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CARTAN–FUBINI TYPE RIGIDITY OF DOUBLE COVERING MORPHISMS OF QUADRATIC MANIFOLDS

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Let $Z \subset \mathbb{P}^N$ be a Fano manifold whose Picard group is generated by the hyperplane section class. Assume that Z is covered by lines and $i(Z) \geq 3$. Let $\phi : X^Z \to Z$ be a double cover, branched along a smooth hypersurface section of degree $2m$, $1 \leq m \leq i(Z) - 2$. We describe the defining ideal of *the variety of minimal rational tangents* at a general point. As an application, we show that if $Z \subset \mathbb{P}^N$ is defined by quadratic equations and $2 \leq m \leq i(Z) - 2$, then the morphism ϕ satisfies the Cartan–Fubini type rigidity property.

1. Introduction

Throughout the paper, we will work over the field of complex numbers. Let X be a Fano manifold of Picard number 1. The index of X is the integer $i(X)$ such that $-K_X = i(X)L$ where L is the ample generator of the Picard group of X. For a general point $x \in X$, a rational curve through x is called a *minimal rational curve* if it has minimal K_X^{-1}-degree among all rational curves through x. Denote by $\mathcal{K}_x$ the normalized space of minimal rational curves through x. It is known (e.g., [Kollár 1996, II.3.11.5]) that $\mathcal{K}_x$ is a disjoint union of finitely many nonsingular projective varieties of dimension $i(X) - 2$. The rational morphism $\mathcal{K}_x \dashrightarrow \mathbb{P}T_x(X)$, sending a member of $\mathcal{K}_x$ which is smooth at x to its tangent direction, can be extended to a birational morphism $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$; see [Hwang and Mok 2004; Kebekus 2002]. We denote the image of τ_x by $\mathcal{C}_x$ and call it *the variety of minimal rational tangents (VMRT)* at x. The projective geometry of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ helps us to understand the geometry of X. This is the motivation for the study of the VMRT for various examples of X. For example, let $\phi : X \to \mathbb{P}^n$ be a double cover branched on a smooth hypersurface of degree $2m$, $2 \leq m \leq n - 1$. Then for a general point $x \in X$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a complete intersection of multidegree $(m+1, \ldots, 2m)$, and this description implies a certain rigidity property of ϕ [Hwang and Kim 2013].

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Let $Z \subset \mathbb{P}^N$ be a Fano manifold whose Picard group is generated by the hyperplane section class. For each point $y \in Z$, we denote by $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ the space of tangent directions of lines in Z through y. We say that $Z \subset \mathbb{P}^N$ is covered by lines if $\mathcal{L}_y(Z)$ is nonempty for each $y \in Z$. If $Z \subset \mathbb{P}^N$ is covered by lines, then minimal rational curves on Z are lines in $\mathbb{P}^N$ contained in Z, and for general $y \in Z$, $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ coincides with the VMRT $\mathcal{C}_y \subset \mathbb{P}T_y(Z)$, which is smooth of dimension $i(Z)-2$; see [Hwang 2001, Proposition 1.5].

Our first result is the following theorem.

Theorem 1.1. *Let $Z \subset \mathbb{P}^N$ be a Fano manifold whose Picard group is generated by the hyperplane section class. Assume that Z is covered by lines and $i(Z) \geq 3$. Let $Y \subset \mathbb{P}^N$ be a hypersurface of degree $2m$, $1 \leq m \leq i(Z)-2$, with smooth intersection $Y \cap Z$. Let $\phi : X^Z \to Z$ be a double cover branched along $Y \cap Z$. Then for a general point $x \in X^Z$, the VMRT $\mathcal{C}_x$ is smooth of dimension $i(Z)-m-2$ and the differential $d\phi_x : \mathbb{P}T_x(X^Z) \to \mathbb{P}T_{\phi(x)}(Z)$ sends the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X^Z)$ isomorphically to an intersection of $\mathcal{L}_{\phi(x)}(Z) \subset \mathbb{P}T_{\phi(x)}(Z)$ and m hypersurfaces in $\mathbb{P}T_{\phi(x)}(Z)$ of degrees $m+1, \ldots, 2m$ respectively.*

In order to prove the above theorem, we first show that for a certain choice of Y, the statements in Theorem 1.1 hold by identifying minimal rational curves on X^Z with ECO (even contact order) lines with respect to Y (see Definition 2.4) contained in Z. For arbitrary Y, we use a flatness argument.

