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The N-body problem with a $1/r^2$ potential has, in addition to translation and rotational symmetry, an effective scale symmetry which allows its zero energy flow to be reduced to a geodesic flow on complex projective $(N-2)$-space, minus a hyperplane arrangement. When $N = 3$ we get a geodesic flow on the 2-sphere minus three points. If, in addition we assume that the three masses are equal, then it was proved in a previous paper that the corresponding metric is hyperbolic: its Gaussian curvature is negative except at two points. Does the negative curvature property persist for $N = 4$, that is, in the equal mass $1/r^2$ potential 4-body problem? Here we prove that it does not by computing that the corresponding Riemannian metric in this $N = 4$ case has positive sectional curvature at some 2-planes. This curvature computation underlines an essential difference between the 3- and 4-body problem, a difference whose consequences remain to be explored.

1. Introduction

In [Montgomery 2005] it was shown that the reduced Jacobi–Maupertuis metric for a certain 3-body problem had negative Gaussian curvature (except at two points where it is zero). This hyperbolicity led to deep dynamical consequences. Does hyperbolicity, i.e., curvature negativity, persist for the analogous N-body problem with $N > 3$? No. We show that the analogous reduced 4-body problem with its metric has 2-planes at which the sectional curvature is positive.

The N-body problem in question has equal masses and the inverse *cube* law attractive force between bodies.

2. Setup

Identify the complex numbers $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$. Then the planar N-body problem has configuration space $\mathbb{C}^N \setminus \Delta$. Here Δ is the "fat diagonal"

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consisting of all collisions:

$$\Delta = \{q = (q_1, q_2, \dots, q_N) \in \mathbb{C}^N : q_i = q_j \text{ for some pair } i \neq j\}.$$

The quotient of $\mathbb{C}^N \setminus \Delta$ by translations and rotations is the "reduced N-body configuration space":

$$C_N = Y_N \times \mathbb{R}^+, \qquad Y_N = \mathbb{CP}^{N-2} \setminus \mathbb{P}\Delta,$$

where $\mathbb{CP}^{N-2}$ is the projectivization of the center of mass subspace

$$\mathbb{C}^{N-1} = \left\{q \in \mathbb{C}^N : \sum m_i q_i = 0\right\}$$

and $\mathbb{P}\Delta \subset \mathbb{CP}^{N-2}$ is the projectivization of $\Delta \cap \mathbb{C}^{N-1}$. The $\mathbb{R}^+$ factor records the overall scale of the planar N-gon and is coordinatized by $\sqrt{I}$ with $I = \Sigma m_i |q_i|^2$ being the total moment of inertia about the center of mass. Y_N is the moduli space of oriented similarity classes of noncollision N-gons and will be called "shape space".

The following considerations reduce the zero angular momentum, zero energy N-body problem to a geodesic flow on shape space Y_N, *provided* the potential V is homogeneous of degree -2. If V is homogeneous of degree $-\alpha$ then the virial identity, also known as the Lagrange–Jacobi identity, asserts that along solutions of energy H we have $\ddot{I} = 4H - (4 - 2\alpha)V$, which implies that the only case in which we can generally guarantee that $\ddot{I} = 0$ is when $\alpha = 2$ and $H = 0$. If in addition $\dot{I} = 0$ then solutions lie on constant levels of I.

Now we recall the Jacobi–Maupertuis (JM) reformulation of mechanics, which asserts that the solutions to Newton's equations at energy H are, after a time reparametrization, precisely the geodesic equations for the *Jacobi–Maupertuis metric*

$$ds^2_{\mathrm{JM}} = 2(H - V)\, ds^2$$

on the *Hill region* $\{H - V \geq 0\} \subset \mathbb{C}^N \setminus \Delta$ with ds^2 the mass metric. We are interested in the case $H = 0$, $-V > 0$ with V homogeneous of degree -2, in which case the Hill region is all of $\mathbb{C}^N \setminus \Delta$ and

$$ds^2_{\mathrm{JM}} = U\, ds^2, \qquad U = -V.$$

The case of prime interest to us is

$$U = -V = \sum_{i \neq j} m_i m_j / r_{ij}^2. \tag{1}$$

This U, and hence the JM metric, is invariant under rotations and translations. Quotienting first by translations, we take representatives in the totally geodesic center-of-mass-zero subspace $\mathbb{C}^{N-1}$, which reduces the dynamics to geodesics of the metric $ds^2_{\mathrm{JM}}|_{\mathbb{C}^{N-1}}$ on $\mathbb{C}^{N-1}$. Moreover, $ds^2_{\mathrm{JM}}|_{\mathbb{C}^{N-1}}$ is also invariant under scaling since the homogeneities of U and the Euclidean mass metric ds^2 on $\mathbb{C}^{N-1}$ cancel. Thus the JM metric admits the group $G = \mathbb{C}^*$ of rotations and scalings as an isometry group.

