THE UNTWISTING NUMBER OF A KNOT

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Dedicated to Tim Cochran

The unknotting number of a knot is the minimum number of crossings one must change to turn that knot into the unknot. The algebraic unknotting number is the minimum number of crossing changes needed to transform a knot into an Alexander polynomial-one knot. We work with a generalization of unknotting number due to Mathieu and Domergue, which we call the untwisting number. The untwisting number is the minimum number (over all diagrams of a knot) of right- or left-handed twists on even numbers of strands of a knot, with half of the strands oriented in each direction, necessary to transform that knot into the unknot. We show that the algebraic untwisting number is equal to the algebraic unknotting number. However, we also exhibit several families of knots for which the difference between the unknotting and untwisting numbers is arbitrarily large, even when we only allow twists on a fixed number of strands or fewer.

1. Introduction

It is a natural knot-theoretic question to seek to measure "how knotted up" a knot is. One such "knottiness" measure is given by the *unknotting number* $u(K)$, the minimum number of crossings, taken over all diagrams of K, one must change to turn K into the unknot. By a *crossing change* we shall mean one of the two local moves on a knot diagram given in Figure 1.

This invariant is quite simple to define but has proven itself very difficult to master. Fifty years ago, Milnor conjectured that the unknotting number for the (p, q)-torus knot was $\frac{1}{2}(p - 1)(q - 1)$; only in 1993, in two celebrated papers, did Kronheimer and Mrowka [1993; 1995] prove this conjecture true. Hence, it is desirable to look at variants of unknotting number which may be more tractable. One natural variant (due to Murakami [1990]) is the *algebraic unknotting number* $u_a(K)$, the minimum number of crossing changes necessary to turn a given knot into an Alexander polynomial-one knot. Alexander polynomial-one knots are significant because they "look like the unknot" to *classical invariants*, knot invariants derived
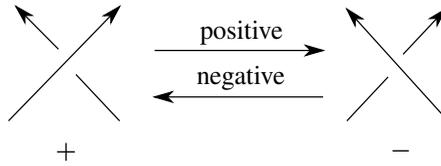

Figure 1. A positive and negative crossing change.

from the Seifert matrix. It is obvious that $u_a(K) \le u(K)$ for any knot K, and there exist knots such that $u_a(K) < u(K)$ (for instance, any nontrivial knot with trivial Alexander polynomial).

Mathieu and Domergue [1988] defined another generalization of unknotting number. Livingston [2002] worked with this definition. He described it as follows:

> "One can think of performing a crossing change as grabbing two parallel strands of a knot with opposite orientation and giving them one full twist. More generally, one can grab $2k$ parallel strands of K with k of the strands oriented in each direction and give them one full twist."

Following Livingston, we call such a twist a *generalized crossing change*. We describe in Section 2A how a crossing change may be encoded as a ± 1-surgery on a nullhomologous unknot $U \subset S^3 - K$ bounding a disk D such that $D \cap K = 2$ points. From this perspective, a generalized crossing change is a relaxing of the previous definition to allow $D \cap K = 2k$ points for any k, provided $\mathrm{lk}(K, U) = 0$ (see Figure 2). In particular, any knot can be unknotted by a finite sequence of generalized crossing changes.

One may then naturally define the *untwisting number* $tu(K)$ to be the minimum length, taken over all diagrams of K, of a sequence of generalized crossing changes beginning at K and resulting in the unknot. By $tu_p(K)$, we will denote the minimum number of twists on $2p$ or fewer strands needed to unknot K; notice that $tu_1(K) = u(K)$ and that

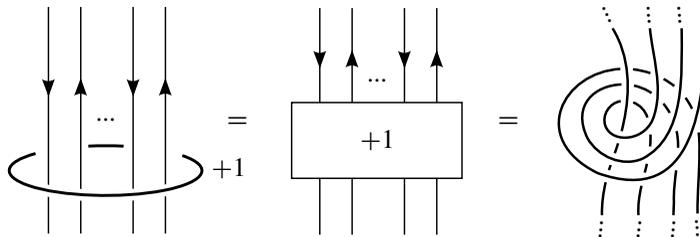$$tu \le \cdots \le tu_{p+1} \le tu_p \le \cdots \le tu_1 = u.$$

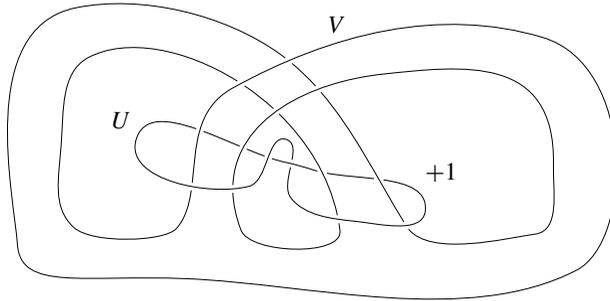**Figure 2.** A right-handed, or positive, generalized crossing change.