We are going to show an application of Theorem 1.1. First, let us introduce the definition of Cartan–Fubini type rigidity (CF-rigidity) which was initially defined by Jun-Muk Hwang.

Definition 1.2 (Cartan–Fubini type rigidity). Let X_1 and X_2 be Fano manifolds of Picard number 1 such that $2 \leq \dim X_1 \leq \dim X_2$, and for general $x_1 \in X_1$ and $x_2 \in X_2$, $0 \leq \dim \mathcal{K}_{x_1} \leq \dim \mathcal{K}_{x_2}$. We say that a morphism $\phi : X_1 \to X_2$ is *CF-rigid* if for any connected open subset (in classical topology) U of X_1 and any biholomorphic immersion $\psi : U \to X_2$ such that for any member C of $\mathcal{K}_x$, $x \in U$, $\psi(C \cap U)$ is contained in a minimal rational curve of X_2, then there exists $\Gamma \in \mathrm{Aut}(X_2)$ such that $\psi = \Gamma \circ \phi|_U$.

The next theorem is on the CF-rigidity of the identity morphism, which was essentially proved in [Hwang and Mok 2001].

Theorem 1.3 (Cartan–Fubini type extension theorem). *Let X be a Fano manifold of Picard number* 1 *and suppose that* $\dim \mathcal{K}_x \geq 1$ *for general $x \in X$. Then the identity morphism on X is CF-rigid, i.e., for any connected open subset* (*in the standard topology*) *U of X and any biholomorphic immersion $\psi : U \to X$ such that for any member C of $\mathcal{K}_x$, $x \in U$, $\psi(C \cap U)$ is contained in a minimal rational curve of X, then there exists* $\Gamma \in \mathrm{Aut}(X)$ *such that* $\psi = \Gamma|_U$.

Let X be a Fano manifold of Picard number 1 and let $\phi : X \to \mathbb{P}^n$ is a surjective holomorphic map sending minimal rational curves on X to lines in $\mathbb{P}^n$. [Hwang and Kim 2013, Theorem 5.4] says that if the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is not contained in a hyperquadric, then $\phi : X \to \mathbb{P}^n$ is CF-rigid. For example, the double covering morphism $\phi : X \to \mathbb{P}^n$ branched along a smooth hypersurface of degree $2m$, with $2 \le m \le n-1$, is CF-rigid.

Next is an application of Theorem 1.1 on CF-rigidity.

Theorem 1.4. *In the setting of Theorem 1.1, assume that $Z \subset \mathbb{P}^N$ is a quadratic manifold (i.e., scheme theoretically defined by quadratic equations) with $i(Z) \ge 4$, and $2 \le m \le i(Z)-2$. Then $\phi : X^Z \to Z$ is CF-rigid. In other words, for any connected open subset (in classical topology) $U \subset X^Z$ and any biholomorphic immersion $\psi : U \to Z$ such that for any member C of $\mathcal{K}_x$, $x \in U$, the image $\psi(C \cap U) \subset Z$ is contained in a line in $\mathbb{P}^N$, there exists $\Gamma \in \mathrm{Aut}(Z)$ such that $\psi = \Gamma \circ \phi|_U$.*

The key point is that for $Z \subset \mathbb{P}^N$ in Theorem 1.4, its VMRT $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ is also a quadratic manifold for a general point $y \in Z$; see [Ionescu and Russo 2013, Theorem 2.4]. Using this observation and Theorem 1.3, we shall prove Theorem 1.4.

The organization of this paper is as follows. In Section 2, we will review some basic facts on ECO lines. In Section 3, we will study minimal rational curves on double covers of certain Fano manifolds covered by lines. Theorem 1.1 will be proved in Section 4. In the final section, we will prove Theorem 1.4 and present its applications.

2. ECO lines

The aim of this section is to give a brief review of basic facts on ECO lines which will be used in the proof of Theorem 1.1. For more details, see [Hwang and Kim 2013].

Definition 2.1. A homogeneous polynomial of degree $2m$, $m \ge 1$, in the polynomial ring $\mathbb{C}[s,t]$ with two variables s and t is an *ECO (even contact order) polynomial* if it can be written as the square of a homogeneous polynomial of degree m in $\mathbb{C}[s,t]$.