Now Y_N is the quotient space: $Y_N = (\mathbb{C}^{N-1} \setminus \Delta)/G = \mathbb{CP}^{N-2} \setminus \Delta$. (By abuse of notation, we continue to use the symbol Δ to denote the image of the collision locus Δ under projectivization and intersection.) Insisting that the quotient map $\pi : \mathbb{C}^{N-1} \setminus \Delta \to Y_N$ is a Riemannian submersion induces a metric on Y_N. Recall that this means that we can define the metric on Y_N by *isometrically* identifying the tangent space to Y_N at a point p with the orthogonal complement (relative to ds^2_{JM} or ds^2, and at any point lying over p in $\mathbb{C}^{N-1}$) to the G-orbit that corresponds to that point. These orthogonality conditions are equivalent to the conditions that the linear momentum, angular momentum, and "scale momentum" $\dot{I}$ are all zero. To summarize, by using the JM metric and forming the Riemannian quotient, the zero angular momentum, zero energy $1/r^2$ potential N-body problem becomes equivalent to the problem of finding geodesics for the metric defined by Riemannian submersion on Y_N.

Remark. The metric quotient procedure just described realizes the Marsden–Weinstein symplectic reduced space of $T^*(\mathbb{C}^N \setminus \Delta)$ by the action of translations, rotations and scalings, $\mathbb{C} \rtimes \mathbb{C}^*$, at momentum values 0, together with the N-body reduced Hamiltonian flow, but valid only at zero energy.

Remark. This metric on Y_N can be expressed as $U\, ds^2_{\text{FS}}$ where ds^2_{FS} is the usual Fubini–Study metric on $\mathbb{CP}^{N-2}$.

Remark. For the standard $1/r^2$ potential of (1), this metric on Y_N is complete, with infinite volume.

The collinear N-body problem defines a totally geodesic submanifold

$$\mathbb{RP}^{N-2} \setminus \Delta \subset \mathbb{CP}^{N-2} \setminus \Delta.$$

We obtain this submanifold by placing the N-masses anywhere along the real axis $\mathbb{R} \subset \mathbb{C}$, arranged so their center of mass is zero and so that there are no collisions, and then taking the quotient. In other words, $\mathbb{RP}^{N-2} \setminus \Delta$ is the quotient of $\mathbb{R}^{N-1} \subset \mathbb{C}^{N-1}$ by dilations and real reflections.

3. Main result

In case $N = 3$, with the potential (1) above, Y_3 is a pair of pants — a sphere minus three points. The point of [Montgomery 2005] was to show that the metric on Y_3 just described is hyperbolic provided $m_1 = m_2 = m_3$. Specifically, in this equal mass case the Gaussian curvature of the metric on the surface Y_3 is negative everywhere except at two points (these being the "Lagrange points" corresponding to equilateral triangles.) What about Y_4?

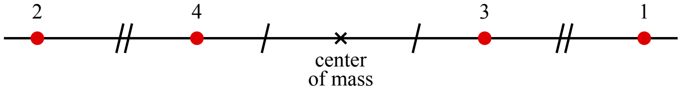

Figure 1. The collinear configurations p which we consider.

Theorem 1. *Consider the Jacobi–Maupertuis metric on Y_4 induced as above for the case of 4 equal masses under the strong force $1/r^2$ potential (1). Then there are 2-planes σ tangent to Y_4 at which the Riemannian sectional curvature $\mathcal{K}(\sigma)$ is positive.*

Remark. The 2-planes σ of the theorem pass through special points $p \in \mathbb{RP}^2 \subset \mathbb{CP}^2$ which represent certain special collinear configurations. See Figure 1. The 2-plane σ at p will be the orthogonal complement to $T_p\mathbb{RP}^2$, the normal 2-plane, and is realized as $\sigma = iT_p\mathbb{RP}^2$, using the standard complex structure on $\mathbb{CP}^2$.