Figure 3. The generalized crossing change for $V \subset S^3$ which results in a knot $K \subset S^3$ with $tu(K) \neq u(K)$.

The *algebraic untwisting number* $tu_a(K)$ is the minimum number of generalized crossing changes, taken over all diagrams of K, needed to transform K into an Alexander polynomial-one knot. It is clear that $tu_a(K) \leq tu(K)$ for all knots K.

It is natural to ask how $tu(K)$ and $u(K)$ are related. We show that these invariants are "algebraically the same" in the following sense:

Theorem 1.1. *For any knot $K \subset S^3$, $tu_a(K) = u_a(K)$.*

Therefore, tu and u cannot be distinguished by classical invariants. By using the Jones polynomial, which is not a classical invariant, we can show that tu and u are not equal in general:

Theorem 1.2. *Let K be the image of $V \subset S^3$ in the manifold $M \cong S^3$ resulting from $+1$-surgery on the unknot $U \subset S^3$ shown in Figure 3. Then $tu(K) = 1$ but $u(K) > 1$.*

Furthermore, using the fact that the absolute value of the Ozsváth–Szabó τ invariant is a lower bound on unknotting number, we show in Section 5A that the difference $u - tu_p$ can be arbitrarily large, and thus so can the difference $u - tu$. Throughout this paper, $K_{p,q}$ will denote the (p, q)-cable of the knot K, where p denotes the longitudinal winding and q the meridional winding.

Theorem 1.3. *Let K be a knot in S^3 such that $u(K) = 1$. If $\tau(K) > 0$ and $p, q > 0$, then*

$$u(K_{p,q}) - tu_p(K_{p,q}) \geq p - 1.$$

In particular, if we take $q = 1$, then $tu_p(K_{p,q}) = 1$, while $u(K_{p,q}) \geq p$.

It may seem that the above examples are "cheating" in some sense, as in each of them the number of strands of K passing through the ± 1-framed unknot U in the generalized crossing change diagram is increasing along with $u(K)$. The following theorem shows that $u(K)$ can be arbitrarily larger than $tu(K)$ even when we restrict to doing q-generalized crossing changes for any fixed integer $q \geq 1$.

Theorem 1.4. *For any knot K with $u(K) = 1$ and $\tau(K) > 0$, the infinite family of knots $J_p^q := \#^p K_{q,1}$ satisfies*

$$u(J_p^q) - tu_q(J_p^q) \geq p$$

for any integers $p > 1, q > 0$.

So far, all of the families of knots we have worked with are quite complicated, in the sense that they are (p, q)-cables for large p or connected sums of such cables. One may wonder whether it is possible to find a "simpler" knot K for which $tu(K) < u(K)$. One measure of "knot simplicity" is *topological sliceness*; a knot K is *topologically slice* if there exists a locally flat disk $D \subset B^4$ such that $\partial D = K \subset S^3 = \partial B^4$.

Theorem 1.5. *For any knot K with $\tau(K) > 0$, let $D_+(K, 0)$ denote the positive-clasped, untwisted Whitehead double of K. Then the knots $S_p^q := \#^p(D_+(K, 0))_{q,1}$ are topologically slice and satisfy*

$$u(S_p^q) - tu_q(S_p^q) \geq p$$

for all integers $p > 0, q > 0$.

This paper is organized as follows. First, we will review the operations of Dehn surgery on knots and knot cabling and define the untwisting number more precisely. Next, we will give some background on the Blanchfield form which is necessary to prove that $tu_a = u_a$. Finally, we will prove that each of the above families of knots gives arbitrarily large gaps between u and tu.

Convention. In this paper, all manifolds are assumed to be compact, orientable, and connected.

2. Preliminaries

2A. *Dehn surgery.* In this section, we will describe the operation of Dehn surgery on knots.