Proposition 2.2 [op. cit., Proposition 3.3]. *For each $m > 0$, there exist m unique polynomials in the variables $t_1, \dots, t_m$,*

$$A_k(t_1, \dots, t_m) \in \mathbb{C}[t_1, \dots, t_m], \quad m+1 \le k \le 2m$$

with the following properties:

(i) *$A_k(t_1, \dots, t_m)$ is weighted homogeneous of degree k with respect to* $\mathrm{wt}(t_i) = i$ *for each $i = 1, \dots, m$;*

(ii) *the polynomial in two variables* (s, t)

$$s^{2d}+a_1 s^{2d-1}t+\cdots+a_{2d-1}st^{2d-1}+a_{2d}t^{2d}$$

is an ECO polynomial if and only if $a_k = A_k(a_1, \dots, a_d)$ *for each* $d+1 \le k \le 2d$. *In particular*, *the polynomial*

$$s^{2d}+a_{d+1}s^{d-1}t^{d+1}+\cdots+a_{2d-1}st^{2d-1}+a_{2d}t^{2d}$$

is an ECO polynomial if and only if $a_{d+1} = \cdots = a_{2d} = 0$.

Definition 2.3. Let $f(t_0, \dots, t_N)$ be a homogeneous polynomial of degree $2m$ in variables $t_0, \dots, t_N$. Write

$$f(1, y_1+\lambda z_1, \dots, y_N+\lambda z_N) = a_0^f(y; z)+a_1^f(y; z)\lambda+\cdots+a_{2m}^f(y; z)\lambda^{2m},$$

where each $a_k^f(y; z) = a_k^f(y_1, \dots, y_N; z_1, \dots, z_N)$ is a polynomial in $2N$ variables $y_1, \dots, y_N, z_1, \dots, z_N$. Let A_k be as in Proposition 2.2 and set

$$B_k^f(y; z) := \frac{a_k^f(y; z)}{a_0^f(y; z)} - A_k\left(\frac{a_1^f(y; z)}{a_0^f(y; z)}, \dots, \frac{a_m^f(y; z)}{a_0^f(y; z)}\right).$$

We remark that for a fixed y, $a_k^f(y; z)$ is a homogeneous polynomial in variables $z_1, \dots, z_N$ of degree k. Furthermore for a fixed y with

$$a_0^f(y; z) = f(1, y_1, \dots, y_N) \neq 0,$$

each $B_k^f(y; z)$ is a homogeneous polynomial of degree k in variables $z_1, \dots, z_N$.

Definition 2.4. Let $Y \subset \mathbb{P}^N$ be a hypersurface of even degree $2m$. A line $\ell \subset \mathbb{P}^N$ is called an *ECO (even contact order) line* with respect to Y if $\ell \not\subset Y$ and the local intersection number at each point of $\ell \cap Y$ is even. For each point $y \in \mathbb{P}^N \setminus Y$, we denote by $\mathcal{E}_y^Y \subset \mathbb{P}T_y(\mathbb{P}^N)$ the space of tangent directions of ECO lines with respect to Y passing through y.

Proposition 2.5 [Hwang and Kim 2013, Proposition 3.8]. *Choose a homogeneous coordinate system* $t_0, \dots, t_N$ *on* $\mathbb{P}^N$. *We denote by* $\mathbb{P}_\infty^{N-1} \subset \mathbb{P}^N$ *the hyperplane defined by* $t_0 = 0$ *and choose a homogeneous coordinate system* $z_1, \dots, z_N$ *on* $\mathbb{P}_\infty^{N-1}$ *given by the restrictions of* $t_1, \dots, t_N$ *respectively. Let* $f(t_0, \dots, t_N)$ *be a homogeneous polynomial of degree* $2m$ *and let* $Y \subset \mathbb{P}^N$ *be its associated hypersurface. For each point* $y = [1 : y_1 : \cdots : y_N] \in \mathbb{P}^N \setminus Y \cup \mathbb{P}_\infty^{N-1}$, *define the projective isomorphism*

$$\upsilon_y : \mathbb{P}_\infty^{N-1} \to \mathbb{P}T_y(\mathbb{P}^N)$$

by sending $[z_1 : \cdots : z_N] \in \mathbb{P}_\infty^{N-1}$ *to the tangent direction of the line*

$$\{(y_1+\lambda z_1, \dots, y_N+\lambda z_N) \mid \lambda \in \mathbb{C}\}$$

at the point y. Then the variety $\upsilon_y^{-1}(\mathcal{E}_y^Y) \subset \mathbb{P}_\infty^{N-1}$ is set-theoretically the intersection of m hypersurfaces in $\mathbb{P}_\infty^{N-1}$ defined by polynomials in $\{B_k^f(y;z) \mid m+1 \le k \le 2m\}$.

