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ON THE RELATIONSHIP OF CONTINUITY AND BOUNDARY REGULARITY IN PRESCRIBED MEAN CURVATURE DIRICHLET PROBLEMS

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In 1976, Leon Simon showed that if a compact subset of the boundary of a domain is smooth and has negative mean curvature, then the nonparametric least area problem with Lipschitz continuous Dirichlet boundary data has a generalized solution which is continuous on the union of the domain and this compact subset of the boundary, even if the generalized solution does not take on the prescribed boundary data. Simon's result has been extended to boundary value problems for prescribed mean curvature equations by other authors. In this note, we construct Dirichlet problems in domains with corners and demonstrate that the variational solutions of these Dirichlet problems are discontinuous at the corner, showing that Simon's assumption of regularity of the boundary of the domain is essential.

1. Introduction

For $n \in \mathbb{N}$ with $n \geq 2$, suppose Ω is a bounded, open set in $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$. Fix $H \in C^2(\mathbb{R}^n \times \mathbb{R})$ such that H is bounded and $H(x,t)$ is nondecreasing in t for $x \in \Omega$. Consider the prescribed mean curvature Dirichlet problem of finding a function $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$\operatorname{div}(Tf) = H(x, f) \quad \text{in } \Omega, \tag{1}$$

$$f = \phi \quad \text{on } \partial\Omega, \tag{2}$$

where $\phi \in C^0(\partial\Omega)$ is a prescribed function and

$$Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}};$$

such a function f, if it exists, is a classical solution of the Dirichlet problem. It has long been known (e.g., Bernstein in 1912) that some type of boundary curvature condition (which depends on H) must be satisfied in order to guarantee that a classical solution exists for each $\phi \in C^0(\partial\Omega)$ (e.g., [Jenkins and Serrin 1968; Serrin

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1969]). When $H \equiv 0$ and $\partial\Omega$ is smooth, this curvature condition is that $\partial\Omega$ must have nonnegative mean curvature (with respect to the interior normal direction of Ω) at each point [Jenkins and Serrin 1968]. However, Leon Simon [1976] has shown that if $\Gamma_0 \subset \partial\Omega$ is smooth (i.e., C^4), the mean curvature Λ of $\partial\Omega$ is negative on Γ_0, and Γ is a compact subset of Γ_0, then the minimal hypersurface $z = f(x)$, $x \in \Omega$, extends to $\Omega \cup \Gamma$ as a continuous function, even though f may not equal ϕ on Γ. Since [Simon 1976] appeared, the requirement that $H \equiv 0$ has been eliminated and the conclusion remains similar to that which Simon reached (see, for example, [Bourni 2011; Lau and Lin 1985; Lin 1987]).

How important is the role of boundary smoothness in the conclusions reached in [Simon 1976]? We shall show, by constructing suitable domains Ω and Dirichlet data ϕ, that the existence of a "nonconvex corner" P in Γ can cause the unique generalized (e.g., variational) solution to be discontinuous at P even if $\Gamma \setminus \{P\}$ is smooth and the generalized mean curvature Λ^* (i.e., [Serrin 1969]) of Γ at P is $-\infty$; this shows that some degree of smoothness of Γ is required to obtain the conclusions in [Simon 1976]. We shall prove the following.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume there exists $\lambda > 0$ such that $|H(x,t)| \leq \lambda$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then there exist a domain $\Omega \subset \mathbb{R}^n$ and a point $P \in \partial\Omega$ such that*

(i) *$\partial\Omega \setminus \{P\}$ is smooth (C^∞),*

(ii) *there is a neighborhood $\mathcal{N}$ of P such that $\Lambda(x) < 0$ for $x \in \mathcal{N} \cap \partial\Omega \setminus \{P\}$, where Λ is the mean curvature of $\partial\Omega$, and*

(iii) *$\Lambda^*(P) = -\infty$, where Λ^* is the generalized mean curvature of $\partial\Omega$,*

and there exists Dirichlet boundary data $\phi \in C^\infty(\mathbb{R}^n)$ such that the minimizer $f \in \mathrm{BV}(\Omega)$ of

$$J(u) = \int_\Omega |Du| + \int_\Omega \int_0^u H(x,t)\,dt\,dx + \int_{\partial\Omega} |u - \phi|\,d\mathcal{H}^{n-1}, \quad u \in \mathrm{BV}(\Omega), \tag{3}$$

exists and satisfies (1), $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\}) \cap L^\infty(\Omega)$, $f \notin C^0(\overline{\Omega})$, and $f \neq \phi$ in a neighborhood of P in $\partial\Omega$.

Since there are certainly many examples of Dirichlet problems which have continuous solutions even though their domains fail to satisfy appropriate smoothness or boundary curvature conditions (e.g., by restricting to a smaller domain a classical solution of a Dirichlet problem on a larger domain), the question of necessary or sufficient conditions for the continuity at P of a generalized solution of a particular Dirichlet problem is of interest and the examples here suggest (to us) that a "Concus–Finn" type condition might yield necessary conditions for the continuity at P of solutions (see Section 5).

We view this note as analogous to other articles (e.g., [Shi and Finn 2004; Huff and McCuan 2006; 2009; Korevaar 1980]) which enhance our knowledge of the behavior of solutions of boundary value problems for prescribed mean curvature equations by constructing and analyzing specific examples. One might also compare Theorem 1 with the behavior of generalized solutions of (1)–(2) when $\partial\Omega \setminus \{P\}$ is smooth and $|H(x, \phi(x))| \le (n-1)\Lambda(x)$ for $x \in \partial\Omega\setminus\{P\}$ (e.g., [Elcrat and Lancaster 1986; Lancaster 1985; 1988]) and with capillary surfaces (e.g., [Lancaster and Siegel 1996]).

2. Nonparametric minimal surfaces in $\mathbb{R}^3$

In this section, we will assume $n = 2$ and $H \equiv 0$; this allows us to use explicit comparison functions and illustrate our general procedure. Let Ω be a bounded, open set in $\mathbb{R}^2$ with locally Lipschitz boundary $\partial\Omega$ such that a point P lies on $\partial\Omega$ and there exist distinct rays $l^\pm$ starting at P such that $\partial\Omega$ is tangent to $l^+ \cup l^-$ at P. By rotating and translating the domain, we may assume $P = (0, 1)$ and there exists a $\sigma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$l^- = \{(r\cos(\sigma), 1 + r\sin(\sigma)) : r \ge 0\},$$
$$l^+ = \{(r\cos(\pi-\sigma), 1 + r\sin(\pi-\sigma)) : r \ge 0\},$$
$$\Omega \cap B(P, \delta) = \{(r\cos(\theta), 1 + r\sin(\theta)) : 0 < r < \delta, \theta^-(r) < \theta < \theta^+(r)\} \tag{4}$$

for some $\delta > 0$ and functions $\theta^\pm \in C^0([0, \delta))$ which satisfy $\theta^- < \theta^+$, $\theta^-(0) = \sigma$ and $\theta^+(0) = \pi - \sigma$; here $B(P, \delta)$ is the open ball in $\mathbb{R}^2$ centered at P of radius δ. If we set $\alpha = \frac{\pi}{2} - \sigma$, then $\alpha \in (0, \pi)$ and the angle at P in Ω of $\partial\Omega$ has size 2α. As $\sigma < 0$ goes to zero, $2\alpha > \pi$ goes to π and the (upper) region between l^- and l^+ becomes "less nonconvex" and approaches a half-plane through P. We will show that for each choice of $\sigma \in \left(-\frac{\pi}{2}, 0\right)$, there is a domain Ω as above and a choice of Dirichlet data $\phi \in C^\infty(\partial\Omega)$ such that the solution of (1)–(2) for Ω and ϕ is discontinuous at P.