Definition 2.1. Let $K \subset S^3$ be an oriented knot and $U \subset S^3$ be an unknot with $\text{lk}(K, U) = 0$. Let W be a closed tubular neighborhood of U in S^3. Let λ be a longitude of W, and let μ be a meridian of W such that $\text{lk}(\mu, \lambda) = 1$. The 3-manifold

$$M = (S^3 - \mathring{W}) \cup_h W,$$

where $h : \partial W \to \partial W$ is a homeomorphism taking a meridian of W onto $\pm \mu + \lambda \subset W$, is the *result of ± 1-surgery on U*, and U is said to be ± 1-*framed*. In this situation, we define a *generalized crossing change diagram for K* to be a diagram of the link $K \cup U$ with the number ± 1 written next to U, indicating that U is ± 1-framed. Figure 3 is an example of a generalized crossing change diagram for the unknot V.
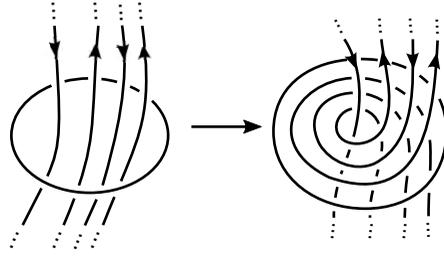
Figure 4. A right-handed twist about an unknotted component.

In the general case, note that the complement of $\mathring{N} \supset U$ in S^3 is a solid torus, which we may modify with a meridional twist. This alters K as follows: if D is a disk bounded by U such that k strands of K pass through D in straight segments, then each of the k straight pieces is replaced by a helix which screws through a neighborhood of D in the right-hand sense (see Figure 4).

If U is -1-framed, the knot obtained by erasing U and twisting the strands of K that pass through U as in Figure 4 represents the image of K under the -1-surgery on U [Rolfsen 1976]. If instead U has framing $+1$, the knot obtained by erasing U and giving K a left-handed meridional twist represents the image of K under the $+1$-surgery on U. The process of performing a $\mp$-meridional twist on the complement of a ±1-framed unknot U, then erasing U from the resulting diagram, is called a *blow-down on U*. The inverse process of introducing an unknotted component U to a surgery diagram consisting of a knot K, then performing a $\pm$-meridional twist on the complement of U to link it with K, is known as a *blow-up on U* and results in a diagram consisting of K and the ∓1-framed unknot U, where $\mathrm{lk}(K, U) = 0$.

Now, it can be easily verified that blowing down the $+1$-framed unknot on the left side of Figure 5 transforms the crossing labeled $+$ into the crossing labeled $-$. The inverse process of introducing an unknot to the right side of Figure 5 and performing a $-$-meridional twist on its complement yields the positive crossing.

2B. *Untwisting number.* We define a $\pm$-*generalized crossing change on K* as the process of blowing down the ±1-framed unknot in a generalized crossing change diagram for K. In this situation, K must pass through U an even number of times, for otherwise $\mathrm{lk}(K, U) \neq 0$. If at most $2p$ strands of K pass through U in a
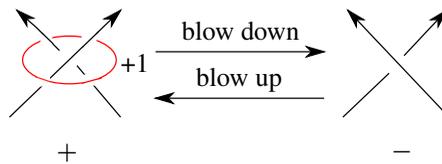

Figure 5. Crossing changes as blow-downs of ±1-framed unknots.

generalized crossing change diagram, we may call the associated $\pm$-generalized crossing change a $\pm p$-*generalized crossing change on* K.

The *result of a* $\pm$-*generalized crossing change on* K is defined to be the image of K under the blow-down. The *untwisting number* $tu(K)$ *of* K is the minimum length of a sequence of generalized crossing changes on K such that the result of the sequence is the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes. Note that by the reasoning on page 58 of [Adams 1994], this definition is equivalent to taking the minimum length, over all diagrams of K, of a sequence of generalized crossing changes beginning with a fixed diagram of K such that the result of the sequence is the unknot, where we do not allow ambient isotopy of the diagram in between generalized crossing changes.

For $p = 1, 2, 3, \ldots$, we define the p-*untwisting number* $tu_p(K)$ to be the minimum length of a sequence of $\pm p$-generalized crossing changes on K resulting in the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes.

