the simple normal crossing divisor of M. Let

$$TM(-\log D) \subset TM \tag{3-3}$$

be the corresponding logarithmic tangent bundle. Recall that $TM(-\log D)$ is characterized as the maximal coherent subsheaf of TM that preserves $\mathcal{O}_M(-D) \subset \mathcal{O}_M$ for the derivation action of TM on $\mathcal{O}_M$.

Lemma 3.1.

(1) *The image of β in (3-2) is contained in the subsheaf $TM(-\log D) \subset TM$.*

(2) *The resulting homomorphism $\beta : \mathcal{V} \to TM(-\log D)$ is an isomorphism.*

Proof. The divisor D is preserved by the action of T on M. Therefore, the action of T on $\mathcal{O}_M$, given by the action of T on M, preserves the subsheaf $\mathcal{O}_M(-D)$. From this it follows immediately that the subsheaf $\mathcal{O}_M(-D) \subset \mathcal{O}_M$ is preserved by the derivation action of the subsheaf

$$\beta(\mathcal{V}) \subset TM.$$

Therefore, we conclude that $\beta(\mathcal{V}) \subset TM(-\log D)$.

It is known that the vector bundle $TM(-\log D)$ is holomorphically trivial. This follows from Proposition 2 in [Fulton 1993, p. 87], which says that $\Omega^1_M(\log D)$ is holomorphically trivial, together with the equality $\Omega^1_M(\log D)^* = TM(-\log D)$.

So, both $\mathcal{V}$ and $TM(-\log D)$ are trivial vector bundles, and β is a homomorphism between them which is an isomorphism over the open subset M^0. From this it can be deduced that β is an isomorphism over entire M. To see this, consider the homomorphism

$$\textstyle\bigwedge^r \beta : \bigwedge^r \mathcal{V} \to \bigwedge^r TM(-\log D)$$

induced by β, where $r = \dim_{\mathbb{C}} T = \operatorname{rank}(\mathcal{V})$. So $\bigwedge^r \beta$ is a holomorphic section of the line bundle $\left(\bigwedge^r TM(-\log D)\right) \otimes \left(\bigwedge^r \mathcal{V}\right)^*$. This line bundle is holomorphically trivial because both $\mathcal{V}$ and $TM(-\log D)$ are holomorphically trivial. Fixing a trivialization of $\left(\bigwedge^r TM(-\log D)\right) \otimes \left(\bigwedge^r \mathcal{V}\right)^*$, we consider $\bigwedge^r \beta$ as a holomorphic function on M. This function is nowhere vanishing because it does not vanish on M^0 and holomorphic functions on M are constants. Since $\bigwedge^r \beta$ is nowhere vanishing, the homomorphism β is an isomorphism. □

A canonical logarithmic connection on equivariant bundles. The Lie algebra of G will be denoted by $\mathfrak{g}$.

Let $p : E_G \to M$ be an algebraic principal G-bundle. Consider the differential

$$dp : TE_G \to p^* TM, \tag{3-4}$$

where TE_G is the algebraic tangent bundle of E_G. The kernel of dp will be denoted by $T_{E_G/M}$. Using the action of G on E_G, the subbundle $T_{E_G/M} \subset TE_G$ is identified with the trivial vector bundle over E_G with fiber $\mathfrak{g}$.

The action of G on E_G produces an action of G on TE_G. So we get an action of G on the quasicoherent sheaf $p_* TE_G$ on M. The invariant part

$$\operatorname{At}(E_G) := (p_* TE_G)^G \subset p_* TE_G$$

is a locally free coherent sheaf; its coherence property follows from the fact that the action of G on the fibers of p is transitive, implying that a G-invariant section of $(TE_G)|_{p^{-1}(x)}$, $x \in M$, is uniquely determined by its evaluation at just one point of the fiber $p^{-1}(x)$. Also note that $\operatorname{At}(E_G) = (TE_G)/G$. This $\operatorname{At}(E_G)$ is known as the *Atiyah bundle* for E_G. Since $T_{E_G/M}$ is identified with $E_G \times \mathfrak{g}$, the invariant direct image $(p_* T_{E_G/M})^G$ is identified with the adjoint vector bundle

