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**MORSE AREA AND SCHARLEMANN–THOMPSON WIDTH  
FOR HYPERBOLIC 3-MANIFOLDS**

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**Scharlemann and Thompson define a numerical complexity for a 3-manifold using handle decompositions of the manifold. We show that for compact hyperbolic 3-manifolds, this is linearly related to a definition of metric complexity in terms of the areas of level sets of Morse functions.**

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## 1. Introduction

Let  $M$  be a closed Riemannian 3-manifold, and let  $f: M \rightarrow \mathbb{R}$  be a Morse function; i.e.,  $f$  is a smooth function, all of whose critical points are nondegenerate, and for which distinct critical points have distinct images in  $\mathbb{R}$ . We define the *area* of  $f$  to be the maximum area of any level set  $F_t = f^{-1}(t)$  over all points  $x \in \mathbb{R}$ . We define the *Morse area* of  $M$  to be the infimum of the area of all Morse functions  $f: M \rightarrow \mathbb{R}$ .

For hyperbolic 3-manifolds, the hyperbolic metric is a topological invariant by Mostow rigidity, and the critical points of a Morse function determine a handle decomposition of the manifold, so one might hope that Morse area is related to a topological measure of complexity defined in terms of handle decompositions of the manifold. We show that Morse area is linearly related to a definition of topological complexity we call *Scharlemann–Thompson width* or *linear width*, and which we now describe.

For a closed (possibly disconnected) surface  $S$ , we define the complexity, or genus, of  $S$  to be the sums of the genera of each connected component. For a compact (possibly disconnected) surface with boundary, we define the genus of  $S$

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to be the genus of the surface obtained by capping off all boundary curves with discs. We shall write  $|\partial S|$  for the number of boundary components of  $S$ .

A *handlebody* is a compact 3-manifold with boundary, homeomorphic to the regular neighborhood of a graph in  $\mathbb{R}^3$ . Up to homeomorphism, a handlebody is determined by the genus  $g$  of its boundary surface. Every 3-manifold  $M$  has a *Heegaard splitting*, which is a decomposition of the manifold into two handlebodies. This immediately gives a notion of complexity for a 3-manifold, called the *Heegaard genus*, which is the smallest genus of any Heegaard splitting of the 3-manifold.

There is a refinement of this, due to Scharlemann and Thompson [1994], which we now describe. Let  $S$  be a closed surface, which need not be connected. A *compression body*  $C$  is a compact 3-manifold with boundary, constructed by attaching some number of 2-handles to one side  $S \times \{0\}$  of  $S \times I$ . We do not require compression bodies to be connected. We shall refer to  $S \times \{1\}$  as the *top boundary*  $\partial_+ C$  of the compression body, and the other boundary components of  $C$  as the *lower boundary*  $\partial_- C$ . The lower boundary may be disconnected, even if  $C$  is connected, and any 2-sphere components are capped off with 3-balls. In particular, if a maximal number of nonparallel 2-handles are attached, then the resulting compression body is a handlebody, so a handlebody is a special case of a compression body. A *generalized Heegaard splitting*, which we shall call a *linear splitting*, is a decomposition of a closed 3-manifold  $M$  into a linearly ordered sequence of compression bodies  $C_1, \dots, C_{2n}$ , which need not be connected, such that the upper boundary of an odd numbered compression body  $C_{2i+1}$  is equal to the top boundary of the compression body  $C_{2i+2}$ , and the lower boundary of  $C_{2i+1}$  is equal to the lower boundary of the previous compression body  $C_{2i}$ . For the even numbered compression bodies  $C_{2i}$ , the top boundary is equal to the upper boundary of  $C_{2i-1}$ , and the lower boundary is equal to the lower boundary of  $C_{2i+1}$ . In the case of the first and last compression bodies  $C_1$  and  $C_{2n}$ , the lower boundaries are empty. Let  $H_i$  be the sequence of surfaces consisting of the upper boundaries of the compression bodies  $C_{2i-1}$  and  $C_{2i}$ ; these are often referred to as the *odd surfaces*, and the surfaces corresponding to the lower boundaries as the *even surfaces*. A linear splitting has a natural height function, i.e., a Morse function onto  $\mathbb{R}$ , in which each odd or even surface, which may not be connected, is the pre-image of a single point, and the compression bodies are the pre-images of the closed intervals determined by these points.

The complexity  $c(H_i)$  of the surface  $H_i$  is the genus of  $H_i$ , i.e., the sum of the genera of each connected component, and the *width* of the linear splitting is the maximum value of  $c(H_i)$  over all upper boundaries. The *Scharlemann–Thompson width*, which we shall also refer to as the *linear width*, of a 3-manifold  $M$  is the minimum width over all possible linear splittings. As a Heegaard splitting is a special case of a linear splitting, the Heegaard genus of  $M$  is an upper bound for the linear width of  $M$ .

There is a refinement of linear width known as *thin position*, which we discuss when we use it in [Section 3](#).

**Results.** In order to bound Morse area in terms of linear width, we shall assume the following result announced by Pitts and Rubinstein [[1986](#)] (see also [[Rubinstein 2005](#)]).

**Theorem 1.1** [[Pitts and Rubinstein 1986](#); [Rubinstein 2005](#)]. *Let  $M$  be a Riemannian 3-manifold with a strongly irreducible Heegaard splitting. Then the Heegaard surface is isotopic to a minimal surface, or to the boundary of a regular neighborhood of a nonorientable minimal surface with a small tube attached vertically in the  $I$ -bundle structure.*

A full proof of this result has not yet appeared in the literature, though recent progress has been made by Colding and De Lellis [[2003](#)], De Lellis and Pellandri [[2010](#)], and Ketover [[2013](#)].

We shall show this:

**Theorem 1.2.** *There is a constant  $K > 0$  such that for any closed hyperbolic 3-manifold,*

$$(1) \quad K(\text{linear width}(M)) \leq \text{Morse area}(M) \leq 4\pi(\text{linear width}(M)),$$

where the right-hand bound holds assuming [Theorem 1.1](#).

Our methods are effective, and the constant  $K$  may be estimated using a bound on the Margulis constant for  $\mathbb{H}^3$ , though we omit the details of this calculation, as our methods seem unlikely to give an optimal constant.

**Outline.** In [Section 2](#), we show how to bound linear width in terms of Morse area. A bound on the Morse area of  $M$  gives a Morse function  $f: M \rightarrow \mathbb{R}$  with bounded area level sets, but with no a priori bound on the topological complexity of the level sets.

