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# A VARIATIONAL CHARACTERIZATION OF FLAT SPACES IN DIMENSION THREE

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**We prove that, in dimension three, flat metrics are the only complete metrics with nonnegative scalar curvature which are critical for the $\sigma_2$-curvature functional.**

## 1. Introduction

Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 3$. To fix the notation, we recall the decomposition of the Riemann curvature tensor of a metric $g$ into the Weyl, Ricci, and scalar curvature components:

$$\mathrm{Rm} = W + \frac{1}{n-2}\mathrm{Ric} \owedge g - \frac{1}{(n-1)(n-2)} R g \owedge g,$$

where $\owedge$ denotes the Kulkarni–Nomizu product. It is well known [Hilbert 1915] that Einstein metrics are critical points for the Einstein–Hilbert functional

$$\mathcal{H} = \int R\, dV$$

on the space of unit volume metrics $\mathcal{M}_1(M^n)$. From this perspective, it is natural to study canonical metrics which arise as solutions of the Euler–Lagrange equations for more general curvature functionals. Berger [1970] commenced the study of Riemannian functionals which are quadratic in the curvature (see [Besse 2008, Chapter 4] and [Smolentsev 2005] for surveys). A basis for the space of quadratic curvature functionals is given by

$$\mathcal{W} = \int |W|^2\, dV, \qquad \rho = \int |\mathrm{Ric}|^2\, dV, \qquad \mathcal{S} = \int R^2\, dV.$$

All such functionals, which also naturally arise as total actions in certain gravitational field theories in physics, have been deeply studied in recent years by many authors, in particular on compact Riemannian manifolds with normalized volume (for instance, see [Berger 1970; Besse 2008; Lamontagne 1994; 1998; Anderson 1997; Gursky and Viaclovsky 2001; 2015; 2013; Catino 2015] and references therein).

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On the other hand, the study of critical metrics for quadratic curvature functionals also has a lot of interest in the noncompact setting. For instance, Anderson [2001] proved that every complete three-dimensional critical metric for the Ricci functional $\rho$ with nonnegative scalar curvature is flat; whereas, Catino [2014] showed a characterization of complete critical metrics for $\mathcal{S}$ with nonnegative scalar curvature in every dimension.

In this paper we focus our attention on the three-dimensional case and consider the $\sigma_2$-curvature functional

$$\mathscr{F}_2 = \int \sigma_2(A)\, dV,$$

where $\sigma_2(A)$ denotes the second elementary symmetric function of the eigenvalues of the Schouten tensor $A = \mathrm{Ric} - \frac{1}{4} R\, g$. This functional was first considered by Gursky and Viaclovsky in the compact three-dimensional case. In [2001] they proved a beautiful characterization theorem of space forms as critical metrics for $\mathscr{F}_2$ on $\mathscr{M}_1(M^3)$ with nonnegative energy $\mathscr{F}_2 \geq 0$.

The main result of this paper is the following variational characterization of three-dimensional flat spaces.

**Theorem 1.1.** *Let $(M^3, g)$ be a complete critical metric for $\mathscr{F}_2$ with nonnegative scalar curvature. Then $(M^3, g)$ is flat.*

We remark the fact that the nonnegativity condition on the scalar curvature cannot be removed. This is clear from the example in [loc. cit.] where the authors exhibit an explicit family of critical metrics for $\mathscr{F}_2$ on $\mathbb{R}^3$. For instance, the metric given in standard coordinates by

$$g = dx^2 + dy^2 + (1 + x^2 + y^2)^2 dz^2$$

is complete, critical and has strictly negative scalar curvature

$$R = -\frac{8}{1 + x^2 + y^2}.$$

## 2. The Euler–Lagrange equation for $\mathcal{F}_t$

In this section we will compute the Euler–Lagrange equation satisfied by critical metrics for $\mathscr{F}_2$. To begin, we observe that, in dimension $n \geq 3$, the second elementary symmetric function of the eigenvalues of the Schouten tensor

$$A = \frac{1}{n-2}\Big(\mathrm{Ric} - \frac{1}{2(n-1)} R\, g\Big)$$

can be written as

$$\sigma_2(A) = -\frac{1}{2(n-2)^2}|\mathrm{Ric}|^2 + \frac{n}{8(n-1)(n-2)^2} R^2.$$