It follows immediately that we have the chain of inequalities

$$(2\text{-}1) \quad tu(K) \leq \cdots \leq tu_{p+1}(K) \leq tu_p(K) \leq \cdots \leq tu_2(K) \leq tu_1(K) = u(K).$$

2C. *Cabling.* In this section, we define satellite and cable knots.

Definition 2.2. A closed subset X of a solid torus $V \cong S^1 \times D^2$ is called *geometrically essential* in V if X intersects every PL meridional disk in V.

Let $P \subset V \subset S^3$ be a knot which is geometrically essential in an unknotted solid torus V. Let $C \subset S^3$ be another knot and let V_1 be a tubular neighborhood of C in S^3. Let $h : V \to V_1$ be a homeomorphism and let K be $h(P)$. Then P is called the *pattern* for the knot K, C is the *companion* of K, and K is called a *satellite of* C *with pattern* P, or just a *satellite knot* for short.

If the homeomorphism h takes the preferred longitude and meridian of V, respectively, to the preferred longitude and meridian of V_1, then h is said to be *faithful*. If P is the (p, q)-torus knot just under ∂V and h is faithful, then K is called the (p, q)-*cable based on* C, denoted $C_{p,q}$, or simply a *cable knot*.

Throughout this paper, we will denote the (p, q)-torus knot by $U_{p,q}$ since it is the (p, q)-cable of the unknot U.

2D. *The Blanchfield form.* Let $K \subset S^3$ be a knot. By Λ we shall denote the ring $\mathbb{Z}[t^{\pm 1}]$, and by Ω we will denote the field $\mathbb{Q}(t)$.

Twisted homology, cohomology groups, and Poincaré duality. Following [Borodzik and Friedl 2014], let X be a manifold with infinite cyclic first homology, and fix a choice of isomorphism of $H_1(X)$ with the infinite cyclic group generated by the indeterminate t. Let $\pi : \widetilde{X} \to X$ be the infinite cyclic cover of X. Given a

submanifold Y of X, let $\widetilde{Y} = \pi^{-1}(Y)$. Since $\mathbb{Z}$ is the deck transformation group of $\widetilde{X}$, Λ acts on the relative chain group $C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})$. If N is any Λ-module, we may define

$$H^*(X, Y; N) := H_*(\mathrm{Hom}_\Lambda(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}), N))$$

and

$$H_*(X, Y; N) := H_*\left(\overline{C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})} \otimes_\Lambda N \right).$$

Here, if H is any Λ-module, $\overline{H}$ denotes the module with the involuted Λ-structure: multiplication by $p(t) \in \Lambda$ in $\overline{H}$ is the same as multiplication by $p(t^{-1})$ in H. When $Y = \varnothing$, we just write $H_*(X; N)$ or $H^*(X; N)$.

Since $\Omega := \mathbb{Q}(t)$ is flat over Λ, we have isomorphisms

$$H_*(X, Y; \Omega) \cong H_*(X, Y; \Lambda) \otimes_\Lambda \Omega$$

and

$$H^*(X, Y; \Omega) \cong H^*(X, Y; \Lambda) \otimes_\Lambda \Omega.$$

If X is an n-manifold, and N is a Λ-module, Poincaré duality gives Λ-module isomorphisms

$$H_i(X, \partial X; N) \cong H^{n-i}(X; N).$$

The Blanchfield form. As above, let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $\Omega = \mathbb{Q}(t)$. Let A be an $n \times n$ invertible hermitian matrix with entries in Λ. We define $\lambda(A)$ to be the pairing

$$\lambda(A) : \Lambda^n/A\Lambda^n \times \Lambda^n/A\Lambda^n \to \Omega/\Lambda$$

sending the pair of column vectors (a, b) to $\bar{a}^t A^{-1} b$. Note that $\lambda(A)$ is a nonsingular, hermitian pairing.

Let $X(K) = S^3 - N(K)$ denote the exterior of K. Consider the sequence of maps

$$\Phi : H_1(X(K); \Lambda) \xrightarrow{\pi_*} H_1(X(K), \partial X(K); \Lambda)$$
$$\xrightarrow{\mathrm{PD}} H^2(X(K); \Lambda) \xleftarrow{\delta} H^1(X(K); \Omega/\Lambda)$$
$$\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_\lambda(H_1(X(K); \Lambda), \Omega/\Lambda)}.$$

