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Let (M, θ) be a compact strictly pseudoconvex CR manifold of real dimension $2n+1$ with a contact form θ. Motivated by the work of Ammann and Humbert, we define the second CR Yamabe invariant, which is a natural generalization of the CR Yamabe invariant, and study its properties in this paper.

1. Introduction

Let (M, g) be an n-dimensional compact Riemannian manifold where $n \geq 3$. The Yamabe problem is to find a Riemannian metric $\tilde{g}$ conformal to g such that the scalar curvature of $\tilde{g}$ is constant. Yamabe [1960] claimed to solve it. However, Trudinger [1968] realized that Yamabe's proof was incomplete, and he was able to solve the Yamabe problem when the scalar curvature of g is nonpositive. When the scalar curvature of g is positive, Aubin [1976] solved the case when $n \geq 6$ and M is not locally conformally flat, and Schoen [1984] solved the remaining cases by using the positive mass theorem.

The method to solve the Yamabe problem was the following. If $\tilde{g} = u^{\frac{4}{n-2}} g$, where $u \in C^\infty(M)$ and $u > 0$, then

$$L_g(u) = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \tag{1-1}$$

where

$$L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g.$$

Here Δ_g is the Laplacian of g, and R_g and $R_{\tilde{g}}$ are the scalar curvatures of g and $\tilde{g}$. The Yamabe problem is to solve (1-1) with $R_{\tilde{g}}$ being constant. The Yamabe invariant $Y(M, g)$ of (M, g) is defined as

$$Y(M, g) = \inf_{u \not\equiv 0,\, u \in C^\infty(M)} E(u),$$

where

$$E(u) = \frac{\int_M u L_g(u)\, dV_g}{\left(\int_M |u|^{\frac{2n}{n-2}}\, dV_g\right)^{\frac{n-2}{n}}}.$$

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The key point of the resolution of the Yamabe problem is the following theorem due to Aubin [1976].

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. If $Y(M, g) < Y(\mathbb{S}^n)$, then there exists a positive smooth function u satisfying* (1-1). *Here $Y(\mathbb{S}^n)$ is the Yamabe invariant of the sphere $\mathbb{S}^n$ with respect to the standard metric.*

The strict inequality was used to show that a minimizing sequence does not concentrate at any point. Aubin [1976] and Schoen [1984] proved the following:

Theorem 1.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then $Y(M, g) \leq Y(\mathbb{S}^n)$. Moreover, the equality holds if and only if (M, g) is conformally diffeomorphic to the sphere.*

These theorems solve the Yamabe problem. See also [Brendle 2005; 2007a; 2007b; Chow 1992; Schwetlick and Struwe 2003; Ye 1994] for using the flow approach to solve the Yamabe problem.

Ammann and Humbert [2006] defined the k-th Yamabe invariant as a generalization of the Yamabe invariant. More precisely, let

$$\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \leq \lambda_k(g) \cdots \to \infty$$

be the eigenvalues of L_g appearing with multiplicities. Let $[g]$ be the conformal class of g. For any positive integer k, the k-th Yamabe invariant $Y_k(M, g)$ is defined by

$$Y_k(M, g) = \inf_{\tilde{g} \in [g]} \lambda_k(\tilde{g}) \operatorname{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

In particular, $Y_1(M, g) = Y(M, g)$ when the Yamabe invariant $Y(M, g)$ is nonnegative.

One can consider the following CR analogue of the Yamabe problem, the CR Yamabe problem. Suppose that (M, θ) is a compact strictly pseudoconvex CR manifold of real dimension $2n + 1$ with a contact form θ. The CR Yamabe problem is to find a contact form $\tilde{\theta}$ conformal to θ such that the Webster scalar curvature of $\tilde{\theta}$ is constant. Jerison and Lee [1987; 1988; 1989] solved the CR Yamabe problem when $n \geq 2$ and M is not locally CR equivalent to the sphere. The remaining cases, namely when $n = 1$ or M is locally CR equivalent to the sphere, were studied respectively by Gamara and Yacoub [2001] and by Gamara [2001]. See also the recent work of Cheng, Chiu and Yang [Cheng et al. 2014] and Cheng, Malchiodi and Yang [Cheng et al. 2013]. See also [Chang and Cheng 2002; Chang et al. 2010; Ho 2012; Zhang 2009] for using the flow approach to solve the Yamabe problem.

Motivated by the result of Ammann and Humbert [2006], we study the k-th CR Yamabe invariant in this paper. In Section 2, we define the k-th CR Yamabe invariant and the generalized contact form. In Section 3, we give the variational characterization of $Y_k(M,\theta)$. In Section 4, we derive the Euler–Lagrange equation for $Y_2(M,\theta)$. Sections 5 and 6 will be devoted to proving a lower bound and an upper bound for $Y_2(M,\theta)$ respectively. In Section 7, we study whether $Y_2(M,\theta)$ is attained by some contact form or generalized contact form. Finally, in Section 8, we study the properties of the k-th CR Yamabe invariant $Y_k(M,\theta)$.

2. Definitions

Suppose that (M,θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ with a given contact form θ. Let $u \in C^\infty(M)$, $u > 0$. Then $\tilde{\theta} = u^{\frac{2}{n}}\theta$ is a contact form conformal to θ, and the Webster scalar curvature $R_{\tilde{\theta}}$ of $\tilde{\theta}$ is given by

$$L_\theta(u) = R_{\tilde{\theta}} u^{1+\frac{2}{n}}. \tag{2-1}$$

Here

$$L_\theta = -\Big(2+\frac{2}{n}\Big)\Delta_\theta + R_\theta, \tag{2-2}$$

where Δ_θ is the sub-Laplacian of θ and R_θ is the Webster scalar curvature of θ. The CR Yamabe invariant is defined as

$$Y(M,\theta) = \inf_{u\not\equiv 0,\, u\in C^\infty(M)} E(u),$$

where

$$E(u) = \frac{\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta u|_\theta^2 + R_\theta u^2 \, dV_\theta}{\big(\int_M |u|^{2+\frac{2}{n}} \, dV_\theta\big)^{\frac{n}{n+1}}}.$$

It is well known that L_θ has discrete spectrum

$$\operatorname{Spec}(L_\theta) = \{\lambda_1(\theta), \lambda_2(\theta), \dots\},$$

where the eigenvalues

$$\lambda_1(\theta) < \lambda_2(\theta) \le \lambda_3(\theta) \le \cdots \le \lambda_k(\theta) \cdots \to \infty$$

appear with multiplicities. The variational characterization of $\lambda_1(\theta)$ is given by

$$\lambda_1(\theta) = \inf_{u\not\equiv 0,\, u\in C^\infty(M)} \frac{\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta u|_\theta^2 + R_\theta u^2 \, dV_\theta}{\int_M u^2 \, dV_\theta}.$$

Let $[\theta]$ be the conformal class of θ, i.e.,

$$[\theta] = \{\tilde{\theta} = u^{\frac{2}{n}}\theta \mid u \in C^\infty(M), u > 0\}.$$

If $Y(M,\theta) \ge 0$, then it is easy to check that

$$Y(M,\theta) = \inf_{\tilde{\theta}\in[\theta]} \lambda_1(\tilde{\theta}) \operatorname{Vol}(M,\tilde{\theta})^{\frac{1}{n+1}}. \tag{2-3}$$

Following the definition of the k-th Yamabe invariant in [Ammann and Humbert 2006], we have the following:

Definition. For any positive integer k, the k-th CR Yamabe invariant is defined by

$$Y_k(M,\theta)=\inf_{\tilde{\theta}\in[\theta]}\lambda_k(\tilde{\theta})\,\mathrm{Vol}(M,\tilde{\theta})^{\frac{1}{n+1}}. \tag{2-4}$$

Then it follows from (2-3) and Theorem 8.2 that

$$Y_1(M,\theta)=\begin{cases}Y(M,\theta) & \text{if } Y(M,\theta)\geq 0,\\ -\infty & \text{if } Y(M,\theta)<0.\end{cases}$$

We write $L_+^{2+\frac{2}{n}}(M)=\{u\in L^{2+\frac{2}{n}}(M)\,|\,u\geq 0, u\not\equiv 0\}$. For $u\in L_+^{2+\frac{2}{n}}(M)$, we define $\mathrm{Gr}_k^u(C^\infty(M))$ to be the set of all k-dimensional subspaces of $C^\infty(M)$ such that the restriction operator to $M\setminus u^{-1}(0)$ is injective. More precisely, we have

$$\begin{aligned}&\mathrm{span}(v_1,\dots,v_k)\in\mathrm{Gr}_k^u(C^\infty(M))\\ &\qquad\Longleftrightarrow v_1|_{M\setminus u^{-1}(0)},\dots,v_k|_{M\setminus u^{-1}(0)} \text{ are linearly independent}\\ &\qquad\Longleftrightarrow u^{\frac{1}{n}}v_1,\dots,u^{\frac{1}{n}}v_k \text{ are linearly independent.}\end{aligned}$$

Similarly, replacing $C^\infty(M)$ by $S_1^2(M)$, we obtain the definition of $\mathrm{Gr}_k^u(S_1^2(M))$. Hereafter, $S_1^2(M)$ denotes the Folland–Stein space, which is the completion of $C^1(M)$ with respect to the norm

$$\|u\|_{S_1^2(M)}=\left(\int_M(|\nabla_\theta u|_\theta^2+u^2)\,dV_\theta\right)^{\frac{1}{2}}.$$

(For more properties about the Folland–Stein space, see [Folland and Stein 1974].)

Proposition 2.1. *Suppose $\tilde{\theta}$ is a contact form conformal to θ. Then we have*

$$\lambda_k(\tilde{\theta})=\inf_{V\in\mathrm{Gr}_k^u(S_1^2(M))}\ \sup_{v\in V\setminus\{0\}}\frac{\int_M vL_\theta v\,dV_\theta}{\int_M u^{\frac{2}{n}}v^2\,dV_\theta}. \tag{2-5}$$

Proof. Let $u\in C^\infty(M)$, $u>0$. For all $f\in C^\infty(M)$, $f\not\equiv 0$, we set $\tilde{\theta}=u^{\frac{2}{n}}\theta$ and

$$F'(u,f)=\frac{\int_M fL_{\tilde{\theta}}f\,dV_{\tilde{\theta}}}{\int_M f^2\,dV_{\tilde{\theta}}}.$$

The operator L_θ is conformally invariant in the following sense:

$$u^{1+\frac{2}{n}}L_{\tilde{\theta}}(u^{-1}f)=L_\theta(f), \tag{2-6}$$

because

$$\begin{aligned} u^{1+\frac{2}{n}} L_{\tilde\theta}(u^{-1}f) &= -\Big(2+\frac{2}{n}\Big) u^{1+\frac{2}{n}} \Delta_{\tilde\theta}(u^{-1}f) + R_{\tilde\theta} u^{1+\frac{2}{n}}(u^{-1}f) \\ &= -\Big(2+\frac{2}{n}\Big)\big(u\Delta_\theta(u^{-1}f) + 2\langle \nabla_\theta u, \nabla_\theta(u^{-1}f)\rangle_\theta\big) \\ &\qquad + \Big(-\Big(2+\frac{2}{n}\Big)\Delta_\theta u + R_\theta u\Big)(u^{-1}f) \\ &= -\Big(2+\frac{2}{n}\Big)\Delta_\theta f + R_\theta f = L_\theta(f), \end{aligned}$$

where we have used (2-1) and (2-2). Combining (2-6) with the fact that

$$dV_{\tilde\theta} = u^{2+\frac{2}{n}}\, dV_\theta, \tag{2-7}$$

we get

$$\begin{aligned} F'(u,f) &= \frac{\int_M f L_{\tilde\theta} f\, dV_{\tilde\theta}}{\int_M f^2\, dV_{\tilde\theta}} \\ &= \frac{\int_M f u^{-(1+\frac{2}{n})} L_\theta(uf) u^{2+\frac{2}{n}}\, dV_\theta}{\int_M f^2 u^{2+\frac{2}{n}}\, dV_\theta} = \frac{\int_M (uf) L_\theta(uf)\, dV_\theta}{\int_M u^{\frac{2}{n}} (uf)^2\, dV_\theta}. \end{aligned} \tag{2-8}$$