We use a Voronoi decomposition of  $M$  to give a polyhedral approximation of the Morse function, which we now describe in a simple case. Let  $\mathcal{V}$  be a Voronoi decomposition of  $M$  in which every Voronoi cell  $V_i$  is a topological ball, and has size bounded above and below; i.e., there is an  $\epsilon > 0$  such that  $B(x_i, \epsilon/2) \subset V_i \subset B(x_i, \epsilon)$ , where  $x_i$  is the center of the Voronoi cell. Let  $M_t$  be the sublevel set of the Morse function, i.e.,  $M_t = f^{-1}((-\infty, t])$ . The sublevel sets are a monotonically increasing collection of subsets of  $M$ , which start off empty, and eventually contain all of  $M$ ; so, in particular, for each Voronoi cell  $V_i$ , there is a  $t_i$  such that the volume of  $M_t \cap B(x_i, \epsilon/2)$  is exactly half the volume of  $B(x_i, \epsilon/2)$ , and we shall call  $t_i$  the *cell splitter* for the Voronoi cell  $V_i$ . Furthermore, we may assume that the  $t_i$  are distinct for distinct Voronoi cells. This gives a linear order to the Voronoi cells, and we wish to show that constructing the manifold by adding the Voronoi cells in this

order gives a bounded linear-width handle decomposition for  $M$ . Let  $P_t$  be the union of the Voronoi cells whose cell splitters  $t_i$  are at most  $t$ . Each Voronoi cell is a ball, with a bounded number of faces, so adding a Voronoi cell corresponds to adding a bounded number of handles. It remains to show that the boundary of each  $P_t$  has genus bounded in terms of the area of the level set  $F_t = f^{-1}(t)$ . Let  $V_i$  and  $V_j$  be two adjacent Voronoi cells, with  $V_i$  contained in  $P_t$  and  $V_j$  outside  $P_t$ , so their common face is a subset of  $\partial P_t$ . Consider the sequence of balls  $B(x, \epsilon/2)$ , as  $x$  runs along the geodesic from  $x_i$  to  $x_j$ . At least half the volume of  $B(x_i, \epsilon/2)$  is contained in  $M_t$ , and at most half the volume of  $B(x_j, \epsilon/2)$  is contained in  $M_t$ ; so there is an  $x$  such that exactly half the volume of  $B(x, \epsilon/2)$  is contained in  $M_t$ , and so there is a lower bound on the area of  $F_t \cap B(x, \epsilon/2)$ . Therefore, a bound on the area of  $F_t$  gives a bound on the number of faces of  $\partial P_t$ . As each face has a bounded number of edges, this gives a bound on the genus of  $\partial P_t$ , and hence a bound on the linear width of  $M$ , though this bound depends on  $\epsilon$ .

In order to produce a bound which works for any compact hyperbolic manifold  $M$ , we use the Margulis lemma and the thick-thin decomposition for hyperbolic manifolds. There is constant  $\mu$ , called a Margulis constant, such that any compact hyperbolic manifold may be decomposed into a thick part  $X_\mu$ , where each point has injectivity radius greater than  $\mu$ , and a thin part, where each point has injectivity radius at most  $\mu$ , and which is a disjoint union of solid tori. If we choose  $\epsilon$  sufficiently small, then we may choose a Voronoi decomposition of the thick part in which each Voronoi cell has size bounded above and below, and run the argument in the previous paragraph to control the genus of  $\partial P_t$  inside the thick part. We do not control the complexity of  $\partial P_t$  in the thin part, but as each component of the thin part is a solid torus, we may cap off  $\partial P_t \cap X_\mu$  with surfaces parallel to  $P_t \cap \partial X_\mu$ , while still obtaining bounds on the genus. In order to bound the number of handles corresponding to adding a Voronoi cell, we use a result of Kobayashi and Rieck [2011] which gives bounds on the topological complexity of the intersection of a Voronoi cell with the thin part.

The key problem for the upper bound is that the techniques of Pitts and Rubinstein use sweepouts, so although their minimax construction produces a sweepout of bounded area, we do not know how to directly replace a bounded area sweepout with a bounded area foliation. However, the upper bound is obtained in recent work of Colding and Gabai [2015], using work of Colding and Minicozzi [2015] on the mean curvature flow, and we describe their results in [Section 3](#).

## 2. Morse area bounds Scharlemann–Thompson width

In this section we show that we can bound the Scharlemann–Thompson width of a hyperbolic manifold in terms of its Morse area. We will approximate level sets by

surfaces which are unions of faces of Voronoi cells, and we start by describing the properties of the Voronoi decompositions that we will use.

**Voronoi cells.** We will approximate the level sets of  $f$  by surfaces consisting of faces of Voronoi cells. We now describe in detail the Voronoi cell decompositions we shall use, and their properties.

A *polygon* in  $\mathbb{H}^3$  is a compact convex subset of a hyperbolic plane whose boundary consists of a finite number of geodesic segments. A *polyhedron* in  $\mathbb{H}^3$  is a convex topological 3-ball in  $\mathbb{H}^3$  whose boundary consists of a finite collection of polygons. A *polyhedral cell decomposition* of  $\mathbb{H}^3$  is a cell decomposition in which every 3-cell is a polyhedron, each 2-cell is a polygon, and the edges are all geodesic segments. We say a cell decomposition of a hyperbolic manifold  $M$  is *polyhedral* if its preimage in the universal cover gives a polyhedral cell decomposition of  $\mathbb{H}^3$ .

Let  $X = \{x_i\}$  be a discrete collection of points in 3-dimensional hyperbolic space  $\mathbb{H}^3$ . The Voronoi cell  $V_i$  determined by  $x_i \in X$  consists of all points of  $M$  which are closer to  $x_i$  than any other  $x_j \in X$ , i.e.,

$$V_i = \{x \in \mathbb{H}^3 \mid d(x, x_i) \leq d(x, x_j) \text{ for all } x_j \in X\}.$$

We shall call  $x_i$  the *center* of the Voronoi cell  $V_i$ , and we shall write  $\mathcal{V} = \{V_i\}$  for the collection of Voronoi cells determined by  $X$ . Voronoi cells are convex sets in  $\mathbb{H}^3$ , and hence topological balls. The set of points equidistant from both  $x_i$  and  $x_j$  is a totally geodesic hyperbolic plane in  $\mathbb{H}^3$ . A *face*  $F$  of the Voronoi decomposition consists of all points which lie in two distinct Voronoi cells  $V_i$  and  $V_j$ , so  $F$  is contained in a geodesic plane. An *edge*  $e$  of the Voronoi decomposition consists of all points which lie in three distinct Voronoi cells  $V_i, V_j$  and  $V_k$ , which is a geodesic segment, and a *vertex*  $v$  is a point lying in four distinct Voronoi cells  $V_i, V_j, V_k$  and  $V_l$ . By general position, we may assume that all edges of the Voronoi decomposition are contained in exactly three distinct faces, the collection of vertices is a discrete set, and there are no points which lie in more than four distinct Voronoi cells. We shall call such a Voronoi decomposition a *regular* Voronoi decomposition, and it is a polyhedral decomposition of  $\mathbb{H}^3$  if every cell is compact. As each edge is 3-valent, and each vertex is 4-valent, this implies that the dual cell structure is a simplicial triangulation of  $\mathbb{H}^3$ , which we shall refer to as the *dual triangulation*. The dual triangulation may be realised in  $\mathbb{H}^3$  by choosing the vertices to be the centers  $x_i$  of the Voronoi cells and the edges to be geodesic segments connecting the vertices, and we shall always assume that we have done this. In this case, the triangles and tetrahedra are geodesic triangles and geodesic tetrahedra in  $\mathbb{H}^3$ .