Remark (negative curvatures). The $\mathbb{RP}^2$ of the previous remark is a totally geodesic surface fixed by an isometric involution. There are other such totally geodesic surfaces defined as fixed loci of symmetries, and computer experiments suggest that these all have negative Gaussian curvature everywhere while their normal 2-planes can have positive sectional curvature at some points, like our special case $\mathbb{RP}^2$. Computer experiments also indicate that in the direction of the normal plane there is positive sectional curvature over all collinear configurations of $\mathbb{RP}^2$ and not just the special configurations verified in the theorem. An analytic proof of these claims beyond our special case, however, looks frightening.

Remark (uniqueness of free homotopy classes). The work in [Montgomery 2005] was chiefly meant as a route for proving the uniqueness (mod symmetries) of the $N = 3$ strong force figure-eight solution. For $N = 4$, hyperbolicity fails and we have no direct "hyperbolic" path for establishing uniqueness of various 4-body choreographies or free homotopy class representatives.

Open Question. A geodesic flow can still be hyperbolic as a flow, without the underlying metric having all sectional curvatures negative. Is geodesic flow on Y_4 hyperbolic as a flow? Is it even partially hyperbolic?

4. Proof of the theorem

We take the case $N = 4$ in the above considerations. When all the masses are equal to 1, the mass metric, used to compute the kinetic energy and moment of inertia, is the standard Hermitian metric in coordinates $(q_1, q_2, q_3, q_4) \in \mathbb{C}^4$, where the q_i represent the positions of the i-th body. We reduce by translations by going to the center-of-mass-zero space, which is a 3-dimensional subspace $\mathbb{C}^3 \subset \mathbb{C}^4$ having

Jacobi coordinates as Hermitian orthonormal coordinates:

$$\mathbb{C}^3 \xrightarrow{L} \mathbb{C}^4 \text{ given by the matrix } \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ in standard bases.}$$

As is well-known, if we start tangent to the center-of-mass-zero subspace $L(\mathbb{C}^3)$, we stay tangent to it. Hence we can restrict the dynamics, potential, metric, etc. to the center-of-mass-zero subspace. We denote the potential restricted to the center-of-mass-zero subspace in Jacobi coordinates as $U_L = U \circ L$ and still write $ds^2_{\text{JM}} = U_L ds^2$ for the restricted JM metric on $\mathbb{C}^3 \setminus \Delta$ where ds^2 is the standard metric on $\mathbb{C}^3$.

Continuing along the outline above, we now quotient by scaling and rotation isometries $\mathbb{C}^*$ of ds^2_{JM} to obtain the "shape space" Y_4 and we label the quotient map $\pi : \mathbb{C}^3 \setminus \Delta \to Y_4$, which takes a configuration q to its orbit $\mathbb{C}^* q$. We denote the vertical and horizontal distributions as

$$\mathcal{V}_p = \ker d_p\pi = \mathbb{C}p \quad \text{and} \quad \mathcal{H}_p = \mathcal{V}_p^{\perp} \overset{d\pi}{\cong} T_{\pi(p)} Y_4.$$

Requiring $d\pi|_{\mathcal{H}, ds^2_{\text{JM}}|_{\mathcal{H}}}$ to be an isometry defines our induced metric on Y_4 whose geodesics correspond to N-body motions in "shape space". Under this induced metric on Y_4 we denote sectional curvature through the plane $\sigma \in T_{\pi(p)} Y_4$ by $\mathcal{K}(\sigma)$.

Suppressing the notation of evaluating at a representative $p \in \pi(q)$, our main tool in the computation of $\mathcal{K}(\sigma)$, the ds^2_{JM} curvature, is the equation

$$\begin{aligned} (2) \quad & U_L^3 \mathcal{K}(\sigma) \\ & = \tfrac{3}{4}\big((\partial_1 U_L)^2 + (\partial_2 U_L)^2\big) - \left\| \frac{\nabla U}{2} \right\|^2 - \frac{U_L}{2}(\partial_1^2 U_L + \partial_2^2 U_L) + 3\frac{U_L^2}{\|p\|^2}(v_1 \cdot i v_2)^2 \end{aligned}$$

Here $\partial_a f$ denotes $df(v_a)$ where $f \in C^\infty(\mathbb{C}^3)$ and where $a = 1, 2$ with $v_1, v_2 \in \mathcal{H}$ being ds^2-orthonormal vectors whose pushforwards $d\pi\, v_a$ span σ. The $\cdot$, $\|\ \|$, and ∇ refer to the norm, metric, and Levi-Civita connection for the Euclidean metric ds^2. For the derivation of (2) see the Appendix.