Fix $\sigma \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$. Let ϵ be a small, fixed parameter, say $\epsilon \in (0, 0.5)$, and let $a = a(\sigma) \in (1, 2)$ be a parameter to be determined. Set $\tau = (1+\epsilon)\cot(-\sigma)$ and $r_1 = \sqrt{\tau^2 + (1+\epsilon)^2}$. Define $h_{2/\pi} \in C^2\big((0, 2) \times (-1, 1)\big)$ by

$$h_{2/\pi}(x_1, x_2) = \frac{2}{\pi} \ln\left(\frac{\cos\left(\frac{\pi x_2}{2}\right)}{\sin\left(\frac{\pi x_1}{2}\right)}\right).$$

Notice that the graph of $h_{2/\pi}$ is part of Scherk's first surface, so $\operatorname{div}(Th_{2/\pi}) = 0$ on $(0, 2) \times (-1, 1)$, and $h_{2/\pi}(t, t-1) = 0$ for each $t \in (0, 2)$. A computation using L'Hospital's Rule shows

$$\lim_{t \to 0^+} h_{2/\pi}\big((t\cos(\theta), 1 + t\sin(\theta))\big) = \tfrac{2}{\pi}\ln(-\tan(\theta)), \quad \theta \in \left(-\tfrac{\pi}{2}, 0\right). \tag{5}$$

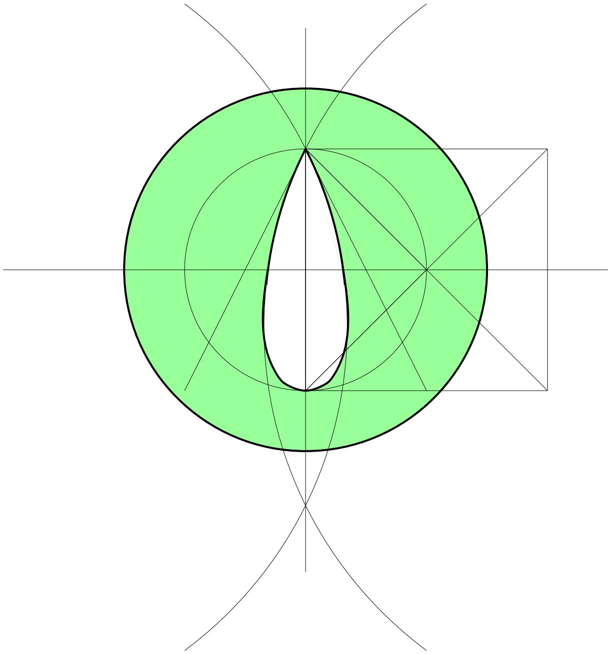

Figure 1. Ω.

Let $D = B(\mathbb{O}, 1) \cap B((\tau, -\epsilon), r_1) \cap B((-\tau, -\epsilon), r_1)$ be the intersection of three open disks and let $E \subset D$ be a strictly convex domain such that $\{x \in \partial E : x_2 < 1\}$ is a C^∞ curve, $E \cap \{x_2 \geq 0\} = D \cap \{x_2 \geq 0\}$, E is symmetric with respect to the x_2-axis, and $(0, -1) \in \partial E$; here $\mathbb{O}$ denotes $(0, 0)$. Define

$$\Omega = B(\mathbb{O}, a) \setminus \overline{E}$$

(see Figure 1); notice that $P \in \partial\Omega$ and (4) holds with the choice of σ above. If we set $C = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, x_1 - 1 < x_2 < 1 - x_1\}$, then (5) implies $\sup_{x \in C \cap \partial E} h_{2/\pi}(x) < \infty$.

Let

$$m > \max\left\{ r_1 \cosh^{-1}\left(\frac{2+\sqrt{\tau^2+\epsilon^2}}{r_1}\right), \sup_{x \in C \cap \partial E} h_{2/\pi}(x) \right\}.$$

Notice that m is independent of the parameter a. Define $\phi \in C^\infty(\partial\Omega)$ by $\phi = 0$ on $\partial B(\mathbb{O}, a)$ and $\phi = m$ on ∂E. Let f be the variational solution of (1)–(2) with ϕ as given here (e.g., [Gerhardt 1974; Giusti 1978]). Since $\phi \geq 0$ on $\partial\Omega$ and $\phi > 0$ on ∂E, $f \geq 0$ in Ω (e.g., Lemma 2 (with $h \equiv 0$)) and so $f > 0$ in Ω (e.g., the Hopf boundary point lemma). Notice that $h_{2/\pi} = 0 < f$ on $\Omega \cap \partial C$ and $h_{2/\pi} < \phi$ on $C \cap \partial E = C \cap \partial\Omega$, and therefore $h_{2/\pi} < f$ on $\Omega \cap C$ (see Figure 2). Together with

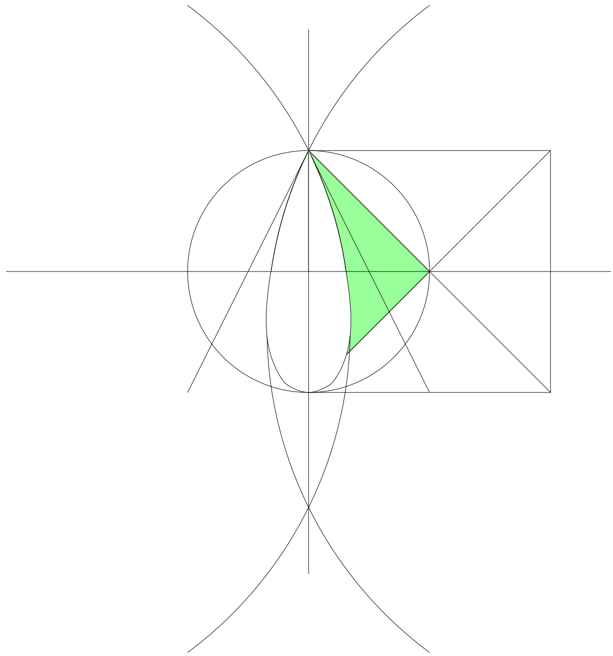

Figure 2. $\Omega \cap C$, the domain of the comparison function for (6).

(5), this implies

$$\liminf_{\Omega\cap C\ni x\to P} f(x) \geq \tfrac{2}{\pi}\ln(\tan(-\sigma)) > 0. \tag{6}$$

Set $W = B(\mathbb{O}, a) \setminus \overline{B(\mathbb{O}, 1)}$ (see Figure 3); then $W \subset \Omega$. Define the function $g \in C^\infty(W) \cap C^0(\overline{W})$ by $g(x) = \cosh^{-1}(a) - \cosh^{-1}(|x|)$ and notice that the graph of g is part of a catenoid, where $g = 0$ on $\partial B(\mathbb{O}, a)$ and $g = \cosh^{-1}(a)$ on $\partial B(\mathbb{O}, 1)$. It follows from the general comparison principle (e.g., [Finn 1986, Theorem 5.1]) that $f \leq g$ on W and therefore

$$f \leq \cosh^{-1}(a) \quad \text{on } W. \tag{7}$$

If we select $a > 1$ so that $\cosh^{-1}(a) < \frac{2}{\pi}\ln(\tan(-\sigma))$, then (6) and (7) imply that f cannot be continuous at P. Notice that [Simon 1976] implies $f \in C^0(\overline{\Omega} \setminus \{P\})$.

This example illustrates the procedure we shall use in Section 4; a somewhat similar approach was used in [Shi and Finn 2004; Korevaar 1980; Lancaster and Siegel 1996; Serrin 1969]. The case $\sigma \in \left[-\frac{\pi}{4}, 0\right)$ has a similar proof with the changes that D is the intersection of the open disk $B(\mathbb{O}, 1)$ with the interiors of two ellipses, and a Scherk surface with rhomboidal domain [Nitsche 1989, pp. 70–71] is used as a comparison surface to obtain the analog of (6); the details can be found in [Melin 2013].

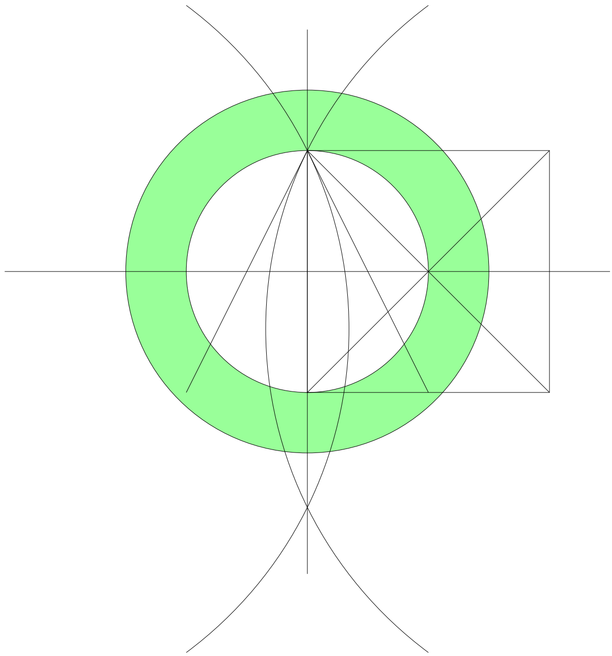

Figure 3. W, the domain of the comparison function for (7).

3. Lemmata

Lemma 1. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with locally Lipschitz boundary and let Γ be an open C^2 subset of $\partial\Omega$. Let $\phi \in L^\infty(\partial\Omega) \cap C^{1,\beta}(\Gamma)$. Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ is the variational solution of (1)–(2) and $g < \phi$ on Γ. Then*

$$\nu \equiv \frac{(\nabla g, -1)}{\sqrt{1+|\nabla g|^2}} \in C^0(\Omega \cup \Gamma)$$

and $\nu \cdot \eta = 1$ on Γ, where $\eta(x) \in S^{n-1}$ is the exterior unit normal to Γ at x.