Given a collection of points  $X = \{x_i\}$  in a hyperbolic 3-manifold  $M$ , let  $\tilde{X}$  be the pre-image of  $X$  in the universal cover of  $M$ , which is isometric to  $\mathbb{H}^3$ . We say a subset of  $\mathbb{H}^3$  is *equivariant* if it is preserved by the covering translations determined

by the quotient  $M$ . As  $\tilde{X}$  is equivariant, the  $k$ -skeleton of the corresponding Voronoi cell decomposition  $\mathcal{V}$  of  $\mathbb{H}^3$  is also equivariant for  $0 \leq k \leq 3$ , as are the  $k$ -skeletons of the dual triangulation.

We now show that the interior of each Voronoi cell  $V$  is mapped down homeomorphically by the covering projection. Suppose  $y$  is a point in the interior of a Voronoi cell  $V$  with center  $x$ , so  $d(x, y) < d(x', y)$  for any other  $x' \in X$ . Let  $g$  be a covering translation, which is an isometry, so  $d(x, y) = d(gx, gy)$ . As covering translations act freely, this implies that  $gy$  lies in the interior of the Voronoi cell corresponding to  $gx \neq x$ . Therefore  $\text{interior}(V)$  has disjoint translates under the group of covering translations, and so is mapped down homeomorphically into  $M$ , though the covering projection may identify distinct faces of a Voronoi cell under projection into  $M$ .

By abuse of notation, we shall refer to the resulting polyhedral decomposition of  $M$  as the Voronoi decomposition  $\mathcal{V}$  of  $M$ . By general position, we may assume that  $\mathcal{V}$  is regular. The dual triangulation also projects down to a triangulation of  $M$ , which we will also refer to as the dual triangulation, though this triangulation may no longer be simplicial.

We say a collection  $X = \{x_i\}$  of points in  $M$  is  $\epsilon$ -separated if the distance between any pair of points is at least  $\epsilon$ , i.e.,  $d(x_i, x_j) \geq \epsilon$  for all  $i \neq j$ .

**Definition 2.1.** Let  $M$  be a compact hyperbolic 3-manifold. We say a Voronoi decomposition  $\mathcal{V}$  is  $\epsilon$ -regular if it is regular and it arises from a maximal collection of  $\epsilon$ -separated points.

We shall write  $B(x, r)$  for the closed metric ball of radius  $r$  about  $x$  in  $M$ ,

$$B(x, r) = \{y \in M \mid d(x, y) \leq r\},$$

which need not be a topological ball. As the cells of an  $\epsilon$ -regular Voronoi decomposition are determined by a maximal collection of  $\epsilon$ -separated points in  $M$ , each Voronoi cell is contained in a metric ball of radius  $\epsilon$  about its center. Furthermore, as the points  $x_i$  are distance at least  $\epsilon$  apart, each Voronoi cell contains a metric ball of radius  $\epsilon/2$  about its center, i.e.,

$$B(x_i, \epsilon/2) \subset V_i \subset B(x_i, \epsilon).$$

One useful property of  $\epsilon$ -regular Voronoi decompositions is that the boundary of any union of Voronoi cells is an embedded surface, in fact an embedded normal surface in the dual triangulation, as we now describe.

A *simple arc* in the boundary of a tetrahedron is a properly embedded arc in a face of the tetrahedron with endpoints in distinct edges. A *triangle* in a tetrahedron is a properly embedded disc whose boundary is a union of three simple arcs, and a *quadrilateral* is a properly embedded disc whose boundary is the union of four

simple arcs. A *normal surface* in a triangulated 3-manifold is a surface that intersects each tetrahedron in a union of normal triangles and quadrilaterals.

**Proposition 2.2.** *Let  $M$  be a compact hyperbolic manifold, and let  $\mathcal{V}$  be an  $\epsilon$ -regular Voronoi decomposition. Let  $P$  be a union of Voronoi cells in  $\mathcal{V}$ , and let  $S$  be the boundary of  $P$ . Then  $S$  is an embedded surface in  $M$ .*

*Proof.* The collection of Voronoi cells  $P$  intersects a tetrahedron  $T$  in the dual triangulation in a regular neighborhood of the vertices of  $T$ . If a tetrahedron  $T$  has one or three vertices corresponding to Voronoi cells in  $P$ , then  $S$  intersects  $T$  in a single normal triangle. If  $T$  has exactly two vertices corresponding to Voronoi cells in  $P$ , then  $S$  intersects  $T$  in a single normal quadrilateral. Therefore  $S$  consists of at most one triangle or quadrilateral in each tetrahedron, and so is an embedded normal surface.  $\square$

We shall write  $\text{inj}_M(x)$  for the injectivity radius of  $M$  at  $x$ , i.e., the radius of the largest embedded ball in  $M$  centered at  $x$ . We shall write  $\text{inj}(M)$  for the injectivity radius of  $M$ , which is defined to be

$$\text{inj}(M) = \inf_{x \in M} \text{inj}_M(x).$$

We shall say a Voronoi cell  $V_i$  with center  $x_i$  is a *deep* Voronoi cell if the injectivity radius at  $x_i$  is at least  $4\epsilon$ , i.e.,  $\text{inj}_M(x_i) \geq 4\epsilon$ , and, in particular, this implies that the metric ball  $B(x_i, 3\epsilon)$  is a topological ball. We shall also call centers, faces, edges and vertices of deep Voronoi cells *deep*. We shall write  $\mathcal{W}$  for the subset of  $\mathcal{V}$  consisting of deep Voronoi cells. The fact that a deep Voronoi cell  $V_i$  has injectivity radius at least  $4\epsilon$  at its center  $x_i$  guarantees that every adjacent Voronoi cell is also a topological ball.

We now show that there are bounds, which only depend on  $\epsilon$ , on the number of faces of a deep Voronoi cell, and the number of edges and faces of a deep Voronoi cell.

**Proposition 2.3.** *Let  $M$  be a compact hyperbolic 3-manifold with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ , and let  $\mathcal{W}$  be the collection of deep Voronoi cells. Then there is a number  $J$ , which only depends on  $\epsilon$ , such that each deep Voronoi cell  $W_i \in \mathcal{W}$  has at most  $J$  faces, edges and vertices.*

*Proof.* Let  $W$  be a deep Voronoi cell with center  $x$ , and with faces  $F_1, \dots, F_n$ . Let  $x_i$  be the center of the Voronoi cell  $W_i$  adjacent to the face  $F_i$ . As  $W$  is deep, the Voronoi cell  $W_i$  is also a topological ball.

If two Voronoi cells share a common face, then the distance between their centers is at most  $2\epsilon$ . Therefore all of the centers of the Voronoi cells corresponding to the faces of  $W$  are contained in the metric ball  $B(x, 2\epsilon)$ . This implies that the balls



of radius  $\epsilon/2$  around the  $x_i$  are contained in the metric ball  $B(x, 5\epsilon/2)$ . As the  $B(x_i, \epsilon/2)$  are all disjoint, this implies that the number of faces is at most

$$J_1 = \frac{\text{vol}_{\mathbb{H}^3}(B(x, 5\epsilon/2))}{\text{vol}_{\mathbb{H}^3}(B(x, \epsilon/2))}.$$

Note that  $J_1$  is also an upper bound for the maximum number of edges in any face of a Voronoi cell because every edge of that face is contained in another face in that cell. So the total number of edges is at most  $J_1^2$ , and by the formula for Euler characteristic, the number of vertices is at most  $J_1^2 + C$ . Therefore we may choose  $J$  to be  $J_1^2 + 2$ .  $\square$

A similar volume bound argument to the one above proves the following:

**Proposition 2.4.** *Let  $M$  be a compact hyperbolic 3-manifold with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ . Then there is a number  $L$ , which depends only on  $\epsilon$ , such that for any deep Voronoi center  $x_i$ , the number of Voronoi centers contained in  $B(x_i, 3\epsilon)$  is at most  $L$ .*

**Polyhedral surfaces.** We may choose a Morse function  $f: M \rightarrow \mathbb{R}$  such that the complexity of  $f$  is within some small  $\delta > 0$  of the infimum, i.e.,

$$\text{area}(F_t) \leq \text{Morse area}(M) + \delta$$

for all  $t \in \mathbb{R}$ . We now describe how to use the Morse function  $f$  to give a linear ordering to the Voronoi cells in  $\mathcal{V}$ .