3. Minimal rational curves on double covers of prime Fano manifolds

Definition 3.1. Let $Z \subset \mathbb{P}^N$ be a Fano manifold whose Picard group is generated by the hyperplane section class H. Assume that Z is covered by lines and $i(Z) \ge 3$. Let $Y \subset \mathbb{P}^N$ be a hypersurface of degree $2m$, $1 \le m \le i(Z)-2$, defining smooth hypersurface section $B := Y \cap Z \subset Z$. Let

$$\phi : X^Z \to Z$$

be a double cover branched along B. From the adjunction formula

$$K_{X^Z} = \phi^*\big(K_Z + \tfrac{1}{2}(B)\big) = \phi^*\big((-i(Z)+m)H\big),$$

X^Z is a Fano manifold of index $i(X^Z) = i(Z) - m$ and its Picard group is generated by $\phi^*(H)$.

Proposition 3.2. *In the setting of* Definition 3.1, *an irreducible reduced curve $C \subset X^Z$ with $\phi(C) \not\subset B$ is a minimal rational curve if and only if its image curve $\phi(C)$ is an ECO line with respect to Y. Moreover for any minimal rational curve $C \subset X^Z$ with $\phi(C) \not\subset B$, $\phi|_C : C \to \phi(C)$ is an isomorphism.*

Proof. We fist observe:

Claim: An irreducible reduced curve $C \subset X^Z$ with $\phi(C) \not\subset B$ has ϕ^*H-degree 1 if and only if its image curve $\phi(C) \subset Z$ is an ECO line with respect to Y. Moreover for any ϕ^*H-degree 1 curve $C \subset X^Z$ with $\phi(C) \not\subset B$, $\phi|_C : C \to \phi(C)$ is an isomorphism.

Proof of the claim. Let $C \subset X^Z$ be an irreducible reduced curve such that the image $\phi(C)$ is an ECO line with respect to Y. Suppose that $\phi|_C : C \to \phi(C)$ is not birational. For a point $z \in \phi(C) \cap Y$, let t be a local uniformizing parameter on $\phi(C)$ at z and let r_z be the local intersection number of $\phi(C)$ and Y at z. Then C is analytically defined by the equation $s^2 = t^{r_z}$. Since r_z is even for any choice of $z \in \phi(C) \cap Y$, the composition of the normalization morphism $\tilde{C} \to C$ and the covering morphism $\phi|_C : C \to \phi(C)$ induces a morphism $\tilde{C} \to \phi(C)$ of degree 2 without ramification point, a contradiction. Thus $\phi|_C : C \to \phi(C)$ is birational and C has ϕ^*H-degree 1.

Conversely, if C is an irreducible reduced curve of ϕ^*H-degree 1, then $\phi(C) \subset Z$ with $\phi(C) \not\subset B$ and $\phi|_C : C \to \phi(C)$ must be birational. Thus $\phi^{-1}(\phi(C))$ has an irreducible component C' different from C with $\phi(C \cap C') = \phi(C) \cap B$. By the same argument as before, if the local intersection number r_z at $z \in \phi(C) \cap Y$ is odd, the germ of $\phi^{-1}(\phi(C))$ over z, defined by $s^2 = t^{r_z}$, is irreducible, a contradiction. Thus

r_z is even for all $z \in \phi(C) \cap Y$ which implies that $\phi(C)$ is an ECO line. Moreover, C must be smooth and the morphism $\phi|_C : C \to \phi(C)$ is an isomorphism. □

Let us go back to the proof of our proposition. By the claim above, we only need to show that for a general point $x \in X^Z$, there exists a ϕ^*H-degree 1 curve through x. From $\dim \mathcal{L}_{\phi(x)}(Z) = i(Z) - 2 \geq m$ and Proposition 2.5, it is induced that there exists an ECO line ℓ with respect to Y through $\phi(x)$ and contained in Z. Take one such ECO line ℓ. The claim above shows that the inverse image $\phi^{-1}(\ell)$ consists of two smooth rational curves of degree 1 with respect to ϕ^*H. Clearly, one of those two curves passes through x. □

4. Defining equations of VMRT

In order to find the defining ideal of the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X^Z)$, we proceed in a manner analogous to [Hwang and Kim 2013].