The collinear configurations form a totally geodesic projective plane $\mathbb{RP}^2 \subset \mathbb{CP}^2$, the image under π of the real 2-sphere in $\mathbb{C}^3$, which we parametrize by

$$p = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi).$$

We evaluate (2) and find positive sectional curvature over the configurations with $\theta = \pi/2$ (see Figure 1) in the direction of the $iT\mathbb{RP}^2$ plane. This plane is spanned by the pushforwards of

$$v_1 = -i\frac{\partial p}{\partial \phi} = i(\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi),$$
$$v_2 = \frac{i}{\cos\phi}\frac{\partial p}{\partial\theta} = i(-\sin\theta, \cos\theta, 0).$$

Terms 1. *Over $\mathbb{RP}^2$ in the $iT\mathbb{RP}^2$ direction, the last and first summands on the second line of (2) vanish*:

$$v_1 \cdot i v_2 = 0, \qquad \partial_a U_L = 0.$$

Proof. That $v_1 \cdot i v_2 = 0$ is clear: i rotates v_2 into purely real coordinates. To evaluate the first partials, note that Lp has purely real coordinates and ∇U has k-th component $\sum_{j\neq k}(q_j - q_k)/r_{jk}^4$, so $\nabla|_{Lp}U$ has purely real coordinates. Now since Lv_a has purely complex coordinates,

$$\partial_a U_L = \nabla|_{Lp}U \cdot Lv_a = 0. \qquad \square$$

Terms 2. *With the notation $Lp = (q_1, q_2, q_3, q_4)$, $Lv_a = i(v_a^1, v_a^2, v_a^3, v_a^4)$, and*

$$\rho_{jk} = 1/(q_j - q_k),\ \ \alpha_{jk} = (v_1^j - v_1^k)^2 + (v_2^j - v_2^k)^2 \in \mathbb{R},$$

the sum of second partials in (2) is given by

$$\partial_1^2 U_L + \partial_2^2 U_L = -2\sum_{j>k}\alpha_{jk}\rho_{jk}^4.$$

Proof. We write our standard coordinates on $\mathbb{C}^4$ as $q_j = x_j + i y_j$. Then since Lv_a is purely imaginary, we have

$$\partial_a^2 U_L = \nabla|_{Lp}(\nabla U \cdot Lv_a)\cdot Lv_a = \left(\nabla|_{Lp}\frac{\partial U}{\partial y_k}v_a^k\right)\cdot Lv_a = \frac{\partial^2 U}{\partial y_j \partial y_k}\bigg|_{Lp} v_a^k v_a^j.$$

Next we compute $\frac{\partial^2 U}{\partial y_j\partial y_k}\Big|_{Lp} = 2\rho_{jk}^4$ for $j\neq k$ and $\frac{\partial^2 U}{\partial y_k^2}\Big|_{Lp} = -2\sum_{j\neq k}\rho_{jk}^4$, so now

$$\begin{aligned}\partial_a^2 U_L &= -2\sum_{j\neq k}\rho_{jk}^4\big((v_a^k)^2 - v_a^j v_a^k\big)\\ &= -2\sum_{j>k}\rho_{jk}^4\big((v_a^k)^2 - 2v_a^k v_a^j + (v_a^j)^2\big)\\ &= -2\sum_{j>k}\rho_{jk}^4(v_a^k - v_a^j)^2. \qquad \square\end{aligned}$$

Result. *Over the circle $\theta = \pi/2$, $\mathcal{K}(iT\mathbb{RP}^2)$ is positive.*

Now, substituting Terms 1 and 2 into formula (2), we see that

$$0 < \mathcal{K} \iff 0 < U_L^3 \mathcal{K} = -\|\nabla U/2\|^2 + U_L \sum_{j>k} \alpha_{jk} \rho_{jk}^4$$

$$\iff \sum_k \Big(\sum_{j \neq k} \rho_{jk}^3 \Big)^2 < \Big(\sum_{j>k} \rho_{jk}^2 \Big) \Big(\sum_{j>k} \alpha_{jk} \rho_{jk}^4 \Big). \tag{3}$$