**Definition 2.5.** Let  $M$  be a compact hyperbolic 3-manifold, and let  $f: M \rightarrow \mathbb{R}$  be a Morse function. Given  $t \in \mathbb{R}$ , define the *sublevel set of  $M$  at  $t$* , which we shall denote  $M_t$ , to be the subset of  $M$  consisting of the union of all level sets  $F_t$  with  $t \in (-\infty, t]$ , i.e.,

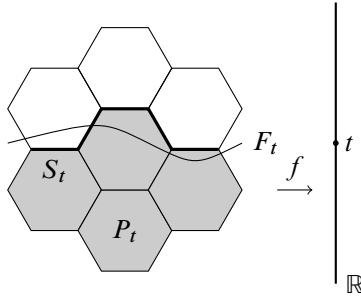
$$M_t = f^{-1}((-\infty, t]).$$

For  $t$  sufficiently small,  $M_t$  is the empty set, and for  $t$  sufficiently large,  $M_t$  is equal to all of  $M$ . The region  $M_t$  varies continuously in  $t$  and is monotonically increasing in  $t$ .

**Definition 2.6.** Let  $M$  be a compact hyperbolic 3-manifold with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ . Let  $f: M \rightarrow \mathbb{R}$  be a Morse function. For each Voronoi cell  $V_i$  with center  $x_i$ , there is a unique  $t_i \in \mathbb{R}$  such that the surface  $F_{t_i}$  divides the metric ball  $B(x_i, \epsilon/2)$  exactly in half by volume, i.e.,

$$\text{vol}(M_t \cap B(x_i, \epsilon/2)) = \frac{1}{2} \text{vol}(B(x_i, \epsilon/2)).$$

We call this  $t_i$  the *cell splitter* of  $V_i$ .



**Figure 1.** A polyhedral surface  $S_t$  determined by a level set  $F_t$ .

**Definition 2.7.** We say that a Morse function  $f: M \rightarrow \mathbb{R}$  is *generic* with respect to a Voronoi decomposition  $\mathcal{V}$  if the cell splitters for distinct Voronoi cells  $V_i$  correspond to distinct points  $t_i \in \mathbb{R}$ , and no cell splitter is also a critical point for the Morse function. We say a point  $t \in \mathbb{R}$  is *generic* if it is not a critical point for the Morse function, and is not a cell splitter.

We may assume that  $f$  is generic by an arbitrarily small perturbation of  $f$ , and we shall always assume that  $f$  is generic from now on.

**Definition 2.8.** Let  $M$  be a compact hyperbolic 3-manifold with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ , and let  $f: M \rightarrow \mathbb{R}$  be a generic Morse function. Let  $\mathcal{V}$  be the Voronoi decomposition ordered by the order inherited from the cell splitters  $t_i$ . Given  $t \in \mathbb{R}$ , Let  $M_t$  be the sublevel set of  $M$  at  $t$ . We define  $P_t$ , the *polyhedral approximation to  $M_t$* , to be the union of the Voronoi cells  $V_i$  with  $t_i \leq t$ , and call  $S_t = \partial P_t$  the *polyhedral surface* determined by  $t \in \mathbb{R}$ .

The polyhedral surface  $S_t$  is a union of faces of the Voronoi cells, and so is a normal surface in the dual triangulation. We shall write  $\|S_t\|$  for the number of Voronoi faces the polyhedral surface  $S_t$  contains. We shall write  $\|S_t \cap \mathcal{W}\|$  for the number of faces in the polyhedral surface  $S_t \cap \mathcal{W}$ , which may have boundary. A schematic picture of a polyhedral surface is given in [Figure 1](#).

In this section, we will show the following bound on the complexity of the polyhedral surface in the deep part  $\mathcal{W}$ .

**Proposition 2.9.** *Let  $M$  be a compact hyperbolic 3-manifold, with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ , deep part  $\mathcal{W}$ , and a generic Morse function  $f: M \rightarrow \mathbb{R}$ . For  $t \in \mathbb{R}$ , let  $S_t$  be the polyhedral surface associated to  $t$ . Then there is a constant  $K$ , which only depends on  $\epsilon$ , such that*

$$|\partial(S_t \cap \mathcal{W})| \leq K \text{ area}(F_t),$$

$$\text{genus}(S_t \cap \mathcal{W}) \leq K \text{ area}(F_t).$$

In particular, this bounds the genus of  $S_t \cap \mathcal{W}$  as a constant times the Morse width of  $M$ , where the constant depends only on  $\epsilon$ . We start by showing that the area of the level sets bounds the number of faces of the polyhedral surface in the deep part  $\mathcal{W}$ .

**Proposition 2.10.** *Let  $M$  be a compact hyperbolic 3-manifold, with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ , deep part  $\mathcal{W}$ , and a generic Morse function  $f: M \rightarrow \mathbb{R}$ . For  $t \in \mathbb{R}$ , let  $S_t$  be the polyhedral surface associated to  $t$ . Then there is a constant  $K$ , which only depends on  $\epsilon$ , such that*

$$\|S_t \cap \mathcal{W}\| \leq K \text{area}(F_t).$$

*Proof.* Let  $P_t$  be the polyhedral approximation to  $M_t$ . Let  $C$  be a face of  $S_t \cap \mathcal{W}$ , and let  $W_i$  and  $W_j$  be the two adjacent Voronoi cells in  $\mathcal{V}$ . Up to relabeling, we may assume that  $W_i$  is contained in  $P_t$ , and  $W_j$  is not. Let  $\gamma$  be a geodesic connecting  $x_i$  to  $x_j$ , and consider  $B(s, \epsilon/2)$  for  $s \in \gamma$ . As  $W_i$  and  $W_j$  are deep, the metric balls  $B(x_i, \epsilon/2)$ ,  $B(x_j, \epsilon/2)$  and  $B(s, \epsilon/2)$  are all topological balls, isometric to the ball  $B(x, \epsilon/2)$  in  $\mathbb{H}^3$ . At least half of the volume of  $B(x_i, \epsilon/2)$  is contained in  $P_t$ , and strictly less than half of the volume of  $B(x_j, \epsilon/2)$  is contained in  $P_t$ , so there is some  $s \in \gamma$  such that exactly half the volume of  $B(s, \epsilon/2)$  is contained in  $P_t$ . There is a constant  $A$ , depending only on  $\epsilon$ , such that any surface dividing a ball in hyperbolic space into regions of equal volume has area at least  $A$ . In fact, we may take  $A$  to be the area of the equatorial disc, which is  $2\pi(\cosh(\epsilon/2) - 1)$ ; see, for example, [Bachman et al. 2004].