Notation 4.1. Let $Y \subset \mathbb{P}^N$ be a hypersurface of even degree $2m$, $m \geq 1$, and let $Z \subset \mathbb{P}^N$ be a projective submanifold which is not contained in Y. For each point $y \in Z \setminus Z \cap Y$, we denote by $\mathcal{E}^Y_y(Z) \subset \mathbb{P}T_y(Z)$ the space of tangent directions of ECO lines with respect to Y contained in Z.

Proposition 4.2. *In the setting of Definition 3.1, for a general point $x \in X^Z$, the tangent morphism $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X^Z)$, sending each member of $\mathcal{K}_x$ to its tangent direction, is an embedding. In particular the VMRT $\mathcal{C}_x = \mathrm{Im}(\tau_x) \subset \mathbb{P}T_x(X^Z)$ is a nonsingular projective variety with finitely many components of dimension $i(Z) - m - 2$, isomorphic to $\mathcal{E}^Y_{\phi(x)}(Z) \subset \mathbb{P}T_{\phi(x)}(Z)$ via the differential morphism $d\phi_x : \mathbb{P}T_x(X^Z) \to \mathbb{P}T_{\phi(x)}(Z)$.*

Proof. From Proposition 3.2, the differential $d\phi_x : \mathbb{P}T_x(X^Z) \to \mathbb{P}T_{\phi(x)}(Z)$ sends the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X^Z)$ isomorphically to the variety $\mathcal{E}^Y_{\phi(x)}(Z) \subset \mathbb{P}T_{\phi(x)}(Z)$.

We note that $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X^Z)$ is the normalization morphism of its image, which is equal to $\mathcal{C}_x$. Thus we only need to show that τ_x is an embedding because $\mathcal{K}_x$ is a smooth projective variety of dimension $i(X) - 2 = i(Z) - m - 2$.

Assume that there are two distinct members C_1 and C_2 of $\mathcal{K}_x$ such that $\tau_x([C_1]) = \tau_x([C_2])$. Thus $\phi(C_1)$ and $\phi(C_2)$ are lines on $\mathbb{P}^N$ passing through $\phi(x)$ with the same tangent direction at $\phi(x)$, which implies that $\phi(C_1) = \phi(C_2)$ is a line; denote it by ℓ. Therefore $\phi^{-1}(\ell) = C_1 \cup C_2$, and hence C_2 and C_2 meets only over the points on $\ell \cap B$, a contradiction because $x \in C_1 \cap C_2$ but $\phi(x) \notin B$ by the general condition on x. Thus we have shown that τ_x is injective.

Since we know that $\mathcal{K}_x$ is nonsingular, to prove that τ_x is an embedding, it remains to show that τ_x is an immersion. By [Hwang 2001, Proposition 1.4], this is equivalent to showing that for any member $C \subset X^Z$ of $\mathcal{K}_x$, the normal bundle N_{C/X^Z} satisfies

$$N_{C/X^Z} = O_{\mathbb{P}^1}(1)^{i(Z)-m-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{n+m-i(Z)+1}.$$

By the generality of x, we can write

$$N_{C/X^Z} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})$$

for integers $a_1 \geq \cdots \geq a_{n-1} \geq 0$ satisfying $\sum_i a_i = i(Z) - m - 2$. Since ϕ is unramified at general points of C and $\phi|_C : C \to \ell := \phi(C)$ is an isomorphism, we have an injective sheaf homomorphism

$$\phi_* : N_{C/X^Z} \to N_{\ell/\mathbb{P}^N} = \mathcal{O}(1)^{N-1}.$$

Thus $a_1 \leq 1$; hence, $a_1 = \cdots = a_{i(Z)-m-2} = 1$ and $a_{i(Z)-m-1} = \cdots = a_{n-1} = 0$. □

Proposition 4.3. *Let $Z \subset \mathbb{P}^N$ be a Fano manifold whose Picard group is generated by the hyperplane section class. Assume that Z is covered by lines and $i(Z) \geq 3$. Then for each m with $1 \leq m \leq i(Z) - 2$, there exists a hypersurface Y of degree $2m$ with smooth $Y \cap Z$ such that for a general point $y \in Z$, $\mathcal{E}_y^Y(Z) \subset \mathbb{P}T_y(Z)$ is scheme-theoretically the intersection of $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ with m hypersurfaces in $\mathbb{P}T_y(Z)$ of degrees $m+1, \ldots, 2m$, respectively.*