Using the min-max principle, we have

$$\lambda_k(\tilde\theta) = \inf_{V\in \mathrm{Gr}_k(S_1^2(M))} \sup_{v\in V\setminus\{0\}} \frac{\int_M v L_{\tilde\theta} v\, dV_{\tilde\theta}}{\int_M v^2\, dV_{\tilde\theta}}. \tag{2-9}$$

Since $u>0$, we have $\mathrm{Gr}_k(S_1^2(M)) = \mathrm{Gr}_k^u(S_1^2(M))$. Therefore, it follows from (2-8) and (2-9) that

$$\lambda_k(\tilde\theta) = \inf_{V\in \mathrm{Gr}_k(S_1^2(M))} \sup_{f\in V\setminus\{0\}} F'(u,f).$$

Now replacing uf by v, we obtain (2-5) by (2-8). □

Now we can define the generalized contact form:

Definition. The generalized contact form $\tilde\theta$ is defined as $\tilde\theta = u^{\frac{2}{n}}\theta$, where u is no longer necessarily positive or smooth, but $u \in L_+^{2+\frac{2}{n}}(M)$.

We enlarge the conformal class $[\theta]$ of θ by including all the generalized contact forms conformal to θ, as follows:

$$[\theta] = \{\tilde\theta = u^{\frac{2}{n}}\theta \mid u \in L_+^{2+\frac{2}{n}}(M)\}.$$

In view of Proposition 2.1, for a generalized contact form $\tilde\theta = u^{\frac{2}{n}}\theta$, $u\in L_+^{2+\frac{2}{n}}(M)$, conformal to θ, we define

$$\lambda_k(\tilde\theta) = \inf_{V\in \mathrm{Gr}_k^u(S_1^2(M))} \sup_{v\in V\setminus\{0\}} \frac{\int_M v L_\theta v\, dV_\theta}{\int_M u^{\frac{2}{n}} v^2\, dV_\theta}. \tag{2-10}$$

Using (2-10), we can generalize the definition of k-th CR Yamabe invariant to the generalized contact form by using (2-4).

3. Variational characterization of $Y_k(M,\theta)$

For all $u \in L_+^{2+\frac{2}{n}}(M)$, $v \in S_1^2(M)$ such that $u^{\frac{1}{n}}v \not\equiv 0$, we set

$$F(u,v) = \frac{\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta v|_\theta^2 + R_\theta v^2\, dV_\theta}{\int_M u^{\frac{2}{n}} v^2\, dV_\theta}\bigg(\int_M u^{2+\frac{2}{n}}\, dV_\theta\bigg)^{\frac{1}{n+1}}.$$

Proposition 3.1. *If $[\theta]$ contains all the contact forms conformal to θ, then*

$$Y_k(M,\theta) = \inf_{\substack{u\in C^\infty(M)\\ V\in \mathrm{Gr}_k^u(S_1^2(M))}} \sup_{v\in V\setminus\{0\}} F(u,v). \tag{3-1}$$

Similarly, if $[\theta]$ contains all the generalized contact forms conformal to θ, then

$$Y_k(M,\theta) = \inf_{\substack{u\in L_+^{2+\frac{2}{n}}(M)\\ V\in \mathrm{Gr}_k^u(S_1^2(M))}} \sup_{v\in V\setminus\{0\}} F(u,v). \tag{3-2}$$

Proof. Using the definition of $Y_k(M,\theta)$ and the fact that $\mathrm{Vol}(M,\tilde\theta) = \int_M u^{2+\frac{2}{n}}\, dV_\theta$, we obtain from (2-5) that

$$\begin{aligned} Y_k(M,\theta) &= \inf_{\tilde\theta\in[\theta]} \lambda_k(\tilde\theta)\,\mathrm{Vol}(M,\tilde\theta)^{\frac{1}{n+1}} \\ &= \inf_{u\in C^\infty(M),u>0} \lambda_k(\tilde\theta)\bigg(\int_M u^{2+\frac{2}{n}}\, dV_\theta\bigg)^{\frac{1}{n+1}} \\ &= \inf_{\substack{u\in C^\infty(M),u>0\\ V\in \mathrm{Gr}_k^u(S_1^2(M))}} \sup_{v\in V\setminus\{0\}} F(u,v), \end{aligned}$$

which proves (3-1). Similarly, we can prove (3-2) by using the same arguments as above, except we need to replace $C^\infty(M)$ by $L_+^{2+\frac{2}{n}}(M)$. □

4. Generalized contact form and the Euler–Lagrange equation

We will need the following:

Lemma 4.1. *Let $u \in L^{2+\frac{2}{n}}(M)$ and $v \in S_1^2(M)$. We assume that*

$$L_\theta v = u^{\frac{2}{n}} v \tag{4-1}$$

holds in the sense of distributions. Then $v \in L^{2+\frac{2}{n}+\varepsilon}(M)$ for some $\varepsilon > 0$.

Proof. Without loss of generality, suppose $v \not\equiv 0$. We define $v_+ = \sup(v, 0)$. We let $q \in (1, (n+1)/n]$ be a fixed number and $l > 0$ be a large real number which will tend to $+\infty$. We let $\beta = 2q - 1$. We then define for $x \in \mathbb{R}$,

$$G_l(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^\beta & \text{if } 0 \le x < l, \\ l^{q-1}(ql^{q-1}x - (q-1)l^q) & \text{if } x \ge l, \end{cases}$$

$$F_l(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^q & \text{if } 0 \le x < l, \\ ql^{q-1}x - (q-1)l^q & \text{if } x \ge l. \end{cases}$$

It is easy to check that for all $x \in \mathbb{R}$,

$$(F_l'(x))^2 \le qG_l'(x), \tag{4-2}$$

$$(F_l(x))^2 \ge xG_l(x), \tag{4-3}$$

$$xG_l'(x) \le \beta G_l(x). \tag{4-4}$$

Since F_l and G_l are uniformly Lipschitz continuous functions, $F_l(v_+)$ and $G_l(v_+)$ belong to $S_1^2(M)$. Let $x_0 \in M$. Denote by η a C^2 nonnegative function supported in $B(x_0, 2\delta)$, where $\delta > 0$ is a small fixed number such that $0 \le \eta \le 1$ and $\eta(B(x_0, \delta)) = \{1\}$. Multiply (4-1) by $\eta^2 G_l(v_+)$ and integrate over M. Since the supports of v_+ and $G_l(v_+)$ coincide, we get

$$\Big(2 + \frac{2}{n}\Big) \int_M \langle \nabla_\theta v_+, \nabla_\theta \eta^2 G_l(v_+) \rangle_\theta \, dV_\theta + \int_M R_\theta v_+ \eta^2 G_l(v_+) \, dV_\theta = \int_M u^{\frac{2}{n}} v_+ \eta^2 G_l(v_+) \, dV_\theta. \tag{4-5}$$

We are going to estimate the terms in (4-5). In the following, C will denote a positive constant depending possibly on η, q, β, δ, but not on l. Note that

$$\begin{aligned} \int_M & \langle \nabla_\theta v_+, \nabla_\theta \eta^2 G_l(v_+) \rangle_\theta \, dV_\theta \\ &= \int_M G_l(v_+) \langle \nabla_\theta v_+, \nabla_\theta \eta^2 \rangle_\theta \, dV_\theta + \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 \, dV_\theta \\ &= -\int_M G_l(v_+) v_+ \Delta_\theta(\eta^2) \, dV_\theta - 2\int_M v_+ G_l'(v_+) \eta \langle \nabla_\theta v_+, \nabla_\theta \eta \rangle_\theta \, dV_\theta \\ &\qquad + \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 \, dV_\theta \\ &\ge -C \int_M v_+ G_l(v_+) \, dV_\theta - 2\int_M v_+^2 G_l'(v_+) |\nabla_\theta \eta|_\theta^2 \, dV_\theta \\ &\qquad + \frac{1}{2} \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 \, dV_\theta, \end{aligned} \tag{4-6}$$

where the last inequality follows from $|\langle\nabla_\theta v_+, \nabla_\theta\eta\rangle_\theta| \le |\nabla_\theta\eta|^2_\theta + \frac{1}{4}|\nabla_\theta v_+|^2_\theta$. Hence, we have

$$
\begin{aligned}
(4\text{-}7)\quad & \int_M \langle\nabla_\theta v_+, \nabla_\theta\eta^2 G_l(v_+)\rangle_\theta\, dV_\theta \\
&\ge -C\int_M v_+G_l(v_+)\, dV_\theta - 2\int_M v_+^2 G_l'(v_+)|\nabla_\theta\eta|^2_\theta\, dV_\theta \\
&\qquad\qquad + \frac{1}{2}\int_M G_l'(v_+)\eta^2|\nabla_\theta v_+|^2_\theta\, dV_\theta \\
&\ge -C\int_M v_+G_l(v_+)\, dV_\theta - 2\beta\int_M v_+ G_l(v_+)|\nabla_\theta\eta|^2_\theta\, dV_\theta \\
&\qquad\qquad + \frac{1}{2}\int_M G_l'(v_+)\eta^2|\nabla_\theta v_+|^2_\theta\, dV_\theta \\
&\ge -C\int_M (F_l(v_+))^2\, dV_\theta + \frac{1}{2q}\int_M (F_l'(v_+))^2\eta^2|\nabla_\theta v_+|^2_\theta\, dV_\theta \\
&= -C\int_M (F_l(v_+))^2\, dV_\theta + \frac{1}{2q}\int_M \eta^2|\nabla_\theta F_l(v_+)|^2_\theta\, dV_\theta \\
&\ge -C\int_M (F_l(v_+))^2\, dV_\theta + \frac{1}{4q}\int_M |\nabla_\theta(\eta F_l(v_+))|^2_\theta\, dV_\theta \\
&\qquad\qquad - \frac{1}{2q}\int_M |\nabla_\theta\eta|^2_\theta (F_l(v_+))^2\, dV_\theta \\
&\ge -C\int_M (F_l(v_+))^2\, dV_\theta + \frac{1}{4q}\int_M |\nabla_\theta(\eta F_l(v_+))|^2_\theta\, dV_\theta,
\end{aligned}
$$

where the first inequality follows from (4-6), the second inequality follows from (4-4), the third inequality follows from (4-2) and (4-3), and the fourth inequality follows from

$$
\begin{aligned}
|\nabla_\theta(\eta F_l(v_+))|^2_\theta &= |F_l(v_+)\nabla_\theta\eta + \eta\nabla_\theta F_l(v_+)|^2_\theta \\
&\le 2\eta^2|\nabla_\theta F_l(v_+)|^2_\theta + 2|\nabla_\theta\eta|^2_\theta(F_l(v_+))^2.
\end{aligned}
$$