Proof. Since g minimizes the functional J in (3) over $\mathrm{BV}(\Omega)$, g also minimizes the functional $K(u) = J(u) - \int_\Gamma \phi \, d\mathcal{H}^{n-1}$. Notice

$$K(u) = \int_\Omega |Du| + \int_\Omega \int_0^u H(x,t)\, dt\, dx + \int_{\partial\Omega \setminus \Gamma} |u - \phi| \, d\mathcal{H}^{n-1} - \int_\Gamma u \, d\mathcal{H}^{n-1}$$

for each $u \in \mathrm{BV}(\Omega)$ with $\mathrm{tr}(u) \leq \phi$ on Γ; in particular, this holds when $u = g$. Therefore, for each $x \in \Gamma$, there exists $\rho > 0$ such that $\partial\Omega \cap B_n(x, \rho) \subset \Gamma$, and the lemma follows as in [Korevaar and Simon 1996]. □

Lemma 2. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with locally Lipschitz boundary, $\phi, \psi \in L^\infty(\partial\Omega)$ with $\psi \leq \phi$ on $\partial\Omega$, $H_0 \in C^2(\Omega \times \mathbb{R})$ with $H_0(x,t)$*

nondecreasing in t for $x \in \Omega$, *and* $H_0 \geq H$ *on* $\Omega \times \mathbb{R}$. *Consider the boundary value problem*

$$\operatorname{div}(Tf) = H_0(x, f) \quad \text{in } \Omega, \tag{8}$$

$$f = \psi \quad \text{on } \partial\Omega. \tag{9}$$

Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ *is the variational solution of* (1)–(2) *and either*

(i) $h \in C^2(\Omega) \cap L^\infty(\Omega)$ *is the variational solution of* (8)–(9), *or*

(ii) $\psi \in C^0(\partial\Omega)$, $h \in C^2(\Omega) \cap C^0(\overline{\Omega})$, *and h satisfies* (8)–(9).

Then $h \leq g$ *in* Ω.

Proof. Let $A = \{x \in \Omega : h(x) > g(x)\}$. In case (i), let $f = hI_{\Omega \setminus A} + gI_A$, where I_B is the characteristic function of a set B; then a simple calculation using $J(g) \leq J(f)$ shows that $J_1(f) \leq J_1(h)$ and therefore $f = h$ and $A = \varnothing$, where

$$J_1(u) = \int_\Omega |Du| + \int_\Omega \int_0^u H_0(x, t)\, dt\, dx + \int_{\partial\Omega} |u - \psi|\, d\mathcal{H}^{n-1}, \quad u \in \mathrm{BV}(\Omega),$$

is the functional which h minimizes. In case (ii), the conclusion follows from Lemma 1 of [Williams 1978]. □

Lemma 3. *Let* $\Omega \subset \{x \in \mathbb{R}^2 : x_2 > 0\}$ *be a bounded open set*, $n \in \mathbb{N}$ *with* $n \geq 2$, *and* $g \in C^2(\Omega)$. *Set* $\widetilde{\Omega} = \{(x_1, x_2\omega) \in \mathbb{R}^n : (x_1, x_2) \in \Omega, \omega \in S^{n-2}\}$ *and define* $\tilde{g} \in C^2(\widetilde{\Omega})$ *by* $\tilde{g}(x_1, x_2\omega) = g(x_1, x_2)$ *for* $(x_1, x_2) \in \Omega$, $\omega \in S^{n-2}$. *Then, for*

$$x = (x_1, \ldots, x_n) = (x_1, r\omega) \in \widetilde{\Omega}$$

with $r = \sqrt{x_2^2 + \cdots + x_n^2}$, $\omega = \frac{1}{r}(x_2, \ldots, x_n)$, *and* $(x_1, r) \in \Omega$, *we have*

$$\operatorname{div}\left(\frac{\nabla \tilde{g}}{\sqrt{1+|\nabla \tilde{g}|^2}}\right)(x) = \operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right)(x_1, r) + \frac{n-2}{r} \frac{g_{x_2}(x_1, r)}{\sqrt{1+|\nabla g(x_1, r)|^2}}.$$

In particular, if $H \geq 0$, $R > 0$, $\Omega \subset \{x \in \mathbb{R}^2 : x_2 \geq R\}$, *and*

$$\operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) \geq H + \frac{n-2}{R} \quad \text{on } \Omega,$$

then

$$\operatorname{div}\left(\frac{\nabla \tilde{g}}{\sqrt{1+|\nabla \tilde{g}|^2}}\right) \geq H \quad \text{on } \widetilde{\Omega}.$$

Proof. Notice that

$$1+|\nabla\tilde{g}|^2 = 1+|\nabla g|^2,$$

$$\big(1+|\nabla\tilde{g}|^2\big)\triangle\tilde{g} = \big(1+|\nabla g|^2\big)\Big(\triangle g + \frac{n-2}{r}g_{x_2}\Big),$$

$$\sum_{i,j=1}^{n}\frac{\partial\tilde{g}}{\partial x_i}\frac{\partial\tilde{g}}{\partial x_j}\frac{\partial^2\tilde{g}}{\partial x_i\partial x_j} = \Big(\frac{\partial g}{\partial x_1}\Big)^2\frac{\partial^2 g}{\partial x_1^2}+2\frac{\partial g}{\partial x_1}\frac{\partial g}{\partial x_2}\frac{\partial^2 g}{\partial x_1\partial x_2}+\Big(\frac{\partial g}{\partial x_2}\Big)^2\frac{\partial^2 g}{\partial x_2^2},$$

and so

$$\big(1+|\nabla\tilde{g}|^2\big)\triangle\tilde{g} - \sum_{i,j=1}^{n}\frac{\partial\tilde{g}}{\partial x_i}\frac{\partial\tilde{g}}{\partial x_j}\frac{\partial^2\tilde{g}}{\partial x_i\partial x_j}$$

$$= \big(1+g_{x_2}^2\big)g_{x_1x_1} - 2g_{x_1}g_{x_2}g_{x_1x_2} + \big(1+g_{x_1}^2\big)g_{x_2x_2} + \frac{n-2}{r}\big(1+g_{x_1}^2+g_{x_2}^2\big)g_{x_2}.$$

The lemma follows from this. □

4. The n-dimensional case

Let $B_k(x,r)$ denote the open ball in $\mathbb{R}^k$ centered at $x\in\mathbb{R}^k$ with radius $r>0$ and $\mathbb{O}_k=(0,\dots,0)\in\mathbb{R}^k$, for $k\in\mathbb{N}$. Now consider $n\ge 2$ and set

$$\lambda = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}|H(x,t)|;$$

if $\lambda=0$, replace it with a positive constant. For each $a\in\big(0,\frac{n}{\lambda}\big)$ and $Q\in\mathbb{R}^n$, we have

$$\int_{B_n(Q,a)}\lambda^n\,dx < n^n\omega_n. \tag{10}$$

By translating our problem in $\mathbb{R}^n$, we may (and will) assume $Q=\mathbb{O}_n$. By Proposition 1.1 and Theorem 2.1 of [Giusti 1976], we see that if Ω is a bounded, connected, and open set in $\mathbb{R}^n$ with Lipschitz-continuous boundary, $\overline{\Omega}\subset B_n\big(\mathbb{O}_n,\frac{n}{\lambda}\big)$, and $\phi\in L^1(\partial\Omega)$, then the functional J in (3) has a minimizer $f\in\mathrm{BV}(\Omega)$, $f\in C^2(\Omega)$ satisfying (1).

The proof in Section 4.1 consists of setting some parameters (e.g., p, r_1, r_2, m_0, b, c, τ, σ, a), determining the domain Ω, finding different comparison functions (e.g., g_1, $g^{[u]}$, $k_\pm$, k_2, k_3, k_4), and mimicking (6) and (7) to show that the variational solution f of (1)–(2) is discontinuous at a nonconvex corner. In particular, we use a torus (i.e., j_a) to obtain (21), unduloids (i.e., $k_\pm$, k_2) to obtain (24) (an analog of (7)), and nodoids (i.e., g_1, $g^{[u]}$), unduloids (i.e., $k_\pm$, k_4), and a helicoidal function (i.e., h_2) to obtain (30) (an analog of (6)) and prove that f is discontinuous at $P=(0,p,0,\dots,0)\in\mathbb{R}^n\in\partial\Omega$.

4.1. ***Codimension* 1 *singular set.*** In this section, we will obtain a domain Ω as above and $\phi \in C^{\infty}(\mathbb{R}^n)$ such that $P \in \partial\Omega$, the minimizer f of (3) is discontinuous at P, $\partial\Omega \setminus T$ is smooth (C^{∞}), and $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus T)$, where T is a smooth set of dimension $n-2$ (i.e., T has codimension 1 in $\partial\Omega$). We will use portions of nodoids, unduloids, and helicoidal surfaces with constant mean curvature as comparison functions. For the convenience of the reader, we will denote functions whose graphs are subsets of nodoids with the letter g (e.g., $g_1(x_1, x_2)$), subsets of CMC helicoids with the letter h, and subsets of unduloids (or onduloids) with the letter k.