Proof. Take a point $\hat{y} \in Z$ such that $\mathcal{L}_{\hat{y}}(Z)$ is smooth of dimension $i(Z)-2$. Choose a homogeneous coordinate system $t_0, \ldots, t_N$ so that $\hat{y} = [1 : 0 : \cdots : 0] \in Z$ and the hyperplane section $Z \cap (t_0 = 0)$ is smooth. Choose homogeneous polynomials $\{b_k(t_1, \ldots, t_N) \mid m+1 \leq k \leq 2m\}$ with $\deg b_k = k$ so that each of the following is smooth:

(i) the intersection of $Z \cap (t_0 \neq 0)$ with the hypersurface in $\mathbb{P}^N$ defined by

$$1 + b_{m+1}(t_1, \ldots, t_N) + \cdots + b_{2m}(t_1, \ldots, t_N) = 0,$$

(ii) the intersection of $Z \cap (t_0 = 0)$ with the hypersurface in $\mathbb{P}^N$ defined by

$$b_{2m}(t_1, \ldots, t_N) = 0.$$

Set

$$f(t_0, t_1, \ldots, t_N) := t_0^{2m} + t_0^{m-1} b_{m+1}(t_1, \ldots, t_N) + \cdots + t_0 b_{2m-1}(t_1, \ldots, t_N) + b_{2m}(t_1, \ldots, t_N).$$

The assumptions (i) and (ii) imply that the hypersurface section of Z defined by $f(t_0, \ldots, t_N)$ is smooth. From Proposition 2.2(ii) we obtain the equalities

$$B_k^f(y; z) = b_k(z_1, \ldots, z_N), \quad m+1 \leq k \leq 2m.$$

Since $\upsilon_y^{-1}(\mathcal{L}_y(Z))$ is smooth of dimension $i(Z) - 2 \geq m$, it follows that for general $\{b_k(z_1, \ldots, z_N) \mid m+1 \leq k \leq 2m\}$, the scheme-theoretical intersection of $\upsilon_y^{-1}(\mathcal{L}_y(Z))$ and the m hypersurfaces defined by $B_k^f(y; z)$, $m+1 \leq k \leq 2m$, is smooth of dimension $i(Z)-m-2$. Therefore our proposition comes from Proposition 2.5. □

Proof of Theorem 1.1. By Propositions 4.2 and 4.3, the theorem holds for a general hypersurface $Y \subset \mathbb{P}^N$ with smooth intersection $Y \cap Z$. In order to prove it for arbitrary hypersurface $Y \subset \mathbb{P}^N$ with smooth intersection $Y \cap Z$, choose a deformation $\{Y_t \mid |t| < \epsilon\}$ of $Y = Y_0$ with smooth $Y_t \cap Z$ such that for a Zariski open subset $U_t \subset Z \setminus Y_t \cap Z$, the varieties $\mathcal{E}_y^{Y_t}(Z) \subset \mathbb{P}T_y(Z)$, $y \in U_t$, are

(i) smooth of dimension $i(Z)-m-2$ for any t, and

(ii) the intersection of m hypersurface sections of $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ defined by hypersurfaces in $\mathbb{P}T_y(Z)$ of degrees $m+1, \ldots, 2m$, respectively. (The intersection is scheme-theoretic for any $t \neq 0$ and set-theoretic for $t = 0$.)

By shrinking ϵ if necessary, the intersection $\bigcap_t U_t$ is nonempty. Let V be a Zariski open subset of Z such that the variety $\mathcal{L}_y(Z)$ is smooth of dimension $i(Z)-2$. Pick a point $y \in \left(\bigcap_t U_t\right) \cap V$. We can construct a smooth family $\{\phi_t : X_t^Z \to Z \mid |t| < \epsilon\}$ of double covers of Z branched along the $Z \cap Y_t$. Choose $x_t \in \phi_t^{-1}(y)$ in a continuous way. The family $\{\mathcal{K}_{x_t} \mid |t| < \epsilon\}$ is a flat family of nonsingular projective subvarieties; see, e.g., [Kollár 1996, II.3.11.5]. Via Proposition 4.2, this implies that $\{\mathcal{E}_y^{Y_t}(Z) \mid |t| < \epsilon\}$ is a flat family of nonsingular projective subvarieties of $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$. From condition (ii) and the flatness, we conclude that $\mathcal{E}_y^{Y_0}(Z)$ is also scheme-theoretically the intersection of $\mathcal{L}_y(Z) \subset \mathbb{P}T_y(Z)$ with m hypersurfaces in $\mathbb{P}T_y(Z)$ of degrees $m+1, \ldots, 2m$, respectively. □

5. Rigidity and Extension

The following fact is obvious, but plays an important role in the proof of Theorem 1.4.