By the Folland–Stein embedding from $S_1^2(M)$ into $L^{2+\frac{2}{n}}(M)$, there exists a constant $A > 0$ depending only on (M, θ) such that

$$\int_M |\nabla_\theta(\eta F_l(v_+))|^2_\theta\, dV_\theta \ge A\left(\int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\, dV_\theta\right)^{\frac{n}{n+1}} - \int_M (\eta F_l(v_+))^2\, dV_\theta.$$

From this, together with (4-7), we obtain

$$
\begin{aligned}
(4\text{-}8)\quad & \int_M \langle\nabla_\theta v_+, \nabla_\theta\eta^2 G_l(v_+)\rangle_\theta\, dV_\theta \\
&\ge -C\int_M (F_l(v_+))^2\, dV_\theta + \frac{A}{4q}\left(\int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\, dV_\theta\right)^{\frac{n}{n+1}}.
\end{aligned}
$$

Independently, we choose $\delta > 0$ small enough such that

$$\int_{B(x_0,2\delta)} u^{2+\frac{2}{n}}\,dV_\theta \le \left(\left(2+\frac{2}{n}\right)\frac{A}{8q}\right)^{n+1}. \tag{4-9}$$

Then it follows from (4-3), (4-9) and Hölder's inequality that

$$\begin{aligned}\int_M u^{\frac{2}{n}} v_+ \eta^2 G_l(v_+)\,dV_\theta &\le \int_M u^{\frac{2}{n}}\eta^2 (F_l(v_+))^2\,dV_\theta\\ &\le \left(\int_{B(x_0,2\delta)} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}}\left(\int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{n}{n+1}}\\ &\le \left(2+\frac{2}{n}\right)\frac{A}{8q}\left(\int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{n}{n+1}}.\end{aligned} \tag{4-10}$$

On the other hand, it follows from (4-3) that

$$\begin{aligned}\int_M R_\theta v_+\eta^2 G_l(v_+)\,dV_\theta &\ge -(\max_M |R_\theta|)\int_M v_+\eta^2 G_l(v_+)\,dV_\theta\\ &\ge -(\max_M |R_\theta|)\int_M \eta^2 (F_l(v_+))^2\,dV_\theta\\ &\ge -C\int_M (F_l(v_+))^2\,dV_\theta.\end{aligned} \tag{4-11}$$

Substituting (4-8), (4-10), (4-11) into (4-5), we obtain

$$\left(2+\frac{2}{n}\right)\frac{A}{8q}\left(\int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{n}{n+1}} \le C\int_M (F_l(v_+))^2\,dV_\theta.$$

Now, by the Folland–Stein embedding, $v_+ \in L^{2+\frac{2}{n}}(M)$. Since $2q \le 2+\frac{2}{n}$ and C does not depend on l, the right-hand side of the inequality is bounded when $l \to \infty$, and we obtain

$$\limsup_{l\to\infty} \int_M (\eta F_l(v_+))^{2+\frac{2}{n}}\,dV_\theta < \infty.$$

This proves that $v_+ \in L^{q(2+\frac{2}{n})}(B(x_0,\delta))$. Since x_0 is arbitrary, we get that $v_+ \in L^{q(2+\frac{2}{n})}(M)$. Doing the same with $v_- = \sup(-v, 0)$ instead of v_+, we get that $v \in L^{q(2+\frac{2}{n})}(M)$. This proves Lemma 4.1. □

Proposition 4.2. *For any generalized contact form $\tilde\theta = u^{\frac{2}{n}}\theta$, $u \in L_+^{2+\frac{2}{n}}(M)$, conformal to θ, there exist two functions $v, w \in S_1^2(M)$ with $v \ge 0$ such that in the* *sense of distributions*

$$L_\theta v = \lambda_1(\tilde{\theta}) u^{\frac{2}{n}} v, \tag{4-12}$$

$$L_\theta w = \lambda_2(\tilde{\theta}) u^{\frac{2}{n}} w. \tag{4-13}$$

Moreover, we can normalize v and w such that

$$\int_M u^{\frac{2}{n}} v^2 \, dV_\theta = \int_M u^{\frac{2}{n}} w^2 \, dV_\theta = 1 \quad \textit{and} \quad \int_M u^{\frac{2}{n}} vw \, dV_\theta = 0. \tag{4-14}$$

Proof. Let $(v_m)_m$ be a minimizing sequence for $\lambda_1(\tilde{\theta})$, i.e., a sequence $v_m \in S_1^2(M)$ such that

$$\lim_{m\to\infty} \frac{\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta v_m|_\theta^2 + R_\theta v_m^2 \, dV_\theta}{\int_M u^{\frac{2}{n}} v_m^2 \, dV_\theta} = \lambda_1(\tilde{\theta}).$$

It is well known that $(|v_m|)_m$ is also a minimizing sequence. Hence we can assume that $v_m \geq 0$. If we normalize v_m by $\int_M u^{\frac{2}{n}} v_m^2 \, dV_\theta = 1$, then $(v_m)_m$ is bounded in $S_1^2(M)$ and after passing to a subsequence, we may assume that there exists $v \in S_1^2(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $S_1^2(M)$ and strongly in $L^2(M)$ almost everywhere. If u is smooth, then

$$\int_M u^{\frac{2}{n}} v^2 \, dV_\theta = \lim_{m\to\infty} \int_M u^{\frac{2}{n}} v_m^2 \, dV_\theta = 1, \tag{4-15}$$

and by standard arguments, v is nonnegative minimizer of the functional associated to $\lambda_1(\tilde{\theta})$.

We must show that (4-15) still holds if $u \in L_+^{2+\frac{2}{n}}(M)$. Let $A > 0$ be a large real number and set $u_A = \inf(u, A)$. Then

$$\begin{aligned} \left|\int_M u^{\frac{2}{n}} (v_m^2 - v^2) \, dV_\theta\right| &\leq \int_M u_A^{\frac{2}{n}} |v_m^2 - v^2| \, dV_\theta + \int_M (u^{\frac{2}{n}} - u_A^{\frac{2}{n}})(|v_m| + |v|)^2 \, dV_\theta \\ &\leq A^{\frac{2}{n}} \int_M |v_m^2 - v^2| \, dV_\theta \\ &\quad + \left(\int_M (u^{\frac{2}{n}} - u_A^{\frac{2}{n}})^{n+1} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M (|v_m| + |v|)^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{n}{n+1}}, \end{aligned} \tag{4-16}$$

where we have used Hölder's inequality in the last inequality. Since

$$|u^{\frac{2}{n}} - u_A^{\frac{2}{n}}|^{n+1} \leq u^{2+\frac{2}{n}} \in L^1(M),$$

by Lebesgue's dominated convergence theorem we have

$$\lim_{A\to\infty} \int_M (u^{\frac{2}{n}} - u_A^{\frac{2}{n}})^{n+1} \, dV_\theta = \int_M \lim_{A\to\infty} (u^{\frac{2}{n}} - u_A^{\frac{2}{n}})^{n+1} \, dV_\theta = 0. \tag{4-17}$$

Since $(v_m)_m$ is bounded in $S_1^2(M)$, it is bounded in $L^{2+\frac{2}{n}}(M)$, and hence there exists $C > 0$ such that

$$\int_M (|v_m| + |v|)^{2+\frac{2}{n}} \, dV_\theta \le C. \tag{4-18}$$

By strong convergence in $L^2(M)$,

$$\lim_{m\to\infty} \int_M |v_m^2 - v^2| \, dV_\theta = 0. \tag{4-19}$$

Combining (4-16)–(4-19), we obtain (4-15). Therefore v is a nonnegative minimizer of the functional associated to $\lambda_1(\tilde{\theta})$. Writing the Euler–Lagrange equation of v, we find that v satisfies (4-12).

Now we define

$$\lambda_1'(\tilde{\theta}) = \inf \frac{\int_M \left(2+\frac{2}{n}\right)|\nabla_\theta w|_\theta^2 + R_\theta w^2 \, dV_\theta}{\int_M u^{\frac{2}{n}} |w|^2 \, dV_\theta},$$

where the infimum is taken over smooth functions w such that $u^{\frac{1}{n}} w \not\equiv 0$ and such that

$$\int_M u^{\frac{2}{n}} v w \, dV_\theta = 0.$$

With the same method, we find a minimizer w of this problem that satisfies (4-13) with $\lambda_2'(\tilde{\theta})$ instead of $\lambda_2(\tilde{\theta})$. However, it is not difficult to see that $\lambda_2'(\tilde{\theta}) = \lambda_2(\tilde{\theta})$ and Proposition 4.2 easily follows. □

Lemma 4.3. *Let $u \in L_+^{2+\frac{2}{n}}(M)$ with $\int_M u^{2+\frac{2}{n}} \, dV_\theta = 1$. Suppose that $w_1, w_2 \in S_1^2(M) \setminus \{0\}$, $w_1, w_2 \ge 0$ satisfy*

$$\int_M \left(\left(2+\frac{2}{n}\right)|\nabla_\theta w_1|_\theta^2 + R_\theta w_1^2\right) dV_\theta \le Y_2(M,\theta) \int_M u^{\frac{2}{n}} w_1^2 \, dV_\theta, \tag{4-20}$$

$$\int_M \left(\left(2+\frac{2}{n}\right)|\nabla_\theta w_2|_\theta^2 + R_\theta w_2^2\right) dV_\theta \le Y_2(M,\theta) \int_M u^{\frac{2}{n}} w_2^2 \, dV_\theta, \tag{4-21}$$

and suppose that $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero. Then u is a linear combination of w_1 and w_2, and we have equality in (4-20) *and* (4-21).