Let $\mathcal{N}_1 \subset \mathbb{R}^3$ be a nodoid which is symmetric with respect to the x_3-axis and has mean curvature 1 (when $\mathcal{N}_1$ is oriented "inward", so that the unit normal $\vec{N}_{\mathcal{N}_1}$ to $\mathcal{N}_1$ points toward the x_3-axis at the points of $\mathcal{N}_1$ which are furthest from the x_3-axis). Let $s_1 = \inf_{(x,t)\in\mathcal{N}_1} |x|$ be the inner neck size of $\mathcal{N}_1$ and let s_3 satisfy the condition that the unit normal to $\mathcal{N}_1$ is vertical (i.e., parallel to the x_3-axis) at each point $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ of $\mathcal{N}_1$ at which $|x| = s_3$; then $s_1 < s_3$. Let $s_2 \in (s_1, s_3)$. (Notice that we can assume s_2/s_1 is close to s_3/s_1 if we wish.)

Let us fix $0 < p < \frac{1}{\lambda}$ and set $w = (0, p) \in \mathbb{R}^2$, $P = (0, p, 0, \ldots, 0) \in \mathbb{R}^n$. Let $m_0 = \lambda/2 + (n-2)/(p/3)$. We shall assume $r_2 = s_2/m_0 < p/3$; if necessary, we may increase m_0 to accomplish this. Let $r_1 = s_1/m_0$ and $r_3 = s_3/m_0$. Let $\mathcal{N} = \{(m_0)^{-1}X \in \mathbb{R}^3 : X \in \mathcal{N}_1\}$; then $\mathcal{N}$ is a nodoid with mean curvature m_0. Set $\Delta_1 = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$. Fix $b \in \left(0, \frac{1}{4m_0}\left(1 + 2m_0 p - \sqrt{1 + 4m_0^2 p^2}\right)\right)$.

Define $g_1 \in C^{\infty}(\Delta_1) \cap C^0(\overline{\Delta_1})$ to be a function whose graph is a subset of $\mathcal{N}$ on which $\vec{N}_{\mathcal{N}} = (n_1, n_2, n_3)$ satisfies $n_3 \geq 0$; then

$$\operatorname{div}\left(\frac{\nabla g_1}{\sqrt{1+|\nabla g_1|^2}}\right) = m_0 \geq \lambda + \frac{2(n-2)}{p/3}. \tag{11}$$

By moving $\mathcal{N}$ vertically, we may assume $g_1(x) = 0$ when $|x| = r_2$; then $g_1 > 0$ in Δ_1. Notice that $\frac{\partial g_1}{\partial x_1}(r_1, 0) = -\infty$ and $\frac{\partial g_1}{\partial x_1}(r_2, 0) < 0$; then there exists a $\beta_0 > 0$ such that, for each $\theta \in \mathbb{R}$,

$$\frac{\partial}{\partial r}(g_1(r\Theta)) < -\beta_0 \quad \text{for } r_1 < r < r_2, \tag{12}$$

where $\Theta = (\cos(\theta), \sin(\theta))$. Fix $\beta \in (0, \beta_0)$. Let

$$0 < \tau < \min\left\{\frac{pr_1}{\sqrt{r_2^2 - r_1^2}}, \frac{2(1-p\lambda)}{\lambda(2-p\lambda)}, \frac{b(4p-b)}{4(2p-b)}\right\}. \tag{13}$$

Consider $\sigma \in \left(-\frac{\pi}{2}, 0\right)$. Notice that the distance between L and the point $(0, p-r_2)$ is $r_2\cos(\sigma)$, where L is the closed sector given by

$$L = \{(r\cos(\theta), p + r\sin(\theta)) : r \geq 0, \sigma \leq \theta \leq \pi - \sigma\}.$$

Define $r_4 = \sqrt{p^2 + \tau^2}$ and

$$M = B_2((\tau, 0), r_4) \cap B_2((-\tau, 0), r_4).$$

Notice that

$$\tau < \frac{b(4p-b)}{4(2p-b)}$$

and therefore $B_2\big(\mathbb{O}_2, \frac{1}{2}(a+p)-b\big) \subset M$ if $p < a < p+b$.

Set $\sigma = -\arctan(\tau/p)$; then $\cos(\sigma) > r_1/r_2$, since

$$\tau < \frac{p\sqrt{r_2^2 - r_1^2}}{r_1},$$

and $L \cap \overline{B_2} = \varnothing$, where $B_2 = B_2((0, p-r_2), r_1)$. Therefore there exists a $\delta_1 > 0$ such that if $u = (u_1, u_2) \in \partial B_2(\mathbb{O}_2, p)$ with $|u - w| < \delta_1$, then

$$B_2\Big(\frac{p-r_2}{p}u, r_1\Big) \subset M. \tag{14}$$

Since

$$\tau < \frac{2(1-p\lambda)}{\lambda(2-p\lambda)},$$

we have $\tau - \big(\frac{2}{\lambda} - r_4\big) < -p$ and so $B_2(\mathbb{O}_2, p) \subset B_2\big((\tau, 0), \frac{2}{\lambda} - r_4\big)$ (see Figure 8, right). Notice that

$$M \setminus \{(0, \pm p)\} = \{(r\cos(\theta), p + r\sin(\theta)) : 0 < r < 2p, \theta^-(r) < \theta < \theta^+(r)\} \tag{15}$$

for some $\theta^\pm \in C^0([0, \delta))$ satisfying $\theta^- < \theta^+$, $\theta^-(0) = -\pi - \sigma$, and $\theta^+(0) = \sigma$.

Let $a > p$ and set

$$\mathcal{T} = \Big\{\Big(\big(\tfrac{1}{2}(a+p) + b\cos v\big)\cos u, \big(\tfrac{1}{2}(a+p) + b\cos v\big)\sin u, b\sin v + c\Big) : (u, v) \in R\Big\},$$

where $R = [0, 2\pi] \times [-\pi, 0]$ and $0 < c < b$; since $b < \frac{1}{4m_0}\big(1 + 2m_0 p - \sqrt{1 + 4m_0^2 p^2}\big)$, we see that

$$\frac{\frac{1}{2}(a+p) - 2b}{4b\big(\frac{1}{2}(a+p) - b\big)} > m_0$$

for all $a \geq p$. We shall assume

$$a \in \big(p, \min\{p+b, \tfrac{1}{\lambda}\}\big) \tag{16}$$

and $c = \sqrt{b^2 - \big(\frac{1}{2}(a-p)\big)^2}$. Notice that $\mathcal{T}$ is the lower half of a torus whose mean curvature (i.e., one half of the trace of the shape operator) at each point is greater than m_0. Let $\mathcal{T}$ be the graph of a function j_a over

$$\Delta_a = \big\{x \in \mathbb{R}^2 : \tfrac{1}{2}(a+p) - b \leq |x| \leq \tfrac{1}{2}(a+p) + b\big\};$$

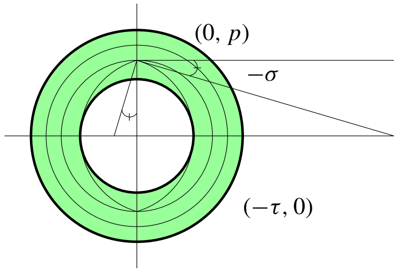

Figure 4. The domain of j_a.

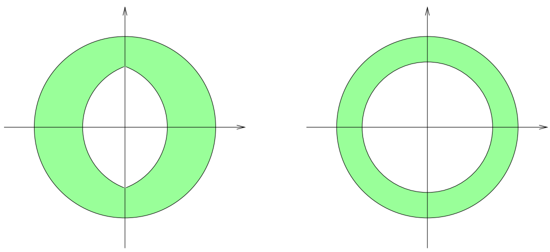

Figure 5. Left: $\Pi_{1,j}(\Omega)$ for $2 \le j \le n$. Right: $\Pi_{i,j}(\Omega)$ for $2 \le i < j \le n$.

then $j_a(x) = 0$ on $|x| = a$ and $|x| = p$, $j_a(x) < 0$ on $p < |x| < a$, and $j_a(x) > 0$ on $\frac{1}{2}(a + p) - b \le |x| < p$ and $a < |x| \le \frac{1}{2}(a + p) + b$ for $x \in \mathbb{R}^2$. Notice that $|j_a(x)| < \frac{1}{2m_0}$ for all $x \in \Delta_a$.