Lemma 5.1. *Let R be the polynomial ring $\mathbb{C}[z_1, \ldots, z_N]$ in variables $z_1, \ldots, z_N$. Consider R as a graded ring with* $\deg z_i = 1$. *Let I, J, and K be homogeneous ideals of R such that I and K are generated by homogeneous polynomials of degree* 2, *and J is generated by homogeneous polynomials of degrees* ≥ 3. *If $h : R \to R$ is an automorphism of the graded ring R with $h(K) \subset I + J$, then $h(K) \subset I$.*

Proof of Theorem 1.4. By shrinking U if necessary, we may assume that

- the restriction $\phi|_U : U \to Z$ is an embedding,
- for any $x \in U$, $\mathcal{L}_{\phi(x)}(Z) \subset \mathbb{P}T_{\phi(x)}(Z)$ is a smooth quadratic manifold of dimension $i(Z)-2$; see [Ionescu and Russo 2013, Theorem 2.4],
- for any $x \in U$, the space $\mathcal{E}_{\phi(x)}^{Y} \subset \mathbb{P}T_{\phi(x)}(Z)$ is scheme-theoretically the intersection of $\mathcal{L}_{\phi(x)}(Z) \subset \mathbb{P}T_{\phi(x)}(Z)$ and m hypersurfaces in $\mathbb{P}T_{\phi(x)}(Z)$ of degrees $m+1, \ldots, 2m$, respectively.

Set $U_1 = \phi(U)$ and $U_2 = \psi(U)$. Then we get a biholomorphism

$$\gamma := \psi \circ \phi|_U^{-1} : U_1 \to U_2.$$

We note that for any $y \in U_1$, the differential $d\gamma_y : \mathbb{P}T_y(Z) \to \mathbb{P}T_{\gamma(y)}(Z)$ is an isomorphism sending $\mathcal{E}_y^Y(Z) \subset \mathbb{P}T_y(Z)$ into $\mathcal{L}_{\gamma(y)}(Z) \subset \mathbb{P}T_{\gamma(y)}(Z)$. From Lemma 5.1 and Theorem 1.1, it follows that $d_y\gamma(\mathcal{L}_y(Z)) \subset \mathcal{L}_{\gamma(y)}(Z)$. By shrinking U if necessary, we may assume that $\mathcal{L}_{\gamma(y)} \subset \mathbb{P}T_{\gamma(y)}(Z)$ is also a smooth quadratic manifold of dimension $i(Z)-2$. Therefore it follows that $d_y\gamma(\mathcal{L}_y(Z)) = \mathcal{L}_{\gamma(y)}(Z)$. We finish the proof by applying [Hwang 2001, Theorem 3.2]. □

The next corollary is an algebraic version of Theorem 1.4

Corollary 5.2. *In the setting of Theorem 1.4, let $\hat{X}$ be a projective variety with generically finite surjective morphisms $g : \hat{X} \to X^Z$ and $h : \hat{X} \to Z$ such that for a minimal rational curve C through a general point of X^Z, there exists an irreducible component C' of $g^{-1}(C)$ whose image $h(C') \subset Z \subset \mathbb{P}^N$ is a line. Then there exists an automorphism $\Gamma : Z \to Z$ such that $h = \Gamma \circ \phi \circ g$.*

Next, Theorems 5.3 and 5.4 can be proved by the same arguments as in the proof of Theorems 1.7 and 1.9 in [Hwang and Kim 2013], respectively. We include their proof for the reader's convenience.