Taking $\theta = \pi/2$ and with the notation introduced in Terms 2, we find the relations

$$\rho_{12} = \frac{1}{\sqrt{2}\cos\phi}, \quad \rho_{34} = \frac{1}{\sqrt{2}\sin\phi} \qquad \alpha_{12} = \frac{1}{\rho_{34}^2}, \quad \alpha_{34} = \frac{1}{\rho_{12}^2}$$

$$\rho_{13} = \frac{\sqrt{2}}{\cos\phi - \sin\phi} = -\rho_{24} \qquad \alpha_{13} = \frac{1}{\rho_{14}^2} + 1 = \alpha_{24}$$

$$\rho_{14} = \frac{\sqrt{2}}{\cos\phi + \sin\phi} = -\rho_{23} \qquad \alpha_{14} = \frac{1}{\rho_{13}^2} + 1 = \alpha_{23}.$$

Now the left side of (3) works out to

$$\begin{aligned}
&2((\rho_{12}^3 + \rho_{13}^3 + \rho_{14}^3)^2 + (\rho_{13}^3 - \rho_{14}^3 - \rho_{34}^3)^2) \\
&\qquad = 2\Big(\sum_{k>j} \rho_{jk}^6 + 2\rho_{12}^3(\rho_{13}^3 + \rho_{14}^3) + 2\rho_{34}^3(\rho_{14}^3 - \rho_{13}^3) \Big) \\
&\qquad = 2\sum_{k>j} \rho_{jk}^6 - 96\frac{1}{\sin^2 2\phi \cos^2 2\phi} = 2\sum_{k>j} \rho_{jk}^6 + \text{negative term},
\end{aligned}$$

and the right side of (3) works out to

$$\begin{aligned}
&\big(\rho_{12}^2 + \rho_{34}^2 + 2(\rho_{13}^2 + \rho_{14}^2)\big)\Big(\frac{\rho_{12}^4}{\rho_{34}^2} + \frac{\rho_{34}^4}{\rho_{12}^2} + 2\Big(\rho_{13}^4 + \rho_{14}^4 + \frac{\rho_{13}^4}{\rho_{14}^2} + \frac{\rho_{14}^4}{\rho_{13}^2} \Big)\Big) \\
&\quad = \Big(\frac{2}{\sin^2 2\phi} + \frac{8}{\cos^2 2\phi} \Big)\Big(\sin^2 2\phi(\rho_{12}^6 + \rho_{34}^6) \\
&\qquad\qquad + \frac{\cos^2 2\phi}{2}(\rho_{13}^6 + \rho_{14}^6) + 2(\rho_{13}^4 + \rho_{14}^4) \Big) \\
&\quad = 2\sum_{k>j} \rho_{jk}^6 + \cot^2 2\phi(\rho_{13}^6 + \rho_{14}^6) + 8\tan^2 2\phi(\rho_{12}^6 + \rho_{34}^6) \\
&\qquad\qquad + (\rho_{13}^4 + \rho_{14}^4)\Big(\frac{4}{\sin^2 2\phi} + \frac{16}{\cos^2 2\phi} \Big) \\
&\quad = 2\sum_{k>j} \rho_{jk}^6 + \text{positive term}.
\end{aligned}$$

Therefore the inequality (3) holds! □

Appendix: Derivation of (2)

Take a ds^2-orthonormal basis $\{v_a\}$ for $\mathbb{C}^3$ with $v_1, v_2 \in \mathcal{H}_p$.

The Kulkarni–Nomizu product formula for conformal curvatures [Sakai 1996, p. 51] reads:

$$\overline{R}_{abcd} - U_L R_{abcd} = -\left\{ds^2_{\mathrm{JM}} \owedge \left(\nabla du - du \otimes du + \tfrac{1}{2}\|du\|^2 ds^2\right)\right\}_{abcd}$$

where $u := \frac{1}{2}\log U_L$, the overbars denote curvature with respect to the ds^2_{JM}-metric, and all other quantities (without overbars) are with respect to the ds^2-metric. Then $R_{abcd} = 0$ since ds^2 is the flat Euclidean metric of $\mathbb{C}^3 = \mathbb{R}^6$. Taking $cd = ab$, we have

$$\begin{aligned} U_L^2 \overline{K}_{ab} = \overline{R}_{abab} &= -U_L\left(\nabla du_{bb} + \nabla du_{aa} - du_b \otimes du_b - du_a \otimes du_a + \|du\|^2\right) \\ &= -U_L\left(\partial_a^2 u + \partial_b^2 u - (\partial_a u)^2 - (\partial_b u)^2 + \|\nabla u\|^2\right). \end{aligned}$$

Next, O'Neill's formula [1983, p. 213] gives

$$\mathcal{K}(d\pi v_1, d\pi v_2) = \overline{K}_{12} + \tfrac{3}{4}\left|[V_1, V_2]^{\mathcal{V}}\right|^2_{ds^2_{\mathrm{JM}}},$$

where $V_a = v_a/\sqrt{U_L(p)}$ and $X^{\mathcal{V}}$ denotes ds^2_{JM} projection of X onto $\mathcal{V}$.