Set

$$\Omega = B_n(\mathbb{O}_n, a) \setminus \overline{\mathcal{M}}, \tag{17}$$

where $\mathcal{M} = \widetilde{M} = \{(x_1, x_2\omega) \in \mathbb{R}^n : (x_1, x_2) \in M, \omega \in S^{n-2}\}$. If we define

$$\Pi_{i,j}(A) = \{(x_i, x_j) : (x_1, \ldots, x_n) \in A, x_k = 0 \text{ for } k \ne i, j\}$$

for $A \subset \mathbb{R}^n$ and $1 \le i < j \le n$, then $\Pi_{1,j}(\Omega) = B_2(\mathbb{O}_2, a) \setminus \overline{M}$ for $2 \le j \le n$ and $\Pi_{i,j}(\Omega) = B_2(\mathbb{O}_2, a) \setminus \overline{B_2(\mathbb{O}_2, 1)}$ for $2 \le i < j \le n$ (see Figure 5).

We wish to select a helicoidal surface in $\mathbb{R}^3$ (e.g., [do Carmo and Dajczer 1982]) with constant mean curvature m_0, axis $\{w\} \times \mathbb{R}$, and pitch $-\beta$ (recall $-\beta \in (-\beta_0, 0)$), which we will denote $\mathcal{S}$; then, for each $t \in \mathbb{R}$, $k_t(\mathcal{S}) = \mathcal{S}$, where $k_t : \mathbb{R}^3 \to \mathbb{R}^3$ is the helicoidal motion given by $k_t(x_1, x_2, x_3) = (l_t(x_1, x_2), x_3 - \beta t)$ with $l_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$l_t(x_1, x_2) = (x_1 \cos(t) + (x_2 - p)\sin(t),\ p - x_1 \sin(t) + (x_2 - p)\cos(t)).$$

Set $c_0 = \frac{1}{4}\beta\sigma < 0$. By vertically translating $\mathcal{S}$, we may assume that there is an open c_0-level curve $\mathcal{L}_0$ of $\mathcal{S}$ with endpoints $w = (0, p)$ and $b = (b_1, b_2)$ such that

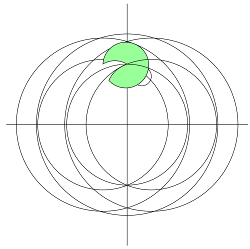

Figure 6. $\mathcal{R}$.

$\mathcal{L}_0 \subset (0,\infty)\times\mathbb{R}$, $\mathcal{L} = \overline{\mathcal{L}_0}$ is tangent to the (horizontal) line $\mathbb{R}\times\{p\}$ at w, and the slope m_v of the tangent line to $\mathcal{L}$ at v satisfies $|m_v| < \tan(-\sigma/5)$ for each $v\in\mathcal{L}_0$; then $\mathcal{L}\times\{c_0\}\subset\mathcal{S}$ and the curves $l_t(\mathcal{L}_0)$, $-\frac{7\pi}{8} < t < \frac{7\pi}{8}$, are mutually disjoint. Notice that the set

$$\mathcal{R} = \left\{l_t(\mathcal{L}_0) : -\tfrac{7\pi}{8} < t < \tfrac{7\pi}{8}\right\} = \bigcup_{-\frac{7\pi}{8}<t<\frac{7\pi}{8}} l_t(\mathcal{L}_0)$$

is an open subset of $\mathbb{R}^2 \setminus ((-\infty,0]\times\{p\})$ (see Figure 6), $w\in\overline{\mathcal{R}}$, and $\mathcal{S}$ implicitly defines the smooth function h_2 on $\mathcal{R}$ given by $h_2(x) = \frac{\beta}{4}(\sigma-4t)$ if $x\in l_t(\mathcal{L}_0)$ for some $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Notice that $B_2(w,b_1)\cap\{x_1>0\}\subset\mathcal{R}$. Now we have $l_t(\mathcal{L}_0)\cap M = \varnothing$ for $t\in\left(\frac{3\sigma}{4},\frac{\sigma}{4}\right)$ and, by making $b_1>0$ sufficiently small, we may assume that

$$l_t(\mathcal{L}_0)\subset B_2(\mathbb{O}_2,p)\setminus M \quad \text{for each } t\in\left(\tfrac{3\sigma}{4},\tfrac{\sigma}{4}\right). \tag{18}$$

Notice that $h_2 < \beta(2\sigma^2-\pi)/(8\sigma)$ on $l_t(\mathcal{L}_0)$ for $-\frac{\pi}{2} < t < \frac{7\pi}{8}$.

Let us fix $u=(u_1,u_2)\in\partial B_2(\mathbb{O}_2,p)$ such that $|u-w| < \min\{\delta_1,b_1\}$ and $u_1>0$. Then there exists $\theta_u\in\left(0,\frac{\pi}{2}\right)$ such that $u=(p\cos(\theta_u), p\sin(\theta_u))$. Define

$$g^{[u]}(x) = g_1\Big(x + \frac{r_2-p}{p}u\Big)$$

and notice that $g^{[u]}(u) = g_1\left(\frac{r_2}{p}u\right) = 0$, since $\left|\frac{r_2}{p}u\right| = r_2$. Note that the domain

$$\mathcal{D}^{[u]} = \Big\{x + \frac{p-r_2}{p}u : x\in\Delta_1\Big\} = B_2\Big(\frac{p-r_2}{p}u, r_2\Big)\setminus\overline{B_2\Big(\frac{p-r_2}{p}u, r_1\Big)}$$

of $g^{[u]}$ is contained in $B_2(\mathbb{O}_2,p)$ since $\partial B_2\left(\frac{p-r_2}{p}u, r_2\right)$ and $\partial B_2(\mathbb{O}_2,p)$ are tangent circles at u and $r_2<p$ (see Figure 7). Notice that

$$h_2(r\cos(\theta_u), r\sin(\theta_u)) < g^{[u]}(r\cos(\theta_u), r\sin(\theta_u)) \tag{19}$$

when $p-r_2+r_1\le r\le p$, because $h_2(u) < 0 = g^{[u]}(u)$, $\beta<\beta_0$, and (12) holds.

Let

$$\mathcal{N}_\pm \subset \{x\in\mathbb{R}^2 : r_4 \le |(x_1\pm\tau, x_2)| \le \tfrac{2}{\lambda} - r_4\}\times\mathbb{R}$$

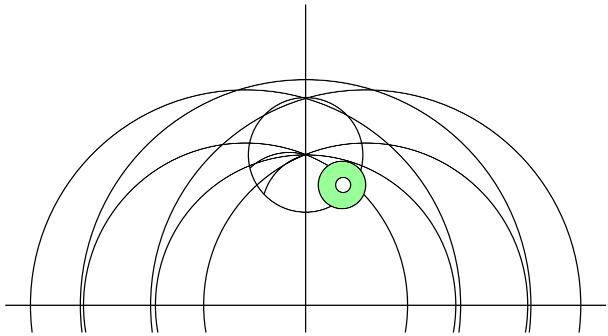

Figure 7. $\mathcal{D}^{[u]}$; $\Omega\cap\widetilde{\mathcal{D}}^{[u]}$ is the domain of the comparison function for (28).

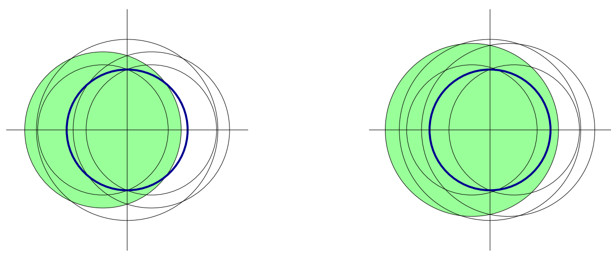

Figure 8. Left: $B_2(\mathcal{O}_2, p) \not\subseteq B_2\big((-\tau, 0), \frac{2}{\lambda} - r_4\big)$. Right: $B_2(\mathcal{O}_2, p) \subset B_2\big((-\tau, 0), \frac{2}{\lambda} - r_4\big)$.

be unduloids in $\mathbb{R}^3$ with mean curvature $\frac{\lambda}{2}$ such that $\{(\mp\tau, 0)\} \times \mathbb{R}$ are the respective axes of symmetry; the minimum and maximum radii (or "neck" and "waist" sizes) of both unduloids are r_4 and $\frac{2}{\lambda} - r_4$, respectively. Set

$$\Delta_\pm = B_2\big((\mp\tau, 0), \tfrac{2}{\lambda} - r_4\big) \setminus \overline{B_2((\mp\tau, 0), r_4)}$$

and define $k_\pm \in C^\infty(\Delta_\pm)$ so that the graphs of $k_\pm$ are subsets of $\mathcal{N}_\pm$, respectively,

$$\operatorname{div}(Tk_\pm) = -\lambda \quad \text{in } \Delta_\pm,$$

$\frac{\partial}{\partial r}\big(k_\pm((\mp p, 0) + r\Theta)\big)\big|_{r=r_4} = -\infty$ and $\frac{\partial}{\partial r}\big(k_\pm((\mp p, 0) + r\Theta)\big)\big|_{r=2/\lambda - r_4} = -\infty$ for each $\theta \in \mathbb{R}$, where $\Theta = (\cos(\theta), \sin(\theta))$. We may vertically translate $\mathcal{N}_\pm$ so that $k_\pm(x) = 0$ for $x \in \mathbb{R}^2$ with $|(x_1 \pm \tau, x_2)| = \frac{2}{\lambda} - r_4$. Notice that $k_+(0, p) = k_-(0, p) = \sup_{\Delta_+} k_+ = \sup_{\Delta_-} k_-$.