In particular, the functional $\mathscr{F}_2$ is proportional to a general quadratic functional of the form

$$\mathcal{F}_t = \int |\mathrm{Ric}|^2 \, dV + t \int R^2 \, dV,$$

with the choice $t = -n/4(n-1)$; see also [Gursky and Viaclovsky 2015; Catino 2015]. The gradients of the functionals $\rho$ and $\mathcal{S}$, computed using compactly supported variations, are given by [Besse 2008, Proposition 4.66]

$$(\nabla\rho)_{ij} = -\Delta R_{ij} - 2R_{ikjl}R_{kl} + \nabla^2_{ij}R - \tfrac{1}{2}(\Delta R)g_{ij} + \tfrac{1}{2}|\mathrm{Ric}|^2 g_{ij}$$

and

$$(\nabla\mathcal{S})_{ij} = 2\nabla^2_{ij}R - 2(\Delta R)g_{ij} - 2RR_{ij} + \tfrac{1}{2}R^2 g_{ij}.$$

Hence, the gradient of $\mathcal{F}_t$ reads

$$(\nabla\mathcal{F}_t)_{ij} = -\Delta R_{ij} + (1+2t)\nabla^2_{ij}R - \tfrac{1}{2}(1+4t)(\Delta R)g_{ij} + \tfrac{1}{2}\big(|\mathrm{Ric}|^2 + tR^2\big)g_{ij} - 2R_{ikjl}R_{kl} - 2tRR_{ij}.$$

Tracing the equation $(\nabla\mathcal{F}_t) = 0$, we obtain

$$\big(n + 4(n-1)t\big)\Delta R = (n-4)\big(|\mathrm{Ric}|^2 + tR^2\big).$$

Defining the tensor $E$ to be the traceless Ricci tensor, $E_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}$, we obtain the Euler–Lagrange equation of critical metrics for $\mathcal{F}_t$.

**Proposition 2.1.** *Let $M^n$ be a complete manifold of dimension $n \geq 3$. A metric $g$ is critical for $\mathcal{F}_t$ if and only if it satisfies*

$$\Delta E_{ij} = (1+2t)\nabla^2_{ij}R - \frac{n+2+4nt}{2n}(\Delta R)g_{ij} - 2R_{ikjl}E_{kl} - \frac{2+2nt}{n}RE_{ij} + \frac{1}{2}\Big(|\mathrm{Ric}|^2 - \frac{4-n(n-4)t}{n^2}R^2\Big)g_{ij}$$

*and*

$$\big(n + 4(n-1)t\big)\Delta R = (n-4)\big(|\mathrm{Ric}|^2 + tR^2\big).$$

In dimension three we recall the decomposition of the Riemann curvature tensor

$$R_{ikjl} = E_{ij}g_{kl} - E_{il}g_{jk} + E_{kl}g_{ij} - E_{kj}g_{il} + \tfrac{1}{6}R(g_{ij}g_{kl} - g_{il}g_{jk}).$$

In particular,

$$R_{ikjl}E_{kl} = -2E_{ip}E_{jp} - \tfrac{1}{6}RE_{ij} + |E|^2 g_{ij}.$$

Hence, if $n = 3$ and $t = -n/4(n-1) = -3/8$, one has

$$\mathscr{F}_2 = -\tfrac{1}{2}\mathcal{F}_{-3/8},$$

and the following formulas hold.