Here π_* is induced by the quotient map $C(X) \to C(X)/C(\partial X)$, PD is the Poincaré duality map, δ is from the long exact sequence in cohomology obtained from the coefficients $0 \to \Lambda \to \Omega \to \Omega/\Lambda \to 0$, and ev is the Kronecker evaluation map. It is well known (see [Hillman 2012, Section 2] for details) that π_* and δ are isomorphisms, PD is the Poincaré duality isomorphism, and ev is also an isomorphism by the universal coefficient spectral sequence (see [Levine 1977, Theorem 2.3] for details on the universal coefficient spectral sequence). Thus, the

above maps define a nonsingular pairing

$$\lambda(K) : H_1(X(K); \Lambda) \times H_1(X(K); \Lambda) \to \Omega/\Lambda,$$
$$(a, b) \mapsto \Phi(a)(b),$$

called the *Blanchfield pairing of K*. This pairing is hermitian.

Now, let V be any $2k \times 2k$ matrix which is S-equivalent to a Seifert matrix for K. Recall that $V - V^T$ is antisymmetric with determinant ± 1. It is well known that, perhaps after replacing V by PVP^T for some $P \in \mathrm{GL}_{2k}(\mathbb{Z})$,

(2-2)
$$V - V^T = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. We define $A_K(t)$ to be the matrix

$$\begin{pmatrix} (1-t^{-1})^{-1} I_k & 0 \\ 0 & I_k \end{pmatrix} V \begin{pmatrix} I_k & 0 \\ 0 & (1-t)I_k \end{pmatrix} + \begin{pmatrix} I_k & 0 \\ 0 & (1-t^{-1})I_k \end{pmatrix} V^T \begin{pmatrix} (1-t)^{-1} I_k & 0 \\ 0 & I_k \end{pmatrix}.$$

Using (2-2), we can write

$$V = \begin{pmatrix} B & C+I \\ C^T & D \end{pmatrix}.$$

One may then compute, as in the proof of [Borodzik and Friedl 2015, Lemma 2.2], that

$$A_K(1) = \begin{pmatrix} B & -I_k \\ -I_k & 0 \end{pmatrix}.$$

Thus, the matrix $A_K(t)$ is a hermitian matrix defined over Λ, and $\det(A_K(1)) = (-1)^k$.

Proposition 2.3 [Borodzik and Friedl 2015, Proposition 2.1]. *Let K be a knot and $A_K(t)$ be as above. Then $\lambda(A_K(t))$ is isometric as a sesquilinear form to $\lambda(K)$.*

2E. *The twisted intersection pairing.* Let W be a topological 4-manifold with boundary M such that $\pi_1(W) = \mathbb{Z}$. Consider the maps

$$H_2(W; \Lambda) \xrightarrow{\pi_*} H_2(W, M; \Lambda) \xrightarrow{\mathrm{PD}} H^2(W; \Lambda) \xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_\Lambda(H_2(W; \Lambda), \Lambda)},$$

where the first map is induced by the quotient, the second map is Poincaré duality, and the third map is the Kronecker evaluation map. The second and third maps are obviously isomorphisms, and the first map is an isomorphism by the long exact sequence of the pair (W, M). Hence this composition defines a pairing

$$H_2(W; \Lambda) \times H_2(W; \Lambda) \to \Lambda,$$

which we call the *twisted intersection pairing on W*.

3. Algebraic untwisting number equals algebraic unknotting number

Our proof that $tu_a(K) = u_a(K)$ generalizes [Borodzik and Friedl 2014; 2015]. Following [Borodzik and Friedl 2014], define a knot invariant $n(K)$ to be the minimum size of a square hermitian matrix $A(t)$ over $\mathbb{Z}[t^{\pm 1}]$ such that $\lambda(A)$ is isometric to $\lambda(K)$ and $A(1)$ is congruent over $\mathbb{Z}$ to a diagonal matrix which has only ± 1 entries. Borodzik and Friedl showed that $u_a(K) = n(K)$. Since $tu_a(K) \leq u_a(K)$, it is obvious that $tu_a(K) \leq n(K)$ as well. After stating Borodzik and Friedl's results, we will show that $n(K) \leq tu_a(K)$; hence $tu_a(K) = n(K) = u_a(K)$ for all knots K. In fact, we will show something stronger.