Let $\mathcal{N} \subset \big\{x \in \mathbb{R}^2 : p \le |x| \le \frac{2}{\lambda} - p\big\} \times \mathbb{R}$ be an unduloid with mean curvature $\frac{\lambda}{2}$ such that the x_3-axis is the axis of symmetry and the minimum and maximum radii (or "neck" and "waist" sizes) are p and $\frac{2}{\lambda} - p$, respectively. Set

$$\Delta_2 = B_2\big(\mathcal{O}_2, \tfrac{2}{\lambda} - p\big) \setminus \overline{B_2(\mathcal{O}_2, p)}$$

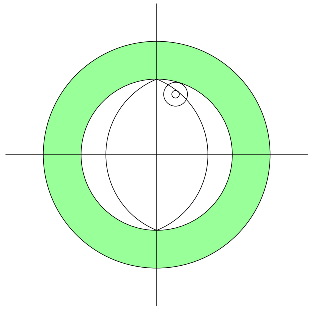

Figure 9. $B_2(\mathbb{O}_2, a) \setminus \overline{B_2(\mathbb{O}_2, p)}$: (22).

and define $k_2 \in C^\infty(\Delta_2)$ so that the graph of k_2 is a subset of $\mathcal{N}$, $\operatorname{div}(Tk_2) = -\lambda$ in Δ_2, and $\frac{\partial}{\partial r}\big(k_2(r\Theta)\big)\big|_{r=p} = \frac{\partial}{\partial r}\big(k_2(r\Theta)\big)\big|_{r=2/\lambda-p} = -\infty$ for each $\theta \in \mathbb{R}$, where $\Theta = (\cos(\theta), \sin(\theta))$.

Define $\phi \in C^\infty(\mathbb{R}^n)$ so that $\phi = 0$ on $\partial B_n(\mathbb{O}_n, a)$ and $\phi = m$ on $\partial\mathcal{M}$, where

(20) $$m > \max\Big\{g_1(0, r_1), \tfrac{1}{2m_0}, k_+(0, r_4 - \tau) + k_2(0, p) - k_2\big(0, \tfrac{2}{\lambda} - p\big)\Big\};$$

recall then that $m > j_a\big(\frac{1}{2}(a+p) - b\big)$. Let f be the variational solution of (1)–(2) with Ω and ϕ as given here; that is, let f minimize the functional given in (3) and notice that the existence of f follows from (10), (16), §1.D of [Giusti 1976], and [Gerhardt 1974; Giusti 1978]. (Notice that there exists $w : B_2(\mathbb{O}_2, a) \setminus M \to \mathbb{R}$ such that $f = \tilde{w}$.) The comparison principle implies $j_a(x) \le f(x)$ for $x \in \Omega$, and so $f(x) \ge j_a(x) \ge 0$ if $x \in \Omega$ with $|x| \le p$ (recall (16) holds). In particular,

(21) $$f(x) \ge 0 \quad \text{when } x \in \Omega \text{ with } |x| \le p.$$

Set $W = (B_2(\mathbb{O}_2, a) \setminus \overline{B_2(\mathbb{O}_2, p)}) \times \mathbb{R}^{n-2}$. Now

$$\Omega \subset B_2(\mathbb{O}_2, a) \times \mathbb{R}^{n-2} \subset B_2\big(\mathbb{O}_2, \tfrac{2}{\lambda} - p\big) \times \mathbb{R}^{n-2}$$

(see Figure 9). Define $k_3(x) = k_2(x_1, x_2) - k_2(0, a)$ for $x = (x_1, x_2, \ldots, x_n) \in W$. Notice that $f = 0 \le k_3$ on $\overline{W} \cap \partial B_n(\mathbb{O}_n, a)$,

$$\operatorname{div}(Tf) = H(x, f(x)) \ge -\lambda = \operatorname{div}(Tk_3) \quad \text{in } \Omega \cap W,$$

and $\frac{\partial}{\partial r}(k_2(r\Theta))|_{r=p} = -\infty$ (so that $\lim_{W \ni y \to x} Tk_3(y) \cdot \xi(x) = 1$, for ξ the unit exterior normal to ∂W and $x \in \partial B_2(\mathbb{O}_2, p) \times \mathbb{R}^{n-2}$). The general comparison principle (e.g., [Finn 1986, Theorem 5.1]) then implies

(22) $$f \le k_3 \quad \text{in } \Omega \cap W$$

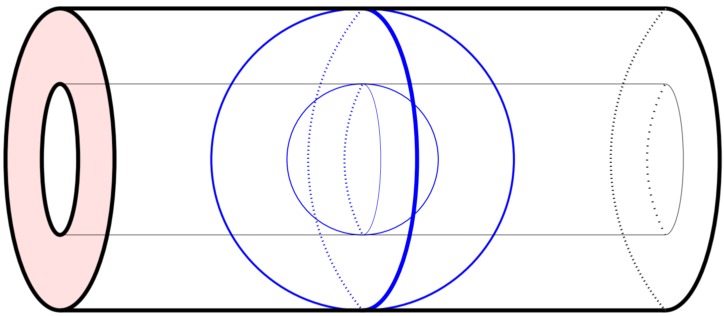

Figure 10. (23): W and $B_n(\mathcal{O}_n, a) \setminus \overline{B_n(\mathcal{O}_n, p)}$ when $n = 3$.

and, in particular,

(23) $$\limsup_{\Omega \cap W \ni y \to x} f(y) \le k_3(x) \quad \text{for } x \in \partial\Omega \cap \overline{W}$$

(see Figure 10). By rotating the axis of symmetry of W through all lines in $\mathbb{R}^n$ containing $\mathcal{O}_n$ (or, equivalently, keeping W fixed and rotating Ω about $\mathcal{O}_n$), we see that

(24) $$\sup\{f(x) : x \in B_n(\mathcal{O}_n, a) \setminus \overline{B_n(\mathcal{O}_n, p)}\} \le k_2(0, p) - k_2(0, a).$$

Now define $k_4 \in C^\infty(\Delta_+ \times \mathbb{R}^{n-2}) \cap C^0(\overline{\Delta_+} \times \mathbb{R}^{n-2})$ by

$$k_4(x) = k_+(x_1, x_2) + k_2(0, p) - k_2(0, a), \quad x = (x_1, x_2, \dots, x_n) \in \overline{\Delta_+} \times \mathbb{R}^{n-2}.$$

Combining (1) and (24) with the facts that $\operatorname{div}(Tk_4) = -\lambda$ in $\Delta_+ \times \mathbb{R}^{n-2}$ and $\lim_{\Delta_+ \times \mathbb{R}^{n-2} \ni y \to x} Tk_4(y) \cdot \xi_+(x) = 1$ for $x \in \partial B_2((-\tau, 0), r_4) \times \mathbb{R}^{n-2}$, where ξ_+ is the inward unit normal to $\partial B_2((-\tau, 0), r_4) \times \mathbb{R}^{n-2}$, we see that

(25) $$f \le k_4 \quad \text{in } \Omega \cap (\Delta_+ \times \mathbb{R}^{n-2}).$$

(If Figure 8 (left) held, then (25) would not be valid.) Now let $L : \mathbb{R}^n \to \mathbb{R}^n$ be any rotation about $\mathcal{O}_n$ which satisfies $L(\Omega) = \Omega$, notice that $f \circ L$ satisfies (1)–(2), and apply the previous argument to obtain $f \circ L \le k_4$ in $\Omega \cap (\Delta_+ \times \mathbb{R}^{n-2})$ and therefore

(26) $$\sup\{f(x) : x \in \partial\mathcal{M}\} \le k_4(p, 0) < m.$$

From Lemma 1, we see that the downward unit normal N_f to the graph of f satisfies $N_f = (\nu, 0)$ on $\partial\mathcal{M} \setminus \{(0, p\omega) : \omega \in S^{n-2}\}$ and

(27) $$\lim_{\Omega \ni y \to x} Tf(y) \cdot \nu(x) = 1 \quad \text{for } x \in \partial\mathcal{M} \setminus \{(0, p\omega) : \omega \in S^{n-2}\}.$$

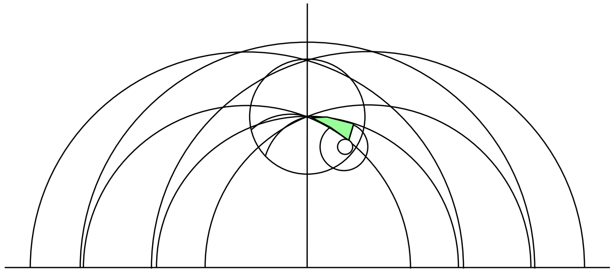

Figure 11. A: (29).