Recall that the Voronoi decomposition has a dual triangulation in which each edge is a geodesic segment, and we shall write  $\Gamma$  for the geodesic graph in  $M$  formed by the 1-skeleton of the dual triangulation. We shall write  $\Gamma_d$  for the subset of  $\Gamma$  consisting of vertices corresponding to deep Voronoi cells, and edges connecting two deep Voronoi cells, and we shall refer to this as the *deep graph*. Each geodesic edge between two deep Voronoi cells has length strictly less than  $2\epsilon$ . Therefore the choice of geodesic is unique for the Voronoi cells in  $\mathcal{W}$ , as its length is smaller than the injectivity radius at each deep Voronoi cell center  $x_i$ . By Proposition 2.3, the geodesic dual graph  $\Gamma_d$  has valence at most  $J$ .

**Claim 2.11.** Consider a collection of points  $\{s_i\}$  such that each point  $s_i$  lies in a distinct edge  $\gamma_i$  of the deep graph  $\Gamma_d$ . Then any ball  $B(s_i, \epsilon/2)$  intersects at most  $L$  other balls  $B(s_j, \epsilon/2)$ , where  $L$  is the constant from Proposition 2.4.

*Proof of Claim 2.11.* If two balls  $B(s_i, \epsilon/2)$  and  $B(s_j, \epsilon/2)$  intersect, then the distance between their corresponding edges  $\gamma_i$  and  $\gamma_j$  is at most  $\epsilon$ , and so there is a pair of vertices,  $x_k \in \gamma_i$  and  $x_l \in \gamma_j$  with  $d(x_k, x_l) \leq 3\epsilon$ . By Proposition 2.4, there are at most  $L$  other vertices within distance  $3\epsilon$  of a given vertex. Therefore the total number of balls intersecting  $B(s_i, \epsilon/2)$  is at most  $L$ , which only depends on  $\epsilon$ .  $\square$

If there are  $N$  faces in  $S_t$  then there are at least  $N/L$  disjoint balls  $B(s_i, \epsilon/2)$ , each containing a part of  $F_t$  of area at least  $A$ . Therefore, the total number of faces is at most

$$(2) \quad \|S_t \cap \mathcal{W}\| \leq \frac{L}{A} \text{area}(F_t),$$

where the constants only depend on  $\epsilon$ , as required.  $\square$

We now show that the bound on the number of faces of  $S_t$  in the deep part  $\mathcal{W}$  gives a bound on the genus of  $S_t \cap \mathcal{W}$ .

**Proposition 2.12.** *Let  $M$  be a compact hyperbolic 3-manifold, with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$ , deep part  $\mathcal{W}$ , and a generic Morse function  $f: M \rightarrow \mathbb{R}$ . For  $t \in \mathbb{R}$ , let  $S_t$  be the polyhedral surface associated to  $t$ . Then there is a constant  $J$ , which only depends on  $\epsilon$ , such that*

$$\begin{aligned} |\partial(S_t \cap \mathcal{W})| &\leq J \|S_t \cap \mathcal{W}\|, \\ \text{genus}(S_t \cap \mathcal{W}) &\leq J \|S_t \cap \mathcal{W}\|, \end{aligned}$$

where  $J$  is the constant from [Proposition 2.3](#).

*Proof.* We shall write  $S$  for  $S_t$  to simplify notation. The first bound follows as each boundary component must contain at least one edge, so the number of boundary components is at most the number of edges in  $S \cap \mathcal{W}$ , which is at most  $J \|S \cap \mathcal{W}\|$  by [Proposition 2.3](#).

We shall write  $\widehat{S}$  for the surface  $S \cap \mathcal{W}$  with all boundary curves capped off with discs. Recall that the genus of a disconnected surface is the sum of the genera of each component, and this in turn is equal to the number of connected components minus half the Euler characteristic, i.e.,

$$\text{genus}(\widehat{S}) = |\widehat{S}| - \frac{1}{2} \chi(\widehat{S}),$$

where  $|\widehat{S}|$  is the number of connected components of  $\widehat{S}$ .

As capping off with discs does not change the number of connected components, this is at most the number of connected components of  $S \cap \mathcal{W}$ , which is at most the number of faces  $\|S \cap \mathcal{W}\|$ . Furthermore, capping off boundary components with discs may only increase the Euler characteristic, so

$$\text{genus}(\widehat{S}) \leq \|S \cap \mathcal{W}\| - \frac{1}{2} \chi(S \cap \mathcal{W}).$$

Therefore

$$\text{genus}(\widehat{S}) \leq \|S \cap \mathcal{W}\| - \frac{1}{2}(V - E + F),$$

where  $V$ ,  $E$  and  $F$  are the numbers of vertices, edges and faces of  $S \cap \mathcal{W}$ . As each face of a deep Voronoi cell has at most  $J$  edges, this implies

$$\text{genus}(\widehat{S}) \leq (1 + J/2) \|S \cap \mathcal{W}\|.$$

As we may assume that  $J$  is at least 2, this gives the second inequality.  $\square$

**Proposition 2.9** now follows immediately from Propositions 2.10 and 2.12.

**Capped surfaces.** We have constructed surfaces with bounded complexity in the deep part. The complement of the deep part is contained in a union of solid tori by the Margulis lemma, and we now explain how to cap off the surfaces in the deep part with surfaces in the solid tori to produce bounded genus surfaces.

We will use the Margulis lemma and the *thick-thin* decomposition for finite volume hyperbolic 3-manifolds, which we now review. Given a number  $\mu > 0$ , let  $X_\mu = M_{[\mu, \infty)}$  be the *thick part* of  $M$ , i.e., the union of all points  $x$  of  $M$  with  $\text{inj}_M(x) \geq \mu$ . We shall refer to the closure of the complement of the thick part as the *thin part* and denote it by  $T_\mu = \overline{M \setminus X_\mu}$ .

The Margulis lemma states that there is a constant  $\mu_0 > 0$ , such that for any compact hyperbolic 3-manifold, the thin part is a disjoint union of solid tori, and each of these solid tori is a regular metric neighborhood of an embedded closed geodesic of length less than  $\mu_0$ . We shall call a number  $\mu_0$  for which this result holds a *Margulis constant* for  $\mathbb{H}^3$ . If  $\mu_0$  is a Margulis constant for  $\mathbb{H}^3$ , then so is  $\mu$  for any  $0 < \mu < \mu_0$ , and furthermore, given  $\mu$  and  $\mu_0$ , there is a number  $\delta > 0$  such that  $N_\delta(T_\mu) \subseteq T_{\mu_0}$ . For the remainder of this section, we shall fix a pair of numbers  $(\mu, \epsilon)$  such that there are Margulis constants  $0 < \mu_1 < \mu < \mu_2$ , a number  $\delta$  such that  $N_\delta(T_\mu) \subseteq T_{\mu_2} \setminus T_{\mu_1}$ , and  $\epsilon = \frac{1}{4} \min\{\mu_1, \delta\}$ . We shall call  $(\mu, \epsilon)$  a choice of *MV-constants* for  $\mathbb{H}^3$ . This choice of constants ensures that the deep part  $\mathcal{W}$  is nonempty.