Theorem 3.1. *Let $K \subset S^3$ be a knot. For every algebraic unknotting sequence for K with u_+ positive crossing changes and u_- negative crossing changes, there exists an algebraic untwisting sequence for K with u_+ positive generalized crossing changes and u_- negative generalized crossing changes. In particular, $u_a(K) = tu_a(K)$.*

In order to prove Theorem 3.1, we must first recall some notation and results used in [Borodzik and Friedl 2015]. The main theorem of that paper implies that $n(K) \leq u_a(K)$:

Theorem 3.2 [Borodzik and Friedl 2015, Theorem 1.1]. *Let K be a knot which can be changed into an Alexander polynomial-one knot by a sequence of u_+ positive crossing changes and u_- negative crossing changes. Then there exists a hermitian matrix $A(t)$ of size $u_+ + u_-$ over Λ such that*

(1) *$\lambda(A(t))$ is isometric to $\lambda(K)$;*

(2) *$A(1)$ is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to 1.*

In particular, $n(K) \leq u_a(K)$.

We need one definition:

Definition 3.3. Let K be a knot and $M(K)$ the result of 0-surgery on K. A 4-manifold W *tamely cobounds* $M(K)$ if

(1) $\partial W = M(K)$;

(2) the inclusion induced map $H_1(M(K); \mathbb{Z}) \to H_1(W; \mathbb{Z})$ is an isomorphism;

(3) $\pi_1(W) = \mathbb{Z}$.

If, in addition, the intersection form on $H_2(W; \mathbb{Z})$ is diagonalizable, we say that W *strictly cobounds* $M(K)$.

Theorem 3.4 [Borodzik and Friedl 2015, Theorem 2.6]. *Let K be a knot and let W be a topological 4-manifold which tamely cobounds $M(K)$. Then $H_2(W; \Lambda)$ is free of rank $b_2(W)$. Moreover, if B is an integral matrix representing the ordinary*

intersection pairing of W, then there exists a basis $\mathcal{B}$ for $H_2(W; \Lambda)$ such that the matrix $A(t)$ representing the twisted intersection pairing with respect to $\mathcal{B}$ satisfies

(1) *$\lambda(A(t))$ is isometric to $\lambda(K)$;*

(2) *$A(1) = B$.*

We generalize Theorem 3.2 as follows:

Theorem 3.5. *Let K be a knot which can be changed into an Alexander polynomial-one knot by a sequence of u_+ positive and u_- negative generalized crossing changes. Then there exists a hermitian matrix of size $u_+ + u_-$ over Λ with the properties*

(1) *$\lambda(A(t))$ is isometric to $\lambda(K)$;*

(2) *$A(1)$ is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to 1.*

In particular, $n(K) \leq tu_a(K)$.

The proof of Theorem 3.5 is similar to that of Theorem 3.2. By Theorem 3.4, in order to prove Theorem 3.5, we only need to show the following proposition.

Proposition 3.6. *Let K be a knot such that u_+ positive generalized crossing changes and u_- negative generalized crossing changes turn K into an Alexander polynomial-one knot. Then there exists an oriented topological 4-manifold W which strictly cobounds $M(K)$. Moreover, the intersection pairing on $H_2(W; \mathbb{Z})$ is represented by a diagonal matrix of size $u_+ + u_-$ such that u_+ entries are equal to -1 and u_- entries are equal to $+1$.*

Proof. Let K be a knot such that u_+ positive generalized crossing changes and u_- negative generalized crossing changes turn K into an Alexander polynomial-one knot J. We write $s = u_+ + u_-$ and $n_i = -1$ for $i = 1, \ldots, u_+$ and $n_i = 1$ for $i = u_+ + 1, \ldots, u_+ + u_-$. Then there exist simple closed curves $c_1, \ldots, c_s$ in $S^3 - N(J)$ such that

(1) $c_1 \cup \cdots \cup c_s$ is the unlink in S^3;

(2) the linking numbers $\text{lk}(c_i, K)$ are zero for all i;

(3) the image of J under the n_i-surgeries is the knot K.

Note that the curves $c_1, \ldots, c_s$ lie in $S^3 - N(J)$; hence we can view them as lying in $M(J)$. The manifold $M(K)$ is then the result of n_i-surgery on all the $c_i \subset M(J)$, where $i = 1, \ldots, s$.