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g} \to M$$

associated to E_G for the adjoint action of G on $\mathfrak{g}$. We note that $\mathrm{ad}(E_G) = T_{E_G/M}/G$. Now the differential dp in (3-4) produces a short exact sequence of holomorphic vector bundles on M

$$0 \to \mathrm{ad}(E_G) \to \mathrm{At}(E_G) \xrightarrow{\phi} TM \to 0, \tag{3-5}$$

which is known as the Atiyah exact sequence. A holomorphic connection on E_G over M is a holomorphic splitting

$$TM \to \mathrm{At}(E_G)$$

of (3-5) [Atiyah 1957].

As before, setting $D = M \setminus M^0$, define

$$\mathrm{At}(E_G)(-\log D) := \phi^{-1}(TM(-\log D)) \subset \mathrm{At}(E_G),$$

where ϕ is the projection in (3-5) and $TM(-\log D)$ is the subsheaf in (3-3). So (3-5) gives the following short exact sequence of holomorphic vector bundles on M:

$$0 \to \mathrm{ad}(E_G) \to \mathrm{At}(E_G)(-\log D) \xrightarrow{\phi} TM(-\log D) \to 0. \tag{3-6}$$

A *logarithmic connection* on E_G, with singular locus D, is a holomorphic homomorphism

$$\delta : TM(-\log D) \to \mathrm{At}(E_G)(-\log D)$$

such that $\phi \circ \delta$ is the identity automorphism of $TM(-\log D)$, where ϕ is the homomorphism in (3-6). Just like the curvature of a connection, the curvature of a logarithmic connection δ on E_G is the obstruction for the homomorphism δ to preserve the Lie algebra structure of the sheaf of sections of $TM(-\log D)$ and $\mathrm{At}(E_G)(-\log D)$ given by the Lie bracket of vector fields. In particular, δ is called *integrable* (or *flat*) if it preserves the Lie algebra structure of the sheaf of sections of $TM(-\log D)$ and $\mathrm{At}(E_G)(-\log D)$ given by the Lie bracket of vector fields.

Proposition 3.2. *Let $(E_G, \tilde{\rho})$ be a T-equivariant principal G-bundle on M. Then E_G admits an integrable logarithmic connection that restricts to the connection $\mathcal{D}^0$ on M^0 constructed in Section 2.*

Proof. Let

$$\tilde{\mathcal{V}} := E_G \times \mathfrak{t} \to E_G$$

be the trivial vector bundle over E_G with fiber $\mathfrak{t}$. Note that $p^*\mathcal{V} = \tilde{\mathcal{V}}$, where $\mathcal{V}$ is the vector bundle in (3-1), and p, as before, is the projection of E_G to M.

The action $\tilde{\rho}$ of T on E_G produces a homomorphism

$$\tilde{\beta} : \tilde{\mathcal{V}} \to TE_G. \tag{3-7}$$

Since $p^{-1}(D)$ is preserved by the action of T on E_G, the induced action of T on $\mathcal{O}_{E_G}$ preserves the subsheaf $\mathcal{O}_{E_G}(-p^{-1}(D))$. Hence the image of $\tilde{\beta}$ lies inside the subsheaf

$$TE_G(-\log p^{-1}(D)) \subset TE_G.$$

Note that $p^{-1}(D)$ is a simple normal crossing divisor on E_G because D is a simple normal crossing divisor on M.