Let  $(\mu, \epsilon)$  be a choice of *MV-constants*, and consider an  $\epsilon$ -regular Voronoi decomposition of  $M$ . The fact that  $N_\delta(T_\mu) \subseteq T_{\mu_2} \setminus T_{\mu_1}$  means that we adjust the boundary of  $T_\mu$  by an arbitrarily small isotopy so that it is transverse to the Voronoi cells, and we will assume that we have done this for the remainder of this section. Our choice of  $\epsilon$  implies that the thick part  $X_\mu$  is contained in the Voronoi cells in the deep part, i.e.,  $X_\mu \subset \bigcup_{W_i \in \mathcal{W}} W_i$ , so, in particular,  $\partial X_\mu = \partial T_\mu$  is contained in the deep part. Furthermore, as  $\epsilon < \delta$ , each deep Voronoi cell hits at most one component of  $T_\mu$ .

Each boundary component of the surface  $S_t \cap X_\mu$  is contained in  $T_\mu$ , so  $S_t \cap X_\mu$  is a properly embedded surface in  $X_\mu$ . We now bound the number of boundary components of  $S_t \cap X_\mu$  in terms of the number of polyhedral faces in the deep part,  $\|S_t \cap \mathcal{W}\|$ .

**Proposition 2.13.** *Let  $(\mu, \epsilon)$  be MV-constants, and let  $M$  be a compact hyperbolic 3-manifold with thin part  $T_\mu$ , an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$  with deep part  $\mathcal{W}$ , and a generic Morse function  $f$ . Let  $S_t$  be a polyhedral surface in  $M$ . Then there is a constant  $J$ , depending only on  $\epsilon$ , such that*

$$\begin{aligned} \text{genus}(S_t \cap X_\mu) &\leq J \|S_t \cap \mathcal{W}\|, \\ |\partial(S_t \cap X_\mu)| &\leq 2J \|S_t \cap \mathcal{W}\|. \end{aligned}$$

*Proof.* The properly embedded surface  $S_t \cap X_\mu$  is obtained from  $S_t \cap \mathcal{W}$  by cutting  $S_t \cap \mathcal{W}$  along simple closed curves and discarding some connected components. This does not increase the genus, which gives the first bound, using [Proposition 2.12](#).

Each face  $C$  of a Voronoi cell is a totally geodesic convex polygon, and a component of  $T_\mu$  lifts to a convex set in the universal cover  $\mathbb{H}^3$ , so  $C \cap T_\mu$  is a convex subset of  $C$ . Therefore  $C \cap \partial T_\mu$  consists either of a simple closed curve, or a collection of properly embedded arcs which have at most two endpoints in each edge of  $C$ , so there are at most as many arcs as the number of edges of  $C$ . Therefore, the number of components of  $C \cap \partial T_\mu$  has at most (number of faces of  $S_t \cap \mathcal{W}$ ) plus (number of edges of  $S_t \cap \mathcal{W}$ ) components, and this gives the second bound, again using [Proposition 2.12](#).  $\square$

We now wish to cap off the properly embedded surfaces  $S_t \cap X_\mu$  with properly embedded surfaces in  $T_\mu$  to form closed surfaces. For each torus  $T_i$  in  $\partial T_\mu$ , let  $U_i$  be the subsurface consisting of  $\partial T_i \cap M_t$ . Let  $S_t^+ = (S \cap X_\mu) \cup \bigcup_i U_i$ , and we shall call the resulting closed surface the *T-capped surface*  $S_t^+$ . We now bound the genus of the resulting *T-capped surfaces*.

**Proposition 2.14.** *Let  $(\mu, \epsilon)$  be MV-constants, and let  $M$  be a compact hyperbolic 3-manifold with thin part  $T_\mu$ , an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$  with deep part  $\mathcal{W}$ , and a generic Morse function  $f$ . Let  $S_t$  be a polyhedral surface in  $M$ , and let  $S_t^+$  be the corresponding *T-capped surface*. Then there is a constant  $K$ , depending only on  $\epsilon$ , such that*

$$\text{genus}(S_t^+) \leq K \text{ area}(F_t).$$

*Furthermore, for any finite collection of generic points  $\{u_i\}$  in  $\mathbb{R}$ , the corresponding *T-capped surfaces*  $\{S_{u_i}^+\}$  may be isotoped to be disjoint.*

*Proof.* By [Proposition 2.10](#), it suffices to bound the genus of the *T-capped surface* in terms of the number of polyhedral faces of the surface in the deep part. We will show

$$\text{genus}(S_t^+) \leq (5J + 1) \|S_t \cap \mathcal{W}\|,$$

where  $J$  is the constant from [Proposition 2.3](#), which only depends on  $\epsilon$ .

Each surface  $U_i$  is a subsurface of a torus, and so consists of a union of planar surfaces, together with at most one surface which is a torus with (possibly many) boundary components.

Capping off components of  $S_t \cap X_\mu$  with planar surfaces cannot increase the genus by more than twice the number of boundary components, and capping off with punctured tori increases the genus by at most the number of boundary components, plus the number of punctured tori. As each Voronoi cell hits at most one component of  $T_\mu$ , there are at most  $\|S_t \cap \mathcal{W}\|$  components of the  $U_i$  surfaces which may be

punctured tori. This implies

$$\text{genus}(S_t^+) \leq \text{genus}(S_t \cap X_\mu) + 2|\partial(S_t \cap X_\mu)| + \|S_t \cap \mathcal{W}\|.$$

Using the bounds from [Proposition 2.13](#), we obtain

$$\text{genus}(S_t^+) \leq (5J + 1)\|S_t \cap \mathcal{W}\|,$$

as required.

Finally, we show that for any finite collection of generic points  $\{u_i\}$  in  $\mathbb{R}$ , we may isotope the corresponding  $T$ -capped surfaces to be disjoint. To simplify notation, given a generic point  $u_i \in \mathbb{R}$ , we will write  $M_i$  and  $S_i$  for the corresponding polyhedral approximation and polyhedral surface determined by  $u_i$ .

For any two distinct points  $u_i < u_j$  in  $\mathbb{R}$ , the polyhedral approximation  $M_i$  is a strict subset of  $M_j$ , so  $S_i$  and  $S_j$  are disjoint normal surfaces. Let  $T$  be a single solid torus component of  $T_\mu$ . Take a small product neighborhood  $\partial T \times [0, 1]$ , and choose the parameterization such that  $\partial T \times \{0\}$  is equal to  $\partial T$ , and the product neighborhood is contained in  $T$ . Let  $U_i$  be the subsurface of  $\partial T$  given by  $\partial T \cap M_i$ . Let  $U_i^+$  be the properly embedded surface in the product  $\partial T \times [0, 1]$  given by placing  $U_i$  at depth  $i/n$ , together with a product neighborhood of the boundary  $\partial U_i$  connecting  $U_i$  to the boundary of  $S_i$ , i.e.,

$$U_i^+ = (U_i \times \{i/n\}) \cup (\partial U_i \times [0, i/n]).$$

As the submanifolds  $M_i$  are strictly nested, the subsurfaces  $U_i$  are also strictly nested, i.e.,  $U_i \subset U_j$  for  $i < j$ , and so the resulting surfaces  $S_i \cup U_i^+$  are disjoint.  $\square$

**Bounded handles.** We now bound the number of handles between a pair of  $T$ -capped surfaces  $S_i^+$  and  $S_j^+$  whose corresponding points in  $u_i$  and  $u_j$  in  $\mathbb{R}$  bound an interval containing a single cell splitter.