Proof. We let $\bar{u} = a w_1 + b w_2$, where $a, b > 0$ are chosen such that

$$\frac{b^{\frac{2}{n}} \int_M u^{\frac{2}{n}} w_1^2 \, dV_\theta}{a^{\frac{2}{n}} \int_M u^{\frac{2}{n}} w_2^2 \, dV_\theta} = \frac{\int_M w_1^{2+\frac{2}{n}} \, dV_\theta}{\int_M w_2^{2+\frac{2}{n}} \, dV_\theta}, \tag{4-22}$$

$$\int_M \bar{u}^{2+\frac{2}{n}} \, dV_\theta = a^{2+\frac{2}{n}} \int_M w_1^{2+\frac{2}{n}} \, dV_\theta + b^{2+\frac{2}{n}} \int_M w_2^{2+\frac{2}{n}} \, dV_\theta = 1. \tag{4-23}$$

Because of the variational characterization of $Y_2(M,\theta)$ in Proposition 3.1, we have

$$Y_2(M,\theta) \le \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(\bar{u}, \lambda w_1 + \mu w_2). \tag{4-24}$$

By (4-20), (4-21), (4-23), and since $(M\setminus w_1^{-1}(0))\cap(M\setminus w_2^{-1}(0))$ has measure zero, we obtain

$$\begin{aligned} F(\bar{u},\lambda w_1+\mu w_2) &= \frac{\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta(\lambda w_1+\mu w_2)|_\theta^2 + R_\theta(\lambda w_1+\mu w_2)^2\,dV_\theta}{\int_M \bar{u}^{\frac{2}{n}}(\lambda w_1+\mu w_2)^2\,dV_\theta} \\ &= \frac{\lambda^2\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta w_1|_\theta^2 + R_\theta w_1^2\,dV_\theta + \mu^2\int_M \big(2+\frac{2}{n}\big)|\nabla_\theta w_2|_\theta^2 + R_\theta w_2^2\,dV_\theta}{\lambda^2\int_M \bar{u}^{\frac{2}{n}} w_1^2\,dV_\theta + \mu^2\int_M \bar{u}^{\frac{2}{n}} w_2^2\,dV_\theta} \\ &\le Y_2(M,\theta)\frac{\lambda^2\int_M u^{\frac{2}{n}} w_1^2\,dV_\theta + \mu^2\int_M u^{\frac{2}{n}} w_2^2\,dV_\theta}{\lambda^2 a^{\frac{2}{n}}\int_M w_1^{2+\frac{2}{n}}\,dV_\theta + \mu^2 b^{\frac{2}{n}}\int_M w_2^{2+\frac{2}{n}}\,dV_\theta}. \end{aligned} \tag{4-25}$$

By (4-22), the right-hand side of (4-25) does not depend on λ and μ. Hence we can choose $\lambda = a$ and $\mu = b$ on the right-hand side of (4-25) to get

$$\begin{aligned} \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} & F(\bar{u},\lambda w_1+\mu w_2) \\ &\le Y_2(M,\theta)\frac{a^2\int_M u^{\frac{2}{n}} w_1^2\,dV_\theta + b^2\int_M u^{\frac{2}{n}} w_2^2\,dV_\theta}{a^{2+\frac{2}{n}}\int_M w_1^{2+\frac{2}{n}}\,dV_\theta + b^{2+\frac{2}{n}}\int_M w_2^{2+\frac{2}{n}}\,dV_\theta} \\ &= Y_2(M,\theta)\int_M u^{\frac{2}{n}}(a^2w_1^2+b^2w_2^2)\,dV_\theta \\ &= Y_2(M,\theta)\int_M u^{\frac{2}{n}}\bar{u}^2\,dV_\theta \\ &\le Y_2(M,\theta)\Big(\int_M u^{2+\frac{2}{n}}\,dV_\theta\Big)^{\frac{1}{n+1}}\Big(\int_M \bar{u}^{2+\frac{2}{n}}\,dV_\theta\Big)^{\frac{n}{n+1}} \\ &= Y_2(M,\theta), \end{aligned} \tag{4-26}$$

where we have used (4-23) in the first equality, the assumption that $(M\setminus w_1^{-1}(0))\cap(M\setminus w_2^{-1}(0))$ has measure zero in the second equality, Hölder's inequality in the second inequality, and the assumption $\int_M u^{2+\frac{2}{n}}\,dV_\theta = 1$ and (4-23) in the last equality.

Combining (4-24) and (4-26), we have

$$\sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(\bar{u},\lambda w_1+\mu w_2) = Y_2(M,\theta).$$

This implies the equality in Holder's inequality in (4-26), which implies that there exists a constant $c > 0$ such that $u = c\bar{u}$ almost everywhere. Since $\int_M u^{2+\frac{2}{n}}\, dV_\theta = \int_M \bar{u}^{2+\frac{2}{n}}\, dV_\theta = 1$ by (4-23), we have $c = 1$, i.e., $u = \bar{u} = aw_1 + bw_2$. Also, equality in (4-25) implies equality in (4-20) and (4-21). This proves the assertion. □

Theorem 4.4 (Euler–Lagrange equation). *Assume $Y_2(M,\theta) \neq 0$ and that $Y_2(M,\theta)$ is attained by a generalized contact form $\tilde{\theta} = u^{\frac{2}{n}}\theta$ with $u \in L_+^{2+\frac{2}{n}}(M)$. Let v and w be as in Proposition 4.2. Then $u = |w|$. In particular,*

$$L_\theta w = Y_2(M,\theta)|w|^{\frac{2}{n}} w. \tag{4-27}$$

Moreover, w has alternating sign and $w \in C^{2,\alpha}(M)$ for all $\alpha \in \left[0, \frac{2}{n}\right]$.

Proof. Without loss of generality, we can assume that $\int_M u^{2+\frac{2}{n}}\, dV_\theta = 1$. By assumption and by Proposition 3.1, we have $\lambda_2(\tilde{\theta}) = Y_2(M,\theta)$. Let $v, w \in S_1^2(M)$ be the functions satisfying (4-12), (4-13), and (4-14).

Step 1. We have $\lambda_1(\tilde{\theta}) < \lambda_2(\tilde{\theta})$.

We prove this by contradiction. Suppose that $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$. After possibly replacing w by a linear combination of v and w, we can assume that the function $u^{\frac{1}{n}} w$ changes sign. If we define $w_1 = \sup(w, 0)$ and $w_2 = \sup(-w, 0)$, then they satisfy the assumption of Lemma 4.3 since w satisfies (4-13) and $\lambda_2(\tilde{\theta}) = Y_2(M,\theta)$. Applying Lemma 4.3, we find $a, b > 0$ such that $u = aw_1 + bw_2$. Now, by Lemma 4.1, $w \in L^{2+\frac{2}{n}+\varepsilon}(M)$. By a standard bootstrap argument, (4-13) shows that $w \in C^{2,\alpha}(M)$ for all $\alpha \in (0,1)$. Since $u = aw_1 + bw_2 = a\sup(w,0) + b\sup(-w,0)$, we have $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0,1)$.

Since $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$ and by the definition of $\lambda_1(\tilde{\theta})$, w is a minimizer of the functional $\bar{w} \mapsto F(u, \bar{w})$ among the functions in $S_1^2(M)$ with $u^{\frac{1}{n}}\bar{w} \not\equiv 0$ by Proposition 3.1. Since $F(u, w) = F(u, |w|)$, we have that $|w|$ is a minimizer for the functional associated to $\lambda_1(\tilde{\theta})$, and $|w|$ satisfies same equation as w. As a consequence, $|w|$ is C^2. By the maximum principle, we have $|w| > 0$ everywhere, which is false since $u^{\frac{1}{n}} w$ changes sign.

Step 2. The function w changes sign.

Assume w does not change sign. Then after possibly replacing w by $-w$, we can assume that $w \geq 0$. Setting $w_1 = v$ and $w_2 = w$, we have (4-20) and (4-21). Using (4-14), we can conclude that $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero. Applying Lemma 4.3, we have equality in (4-20). On the other hand, Step 1 implies that inequality (4-20) is strict since $\lambda_1(\tilde{\theta}) < \lambda_2(\tilde{\theta}) = Y_2(M,\theta)$. This contradiction shows that w changes sign.

Step 3. There exist $a, b > 0$ such that $u = a\sup(w,0) + b\sup(-w,0)$. Moreover, $w \in C^{2,\alpha}(M)$ and $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0,1)$.

As in the proof of Step 1, we apply Lemma 4.3 with $w_1 = \sup(w, 0)$ and $w_2 = \sup(-w, 0)$. We get $a, b > 0$ such that $u = aw_1 + bw_2$. As in Step 1, we get $w \in C^{2,\alpha}(M)$ and $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0, 1)$.

Step 4. Conclusion.

Let $h \in C^\infty(M)$ such that $\operatorname{supp}(h) \subseteq M \setminus u^{-1}(0)$. For t close to 0, set $u_t = |u + th|$. Since $u > 0$ on the support of h, and since u is continuous, we have for t close to 0, $u_t = u + th$. As $\operatorname{span}(v, w) \in \operatorname{Gr}_2^u(S_1^2(M))$, by Proposition 3.1 we have

$$Y_2(M, \theta) \le \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_t, \lambda v + \mu w).$$

Note that

$$\begin{aligned} F(u_t, \lambda v + \mu w) &= \frac{\int_M \left(2 + \frac{2}{n}\right)|\nabla_\theta(\lambda v + \mu w)|_\theta^2 + R_\theta(\lambda v + \mu w)^2 \, dV_\theta}{\int_M u_t^{\frac{2}{n}} (\lambda v + \mu w)^2 \, dV_\theta} \left(\int_M u_t^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \\ &= \frac{\lambda^2 \lambda_1(\tilde\theta) \int_M u^{\frac{2}{n}} v^2 \, dV_\theta + \mu^2 \lambda_2(\tilde\theta) \int_M u^{\frac{2}{n}} w^2 \, dV_\theta}{\lambda^2 a_t + \lambda\mu b_t + \mu^2 c_t} \left(\int_M u_t^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \\ &= \frac{\lambda^2 \lambda_1(\tilde\theta) + \mu^2 \lambda_2(\tilde\theta)}{\lambda^2 a_t + \lambda\mu b_t + \mu^2 c_t} \left(\int_M u_t^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}}, \end{aligned} \tag{4-28}$$

where we have used (4-12), (4-13), and (4-14). Here

$$a_t = \int_M u_t^{\frac{2}{n}} v^2 \, dV_\theta, \quad b_t = 2\int_M u_t^{\frac{2}{n}} vw \, dV_\theta \quad \text{and} \quad c_t = \int_M u_t^{\frac{2}{n}} w^2 \, dV_\theta.$$

Note also that the functions a_t, b_t, and c_t are smooth for t close to 0. Furthermore, $a_0 = c_0 = 1$ and $b_0 = 0$ by (4-14). Define $f(t, \alpha) = F(u_t, \sin(\alpha)v + \cos(\alpha)w)$, which is smooth for small t. By (4-28), we have

$$\begin{aligned} f(t, \alpha) &= F(u_t, \sin(\alpha)v + \cos(\alpha)w) \\ &= \frac{\sin^2(\alpha)\lambda_1(\tilde\theta) + \cos^2(\alpha)\lambda_2(\tilde\theta)}{\sin^2(\alpha)a_t + \sin(\alpha)\cos(\alpha)b_t + \cos^2(\alpha)c_t} \left(\int_M u_t^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}}. \end{aligned} \tag{4-29}$$

Hence, using $\lambda_1(\tilde\theta) < \lambda_2(\tilde\theta)$, we can see that $f\left(0, \left(n + \frac{1}{2}\right)\pi\right)$ is minimum and $f(0, n\pi)$ is maximum for any integer n. This implies that

$$\frac{\partial}{\partial\alpha} f(0, \alpha) = 0 \text{ if and only if } \alpha \in \frac{\pi}{2}\mathbb{Z},$$

$$\frac{\partial^2}{\partial\alpha^2} f(0, \alpha) < 0 \text{ if } \alpha \in \pi\mathbb{Z} \quad \text{and} \quad \frac{\partial^2}{\partial\alpha^2} f(0, \alpha) > 0 \text{ if } \alpha \in \pi\mathbb{Z} + \frac{\pi}{2}.$$

Applying the implicit function theorem to $\partial f/\partial\alpha$ at the point $(0, 0)$, we see that there exists a smooth function $t \mapsto \alpha(t)$, defined on a neighborhood of 0 with

$\alpha(0) = 0$ such that

$$f(t, \alpha(t)) = \sup_{\alpha \in \mathbb{R}} f(t, \alpha) = \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_t, \lambda v + \mu w),$$

where the last equality follows from the fact that

$$F(u_t, c\lambda v + c\mu w) = F(u_t, \lambda v + \mu w)$$

for any nonzero constant c by (4-28). Since $\alpha(0) = 0$, we have

$$\begin{aligned}\frac{d}{dt}\sin^2\alpha(t)\Big|_{t=0} = \frac{d}{dt}\cos^2\alpha(t)\Big|_{t=0} &= \frac{d}{dt}(a_t \sin^2\alpha(t))\Big|_{t=0} \\ &= \frac{d}{dt}(b_t \sin\alpha(t)\cos\alpha(t))\Big|_{t=0} = 0.\end{aligned}$$