In Lemma 3.1(2) we saw that β is an isomorphism. Consider

$$p^*\beta^{-1} : p^*(TM(-\log D)) \to p^*\mathcal{V} = \tilde{\mathcal{V}}.$$

Precomposing this with $\tilde{\beta}$ in (3-7), we have

$$\tilde{\beta} \circ (p^*\beta^{-1}) : p^*(TM(-\log D)) \to TE_G(-\log p^{-1}(D)).$$

We observe that the homomorphism $\tilde{\beta} \circ (p^*\beta^{-1})$ is G-equivariant for the trivial action of G on $p^*(TM(-\log D))$ and the action of G on $TE_G(-\log p^{-1}(D))$ induced by the action of G on E_G. Therefore, taking the G-invariant parts of the direct images by p, the above homomorphism $\tilde{\beta} \circ (p^*\beta^{-1})$ produces a homomorphism

$$\beta' : TM(-\log D) = \big(p_* p^*(TM(-\log D))\big)^G \to \big(p_* TE_G(-\log p^{-1}(D))\big)^G = \mathrm{At}(E_G)(-\log D).$$

It is now straightforward to check that the homomorphism β' produces a holomorphic splitting of the exact sequence in (3-6). Therefore, β' defines a logarithmic connection on E_G singular on D. The restriction of this logarithmic connection to M^0 clearly coincides with the connection $\mathcal{D}^0$ constructed in Section 2. □

4. A criterion for equivariance

For each point $t \in T$, define the automorphism

$$\rho_t : M \to M, \quad x \mapsto \rho(t, x),$$

where ρ is the action in (2-1). If $(E_G, \tilde{\rho})$ is a T-equivariant principal G-bundle on M, then clearly the map

$$E_G \to E_G, \quad z \mapsto \tilde{\rho}(t, z)$$

is an isomorphism of the principal G-bundle $\rho_t^* E_G$ with E_G. The aim in this section is to prove a converse of this statement.

Take an algebraic principal G-bundle

$$p : E_G \to M.$$

Let $\mathcal{G}$ be the set of all pairs of the form (t, f), where $t \in T$ and where

$$f : E_G \to E_G$$

is an algebraic automorphism of the variety E_G satisfying the following conditions:

- $p \circ f = \rho_t \circ p$, and
- f intertwines the action of G on E_G.

Note that the above two conditions imply that f is an algebraic isomorphism of the principal G-bundle $\rho_t^* E_G$ with E_G.

We have the following composition on the set $\mathcal{G}$:

$$(t_1, f_1) \cdot (t_2, f_2) := (t_1 \circ t_2, f_1 \circ f_2).$$

The inverse of (t, f) is (t^{-1}, f^{-1}). These operations make $\mathcal{G}$ a group. In fact, $\mathcal{G}$ has the structure of an affine algebraic group defined over $\mathbb{C}$. Let $\mathcal{A}$ denote the group of all algebraic automorphisms of the principal G-bundle E_G. So $\mathcal{A}$ is a subgroup of $\mathcal{G}$ with the inclusion map being $f \mapsto (e, f)$. We have a natural projection

$$h : \mathcal{G} \to T, \quad (t, f) \mapsto t$$

which fits in the following exact sequence of complex affine algebraic groups:

$$0 \to \mathcal{A} \to \mathcal{G} \xrightarrow{h} T. \tag{4-1}$$

We note that there is a tautological action of $\mathcal{G}$ on E_G; the action of any $(t, f) \in \mathcal{G}$ on E_G is given by the map defined by $y \mapsto f(y)$.

Now assume that E_G satisfies the condition that, for every $t \in T$, the pulled-back principal G-bundle $\rho_t^* E_G$ is isomorphic to E_G. This assumption is equivalent to the statement that the homomorphism h in (4-1) is surjective.

In view of the above assumption, the sequence in (4-1) becomes the following short exact sequence of complex affine algebraic groups:

$$0 \to \mathcal{A} \to \mathcal{G} \xrightarrow{h} T \to 0. \tag{4-2}$$

Let $\mathcal{G}^0 \subset \mathcal{G}$ be the connected component containing the identity element. Since T is connected and h is surjective, the restriction of h to $\mathcal{G}^0$ is also surjective. Therefore, from (4-2) we have the short exact sequence of affine complex algebraic groups

$$0 \to \mathcal{A}^0 \xrightarrow{\iota_{\mathcal{A}}} \mathcal{G}^0 \xrightarrow{h^0} T \to 0, \tag{4-3}$$

where $\mathcal{A}^0 := \mathcal{A} \cap \mathcal{G}^0$ and $h^0 := h|_{\mathcal{G}^0}$.