**Proposition 2.15.** *Let  $(\mu, \epsilon)$  be MV-constants and  $M$  be a hyperbolic 3-manifold with an  $\epsilon$ -regular Voronoi decomposition  $\mathcal{V}$  and a generic Morse function  $f: M \rightarrow \mathbb{R}$ . Let  $u_1 < u_2$  be a pair of points in  $\mathbb{R}$ , which bound an interval containing a single cell splitter  $t$ . Let  $S_1^+$  and  $S_2^+$  be  $T$ -capped surfaces corresponding to the level sets for  $u_1$  and  $u_2$ , bounding regions  $P_1$  and  $P_2$ , with  $P_1 \subset P_2$ . Then  $P_2$  is homeomorphic to a manifold obtained from  $P_1$  by adding at most  $60J^2 \max\{\|S_i \cap \mathcal{W}\|\}$  handles, where  $J$  is the constant from [Proposition 2.3](#), which depend only on  $\epsilon$ .*

We start by observing that attaching a compression body  $P$  to a 3-manifold  $Q$  by a subsurface  $S$  of the upper boundary component of  $P$ , requires a number of handles which is bounded in terms of the Heegaard genus of  $P$ , and the number of boundary components of the attaching surface.

**Proposition 2.16.** *Let  $Q$  be a compact 3-manifold with boundary and let  $R = Q \cup P$ , where  $P$  is a compression body of genus  $g$ , attached to  $Q$  by a homeomorphism*

along a (possibly disconnected) subsurface  $S$  contained in the upper boundary component of  $P$  of genus  $g$ . Then  $R$  is homeomorphic to a 3-manifold obtained from  $Q$  by the addition of at most  $(4 \text{ genus}(P) + 2|\partial S|)$  1- and 2-handles, where  $|\partial S|$  is the number of boundary components of  $S$ .

*Proof.* Recall that the genus of a disconnected surface with boundary is the sum of the genus of each closed component obtained by capping off all boundary components with discs. Therefore, the genus of  $S$  is at most the genus of  $P$ . For a connected surface of genus  $g$  with  $b$  boundary components, cutting along a nonseparating arc with endpoints in the same boundary component produces a surface of genus  $g - 1$  with  $b + 1$  boundary components. A planar surface with  $b$  boundary components may be cut into at most  $b$  discs by  $b - 1$  nonseparating arcs. Therefore we may choose at most  $2g + b$  arcs which cut the surface  $S$  into at most  $g + b$  discs. We can add a 1-handle to  $Q$  for each arc, and then a 2-handle for each disc, to produce a manifold  $Q_+$  which is homeomorphic to  $Q$  union a regular neighborhood of  $\partial P$ . We may then form  $R$  by adding at most  $g$  2-handles. The total number of 1- and 2-handles required is at most  $4g + 2b$ .  $\square$

*Proof of Proposition 2.15.* Let  $V$  be the Voronoi cell corresponding to the single cell splitter  $t$  contained in the interval  $[u_1, u_2]$ . The surfaces  $S_1^+$  and  $S_2^+$  are parallel everywhere, except in a regular neighborhood of  $V$ . If the Voronoi cell  $V$  is disjoint from  $T_\mu$ , then it is a ball, and is attached to  $P_1$  along a subsurface consisting of a union of faces of  $V$ . Therefore the number of boundary components of the attaching surface is at most  $J$ , where  $J$  is the constant from Proposition 2.3, so by Proposition 2.16,  $P_2$  is obtained from  $P_1$  by attaching at most  $2J$  handles.

If the Voronoi cell  $V$  intersects  $T_\mu$ , then  $P_2$  is obtained from  $P_1$  by adding regions of  $V \setminus T_\mu$ , which we shall refer to as the *complementary regions*, together with regions of  $T_\mu \cap (P_2 \setminus P_1)$ . The complementary regions may not be topological balls, but Kobayashi and Rieck [2011] show that they are handlebodies of bounded genus.

**Proposition 2.17** [Kobayashi and Rieck 2011]. *Let  $\mu$  be a Margulis constant for  $\mathbb{H}^3$  and  $M$  be a finite volume hyperbolic 3-manifold; let  $0 < \epsilon < \mu$ , and let  $\mathcal{V}$  be a regular Voronoi decomposition of  $M$  arising from a maximal collection of  $\epsilon$ -separated points. Then there is a number  $G$ , depending only on  $\mu$  and  $\epsilon$ , such that for any Voronoi cell  $V_i$ , there are at most  $G$  connected components of  $V_i \cap X_\mu$ , each of which is a handlebody of genus at most  $G$ , attached to  $T_\mu$  by a surface with at most  $G$  boundary components.*

We state a simplified version of their result which suffices for our purposes. Their stated result involves extra parameters  $d$  and  $R$ , but if  $d$  is chosen close to 0, then  $R$  is close to  $\mu$ , and we obtain the result above. Their proof involves showing that in the universal cover, for any point  $p$  in  $T_\mu \cap V_i$ , projection to  $\partial T_\mu$  along geodesic rays



based at  $p$  gives a topological product structure to  $V_i \cap X_\mu$  as  $(V_i \cap \partial T_\mu) \times I$ . An examination of their proof shows that we may choose  $G = 3J$ , where  $J$  is the constant from [Proposition 2.3](#). Then [Proposition 2.16](#) implies that adding the complementary regions of a Voronoi cell which intersects  $\partial T_\mu$  requires at most  $6G^2 = 54J^2$  handles.

However, if the Voronoi cell intersects a solid torus component  $T$  of  $T_\mu$ , then the surfaces  $S_1^+$  and  $S_2^+$  need not be parallel inside  $T$ , and so we now bound the number of handles needed to add the region corresponding to  $(P_2 \setminus P_1) \cap T$ . If  $U_2$  is equal to all of  $\partial T$ , then the additional region is a solid torus attached along  $\partial T \setminus U_1$ , so adding this region requires at most  $4 + 2|\partial U_1|$  handles, by [Proposition 2.15](#). If  $U_2$  is not equal to all of  $\partial T$ , then this region is homeomorphic to  $(U_2 \times [0, 1]) \setminus (U_1 \times [0, \frac{1}{2}])$ , and so is homeomorphic to  $U_2 \times I$ , which is a handlebody of genus at most  $|\partial U_2|$ . The region is attached along  $U_2$ , so adding this region requires at most  $4|\partial U_2| + 2|\partial U_1|$  handles, and so in either case, at most  $4J\|S_2 \cap \mathcal{W}\| + 2J\|S_1 \cap \mathcal{W}\|$  are required.