Hence, by (4-29), we have

(4-30)

$$\begin{aligned}&\frac{d}{dt} f(t,\alpha(t))\Big|_{t=0} \\ &= \frac{d}{dt}\Bigg(\frac{\sin^2(\alpha(t))\lambda_1(\tilde{\theta})+\cos^2(\alpha(t))\lambda_2(\tilde{\theta})}{\sin^2(\alpha(t))a_t+\sin(\alpha(t))\cos(\alpha(t))b_t+\cos^2(\alpha(t))c_t} \\ &\qquad\qquad \times\Big(\int_M u_t^{2+\frac{2}{n}}\,dV_\theta\Big)^{\frac{1}{n+1}}\Bigg)\Bigg|_{t=0} \\ &= \lambda_2(\tilde{\theta})\Bigg(\Big(-\frac{d}{dt}c_t\Big|_{t=0}\Big)\Big(\int_M u^{2+\frac{2}{n}}\,dV_\theta\Big)^{\frac{1}{n+1}} + \frac{d}{dt}\Big(\int_M u_t^{2+\frac{2}{n}}\,dV_\theta\Big)^{\frac{1}{n+1}}\Big|_{t=0}\Bigg) \\ &= \lambda_2(\tilde{\theta})\frac{2}{n}\Big(-\int_M u^{-1+\frac{2}{n}} h w^2\,dV_\theta + \int_M u^{1+\frac{2}{n}} h\,dV_\theta\Big).\end{aligned}$$

By the definition of $Y_2(M,\theta)$ and $\lambda_2(\tilde{\theta}) = Y_2(M,\theta)$, f admits a minimum at $t = 0$ because

$$f(0, \alpha(0)) = f(0, 0) = F(u, w)$$

and w satisfies (4-13). Since $\lambda_2(\tilde{\theta}) = Y_2(M,\theta) \neq 0$, it follows from (4-30) that

$$\int_M u^{-1+\frac{2}{n}} h w^2\,dV_\theta = \int_M u^{1+\frac{2}{n}} h\,dV_\theta.$$

Since h is arbitrary (we just have to ensure that its support is contained in $M\setminus u^{-1}(0)$), we get

$$u^{-1+\frac{2}{n}} w^2 = u^{1+\frac{2}{n}}$$

and hence $u = |w|$ on $M \setminus u^{-1}(0)$. Together with Step 3, we have $u = |w|$ everywhere. □

5. Lower bound for $Y_2(M,\theta)$

For any compact CR manifold (M,θ) of the real dimension $2n+1$, by the definition of the CR Yamabe invariant $Y_1(M,\theta)$, we have

$$Y_1(M,\theta)=\inf_{u\in S_1^2(M)\setminus\{0\}}\frac{\int_M\big(2+\frac{2}{n}\big)|\nabla_\theta u|_\theta^2+R_\theta u^2\,dV_\theta}{\big(\int_M|u|^{2+\frac{2}{n}}\,dV_\theta\big)^{\frac{n}{n+1}}}. \tag{5-1}$$

Theorem 5.1. *We have*

$$Y_2(M,\theta)\geq 2^{\frac{1}{n+1}}Y_1(M,\theta). \tag{5-2}$$

Moreover, if M is connected and if $Y_2(M,\theta)$ is attained by a generalized contact form, then this inequality is strict.

Proof. The functional

$$F(u,v)=\frac{\int_M\big(2+\frac{2}{n}\big)|\nabla_\theta v|_\theta^2+R_\theta v^2\,dV_\theta}{\int_M u^{\frac{2}{n}}v^2\,dV_\theta}\bigg(\int_M u^{2+\frac{2}{n}}\,dV_\theta\bigg)^{\frac{1}{n+1}}$$

is continuous on $L_+^{2+\frac{2}{n}}(M)\times(S_1^2(M)\setminus\{0\})$. As a consequence, $I(u,V):=\sup_{v\in V\setminus\{0\}}F(u,v)$ depends continuously on $u\in L_+^{2+\frac{2}{n}}(M)$ and $V\in\mathrm{Gr}_2^u(S_1^2(M))$. To prove Theorem 5.1, it suffices to show that $I(u,V)\geq 2^{\frac{1}{n+1}}Y_1(M,\theta)$ for all smooth $u>0$ and $V\in\mathrm{Gr}_2^u(S_1^2(M))$ thanks to Proposition 3.1. Without loss of generality, we can assume that

$$\int_M u^{2+\frac{2}{n}}\,dV_\theta=1. \tag{5-3}$$

The operator

$$v\mapsto P(v):=-\Big(2+\frac{2}{n}\Big)u^{-\frac{1}{n}}\Delta_\theta(u^{-\frac{1}{n}}v)+R_\theta u^{-\frac{2}{n}}v$$

is self-adjoint with respect to the L^2-scalar product and elliptic. Hence, P has discrete spectrum $\lambda_1\leq\lambda_2\leq\cdots$ and the corresponding eigenfunctions $\varphi_1,\varphi_2,\dots$ are smooth. Setting $v_i=u^{-\frac{1}{n}}\varphi_i$, we obtain

$$\begin{aligned}\Big(-\Big(2+\frac{2}{n}\Big)\Delta_\theta+R_\theta\Big)(v_i)&=-\Big(2+\frac{2}{n}\Big)\Delta_\theta(u^{-\frac{1}{n}}\varphi_i)+R_\theta u^{-\frac{1}{n}}\varphi_i\\&=u^{\frac{1}{n}}P(\varphi_i)=\lambda_i u^{\frac{1}{n}}\varphi_i=\lambda_i u^{\frac{2}{n}}v_i\end{aligned} \tag{5-4}$$

and

$$\int_M u^{\frac{2}{n}}v_iv_j\,dV_\theta=\int_M\varphi_i\varphi_j\,dV_\theta=0 \text{ if } i\neq j.$$

The maximum principle implies that an eigenfunction to the smallest eigenvalue λ_1 has no zeros. Hence, $\lambda_1<\lambda_2$ and we can assume that $v_1>0$.

We define $w_+ = a_+ \sup(v_2, 0)$ and $w_- = a_- \sup(-v_2, 0)$, where $a_+, a_- > 0$ are chosen such that

$$\int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta = \int_M u^{\frac{2}{n}} w_-^2 \, dV_\theta = 1. \tag{5-5}$$

We let $\Omega_- = \{v_2 < 0\}$ and $\Omega_+ = \{v_2 \geq 0\}$. By Hölder's inequality, we have

$$\begin{aligned}2 &= \int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta + \int_M u^{\frac{2}{n}} w_-^2 \, dV_\theta \\ &\leq \left(\int_{\Omega_+} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M w_+^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{n}{n+1}} \\ &\qquad + \left(\int_{\Omega_-} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M w_-^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{n}{n+1}}.\end{aligned} \tag{5-6}$$

Using the inequality (5-1), we get

$$\int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta \geq Y_1(M,\theta) \left(\int_M w_+^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{n}{n+1}},$$

which implies that

$$\begin{aligned}&\left(\int_{\Omega_+} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta\right) \\ &\qquad \geq Y_1(M,\theta) \left(\int_M w_+^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{n}{n+1}} \left(\int_{\Omega_+} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \\ &\qquad \geq Y_1(M,\theta) \int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta = Y_1(M,\theta),\end{aligned} \tag{5-7}$$

where we have used Hölder's inequality in the last inequality, and (5-5) in the last equality. Similarly, we have

$$\left(\int_{\Omega_-} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M u^{\frac{1}{n}} w_- P(u^{\frac{1}{n}} w_-) \, dV_\theta\right) \geq Y_1(M,\theta). \tag{5-8}$$

Adding (5-7) and (5-8) together, we obtain

$$\begin{aligned}2Y_1(M,\theta) \leq &\left(\int_{\Omega_+} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta\right) \\ &+ \left(\int_{\Omega_-} u^{2+\frac{2}{n}} \, dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M u^{\frac{1}{n}} w_- P(u^{\frac{1}{n}} w_-) \, dV_\theta\right).\end{aligned} \tag{5-9}$$

Since w_-, respectively w_+, are multiples of v_2 on Ω_-, respectively Ω_+, they satisfy the same equation as v_2. Hence, we obtain from (5-4) and (5-9) that

$$
\begin{aligned}
(5\text{-}10)\quad 2Y_1(M,\theta) &\le \left(\int_{\Omega_+} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M \lambda_2 u^{\frac{2}{n}} w_+^2\,dV_\theta\right) \\
&\quad + \left(\int_{\Omega_-} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}} \left(\int_M \lambda_2 u^{\frac{2}{n}} w_-^2\,dV_\theta\right) \\
&= \lambda_2\left(\left(\int_{\Omega_+} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}} + \left(\int_{\Omega_-} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}}\right),
\end{aligned}
$$

where the last equality follows from (5-5). Now, for any nonnegative numbers $a, b \ge 0$, Hölder's inequality yields

$$a + b \le 2^{\frac{n}{n+1}}(a^{n+1} + b^{n+1})^{\frac{1}{n+1}}.$$

Applying this inequality with

$$a = \left(\int_{\Omega_+} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}} \quad \text{and} \quad b = \left(\int_{\Omega_-} u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}},$$

we derive from (5-10) that

$$
\begin{aligned}
2Y_1(M,\theta) &\le \lambda_2 2^{\frac{n}{n+1}}\left(\left(\int_{\Omega_+} u^{2+\frac{2}{n}}\,dV_\theta\right) + \left(\int_{\Omega_-} u^{2+\frac{2}{n}}\,dV_\theta\right)\right)^{\frac{1}{n+1}} \\
&= \lambda_2 2^{\frac{n}{n+1}}\left(\int_M u^{2+\frac{2}{n}}\,dV_\theta\right)^{\frac{1}{n+1}} = \lambda_2 2^{\frac{n}{n+1}},
\end{aligned}
$$

where the last equality follows from (5-3). This implies that $\lambda_2 \ge 2^{\frac{1}{n+1}} Y_1(M,\theta)$. Since $\lambda_2 = I(u, \mathrm{span}(v_1, v_2))$, this finishes the proof of the first part of Theorem 5.1.

Moreover, if M were connected and if $Y_2(M,\theta)$ were attained by a generalized contact form, then inequality (5-9) would be an equality and we would have that w_+ or w_- is a function for which equality in (5-1) is attained. By the maximum principle, we would get that w_+ or w_- is positive on M, which is impossible. □

6. Upper bound for $Y_2(M, \theta)$

Hereafter, we denote $Y_k(\mathbb{S}^{2n+1})$ the k-th Yamabe invariant of $(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})$, where $\theta_{\mathbb{S}^{2n+1}}$ is the standard contact form on $\mathbb{S}^{2n+1}$ given by

$$\theta_{\mathbb{S}^{2n+1}} = \sqrt{-1}\sum_{j=1}^{n+1}(z_j\,d\bar{z}_j - \bar{z}_j\,dz_j),$$

where $(z_1, \ldots, z_{n+1}) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$.