**Proposition 2.2.** *Let $M^3$ be a complete manifold of dimension three. A metric $g$ is critical for $\mathfrak{F}_2$ if and only if it satisfies*

$$\Delta E_{ij} = \tfrac{1}{4}\nabla^2_{ij} R - \tfrac{1}{12}(\Delta R) g_{ij} + 4E_{ip}E_{jp} + \tfrac{5}{12} R E_{ij} - \tfrac{1}{2}\big(3|E|^2 - \tfrac{1}{72}R^2\big) g_{ij} \tag{2-1}$$

*and*

$$-2\sigma_2(A) = |\mathrm{Ric}|^2 - \tfrac{3}{8}R^2 = |E|^2 - \tfrac{1}{24}R^2 = 0. \tag{2-2}$$

Now, contracting (2-1) with $E$, we obtain the following Weitzenböck formula.

**Corollary 2.3.** *Let $M^3$ be a complete manifold of dimension three. If $g$ is a critical metric for $\mathfrak{F}_2$, then the following formula holds*

$$\tfrac{1}{2}\Delta|E|^2 = |\nabla E|^2 + \tfrac{1}{4}E_{ij}\nabla^2_{ij}R + 4E_{ip}E_{jp}E_{ij} + \tfrac{5}{12}R|E|^2. \tag{2-3}$$

## 3. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. We assume that $(M^3, g)$ is a critical metric for $\mathfrak{F}_2$ with nonnegative scalar curvature $R \geq 0$. In particular, $g$ has zero $\sigma_2$-curvature, i.e., $|E|^2 = \frac{1}{24}R^2$ and we obtain

$$\tfrac{1}{2}\Delta|E|^2 = \tfrac{1}{48}\Delta R^2 = \tfrac{1}{24}R\Delta R + \tfrac{1}{24}|\nabla R|^2.$$

Putting together this equation with (2-3), we obtain that the scalar curvature $R$ satisfies the PDE

$$\tfrac{1}{24}\big(Rg_{ij} - 6E_{ij}\big)\nabla^2_{ij}R = |\nabla E|^2 - \tfrac{1}{24}|\nabla R|^2 + 4E_{ip}E_{jp}E_{ij} + \tfrac{5}{12}R|E|^2. \tag{3-1}$$

To begin, we need the following purely algebraic lemmas.

**Lemma 3.1.** *Let $(M^3, g)$ be a Riemannian manifold with $R \geq 0$ and $\sigma_2(A) \geq 0$. Then,*

$$Rg_{ij} \geq 6E_{ij}$$

*and $g$ has nonnegative sectional curvature.*

*Proof.* Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of the Schouten tensor $A = E + \frac{1}{12}Rg$ at some point. Then, by the assumptions, we have

$$4R = \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 \geq 0 \quad \text{and} \quad \sigma_2(A) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \geq 0.$$

We want to show that $E \leq \frac{1}{6}Rg$ or, equivalently, that

$$A \leq \tfrac{1}{4}Rg = \mathrm{tr}(A)g.$$

Hence, it suffices to prove that $\lambda_3 \leq \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$, i.e., that $\lambda_1 + \lambda_2 \geq 0$. But this follows by

$$0 \leq \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = (\lambda_1 + \lambda_2)\,\mathrm{tr}(A) - (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) \leq (\lambda_1 + \lambda_2)\,\mathrm{tr}(A).$$

The fact that $g$ has nonnegative sectional curvature follows from the decomposition of the Riemann tensor in dimension three and the curvature condition $\mathrm{Ric} \le \frac{1}{2} R g$ (for instance see [Hamilton 1982, Corollary 8.2]). □

**Lemma 3.2.** *Let* $(M^3, g)$ *be a Riemannian manifold with* $R \ge 0$ *and* $\sigma_2(A) = \mathrm{const} \ge 0$. *Then*,

$$|\nabla E|^2 \ge \tfrac{1}{24} |\nabla R|^2.$$

*Proof.* We will follow the proof in [Gursky and Viaclovsky 2001, Lemma 4.1]. Let $p$ be a point in $M^3$. If $R(p) = 0$, then $\nabla R = 0$ and the lemma follows. So we can assume that $R(p) > 0$. Since $-2\sigma_2(A) = |E|^2 - \frac{1}{24} R^2 = \mathrm{const}$,

$$|E|^2 \big|\nabla |E|\big|^2 = \tfrac{1}{576} R^2 |\nabla R|^2. \tag{3-2}$$

By Kato's inequality $\big|\nabla |E|\big|^2 \le |\nabla E|^2$ and the fact that $|E|^2 \le \frac{1}{24} R^2$,

$$|E|^2 |\nabla E|^2 \ge \tfrac{1}{576} R^2 |\nabla R|^2 \ge \tfrac{1}{24} |E|^2 |\nabla R|^2.$$