Therefore  $P_2$  may be constructed from  $P_1$  by adding at most

$$54J^2 + 4J\|S_2 \cap \mathcal{W}\| + 2J\|S_1 \cap \mathcal{W}\| \leq 60J^2 \max\{\|S_i \cap \mathcal{W}\|\}$$

handles, as required.  $\square$

The manifold  $M$  may be constructed by adding the Voronoi cells in the order arising from the cell splitters  $t_i$  in  $\mathbb{R}$ . Choose a finite collection of generic points  $\{u_i\}$ , so that each pair of adjacent cell splitters is separated by one of the  $u_i$ , and let  $\{S_i^+\}$  be the corresponding collection of  $T$ -capped surfaces. The linear width is at most the largest genus of any surface in the collection  $\{S_i^+\}$ , plus the maximum number of handles added by attaching a single Voronoi cell. Therefore the bounds from [Propositions 2.14](#) and [2.15](#) imply

$$\text{linear width}(M) \leq (5J + 1)K(\text{Morse area}(M)) + 60J^2 K(\text{Morse area}(M)).$$

As  $J$  is at least 1, this gives

$$\text{linear width}(M) \leq (66J^2 K) \text{Morse width}(M).$$

The constants  $J$  and  $K$  only depend on the choice of  $MV$ -constants, which may be chosen independently of the hyperbolic 3-manifold  $M$ , and so this completes the proof of the left-hand bound of [Theorem 1.2](#).

### 3. Scharlemann–Thompson width bounds Morse area

We will show linear bounds for Morse area in terms of Scharlemann–Thompson width, assuming the Pitts and Rubinstein result, [Theorem 1.1](#); i.e., we will show the right-hand bound of [Theorem 1.2](#). This result is due to Gabai and Colding [[2015](#), Appendix A], using recent work of Colding and Minicozzi [[2015](#)], but we give a brief description for the convenience of the reader, as they do not state this result explicitly.

We will use properties of a refinement of linear width, known as *thin position*, which we now describe. Let  $\{H_i\}$  be the collection of upper boundaries of compression bodies in the linear splitting, and let  $c(H_i)$  be the complexity of the surface  $H_i$ , i.e., the sum of the genera of its connected components. We say that the complexity of the linear splitting is the collection of integers  $\{c(H_i)\}$ , arranged in decreasing order. A linear splitting which gives the minimum complexity of all possible linear splittings in the lexicographic ordering on sets of integers is called a *thin position* linear splitting. Scharlemann and Thompson [1994] showed that thin position linear splittings have the following property.

**Theorem 3.1** [Scharlemann and Thompson 1994]. *Let  $H$  be a linear splitting that is in thin position. Then every even surface is incompressible in  $M$  and the odd surfaces form strongly irreducible Heegaard surfaces for the components of  $M$  cut along the even surfaces.*

It follows from [Freedman et al. 1983] that the incompressible surfaces may be chosen to be disjoint least area minimal surfaces, and in fact the odd surfaces may also be chosen to be disjoint minimal surfaces, possibly up to compression; see, for example, [Lackenby 2006] or [Renard 2014] for a detailed statement of the result in this case. In a hyperbolic manifold, the intrinsic curvature of a minimal surface is at most  $-1$ , so the Gauss–Bonnet formula gives an upper bound for the area of the minimal surface. Therefore the area of a minimal surface of genus  $g$  is at most  $-2\pi\chi(S) \leq 4\pi g$ .

We say that a hyperbolic 3-manifold  $M$  has *least area boundary* if its boundary components are (possibly empty) least area minimal surfaces, and we say that a Heegaard splitting  $H$  for  $M$  is *minimal* if it is isotopic to an unstable minimal surface. The right-hand bound of Theorem 1.2 is a consequence of the following result of Colding and Gabai [2015], which constructs bounded area foliations for a pair of compression bodies with least area lower boundaries, sharing a common minimal Heegaard splitting surface.

**Theorem 3.2** [Colding and Gabai 2015]. *Let  $M$  be a hyperbolic manifold, with (possibly empty) least area boundary, with a minimal Heegaard splitting  $H$  of genus  $g$ . Then, assuming Theorem 1.1, the manifold  $M$  has a (possibly singular) foliation by compact leaves, containing the boundary surfaces as leaves, such that each leaf has area at most  $4\pi g$ .*

As they do not state this explicitly in their paper, we give a brief outline for the convenience of the reader.

**Definition 3.3.** A mean convex foliation on a Riemannian 3-manifold with boundary is a smooth codimension-1 foliation, possibly with singularities of standard type, such that each leaf is mean convex.

In a 3-manifold, a foliation with singularities of “standard type” means that almost all leaves are completely smooth (i.e., without any singularities). In particular, any connected subset of the singular set is completely contained in a leaf. Furthermore, the entire singular set is contained in finitely many (compact) embedded Lipschitz curves with cylinder singularities together with a countable set of spherical singularities.

The following result is shown by Colding and Gabai [2015].

**Theorem 3.4** [Colding and Gabai 2015, Appendix A]. *Let  $\Sigma$  be an unstable minimal surface in a hyperbolic manifold  $M$ . Then there is a regular neighborhood of  $\Sigma$  with a smooth mean convex product foliation  $\Sigma_t$ ,  $t \in [-\epsilon, \epsilon]$ , with nonminimal boundary leaves  $\Sigma_{-\epsilon}$  and  $\Sigma_\epsilon$ .*

In particular, each leaf in the foliation has area at most  $4\pi g$ . As the boundary leaves  $\Sigma_{-\epsilon}$  and  $\Sigma_\epsilon$  are nonminimal mean convex surfaces, we may apply the mean curvature flow results of Colding and Minicozzi [2015], which show that the mean curvature flow gives rise to a mean convex foliation with standard singularities. As the mean curvature flow gives a foliation by surfaces of decreasing area, the only possible singularities which may arise are disc compressions, 2-spheres collapsing to a point or tori collapsing to circles. In particular, each nonsingular leaf bounds a compression body in the interior of the compression body it is contained in.

If all leaves eventually collapse, then the compression body has empty lower boundary, i.e., it is a handlebody, and this gives a mean convex foliation, and hence area-decreasing foliation, of the handlebody. Otherwise, the mean curvature flow limits to a stable minimal surface  $\Gamma$  whose components bound compression bodies together with the lower boundary of the original compression body.

If the stable minimal surface  $\Gamma$  is not equal to the stable boundary of the compression body, then it bounds a subcompression body with stable boundary, whose standard Heegaard splitting is strongly irreducible, so we may apply the argument again. Anderson [1985] and White [1987] showed that there are only finitely many minimal surfaces of bounded genus in a compact Riemannian manifold, and so this process may occur only finitely many times, resulting in a foliation of the entire compression body. This completes the proof of Theorem 3.2.

Finally we deduce the right-hand bound of Theorem 1.2 from Theorem 3.2.

*Proof of right-hand bound of Theorem 1.2.* By Theorem 1.1, the irreducible Heegaard surface for a hyperbolic 3-manifold  $M$  with stable boundary is either isotopic to an unstable minimal surface  $\Sigma$ , to which we may apply Theorem 3.2 directly, or isotopic to a regular neighborhood of a one-sided stable minimal surface union a small tube parallel to one of the normal fibers. In the latter case, the Heegaard surface bounds a handlebody on at least one side, and cutting along the stable one-sided surface leaves a compression body homeomorphic to the Heegaard surface

cut along the disc corresponding to the tube, where all boundary components are stable minimal surfaces. As the standard Heegaard splitting of a compression body is strongly irreducible, we may now apply [Theorem 3.2](#) in this case as well.  $\square$

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