Theorem 6.1. *Suppose (M,θ) is a compact CR manifold of real dimension $2n+1$ with $Y_1(M,\theta)\geq 0$. Then*

$$Y_2(M,\theta)\leq \big(Y_1(M,\theta)^{n+1}+Y_1(\mathbb{S}^{2n+1})^{n+1}\big)^{\frac{1}{n+1}} \tag{6-1}$$

when $Y_1(M,\theta)>0$ and $n\geq 3$, or $Y_1(M,\theta)=0$ and $n\geq 4$. On the other hand, the inequality in (6-1) *is strict when*

(i) *$Y_1(M,\theta)>0$, $n\geq 7$ and M is not locally CR equivalent to $\mathbb{S}^{2n+1}$, or*

(ii) *$Y_1(M,\theta)=0$, $n\geq 4$ and M is not locally CR equivalent to $\mathbb{S}^{2n+1}$.*

To prove Theorem 5.4, we have the following:

Lemma 6.2. *For any $\alpha>2$, there exists a constant $C>0$ such that*

$$|a+b|^\alpha\leq a^\alpha+b^\alpha+C(a^{\alpha-1}b+ab^{\alpha-1})$$

for all $a,b>0$.

Proof. Dividing both sides by a, without loss of generality, we can assume that $a=1$. Then we set for $x>0$,

$$f(x)=\frac{|1+x|^\alpha-(1+x^\alpha)}{x^{\alpha-1}+x}.$$

By L'Hôpital's rule, we have

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\frac{\alpha(1+x)^{\alpha-1}-\alpha x^{\alpha-1}}{(\alpha-1)x^{\alpha-2}+1}=\alpha,$$
$$\lim_{x\to \infty}f(x)=\lim_{x\to \infty}\frac{\alpha(1+x)^{\alpha-1}-\alpha x^{\alpha-1}}{(\alpha-1)x^{\alpha-2}+1}=\alpha.$$

Since f is continuous, f is bounded by a constant C on $(0,\infty)$. Clearly, this constant is the desired C is the inequality of Lemma 6.2. □

Proof of Theorem 6.1. For $u\in S_1^2(M)\setminus\{0\}$, let

$$E(u)=\frac{\int_M\big(2+\frac{2}{n}\big)|\nabla_\theta u|_\theta^2+R_\theta u^2\,dV_\theta}{\big(\int_M|u|^{2+\frac{2}{n}}\,dV_\theta\big)^{\frac{n}{n+1}}}.$$

The solution of the CR Yamabe problem provides the existence of a smooth positive minimizer v of E, and we can assume

$$\int_M v^{2+\frac{2}{n}}\,dV_\theta=1. \tag{6-2}$$

Then v satisfies the CR Yamabe equation

$$L_\theta(v)=Y_1(M,\theta)v^{1+\frac{2}{n}}. \tag{6-3}$$

Let $x_0 \in M$ be fixed and choose pseudohermitian normal coordinates (z, t) near x_0. Let $\delta > 0$ be a fixed number. If θ is well chosen in the conformal class and if x_0 is well chosen in M, it was proved by Jerison and Lee [1989, Theorem 4.1] that when $n \geq 3$, there exists a function $v_\varepsilon \geq 0$ with $\operatorname{supp}(v_\varepsilon) \subseteq B(x_0, 2\delta)$ such that

$$E(v_\varepsilon) = Y_1(\mathbb{S}^{2n+1}) - c(M)\varepsilon^4 + O(\varepsilon^5), \tag{6-4}$$

where $c(M) \geq 0$ is a positive constant. In fact, $c(M)$ is the square of the norm of the Chern tensor at x_0 up to a dimensional constant. Therefore, we can assume that the constant $c(M)$ in (6-4) satisfies

$$c(M) > 0 \tag{6-5}$$

if (M, θ) is not locally CR equivalent to $\mathbb{S}^{2n+1}$. It follows from (6-4) that

$$\lim_{\varepsilon \to 0} E(v_\varepsilon) = Y_1(\mathbb{S}^{2n+1}). \tag{6-6}$$

More precisely, v_ε is given by (see [Jerison and Lee 1989, p. 326])

$$v_\varepsilon = C_\varepsilon \eta \left(\frac{\varepsilon^2}{t^2 + (|z|^2 + \varepsilon^2)^2} \right)^{\frac{n}{2}},$$

where η is a smooth cut-off function such that

$$0 \leq \eta \leq 1, \quad \eta(x) = \begin{cases} 1 & \text{if } x \in B(x_0, \delta), \\ 0 & \text{if } x \notin B(x_0, 2\delta), \end{cases}$$

and $C_\varepsilon > 0$ is a constant chosen such that

$$\int_M v_\varepsilon^{2+\frac{2}{n}} \, dV_\theta = 1. \tag{6-7}$$

It follows from [Jerison and Lee 1989, Proposition 4.2] that

$$C_\varepsilon = c(n) + O(\varepsilon^4) \tag{6-8}$$

for some positive constant $c(n)$ depending only on n. In the following, C will denote a positive constant depending possibly on δ, n, but not on ε. Let

$$\delta_\varepsilon(z, t) = (\varepsilon z, \varepsilon^2 t).$$

Note that

$$\delta_\varepsilon^* \left(\frac{1}{t^2 + (\varepsilon^2 + |z|^2)^2} \right) = \varepsilon^{-4} \left(\frac{1}{t^2 + (1 + |z|^2)^2} \right)$$

and $\delta_\varepsilon^* \, dz\, dt = \varepsilon^{2n+2}\, dz\, dt$. Hence,

$$
\begin{aligned}
\text{(6-9)}\quad \int_M |v_\varepsilon|^p \, dV_\theta &\le C_\varepsilon^p \int_{\left\{\sqrt[4]{t^2+|z|^4}\le 2\delta\right\}} \frac{\varepsilon^{np}\, dz\, dt}{(t^2+(\varepsilon^2+|z|^2)^2)^{\frac{np}{2}}} \\
&= C_\varepsilon^p \int_{\left\{\sqrt[4]{t^2+|z|^4}\le 2\delta/\varepsilon\right\}} \frac{\varepsilon^{2n+2-np}\, dz\, dt}{(t^2+(1+|z|^2)^2)^{\frac{np}{2}}} \\
&\le C_\varepsilon^p \varepsilon^{2n+2-np} \int_{\{|z|\le 2\delta/\varepsilon\}} \left(\int_{-\infty}^{\infty} \frac{dt}{1+t^2}\right) \frac{dz}{(1+|z|^2)^{np-2}} \\
&= C_\varepsilon^p \pi \varepsilon^{2n+2-np} \int_{\{|z|\le 2\delta/\varepsilon\}} \frac{dz}{(1+|z|^2)^{np-2}} \\
&= C\varepsilon^{2n+2-np} \int_0^{2\delta/\varepsilon} \frac{r^{2n-1}\, dr}{(1+r^2)^{np-2}},
\end{aligned}
$$

where we have used (6-8). Note that for $\varepsilon \ll 1$,

$$
\int_0^{2\delta/\varepsilon} \frac{r^{2n-1}\, dr}{(1+r^2)^{np-2}} \le \int_0^{2\delta/\varepsilon} r^{2n+3-2np}\, dr \le \frac{C}{\varepsilon^{2n+4-2np}}
$$

if $p \le 1 + \frac{3}{2n}$, and

$$
\begin{aligned}
\int_0^{2\delta/\varepsilon} \frac{r^{2n-1}\, dr}{(1+r^2)^{np-2}} &\le \int_0^1 r^{2n-1}\, dr + \int_1^{2\delta/\varepsilon} \frac{dr}{r^{2np-2n-3}} \\
&= \int_0^1 r^{2n-1}\, dr + \int_1^{2\delta/\varepsilon} \frac{dr}{r} = \frac{1}{2n} + \log \varepsilon
\end{aligned}
$$

if $p = 1 + \frac{2}{n}$. Combining these with (6-9), we obtain

$$
\text{(6-10)}\qquad \int_M |v_\varepsilon|^p \, dV_\theta \le \begin{cases} C\varepsilon^{np-2} & \text{if } p \le 1+\frac{3}{2n}, \\ C\varepsilon^n \log \varepsilon & \text{if } p = 1 + \frac{2}{n}. \end{cases}
$$

Similarly, for $\varepsilon \ll 1$, we have

$$
\begin{aligned}
\text{(6-11)}\quad \int_M |v_\varepsilon|^p \, dV_\theta &\ge C_\varepsilon^p \int_{\left\{\sqrt[4]{t^2+|z|^4}\le \delta\right\}} \frac{\varepsilon^{np}\, dz\, dt}{(t^2+(\varepsilon^2+|z|^2)^2)^{\frac{np}{2}}} \\
&= C_\varepsilon^p \int_{\left\{\sqrt[4]{t^2+|z|^4}\le \delta/\varepsilon\right\}} \frac{\varepsilon^{2n+2-np}\, dz\, dt}{(t^2+(1+|z|^2)^2)^{\frac{np}{2}}} \\
&\ge C_\varepsilon^p \varepsilon^{2n+2-np} \int_{\{|z|\le \delta/2\varepsilon\}} \left(\int_{-\delta/2\varepsilon}^{\delta/2\varepsilon} \frac{dt}{1+t^2}\right) \frac{dz}{(1+|z|^2)^{np}} \\
&\ge 2C_\varepsilon^p \tan^{-1}(\delta/2)\varepsilon^{2n+2-np} \int_{\{|z|\le \delta/2\varepsilon\}} \frac{dz}{(1+|z|^2)^{np}} \\
&= C\varepsilon^{2n+2-np} \int_0^{\delta/2\varepsilon} \frac{r^{2n-1}\, dr}{(1+r^2)^{np}},
\end{aligned}
$$

where we have used

$$t^2+(1+|z|^2)^2 \le (1+t^2)(1+|z|^2)^2$$

and

$$\{|z|\le \delta/2\varepsilon\}\cap\{|t|\le \delta/2\varepsilon\}\subset\left\{\sqrt[4]{t^2+|z|^4}\le \delta/\varepsilon\right\}$$

in the second inequality, and (6-8) in the last equality. Note that for $\varepsilon \ll 1$,

$$\int_0^{\delta/2\varepsilon}\frac{r^{2n-1}\,dr}{(1+r^2)^{np}}\ge\int_0^1\frac{r^{2n-1}\,dr}{2^{np}}+\int_1^{\delta/2\varepsilon}\frac{r^{2n-1}\,dr}{(2r^2)^{np}}=C+\frac{C}{\varepsilon^{2n-2np}}$$

if $\le 1-\frac{1}{2n}$, and

$$\begin{aligned}\int_0^{\delta/2\varepsilon}\frac{r^{2n-1}\,dr}{(1+r^2)^{np}}&\ge\int_0^1\frac{r^{2n-1}\,dr}{2^{np}}+\int_1^{\delta/2\varepsilon}\frac{r^{2n-1}\,dr}{(2r^2)^{np}}\\&\ge\frac{1}{2^{np}}\left(\int_0^1 r^{2n-1}\,dr+\int_1^{\delta/2\varepsilon}\frac{dr}{r^{2np-2n+1}}\right)=C+C\varepsilon^{2np-2n}\end{aligned}$$

if $p>1$. Combining these with (6-11), we obtain

$$\int_M |v_\varepsilon|^p\,dV_\theta\ge\begin{cases}C\varepsilon^{np+2} & \text{if } p\le 1-\frac{1}{2n},\\ C\varepsilon^{2n+2-np} & \text{if } p>1.\end{cases}\tag{6-12}$$