By dividing by $|E|^2(p) \ne 0$, the result follows; otherwise, if $|E|(p) = 0$, then $(\nabla R)(p) = 0$ from (3-2), and we conclude. □

**Lemma 3.3.** *Let* $(M^3, g)$ *be a Riemannian manifold. Then*,

$$E_{ip} E_{jp} E_{ij} \ge -\frac{1}{\sqrt{6}} |E|^3.$$

*Proof.* For a proof of this lemma, for instance, see [op. cit., Lemma 4.2]. □

**Corollary 3.4.** *Let* $(M^3, g)$ *be a complete critical metric for* $\mathfrak{F}_2$ *with nonnegative scalar curvature. Then*, $R g_{ij} \ge 6 E_{ij}$, $g$ *has nonnegative sectional curvature, and the scalar curvature satisfies the differential inequality*

$$\big(R g_{ij} - 6 E_{ij}\big) \nabla^2_{ij} R \ge \tfrac{1}{12} R^3.$$

*Proof.* From (3-1), combining Lemmas 3.1, 3.2, and 3.3, we obtain

$$\frac{1}{24}\big(R g_{ij} - 6E_{ij}\big)\nabla^2_{ij} R \ge \frac{5}{12} R |E|^2 - \frac{4}{\sqrt{6}} |E|^3 = |E|^2 \Big(\frac{5}{12} R - \frac{4}{\sqrt{6}} |E|\Big) = \frac{1}{288} R^3,$$

where in the last equality we have used the fact that $|E|^2 = \frac{1}{24} R^2$. □

Now we can prove Theorem 1.1. Clearly, if $M^3$ is compact, from Corollary 3.4, at a maximum point of $R$ we obtain $R \le 0$. Hence, $R \equiv 0$ on $M^3$, and from (2-2), $\mathrm{Ric} \equiv 0$ and the metric is flat. So, from now on, we will assume the manifold $M^3$ to be noncompact.

Choose now $\phi = \phi(r)$ to be a function of the distance $r$ to a fixed point $O \in M^3$ and let $B_s(O)$ be a geodesic ball of radius $s > 0$. We denote by $C_O$ the cut locus at the point $O$ and we choose $\phi$ satisfying the following properties: $\phi = 1$ on $B_s(O)$, $\phi = 0$ on $M^3 \setminus B_{2s}(O)$,

$$-\frac{c}{s}\phi^{3/4} \le \phi' \le 0 \quad \text{and} \quad |\phi''| \le \frac{c}{s^2}\phi^{1/2}$$

on $B_{2s}(O) \setminus B_s(O)$ for some positive constant $c > 0$. In particular, $\phi$ is $C^3$ in $M^3 \setminus C_O$. Let $u := R\phi$ and $a_{ij} := (Rg_{ij} - 6E_{ij})$. From Corollary 3.4, we know that $a_{ij} \ge 0$ and we obtain

$$\begin{aligned}(3\text{-}3)\qquad a_{ij}\nabla^2_{ij}u &= a_{ij}\big(\phi\nabla^2_{ij}R + R\nabla^2_{ij}\phi + 2\nabla_i R\nabla_j\phi\big)\\ &\ge \tfrac{1}{12}R^3\phi + R\phi' a_{ij}\nabla^2_{ij}r + R\phi'' a(\nabla r, \nabla r) + 2a(\nabla R, \nabla\phi).\end{aligned}$$