Let us write $B = B_2\big(\frac{p-r_2}{p}u, r_2\big)$; then $\tilde{g}^{[u]} = 0 \le f$ on $\Omega \cap \partial\widetilde{B}$ and $\tilde{g}^{[u]} \le g_1(r_1, 0) < \phi$ on $\widetilde{B} \cap \partial M$. It follows from (1), (11), and Lemma 3 that

$$\tilde{g}^{[u]} < f \quad \text{on } \Omega \cap \widetilde{\mathcal{D}}^{[u]} = \Omega \cap \widetilde{B}. \tag{28}$$

Set $U = \{r(\cos(\theta), \sin(\theta)\omega) \in \Omega : r \in (0, p), \theta \in (0, \theta_u), \omega \in S^{n-2}\}$. If we write

$$\begin{aligned}
\partial_1 U &= \{(p\cos(\theta), p\sin(\theta)\omega) : \theta \in (0, \theta_u], \omega \in S^{n-2}\},\\
\partial_2 U &= \partial\mathcal{M} \cap \partial U,\\
\partial_3 U &= \{(r\cos(\theta_u), r\sin(\theta_u)\omega) \in \overline{\Omega} : r \in [0, p], \omega \in S^{n-2}\},
\end{aligned}$$

then $\partial U = \partial_1 U \cup \partial_2 U \cup \partial_3 U$, $\tilde{h}_2 \le 0 \le f$ on $\partial_1 U \setminus \{P\}$, and $\tilde{h}_2 < \tilde{g}^{[u]} < f$ on $\partial_3 U$ (see (19)); then (27) and the general comparison principle imply

$$\tilde{h}_2 < f \quad \text{in } U = \widetilde{A}, \tag{29}$$

where $A = \{r(\cos(\theta), \sin(\theta)) \in B_2(\mathbb{O}_2, p) \setminus \overline{M} : r \in (0, p), \theta \in (0, \theta_u)\}$ (see Figure 11). Set $\mathcal{R}_2 = \bigcup_{t=3\sigma/4}^{2\sigma/4} l_t(\mathcal{L}_0)$. Now (18) implies $\widetilde{\mathcal{R}}_2 \subset U$ and so

$$f > \tilde{h}_2 \ge -\frac{\beta\sigma}{4} \quad \text{on } \mathcal{R}_2. \tag{30}$$

Using (24) and (30), we see that if $a \in \big(p, \frac{2}{\lambda} - p\big)$ is close enough to p, then $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$ and therefore f cannot be continuous at P or at any point of $T = \{(0, p\omega) \in \mathbb{R}^n : \omega \in S^{n-2}\}$. Note that $f \in C^0(\overline{\Omega} \setminus T)$ (e.g., [Lin 1987]).

4.2. ***One singular point.*** In this section, we obtain a domain Ω and $\phi \in C^\infty(\mathbb{R}^n)$ such that $P \in \partial\Omega$, the minimizer f of (3) is discontinuous at P, $\partial\Omega \setminus \{P\}$ is smooth (C^∞), and $f \in C^0(\overline{\Omega} \setminus \{P\})$. This is accomplished by replacing $\mathcal{M}$ by a convex set $\mathcal{G}$ such that $\partial\mathcal{G} \setminus \{P\}$ is smooth (C^∞) and $\mathcal{G} \subset B_n(\mathbb{O}_n, p)$. We shall use the notation of Section 4.1 throughout this section. We assume $p \in \big(0, \frac{1}{\lambda}\big)$ and set $P = (0, p, 0, \ldots, 0)$. (We will no longer require Figure 8 (right) to hold.)

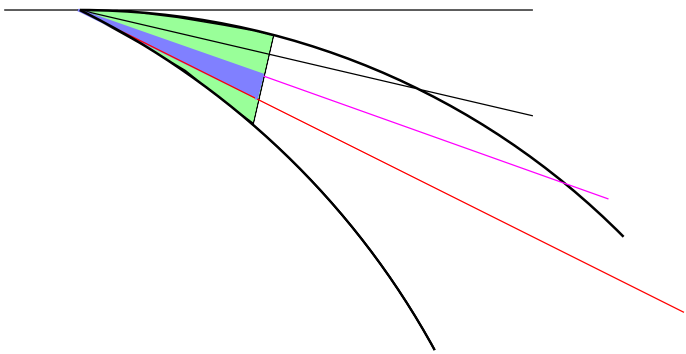

Figure 12. An illustration of $\mathcal{R}_2$ (blue region) and A (green and blue regions).

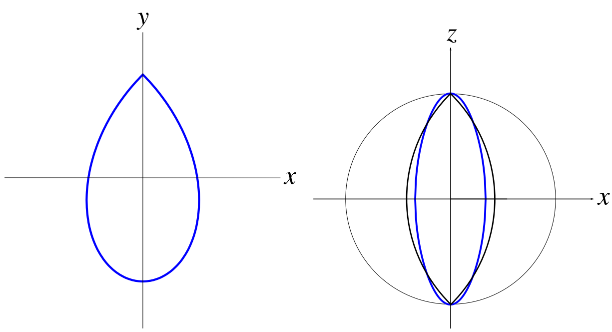

Figure 13. Left: $X\big(\theta, \frac{\pi}{2}, 1\big)$. Right: $X\big(\theta, \frac{1}{2}\arccos(1-\sec(\theta)\sec(2\theta)), 1\big)$.

Let $\alpha > 1$, $n \geq 3$, and $Y : \big[-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\big] \times [0,\pi] \times S^{n-3} \to \mathbb{R}^n$ be defined by

$$Y(\theta,\phi,\omega) = 2\cos(\alpha\theta)\sin(\phi)\big(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\omega\big).$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$F(x_1,\ldots,x_n) = \left(\tfrac{x_2}{p}, \tfrac{1-x_1}{p}, \tfrac{x_3}{p}, \ldots, \tfrac{x_n}{p}\right)$$

and define $X(\theta,\phi,\omega) = F(Y(\theta,\phi,\omega))$ for $-\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha}$, $0 \leq \phi \leq \pi$, $\omega \in S^{n-3}$ (see Figures 13 and Figure 14 with $n = 3$, $\alpha = 2$; the axes are labeled x, y, z for x_1, x_2, x_3, respectively). Let $\mathcal{G}$ be the open, convex set whose boundary is the image of X; that is,

$$\partial\mathcal{G} = \left\{X(\theta,\phi,\omega) : -\tfrac{\pi}{2\alpha} \leq \theta \leq \tfrac{\pi}{2\alpha}, 0 \leq \phi \leq \pi, \omega \in S^{n-3}\right\}.$$

Notice that $\partial\mathcal{G} \setminus \{P\}$ is a C^∞ hypersurface in $\mathbb{R}^n$ and $\partial\mathcal{G} \subset \overline{B_n(\mathbb{O}_n, p)}$.

Let τ satisfy

$$0 < \tau < \min\left\{ \frac{pr_1}{\sqrt{r_2^2 - r_1^2}}, \frac{b(4p-b)}{4(2p-b)} \right\}.$$

Set $\sigma = -\arctan(\tau/p)$ and $\alpha = \pi/(\pi+2\sigma)$. Then the tangent cones to $\partial\mathcal{G}$ and $\partial\mathcal{M}$ at P are identical, $\cos(\sigma) > r_1/r_2$, and (14) holds for $u = (u_1, u_2) \in \partial B_2(\mathbb{O}_2, p)$ with $|u - w| < \delta_1$. By making $\tau > 0$ smaller if necessary, we may assume $B_n\big(\mathbb{O}_n, \frac{1}{2}(a+p) - b\big) \subset \mathcal{G}$ if $p < a < p + b$.

Now pick $a \in \big(p, \min\{p+b, \frac{1}{\lambda}\}\big)$ such that $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$, as in (30), and define

$$\Omega = B_n(\mathbb{O}_n, a) \setminus \overline{\mathcal{G}}. \tag{31}$$

Let

$$m > \max\left\{ g_1(0, r_1), \frac{1}{2m_0}, \frac{\beta(2\sigma^2 - \pi)}{8\sigma} \right\}$$

and define $\phi \in C^\infty(\mathbb{R}^n)$ so that $\phi = 0$ on $\partial B_n(\mathbb{O}_n, a)$ and $\phi = m$ on $\partial\mathcal{G}$, and let f be the variational solution of (1)–(2). Notice that $f \in C^2(\Omega)$ satisfies (1) and $f \in C^0(\overline{\Omega} \setminus \{P\})$ (e.g., [Lin 1987]).