First we assume that $Y_1(M,\theta)>0$. We set

$$u_\varepsilon=E(v_\varepsilon)^{\frac{n}{2}}v_\varepsilon+Y_1(M,\theta)^{\frac{n}{2}}v.$$

Let us find estimates for $F(u_\varepsilon,\lambda v_\varepsilon+\mu v)$. Let $(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}$. Then

$$\begin{aligned}&F(u_\varepsilon,\lambda v_\varepsilon+\mu v)\\&=\frac{\lambda^2\int_M v_\varepsilon L_\theta v_\varepsilon\,dV_\theta+\mu^2\int_M vL_\theta v\;dV_\theta+2\lambda\mu\int_M v_\varepsilon L_\theta v\,dV_\theta}{\int_M |u_\varepsilon|^{\frac{2}{n}}(\lambda v_\varepsilon+\mu v)^2\,dV_\theta}\cdot U\\&=\frac{\lambda^2E(v_\varepsilon)+\mu^2Y_1(M,\theta)+2\lambda\mu Y_1(M,\theta)\int_M v^{1+\frac{2}{n}}v_\varepsilon\,dV_\theta}{\lambda^2\int_M|u_\varepsilon|^{\frac{2}{n}}v_\varepsilon^2\,dV_\theta+\mu^2\int_M|u_\varepsilon|^{\frac{2}{n}}v^2\,dV_\theta+2\lambda\mu\int_M|u_\varepsilon|^{\frac{2}{n}}v_\varepsilon v\,dV_\theta}\cdot U,\end{aligned}\tag{6-13}$$

where $U=\left(\int_M u_\varepsilon^{2+2/n}\,dV_\theta\right)^{1/(n+1)}$ and where we have used (6-2), (6-3) and (6-7). Using the definition of u_ε, we have

$$u_\varepsilon\ge E(v_\varepsilon)^{\frac{n}{2}}v_\varepsilon\quad\text{and}\quad u_\varepsilon\ge Y_1(M,\theta)^{\frac{n}{2}}v,\tag{6-14}$$

which implies that

$$(6\text{-}15)\quad \begin{aligned} \lambda^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2 \, dV_\theta + \mu^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v^2 \, dV_\theta + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta \\ \geq \lambda^2 E(v_\varepsilon) \int_M v_\varepsilon^{2+\frac{2}{n}} \, dV_\theta + \mu^2 Y_1(M,\theta) \int_M v^{2+\frac{2}{n}} \, dV_\theta \\ + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta \\ = \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta, \end{aligned}$$

where the last equality follows from (6-2) and (6-7).

If $\lambda\mu \geq 0$, then we have

$$(6\text{-}16)\qquad 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta \geq 2\lambda\mu Y_1(M,\theta) \int_M v^{1+\frac{2}{n}} v_\varepsilon \, dV_\theta$$

by (6-14). Therefore, (6-15) and (6-16) imply that

$$\frac{\lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) + 2\lambda\mu Y_1(M,\theta) \int_M v^{1+\frac{2}{n}} v_\varepsilon \, dV_\theta}{\lambda^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2 \, dV_\theta + \mu^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v^2 \, dV_\theta + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta} \leq 1.$$

If $\lambda\mu < 0$, then

$$|u_\varepsilon|^{\frac{2}{n}} \leq \big(E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon + Y_1(M,\theta)^{\frac{n}{2}} v\big)^{\frac{2}{n}} \leq E(v_\varepsilon) v_\varepsilon^{\frac{2}{n}} + Y_1(M,\theta) v^{\frac{2}{n}}$$

when $n \geq 2$. Combining this with (6-14) and (6-15), we get

$$\begin{aligned} \lambda^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2 \, dV_\theta + \mu^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v^2 \, dV_\theta + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta \\ \geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) - C\left(\int_M v_\varepsilon^{1+\frac{2}{n}} v \, dV_\theta + \int_M v^{1+\frac{2}{n}} v_\varepsilon \, dV_\theta\right) \\ \geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) - C\left(\int_M v_\varepsilon^{1+\frac{2}{n}} \, dV_\theta + \int_M v_\varepsilon \, dV_\theta\right), \end{aligned}$$

where $C > 0$ is a positive real number independent of ε. This, together with (6-10), gives

$$\begin{aligned} \lambda^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2 \, dV_\theta + \mu^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v^2 \, dV_\theta + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta \\ \geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) - O(\varepsilon^n \log \varepsilon) - O(\varepsilon^{n-2}). \end{aligned}$$

This, together with the assumption that $\lambda\mu < 0$, implies that

$$\frac{\lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M,\theta) + 2\lambda\mu Y_1(M,\theta) \int_M v^{1+\frac{2}{n}} v_\varepsilon \, dV_\theta}{\lambda^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2 \, dV_\theta + \mu^2 \int_M |u_\varepsilon|^{\frac{2}{n}} v^2 \, dV_\theta + 2\lambda\mu \int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v \, dV_\theta} \leq 1 + O(\varepsilon^{n-2}).$$

In any case, we have

(6-17)
$$\sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} \frac{\lambda^2 E(v_\varepsilon)+\mu^2 Y_1(M,\theta)+2\lambda\mu Y_1(M,\theta)\int_M v^{1+\frac{2}{n}} v_\varepsilon\, dV_\theta}{\lambda^2\int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2\, dV_\theta+\mu^2\int_M |u_\varepsilon|^{\frac{2}{n}} v^2\, dV_\theta+2\lambda\mu\int_M |u_\varepsilon|^{\frac{2}{n}} v_\varepsilon v\, dV_\theta} \leq 1+O(\varepsilon^{n-2}).$$

On the other hand,

$$\begin{aligned}\int_M u_\varepsilon^{2+\frac{2}{n}}\, dV_\theta &= \int_M (E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon + Y_1(M,\theta)^{\frac{n}{2}} v)^{2+\frac{2}{n}}\, dV_\theta\\ &\leq E(v_\varepsilon)^{n+1}\int_M v_\varepsilon^{2+\frac{2}{n}}\, dV_\theta + Y_1(M,\theta)^{n+1}\int_M v^{2+\frac{2}{n}}\, dV_\theta\\ &\qquad + C\left(\int_M v_\varepsilon^{1+\frac{2}{n}} v\, dV_\theta + \int_M v^{1+\frac{2}{n}} v_\varepsilon\, dV_\theta\right)\\ &= E(v_\varepsilon)^{n+1}+Y_1(M,\theta)^{n+1}+C\left(\int_M v_\varepsilon^{1+\frac{2}{n}} v\, dV_\theta+\int_M v^{1+\frac{2}{n}} v_\varepsilon\, dV_\theta\right),\end{aligned}$$

where the first inequality follows from Lemma 6.2 with

$$a = E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon \quad\text{and}\quad b = Y_1(M,\theta)^{\frac{n}{2}} v,$$

and the last equality follows from (6-2) and (6-7). This, together with (6-4) and (6-10), implies that

(6-18)
$$\left(\int_M u_\varepsilon^{2+\frac{2}{n}}\, dV_\theta\right)^{\frac{1}{n+1}} \leq (Y_1(\mathbb{S}^{2n+1})^{n+1} + Y_1(M,\theta)^{n+1})^{\frac{1}{n+1}} - c(M)\varepsilon^4 + o(\varepsilon^4) + O(\varepsilon^{n-2}).$$

If $\varepsilon > 0$ is small enough, it follows from (6-13), (6-17), and (6-18) that

(6-19)
$$\begin{aligned}Y_2(M,\theta) &\leq \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_\varepsilon, \lambda v_\varepsilon+\mu v)\\ &\leq \left(Y_1(\mathbb{S}^{2n+1})^{n+1}+Y_1(M,\theta)^{n+1}\right)^{\frac{1}{n+1}} - c(M)\varepsilon^4+o(\varepsilon^4)+O(\varepsilon^{n-2}).\end{aligned}$$

Since $n \geq 3$, (6-1) follows from (6-19) by letting ε go to zero. On the other hand, if (M,θ) is not locally CR equivalent to $\mathbb{S}^{2n+1}$, then (6-5) holds. Hence, if $n \geq 7$, the strict inequality in (6-1) follows from (6-19) by letting ε go to zero.

Now we assume that $Y_1(M,\theta)=0$. We set $u_\varepsilon = v_\varepsilon$. Then we obtain for $(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}$,

(6-20) $F(u_\varepsilon,\lambda v_\varepsilon+\mu v)$

$$= \frac{\lambda^2 E(v_\varepsilon)\big(\int_M v_\varepsilon^{2+\frac{2}{n}}\,dV_\theta\big)^{\frac{1}{n+1}}}{\lambda^2\int_M |v_\varepsilon|^{\frac{2}{n}} v_\varepsilon^2\,dV_\theta+\mu^2\int_M v_\varepsilon^{\frac{2}{n}} v^2\,dV_\theta+2\lambda\mu\int_M |v_\varepsilon|^{\frac{2}{n}} v_\varepsilon v\,dV_\theta}$$
$$= \frac{\lambda^2 E(v_\varepsilon)}{\lambda^2+\mu^2\int_M v_\varepsilon^{\frac{2}{n}} v^2\,dV_\theta+2\lambda\mu\int_M v_\varepsilon^{1+\frac{2}{n}} v\,dV_\theta}$$

by (6-7) and (6-13). Let λ_ε, μ_ε such that $\lambda_\varepsilon^2+\mu_\varepsilon^2=1$ and

$$F(u_\varepsilon,\lambda_\varepsilon v_\varepsilon+\mu_\varepsilon v)=\sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_\varepsilon,\lambda v_\varepsilon+\mu v).$$

If $\lambda_\varepsilon=0$, we obtain that $F(u_\varepsilon,\lambda_\varepsilon v_\varepsilon+\mu_\varepsilon v)=0$ and the theorem would be proved. Then we assume that $\lambda_\varepsilon\neq 0$ and we can write

$$F(u_\varepsilon,\lambda_\varepsilon v_\varepsilon+\mu_\varepsilon v)=\frac{E(v_\varepsilon)}{1+2x_\varepsilon b_\varepsilon+x_\varepsilon^2 a_\varepsilon},$$

where $x_\varepsilon=\mu_\varepsilon/\lambda_\varepsilon$ and

$$C\varepsilon^n\le b_\varepsilon=\int_M v_\varepsilon^{1+\frac{2}{n}} v\,dV_\theta\le C\varepsilon^{n-1}\log\varepsilon \quad \text{as } \varepsilon\to 0,$$
$$a_\varepsilon=\int_M v_\varepsilon^{\frac{2}{n}} v^2\,dV_\theta\ge C\varepsilon^4 \quad \text{as } \varepsilon\to 0$$

by (6-10) and (6-12). Maximizing this expression in x_ε and using (6-4), we obtain (6-21)

$$F(u_\varepsilon,\lambda_\varepsilon v_\varepsilon+\mu_\varepsilon v)\le\frac{Y_1(\mathbb{S}^{2n+1})-c(M)\varepsilon^4+o(\varepsilon^4)}{1-b_\varepsilon^2/a_\varepsilon}=\frac{Y_1(\mathbb{S}^{2n+1})-c(M)\varepsilon^4+o(\varepsilon^4)}{1-C\varepsilon^{2n-6}\log^2\varepsilon},$$

since $\varepsilon\log\varepsilon\to 0$ as $\varepsilon\to 0$. For $n\ge 4$, it follows from (6-21) that

$$F(u_\varepsilon,\lambda_\varepsilon v_\varepsilon+\mu_\varepsilon v)\le Y_1(\mathbb{S}^{2n+1}),$$

which proves (6-1) for the case $Y_1(M,\theta)=0$. On the other hand, if (M,θ) is not locally CR equivalent to $\mathbb{S}^{2n+1}$, then (6-5) holds. Hence, the strictly inequality in (6-1) follows from (6-21) by letting ε go to zero. This proves Theorem 6.1. □

7. Some properties of $Y_2(M,\theta)$

We have the following questions:

(1) Is $Y_2(M,\theta)$ attained by a contact form?