Now, let $p_0$ be a maximum point of $u$ and assume that $p_0 \notin C_O$. If $\phi(p_0) = 0$, then $u \equiv 0$ and then $R \equiv 0$ on $B_{2s}(O)$. Hence, from now on we will assume $\phi(p_0) > 0$. Then, at $p_0$, we have $\nabla u(p_0) = 0$ and $\nabla^2_{ij}u(p_0) \le 0$. In particular, at $p_0$,

$$\nabla R(p_0) = -\frac{R(p_0)}{\phi(p_0)}\nabla\phi(p_0).$$

Moreover, since $a_{ij} \ge 0$, for every vector field $X$, $a(X, X) \le \mathrm{tr}(a)|X|^2 = 3R|X|^2$. On the other hand, from the standard Hessian comparison theorem, since $g$ has nonnegative sectional curvature, we know that on $M^3 \setminus C_O$, one has $\nabla^2_{ij}r \le \frac{1}{r}g_{ij}$. Thus, from (3-3), at $p_0$, we get

$$\begin{aligned}0 &\ge \frac{1}{12}R^3\phi + R\phi' a_{ij}\nabla^2_{ij}r + R\phi'' a(\nabla r, \nabla r) - 2\frac{R}{\phi}a(\nabla\phi, \nabla\phi)\\ &\ge \frac{1}{12}R^3\phi - \Big(\frac{|\phi'|}{r} + |\phi''| + 2\frac{(\phi')^2}{\phi}\Big)R\,\mathrm{tr}(a)\\ &\ge \frac{1}{12}R^3\phi - 3\Big(\frac{|\phi'|}{s} + |\phi''| + 2\frac{(\phi')^2}{\phi}\Big)R^2,\end{aligned}$$

where, in the last inequality, we have used the fact that $r \ge s$ on $B_{2s}(O) \setminus B_s(O)$, i.e., where $\phi' \ne 0$. From the assumptions on the cut-off function $\phi$, we obtain, at the maximum point $p_0$,

$$0 \ge \frac{1}{12}R^2\phi^{1/2}\Big(R\phi^{1/2} - \frac{c'}{s^2}\Big)$$

for some positive constant $c' > 0$. Thus, we have proved that, if $p_0 \notin C_O$, then for every $p \in B_{2s}(O)$

$$u(p) \le u(p_0) = R(p_0)\phi(p_0) \le \frac{c'}{s^2}.$$

If $p_0 \in C_O$ we argue as follows (this trick is usually referred to Calabi). Let $\gamma : [0, L] \to M^3$, where $L = d(p_0, O)$, be a minimal geodesic joining $O$ to $p_0$, the maximum point of $u$. Let $p_\varepsilon = \gamma(\varepsilon)$ for some $\varepsilon > 0$. Define now

$$u_\varepsilon(x) = R(x)\phi\big(d(x, p_\varepsilon) + \varepsilon\big).$$

Since $d(x, p_\varepsilon) + \varepsilon \geq d(x, O)$ and $d(p_0, p_\varepsilon) + \varepsilon = d(p_0, O)$, it is easy to see that $u_\varepsilon(p_0) = u(p_0)$ and

$$u_\varepsilon(x) \leq u(x) \quad \text{for all } x \in M^3,$$

since $\phi' \leq 0$. Hence $p_0$ is also a maximum point for $u_\varepsilon$. Moreover, $p_0 \notin C_{p_\varepsilon}$, so the function $d(x, p_\varepsilon)$ is smooth in a neighborhood of $p_0$ and we can apply the maximum principle argument as before to obtain an estimate for $u_\varepsilon(p_0)$ which depends on $\varepsilon$. Taking the limit as $\varepsilon \to 0$, we obtain the desired estimate on $u$.

By letting $s \to +\infty$ we obtain $u \equiv 0$, so $R \equiv 0$. From (2-2) we have $E \equiv 0$ and so $\mathrm{Ric} \equiv 0$ and Theorem 1.1 follows.

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