As in (28), let $B = B_2\big(\frac{p-r_2}{p}u, r_2\big)$. Set $U_0 = \{x \in \Omega : x \in \widetilde{B}, x_1 > 0\}$ and $U = \{r(\cos(\theta), \sin(\theta)\omega) \in \Omega : r \in (0, p), \theta \in (0, \theta_u), \omega \in S^{n-2}\}$. Now $\tilde{g}^{[u]} = 0$ on $\partial U_0 \cap \partial\widetilde{B}$ and $\tilde{g}^{[u]} \le g_1(0, r_1) < m$ on $\partial U_0 \cap \partial\mathcal{G}$ and so Lemma 2, Lemma 3, and (1) imply $\tilde{g}^{[u]} \le f$ in U_0 since f minimizes the functional in (3).

As before, set

$$\begin{aligned}
\partial_1 U &= \big\{\big(p\cos(\theta), p\sin(\theta)\omega\big) : \theta \in [0, \theta_u], \omega \in S^{n-2}\big\},\\
\partial_2 U &= \partial\mathcal{G} \cap \partial U,\\
\partial_3 U &= \big\{\big(r\cos(\theta_u), r\sin(\theta_u)\omega\big) \in \overline{\Omega} : r \in [0, p], \omega \in S^{n-2}\big\}.
\end{aligned}$$

Then $f \ge 0$ on $\partial_1 U \setminus \{P\}$, $\partial U = \partial_1 U \cup \partial_2 U \cup \partial_3 U$, $\tilde{h}_2 \le 0 \le f$ on $\partial_1 U$, $\tilde{h}_2 < m = \phi$ on $\partial_2 U$, and $\tilde{h}_2 < \tilde{g}^{[u]} < f$ on $\partial_3 U$; Lemma 2 implies that (30) continues to hold. Then (24) and (30) imply f is discontinuous at P since $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$.

5. The Concus–Finn conjecture

For the moment, assume $n = 2$. Around 1970, Paul Concus and Robert Finn conjectured that if $\kappa \ge 0$, $\Omega \subset \mathbb{R}^2$ has a corner at $P \in \partial\Omega$ of (angular) size 2α, $\alpha \in \big(0, \frac{\pi}{2}\big)$, $\gamma : \partial\Omega \setminus \{P\} \to [0, \pi]$, and $\big|\frac{\pi}{2} - \gamma_0\big| > \alpha$, where

$$\lim_{\partial\Omega \ni x \to P} \gamma(x) = \gamma_0, \tag{32}$$

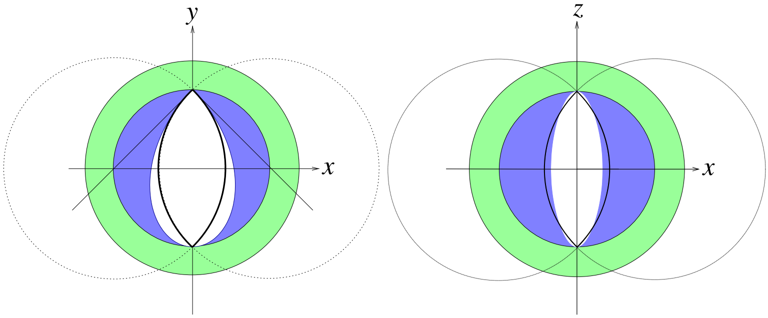

Figure 14. Left: $\Pi_{1,2}(\Omega)$. Right: $\Pi_{1,3}(\Omega)$.

then a function $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{P\})$ which satisfies

$$\operatorname{div}(Tf) = \kappa f \quad \text{in } \Omega, \tag{33}$$

$$Tf \cdot \eta = \cos(\gamma) \quad \text{on } \partial\Omega \setminus \{P\} \tag{34}$$

must be discontinuous at P; here $\eta(x)$ is the exterior unit normal to Ω at $x \in \partial\Omega\setminus\{P\}$.

In the situation above with $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, the "nonconvex Concus–Finn conjecture" states that if $\left|\frac{\pi}{2} - \gamma_0\right| > \pi - \alpha$, then the capillary surface f with contact angle γ must be discontinuous at P. A generalization (including the replacement of (33) by (1)) of this extension of the Concus–Finn conjecture in the case $\gamma_0 \in (0, \pi)$ was proven in [Lancaster 2012]. Both [Lancaster 2010] and [Lancaster 2012] include the possibility of differing limiting contact angles; that is, the limits

$$\lim_{\partial^+\Omega \ni x \to P} \gamma(x) = \gamma_1 \quad \text{and} \quad \lim_{\partial^-\Omega \ni x \to P} \gamma(x) = \gamma_2$$

exist, $\gamma_1, \gamma_2 \in (0, \pi)$, and $\gamma_1 \neq \gamma_2$. Here $\partial^+\Omega$ and $\partial^-\Omega$ are the two components of $\partial\Omega \setminus \{P, Q\}$, where $Q \in \partial\Omega \setminus \{P\}$. When $\gamma_1 \neq \gamma_2$, the necessary and sufficient (when $\alpha \le \frac{\pi}{2}$) or necessary (when $\alpha > \frac{\pi}{2}$) conditions for the continuity of f at P become slightly more complicated.

The cases where $\gamma_0 = 0$, $\gamma_0 = \pi$, $\min\{\gamma_1, \gamma_2\} = 0$, and $\max\{\gamma_1, \gamma_2\} = \pi$ remain unresolved. If we suppose for a moment that the nonconvex Concus–Finn conjecture with limiting contact angles of 0 or π is proven, then the discontinuity of f at P in Section 2 follows immediately from the fact that $f < \phi$ in a neighborhood in $\partial\Omega \setminus \{P\}$ of P, since then Lemma 1 implies $\gamma_0 = 0$ and therefore $\left|\frac{\pi}{2} - \gamma_0\right| > \pi - \alpha$. In this situation (i.e., the solution f of a Dirichlet problem satisfies a 0 (or π) contact angle boundary condition near P), establishing the discontinuity of f at P would be much easier and a much larger class of domains Ω with a nonconvex corner (i.e., $\alpha > \frac{\pi}{2}$) at P would have this property. For example, if Ω is a bounded locally Lipschitz domain in $\mathbb{R}^2$ for which (4) holds, $f \in C^2(\Omega)$ is a generalized solution of (1)–(2) (and H need not vanish), and ϕ is large enough near P (depending on H and the maximum of ϕ outside some neighborhood of P) that $f < \phi$ on $\partial\Omega \setminus \{P\}$ near P, then the fact that $\gamma_0 = 0$ (Lemma 1) together with the nonconvex Concus–Finn conjecture would imply that f is discontinuous at P.

Now consider $n \in \mathbb{N}$ with $n \geq 3$. Formulating generalizations of the Concus–Finn conjecture in the "convex corner case" (i.e., $\Omega \cap B_n(P, r) \subset \{X \in \mathbb{R}^n : (X - P) \cdot \mu > 0\}$ for some $\mu \in S^{n-1}$, $P \in \partial\Omega$ and $r > 0$) and in other cases where $\partial\Omega$ is not smooth at a point $P \in \partial\Omega$ may be complicated because the geometry of $\partial\Omega \setminus \{P\}$ is much more interesting when $n > 2$. Establishing the validity of a generalization of the Concus–Finn conjecture for solutions of (1) and (34) when $n > 2$ is probably significantly harder than doing so when $n = 2$.

Suppose we knew that a solution f of (1) and (34) is necessarily discontinuous at a "nonconvex corner" $P \in \partial\Omega$ when $\gamma_0 = 0$, where γ_0 is given by (32). In this case, a necessary condition for the continuity of f at P would be that

$$\limsup_{\partial\Omega \ni X \to P} Tf(X) \cdot \eta(X) > 0,$$

$$\liminf_{\partial\Omega \ni X \to P} Tf(X) \cdot \eta(X) < \pi.$$

Then the arguments in Section 4 could be made more easily and the conclusion that f is discontinuous at P would hold in a much larger class of domains Ω; here, of course, we use the ridge point P in Section 4 as an example of a "nonconvex corner" of a domain in $\mathbb{R}^n$. The primary difficulty in proving in Section 4 that f is discontinuous at P is establishing (30); a more "natural" generalization of $\Omega \subset \mathbb{R}^2$ in Section 2 would be

$$\Omega^* = \{(x\omega_1, y, \omega_2, \dots, \omega_{n-1}) \in \mathbb{R}^n : (x, y) \in B_2(\mathbb{O}_2, a) \setminus \overline{M}, \omega \in S^{n-1}\}.$$

However, the use of Lemma 3 to help establish (30) in Ω^* is highly problematic. On the other hand, an n-dimensional "Concus–Finn theorem" for a nonconvex conical point (e.g., $P \in \partial\Omega^*$) would only require an inequality like (26) to prove that $f < \phi$ on $\partial\Omega \setminus \{P\}$ near P, and hence that f is discontinuous at P; the replacement of (17) by (31) in order to obtain Ω such that $\partial\Omega \setminus \{P\}$ is C^∞ would be unnecessary.

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