Take a maximal torus $T_{\mathcal{G}} \subset \mathcal{G}^0$. From (4-3) it follows that the restriction

$$h' := h^0|_{T_{\mathcal{G}}} : T_{\mathcal{G}} \to T$$

is surjective. Define $T_{\mathcal{A}} := \mathcal{A}^0 \cap T_{\mathcal{G}} \subset T_{\mathcal{G}}$ using the homomorphism $\iota_{\mathcal{A}}$ in (4-3). Therefore, from (4-3) we have the short exact sequence of algebraic groups

$$0 \to T_{\mathcal{A}} \xrightarrow{\iota_{\mathcal{A}}|_{T_{\mathcal{A}}}} T_{\mathcal{G}} \xrightarrow{h'} T \to 0. \tag{4-4}$$

Recall that $\mathcal{G}$ has a tautological action on E_G. Therefore, the subgroup $T_{\mathcal{G}}$ has a tautological action on E_G which is the restriction of the tautological action of $\mathcal{G}$.

Now we assume that the group G is reductive.

A parabolic subgroup of G is a connected Zariski closed subgroup $P \subset G$ such that the variety G/P is projective. For a parabolic subgroup P, its unipotent radical will be denoted by $R_u(P)$. A Levi subgroup of P is a connected reductive subgroup $L(P) \subset P$ such that the composition

$$L(P) \hookrightarrow P \to P/R_u(P)$$

is an isomorphism. Levi subgroups exist, and any two Levi subgroups of P differ by conjugation by an element of $R_u(P)$ [Humphreys 1975, p. 184–185, §30.2; Borel 1991, p. 158, 11.22 and 11.23].

Let $\mathrm{Ad}(E_G) := E_G \times^G G \to M$ be the adjoint bundle associated to E_G for the adjoint action of G on itself. The fibers of $\mathrm{Ad}(E_G)$ are groups identified with G up to an inner automorphism; the corresponding Lie algebra bundle is $\mathrm{ad}(E_G)$. We note that $\mathcal{A}$ in (4-2) is the space of all algebraic sections of $\mathrm{Ad}(E_G)$.

Using the action of $T_{\mathcal{A}}$ on E_G, we have

- a Levi subgroup $L(P)$ of a parabolic subgroup P of G, and
- an algebraic reduction of structure group $E_{L(P)} \subset E_G$ of E_G to $L(P)$ which is preserved by the tautological action of $T_{\mathcal{G}}$ on E_G,

such that the image of $T_{\mathcal{A}}$ in $\mathrm{Ad}(E_G)$ (recall that the elements of $\mathcal{A}$ are sections of $\mathrm{Ad}(E_G)$) lies in the connected component, containing the identity element, of the center of each fiber of $\mathrm{Ad}(E_{L(P)}) \subset \mathrm{Ad}(E_G)$ (see [Balaji et al. 2005; Biswas and Parameswaran 2006] for the construction of $E_{L(P)}$). The construction of $E_{L(P)}$ requires fixing a point z_0 of E_G, where $E_{L(P)}$ contains z_0. Using z_0, the fiber $(E_{L(P)})_{p(z_0)}$ is identified with $L(P)$. Moreover, the evaluation, at $p(z_0)$, of the sections of $\mathrm{Ad}(E_G)$ corresponding to the elements of $T_{\mathcal{A}}$ makes $T_{\mathcal{A}}$ a subgroup of the connected component, containing the identity element, of the center of $E_{L(P)}$; in particular, this evaluation map on $T_{\mathcal{A}}$ is injective (see the second paragraph in [Balaji et al. 2005, p. 230, Section 3]). We briefly recall (from [Balaji et al. 2005; Biswas and Parameswaran 2006]) the argument that the evaluation map on semisimple elements of $\mathcal{A}$ is injective. Let ξ be a semisimple element of $\mathcal{A} = \Gamma(M, \mathrm{Ad}(E_G))$.