We then compute

$$\partial_a u = \frac{\partial_a U_L}{2U_L} = \frac{\nabla|_{Lp} U \cdot Lv_a}{2U_L(p)}$$

and

$$\begin{aligned} \partial_a^2 u &= \frac{\partial_a^2 U_L}{2U_L} - \frac{(\partial_a U_L)^2}{2U_L^2} \\ &= \frac{\nabla|_{Lp}(\nabla U \cdot Lv_a)\cdot Lv_a}{2U_L(p)} - \frac{(\partial_a U_L)^2}{2U_L(p)^2}. \end{aligned}$$

Note that $\nabla U \in \left\{q \in \mathbb{C}^4 : \sum q_j = 0\right\}$ and Lv_a is a ds^2 orthonormal basis for this center-of-mass-zero subspace, hence

$$\begin{aligned} \|\nabla U\|^2 &= \sum (\nabla U \cdot Lv_a)^2 \\ &= \sum (\partial_a U_L)^2 = 4U_L^2 \|\nabla u\|^2. \end{aligned}$$

Substitution into the Kulkarni–Nomizu formula gives

$$\overline{K}_{12} = -\frac{1}{U_L^3}\left(\frac{U_L}{2}(\partial_1^2 U_L + \partial_2^2 U_L) - \frac{3}{4}(\partial_1 U_L^2 + \partial_2 U_L^2) + \left\|\frac{\nabla U}{2}\right\|^2\right). \tag{4}$$

To compute O'Neill's Lie bracket term we write our standard coordinates on $\mathbb{C}^3$ as $(x^1 + ix^2, \ldots, x^5 + ix^6)$.

Let $H_1 = X^j \partial_{x^j}$, $H_2 = Y^j \partial_{x^j} \in \mathcal{H}$ be any horizontal vector fields. The vertical vector fields are spanned by the Euler vector field $E = x^j \partial_{x^j}$ and iE. Then $H_j \cdot E = H_j \cdot iE = 0$ and

$$\begin{aligned}[H_1, H_2] \cdot E &= \sum_k X^j x^k \partial_{x^j} Y^k - Y^j x^k \partial_{x^j} X^k \\ &= \sum_k X^j (\partial_{x^j}(x^k Y^k) - \delta_j^k Y^k) - Y^j (\partial_{x^j}(x^k X^k) - \delta_j^k X^k) \\ &= \sum_k X^k Y^k - Y^k X^k = 0,\end{aligned}$$

and likewise,

$$\begin{aligned}[H_1, H_2] \cdot iE &= \sum_{k \text{ odd}} (Y^j \partial_{x^j} X^k - X^j \partial_{x^j} Y^k) x^{k+1} + (X^j \partial_{x^j} Y^{k+1} - Y^j \partial_{x^j} X^{k+1}) x^k \\ &= 2 \sum_{k \text{ odd}} -X^k Y^{k+1} + X^{k+1} Y^k = 2 H_1 \cdot i H_2.\end{aligned}$$

Then

$$\begin{aligned}\left|[V_1, V_2]^{\mathcal{V}_p}\right|^2 &= ds^2_{\mathrm{JM}}\left([V_1, V_2], \frac{E_p}{|p|\sqrt{U_L(p)}}\right)^2 + ds^2_{\mathrm{JM}}\left([V_1, V_2], \frac{iE_p}{|p|\sqrt{U_L(p)}}\right)^2 \\ &= \frac{U_L^2}{|p|^2 U_L}\left(([V_1, V_2] \cdot E)^2 + ([V_1, V_2] \cdot iE)^2\right) \\ &= \frac{4U_L(p)(V_1 \cdot iV_2)^2}{|p|^2} = \frac{4}{U_L(p)|p|^2}(v_1 \cdot iv_2)^2.\end{aligned}$$

Now substitution of this Lie bracket expression and (4) into O'Neill's formula and multiplying by U_L^3 yields (2). □

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