(2) Is $Y_2(M,\theta)$ attained by a generalized contact form?

For question 1, we have the following:

Proposition 7.1. *Let $\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1}$ be the disjoint union of two copies of the sphere equipped with the standard contact form induced from $\theta_{\mathbb{S}^{2n+1}}$. Then $Y_2(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1})=2^{\frac{1}{n+1}}Y_1(\mathbb{S}^{2n+1})$ and it is attained by the standard contact form.*

Proof. Let $\tilde{\theta}$ be an arbitrary smooth contact form on $\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1}$. We write $\mathbb{S}_1^{2n+1}$ for the first $\mathbb{S}^{2n+1}$ and $\mathbb{S}_2^{2n+1}$ for the second $\mathbb{S}^{2n+1}$. Then we have

$$\lambda_2(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1},\tilde{\theta}) = \min\Big\{\lambda_2(\mathbb{S}_1^{2n+1},\tilde{\theta}),\lambda_2(\mathbb{S}_2^{2n+1},\tilde{\theta}),\max\{\lambda_1(\mathbb{S}_1^{2n+1},\tilde{\theta}),\lambda_1(\mathbb{S}_2^{2n+1},\tilde{\theta})\}\Big\}. \tag{7-1}$$

Therefore,

$$\begin{aligned} Y_2(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1}) &\le \lambda_2(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1})\,\mathrm{Vol}(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1})^{\frac{1}{n+1}} \\ &= \lambda_2(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1})(2\,\mathrm{Vol}(\mathbb{S}^{2n+1}))^{\frac{1}{n+1}} \\ &= 2^{\frac{1}{n+1}}\lambda_1(\mathbb{S}^{2n+1})\,\mathrm{Vol}(\mathbb{S}^{2n+1})^{\frac{1}{n+1}} \\ &= 2^{\frac{1}{n+1}}Y_1(\mathbb{S}^{2n+1}), \end{aligned} \tag{7-2}$$

where we have used (7-1) in the second equality.

On the other hand, we have

$$\begin{aligned} \lambda_2(\mathbb{S}_1^{2n+1},\tilde{\theta})\,\mathrm{Vol}(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1},\tilde{\theta})^{\frac{1}{n+1}} &\ge \lambda_2(\mathbb{S}_1^{2n+1},\tilde{\theta})\,\mathrm{Vol}(\mathbb{S}_1^{2n+1},\tilde{\theta})^{\frac{1}{n+1}} \\ &\ge Y_2(\mathbb{S}_1^{2n+1}) \\ &= 2^{\frac{1}{n+1}}Y_1(\mathbb{S}^{2n+1}), \end{aligned} \tag{7-3}$$

where the last equality follows from Corollary 7.3. Similarly, we have

$$\lambda_2(\mathbb{S}_2^{2n+1},\tilde{\theta})\,\mathrm{Vol}(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1},\tilde{\theta})^{\frac{1}{n+1}} \ge 2^{\frac{1}{n+1}}Y_1(\mathbb{S}^{2n+1}). \tag{7-4}$$

By the definition of $Y_1(\mathbb{S}^{2n+1})$, we have

$$\lambda_1(\mathbb{S}_i^{2n+1},\tilde{\theta})\,\mathrm{Vol}(\mathbb{S}^{2n+1},\tilde{\theta})^{\frac{1}{n+1}} \ge Y_1(\mathbb{S}^{2n+1}) \quad \text{for } i=1,2,$$

which implies

$$\begin{aligned} 2Y_1(\mathbb{S}^{2n+1})^{n+1} &\le \sum_{i=1}^{2}\lambda_1(\mathbb{S}_i^{2n+1},\tilde{\theta})^{n+1}\,\mathrm{Vol}(\mathbb{S}_i^{2n+1},\tilde{\theta}) \\ &\le \max\{\lambda_1(\mathbb{S}_1^{2n+1},\tilde{\theta})^{n+1},\lambda_1(\mathbb{S}_2^{2n+1},\tilde{\theta})^{n+1}\}\sum_{i=1}^{2}\mathrm{Vol}(\mathbb{S}_i^{2n+1},\tilde{\theta}) \\ &= \max\{\lambda_1(\mathbb{S}_1^{2n+1},\tilde{\theta})^{n+1},\lambda_1(\mathbb{S}_2^{2n+1},\tilde{\theta})^{n+1}\}\,\mathrm{Vol}(\mathbb{S}^{2n+1}\cup\mathbb{S}^{2n+1},\tilde{\theta}), \end{aligned}$$

which gives

$$2^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}) \le \max\{\lambda_1(\mathbb{S}_1^{2n+1}, \tilde{\theta}), \lambda_1(\mathbb{S}_2^{2n+1}, \tilde{\theta})\} \operatorname{Vol}(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}}. \tag{7-5}$$

Combining (7-3), (7-4), and (7-5), we can derive from (7-1) that

$$2^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}) \le \lambda_2(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta}) \operatorname{Vol}(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}}.$$

Since $\tilde{\theta}$ is an arbitrary smooth contact form on $\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}$, we have

$$2^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}) \le Y_2(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}). \tag{7-6}$$

Now Proposition 7.1 follows from combining (7-2) and (7-6). □

On the other hand, we have the following:

Proposition 7.2. *If M is connected, then $Y_2(M, \theta)$ cannot be attained by a contact form.*

Proof. Otherwise, if $Y_2(M, \theta)$ were attained by a contact form $\tilde{\theta} = u^{\frac{2}{n}} \theta$, then by Theorem 4.4, we would have $u = |w|$, and hence u cannot be positive since w has alternating sign. □

For question 2, we have the following:

Corollary 7.3. *We have*

$$Y_2(\mathbb{S}^{2n+1}) = 2^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}).$$

Proof. This follows from (6-1) and Theorem 5.1. □

Corollary 7.4. *$Y_2(\mathbb{S}^{2n+1})$ is not attained by a generalized contact form.*

Proof. This follows from Theorem 5.1 and Corollary 7.3. □

8. The k-th CR Yamabe invariant $Y_k(M, \theta)$

In view of Corollary 7.3, it is natural to conjecture that

$$Y_k(\mathbb{S}^{2n+1}) = k^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1})$$

for all k. However, the following result shows that it is false.

Proposition 8.1. *For $n \ge 3$, we have*

$$Y_{2n+3}(\mathbb{S}^{2n+1}) < (2n+3)^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}).$$

Proof. Consider $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$. Let z_i, where $i = 1, 2, \dots, n+1$, be the coordinates of $\mathbb{C}^{n+1}$. Since $-\Delta_{\theta_{\mathbb{S}^{2n+1}}} z_i = \frac{n}{2} z_i$ and $-\Delta_{\theta_{\mathbb{S}^{2n+1}}} \bar{z}_i = \frac{n}{2} \bar{z}_i$,

$$L_{\theta_{\mathbb{S}^{2n+1}}}(z_i) = \frac{(n+2)(n+1)}{2} z_i \quad \text{and} \quad L_{\theta_{\mathbb{S}^{2n+1}}}(\bar{z}_i) = \frac{(n+2)(n+1)}{2} \bar{z}_i$$

for $i = 1, 2, \dots, n+1$, and hence

$$\lambda_{2n+3}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \leq \frac{(n+2)(n+1)}{2}.$$

This shows by the definition of Y_{2n+3} that

$$\begin{aligned} Y_{2n+3}(\mathbb{S}^{2n+1}) &\leq \lambda_{2n+3}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})^{\frac{1}{n+1}} \\ &\leq \frac{(n+2)(n+1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})^{\frac{1}{n+1}}. \end{aligned} \tag{8-1}$$

Since

$$\begin{aligned} &\frac{(n+2)(n+1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})^{\frac{1}{n+1}} \\ &\qquad < (2n+3)^{\frac{1}{n+1}} \frac{n(n+1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})^{\frac{1}{n+1}} \\ &\qquad = (2n+3)^{\frac{1}{n+1}} Y_1(\mathbb{S}^{2n+1}) \end{aligned}$$

when $n \geq 3$, Proposition 8.1 follows from (8-1) . □

For the case when the k-th CR Yamabe invariant is negative, we have this:

Theorem 8.2. *Let k be an positive integer. Assume that $Y_k(M, \theta) < 0$. Then $Y_k(M, \theta) = -\infty$.*

Proof. After a possible change of contact form in the conformal class, we can assume that $\lambda_k(\theta) < 0$. This implies that we can find smooth functions $v_1, \dots, v_k$ satisfying

$$L_\theta(v_i) = \lambda_i(\theta) v_i \quad \text{for all } i = 1, 2, \dots, k$$

and such that

$$\int_M v_i v_j \, dV_\theta = 0 \quad \text{for all } i, j = 1, 2, \dots, k \text{ and } i \neq j.$$

Let v_k be defined as in the proof of Theorem 6.1. We define $u_\varepsilon = v_\varepsilon + \varepsilon$. We set $V = \operatorname{span}\{v_1, \dots, v_k\}$. For $v \in V$, we have

$$\begin{aligned} \int_M u_\varepsilon^{\frac{2}{n}} v^2 \, dV_\theta &\leq \varepsilon^{\frac{2}{n}} \int_M v^2 \, dV_\theta + \int_M v_\varepsilon^{\frac{2}{n}} v^2 \, dV_\theta \\ &\leq C\varepsilon^{\frac{2}{n}} + C \int_M v_\varepsilon^{\frac{2}{n}} \, dV_\theta \\ &\leq \begin{cases} C\varepsilon^{\frac{2}{n}} + C\left(\int_M v_\varepsilon^{\frac{3}{n}} \, dV_\theta\right)^{\frac{2}{3}} \operatorname{Vol}(M, \theta)^{\frac{1}{3}} = C\varepsilon^{\frac{2}{n}} + C\varepsilon^{\frac{2}{3}} & \text{if } n \geq 2, \\ C\varepsilon^2 + C\left(\int_M v_\varepsilon^{\frac{5}{2}} \, dV_\theta\right)^{\frac{1}{5}} \operatorname{Vol}(M, \theta)^{\frac{4}{5}} = C\varepsilon^2 + C\varepsilon^{\frac{1}{10}} & \text{if } n = 1 \end{cases} \end{aligned}$$

by (6-10) and Hölder's inequality. From this, we have

$$\lim_{\varepsilon\to 0}\int_M u_\varepsilon^{\frac{2}{n}} v^2\, dV_\theta = 0$$

uniformly in $v \in V$. Since $\lambda_k(\theta) < 0$, it is then easy to see that

$$\sup_{v\in V} F(u_\varepsilon, v) = -\infty.$$

Together with the variational characterization of $Y_k(M,\theta)$ in Proposition 3.1, we get that $Y_k(M,\theta) = -\infty$. □

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