Since ξ is semisimple, for each point $x \in M$, the evaluation $\xi(x)$ is a semisimple element of $\mathrm{Ad}(E_G))_x$. The group $\mathrm{Ad}(E_G))_x$ is identified with G up to an inner automorphism of G. All conjugacy classes of a semisimple element of G are parametrized by T_G/W_{T_G}, where T_G is a maximal torus in G, and $W_{T_G} = N(T_G)/T_G$ is the Weyl group with $N(T_G)$ being the normalizer of T_G in G. We note that T_G/W_{T_G} is an affine variety. Therefore, we get a morphism $\xi' : M \to T_G/W_{T_G}$ that sends any $x \in M$ to the conjugacy class of $\xi(x)$. Since M is a projective variety and T_G/W_{T_G} is an affine variety, we conclude that ξ' is a constant map. So if $\xi(x) = e$ for some $x \in M$, then $\xi = e$ identically.

Let $Z^0_{L(P)} \subset L(P)$ be the connected component, containing the identity element, of the center. We note that $Z^0_{L(P)}$ is a product of copies of $\mathbb{G}_m$. Therefore, the above injective homomorphism $T_{\mathcal{A}} \to Z^0_{L(P)}$ extends to a homomorphism

$$\eta : T_{\mathcal{G}} \to Z^0_{L(P)}.$$

Define

$$\eta' := \tau \circ \eta, \tag{4-5}$$

where τ is the inversion homomorphism of $Z^0_{L(P)}$ defined by $g \mapsto g^{-1}$.

Consider the action of $T_{\mathcal{G}}$ on $E_{L(P)}$; recall that $E_{L(P)}$ is preserved by the tautological action of $T_{\mathcal{G}}$ on E_G. We can twist this action on $E_{L(P)}$ by η' in (4-5), because the actions of $Z^0_{L(P)}$ and $L(P)$ on $E_{L(P)}$ commute. For this new action, the group $T_{\mathcal{A}}$ clearly acts trivially on $E_{L(P)}$.

Consider the above action of $T_{\mathcal{G}}$ on $E_{L(P)}$ constructed using η'. Since $T_{\mathcal{A}}$ acts trivially on $E_{L(P)}$, the action of $T_{\mathcal{G}}$ on $E_{L(P)}$ descends to an action of T on $E_{L(P)}$ (see (4-4)). The principal G-bundle E_G is the extension of the structure group of $E_{L(P)}$ using the inclusion of $L(P)$ in G. Therefore, the above action of T on $E_{L(P)}$ produces an action of T on E_G. More precisely, the total space of E_G is the quotient of $E_{L(P)} \times G$ where two elements (z_1, g_1) and (z_2, g_2) of $E_{L(P)} \times G$ are identified if there is an element $g \in L(P)$ such that $z_2 = z_1 g$ and $g_2 = g^{-1} g_1$. Now the action of T on $E_{L(P)} \times G$, given by the above action of T on $E_{L(P)}$ and the trivial action of T on G, descends to an action of T on the quotient space E_G. Consequently, E_G admits a T-equivariant structure.

Therefore, we have proved the following:

Proposition 4.1. *Let G be reductive, and let $E_G \to M$ be a principal G-bundle such that, for every $t \in T$, the pulled-back principal G-bundle $\rho_t^* E_G$ is isomorphic to E_G. Then E_G admits a T-equivariant structure.*

For vector bundles on M, Proposition 4.1 was proved by Klyachko [1989, p. 342, Proposition 1.2.1].

Equivariance property from a logarithmic connection.