Theorem 5.3. *Let $Z \subset \mathbb{P}^N$ be a quadratic Fano manifold such that its Picard group is generated by the hyperplane section class and $i(Z) \geq 4$. Let $Y_1, Y_2 \subset \mathbb{P}^N$, $N \geq 3$, be two hypersurfaces of degree $2(i(Z)-2)$ with smooth intersections $Y_1 \cap Z$ and $Y_2 \cap Z$. Let $\phi_1 : X_1 \to Z$ and $\phi_2 : X_2 \to Z$ be double covers of Z branched along $Y_1 \cap Z$ and $Y_2 \cap Z$, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.*

Proof. Put $m = i(Z)-2$ in the proof of Proposition 4.2. Then minimal rational curves on X_i, $i = 1, 2$, have trivial normal bundles and rational curves through general points with trivial normal bundles are minimal rational curves. By [Hwang and Mok 2003, Proposition 6], for a general minimal rational curve $C \subset X_2$, each irreducible component of $f^{-1}(C)$ is a minimal rational curve in X_1. In other words, f sends minimal rational curves of X_1 through a general point to those of X_2. Putting

$$\hat{X} = X_1, \quad X = X_2, \quad g = f, \quad \phi = \phi_2, \quad \text{and} \quad h = \phi_1$$

in Corollary 5.2, we see that $\phi_1 = \Gamma \circ \phi_2 \circ f$ for some automorphism Γ of Z. Thus f must be birational, and hence an isomorphism. □

The next theorem is a stronger version of Theorem 1.3.

Theorem 5.4. *Let $Z \subset \mathbb{P}^N$ be a quadratic Fano manifold such that its Picard group is generated by the hyperplane section class and $i(Z) \geq 4$. Let $Y_1, Y_2 \subset \mathbb{P}^N$ be two* *hypersurfaces of degree* $2m$, $2 \le m \le i(Z)-2$, *with smooth* $Y_1 \cap Z$ *and* $Y_2 \cap Z$. *Let* $\phi_1 : X_1^Z \to Z$ *and* $\phi_2 : X_2^Z \to Z$ *be double covers of* Z *branched along* $Y_1 \cap Z$ *and* $Y_2 \cap Z$ *respectively, and let* $U_1 \subset X_1^Z$ *and* $U_2 \subset X_2^Z$ *be two connected open subsets. Suppose that we are given a biholomorphic map* $\gamma : U_1 \to U_2$ *such that for any minimal rational curve* $C_1 \subset X_1^Z$, *there exists a minimal rational curve* $C_2 \subset X_2^Z$ *with* $\gamma(U_1 \cap C_1) = U_2 \cap C_2$. *Then we can find a biregular morphism* $\Gamma : X_1^Z \to X_2^Z$ *with* $\Gamma|_{U_1} = \gamma$.

Sketch of the proof. Applying Theorem 1.4 to $\psi := \phi_2 \circ \gamma : U_1 \to \phi_2(U_2) \subset Z$ and $\phi := \phi_1$, we have $\Gamma' \in \mathrm{Aut}(Z)$ such that $\Gamma' \circ \phi_1|_{U_1} = \phi_2 \circ \gamma$. By the assumption on γ and Proposition 3.2, for a general point $y \in \phi_1(U_1)$, we have $d\Gamma'(\mathcal{E}_y^{Y_1}) = \mathcal{E}_{\Gamma'(y)}^{Y_2}$, which implies that a general ECO line with respect to Y_2 contained in Z should be an ECO line with respect to Y'.

Since the Picard group of Z is generated by the hyperplane section class and $\Gamma' \in \mathrm{Aut}(Z)$, there exists a hypersurface $Y' \subset \mathbb{P}^N$ of degree $2m$ such that $\Gamma'(Y_1 \cap Z) = Y' \cap Z$. Suppose $Y' \cap Z \neq Y_2 \cap Z$. By the similar arguments in [Hwang and Kim 2013, Proposition 2.5], we can show that a general ECO line with respect to Y_2 contained in Z cannot be an ECO line with respect to Y', a contradiction. Therefore $Y' \cap Z = Y_2 \cap Z$.

Thus replacing $Y_1 \cap Z$ by $\Gamma(Y_1 \cap Z)$ and ϕ_1 by $\Gamma' \circ \phi_1$, we may assume that $Y_1 \cap Z = Y_2 \cap Z$ and $\phi_1(U_1) = \phi_2(U_2)$. By the uniqueness of double covering, it follows that there exists a biregular morphism $\Gamma : X_1^Z \to X_2^Z$ with $\Phi|_{U_1} = \gamma$. □

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