Theorem 4.2. *Let G be reductive, and let $p : E_G \to M$ be a principal G-bundle admitting a logarithmic connection whose singularity locus is contained in the divisor $D = M \setminus M^0$. Then E_G admits a T-equivariant structure.*

Proof. Since E_G admits a logarithmic connection, by definition, there is a homomorphism of coherent sheaves

$$\delta : TM(-\log D) \to \mathrm{At}(E_G)(-\log D)$$

such that $\phi \circ \delta$ is the identity automorphism of $TM(-\log D)$, where ϕ is the homomorphism in (3-6). Let

$$\hat{\delta} : H^0(M, TM(-\log D)) \to H^0(M, \mathrm{At}(E_G)(-\log D))$$

be the homomorphism of global sections given by δ. From Lemma 3.1(2) we know that $H^0(M, TM(-\log D))$ is the Lie algebra $\mathfrak{t}$ of T.

We will now show that there is a natural injective homomorphism

$$\theta : H^0(M, \mathrm{At}(E_G)(-\log D)) \to \mathrm{Lie}(\mathcal{G}), \tag{4-6}$$

where $\mathrm{Lie}(\mathcal{G})$ is the Lie algebra of the group $\mathcal{G}$ in (4-1).

The elements of $\mathrm{Lie}(\mathcal{G})$ are all holomorphic sections $s \in H^0(M, \mathrm{At}(E_G))$ such that the vector field $\phi(s)$, where ϕ is the projection in (3-5), is of the form $\beta(s')$, where $s' \in \mathfrak{t}$ and where β is the homomorphism in (3-2). Now, if

$$s \in H^0(M, \mathrm{At}(E_G)(-\log D)) \subset H^0(M, \mathrm{At}(E_G)),$$

then $\phi(s)$ is a holomorphic section of $TM(-\log D)$ (see (3-6)). From Lemma 3.1(2) it now follows that $\phi(s)$ is of the form $\beta(s')$, where $s' \in \mathfrak{t}$. This gives us the injective homomorphism in (4-6).

Finally, consider the composition

$$\theta \circ \hat{\delta} : \mathfrak{t} = H^0(M, TM(-\log D)) \to \mathrm{Lie}(\mathcal{G}).$$

From its construction it follows that

$$(dh) \circ \theta \circ \hat{\delta} = \mathrm{Id}_{\mathfrak{t}},$$

where $dh : \mathrm{Lie}(\mathcal{G}) \to \mathfrak{t}$ is the homomorphism of Lie algebras given by h in (4-1). In particular, dh is surjective. Since T is connected, this immediately implies that the homomorphism h is surjective. Now from Proposition 4.1 it follows that E_G admits a T-equivariant structure. □

References

[Atiyah 1957] M. F. Atiyah, "Complex analytic connections in fibre bundles", *Trans. Amer. Math. Soc.* **85** (1957), 181–207. MR 19,172c Zbl 0078.16002

[Balaji et al. 2005] V. Balaji, I. Biswas, and D. S. Nagaraj, "Krull–Schmidt reduction for principal bundles", *J. Reine Angew. Math.* **578** (2005), 225–234. MR 2006e:14061 Zbl 1084.14041

[Biswas and Parameswaran 2006] I. Biswas and A. J. Parameswaran, "On the equivariant reduction of structure group of a principal bundle to a Levi subgroup", *J. Math. Pures Appl.* (9) **85**:1 (2006), 54–70. MR 2006k:14081 Zbl 1159.14307

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, New York, 1991. MR 92d:20001 Zbl 0726.20030

[Fulton 1993] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton Univ. Press, 1993. MR 94g:14028 Zbl 0813.14039

[Humphreys 1975] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, New York, 1975. MR 53 #633 Zbl 0325.20039

[Klyachko 1989] A. A. Klyachko, "Equivariant bundles on toral varieties", *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:5 (1989), 1001–1039. In Russian; translated in *Math. USSR Izv.* **35**:2 (1990), 337–375. MR 91c:14064 Zbl 0706.14010

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INDRANIL BISWAS
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
MUMBAI 400005
INDIA
indranil@math.tifr.res.in

ARIJIT DEY
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY MADRAS
CHENNAI 600036
INDIA
arijitdey@gmail.com

MAINAK PODDAR
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE LOS ANDES
BOGOTA
COLOMBIA
mainakp@gmail.com

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