Since J is a knot with trivial Alexander polynomial, by Freedman's theorem [Freedman and Quinn 1990] J is topologically slice and there exists a locally flat slice disk $D \subset B^4$ for J such that $\pi_1(B^4 - D) = \mathbb{Z}$. Let $X := B^4 - N(D)$. Then X is an oriented topological 4-manifold such that

(1) $\partial X \cong M(J)$ as oriented manifolds;

(2) $\pi_1(X) \cong \mathbb{Z}$;

(3) the inclusion induced map $H_1(M(J); \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is an isomorphism;

(4) $H_2(X; \mathbb{Z}) = 0$.

Let W be the 4-manifold which is obtained by adding 2-handles along $c_1, \ldots, c_s \subset M(J)$ with framings $n_1, \ldots, n_s$ to X. Then $\partial W \cong M(K)$ as oriented manifolds. From now on, we write $M := M(K)$. Since the curves $c_1, \ldots, c_s$ are nullhomologous, the map $H_1(M; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ is an isomorphism and $\pi_1(W) \cong \mathbb{Z}$. It thus remains to prove the following lemma:

Lemma 3.7. *The ordinary intersection pairing on W is represented by a diagonal matrix of size $s = u_+ + u_-$ with u_+ diagonal entries equal to -1 and u_- diagonal entries equal to 1.*

Recall that the curves $c_1, \ldots, c_s$ form the unlink in S^3 and that the linking numbers $\text{lk}(c_i, J)$ are zero. Therefore, the curves $c_1, \ldots, c_s$ are also nullhomologous in $M(J)$. Thus we can now find disjoint surfaces $F_1, \ldots, F_s$ in $M(J) \times [0, 1]$ such that $\partial F_i = c_i \times \{1\}$. By adding the cores of the 2-handles attached to the c_i, we obtain closed surfaces $C_1, \ldots, C_s$ in W. It is clear that $C_i \cdot C_j = 0$ for $i \neq j$ and $C_i \cdot C_i = n_i$.

We argue using Mayer–Vietoris that the surfaces $C_1, \ldots, C_s$ present a basis for $H_2(W; \mathbb{Z})$. Write $W := X \cup H$, where $H \cong \bigsqcup_{i=1}^s (B^2 \times B^2)$ is the set of 2-handles attached to $c_1, \ldots, c_s$. Then write $Y := X \cap H$, so that

$$Y = \bigsqcup_{i=1}^s N(c_i) \cong \bigsqcup_{i=1}^s (S^1 \times D^2).$$

We have the Mayer–Vietoris sequence

$$\cdots \to H_2(X) \oplus H_2(H) \xrightarrow{\psi_*} H_2(W) \xrightarrow{\partial_*} H_1(Y) \xrightarrow{\phi_*} H_1(X) \oplus H_1(H) \xrightarrow{\psi_*} H_1(W) \to 0.$$

Now, since $H_1(Y)$ is generated by all the S^1-factors, or the longitudes $c_1, \ldots, c_s$, and $H_1(H) = H_2(H) = H_2(X) = 0$, the sequence becomes

$$0 \to H_2(W) \xrightarrow{\partial_*} \langle c_1, \ldots, c_s \rangle \xrightarrow{i_*} H_1(X) \xrightarrow{\psi_*} H_1(W) \to 0.$$

From [Livingston 1993, Lemma 8.12], for example, we have:

Lemma 3.8. *Suppose that for some knot K in S^3, there is a locally flat surface F in B^4 with $F \cap S^3 = \partial F \cap S^3 = K$. Then the inclusion map induces an isomorphism $H_1(S^3 - K) \to H_1(B^4 - F) \cong \mathbb{Z}$.*

In our case, the inclusion $S^3 - K \hookrightarrow X$ induces an isomorphism $H_1(S^3 - K) \to H_1(X)$. Since i_* is induced by inclusion and the longitudes $c_1, \ldots, c_s$ are nullhomologous in $S^3 - K$, we see that i_* must be the zero map in X. Hence ∂_* is an isomorphism $H_2(W) \cong H_1(Y)$, and $H_2(W) = \langle C_1, \ldots, C_s \rangle$.

In particular, the intersection matrix on W with respect to this basis is given by $(C_i \cdot C_j)$, i.e., it is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to $+1$. This concludes the proof of Lemma 3.7. Proposition 3.6 follows. Together with Theorem 3.4, this completes the proof of Theorem 3.5.   □

We have shown that, for every untwisting sequence for K with u_+ positive generalized crossing changes and u_- negative generalized crossing changes, there exists a hermitian matrix $A(t)$ of size $u_+ + u_-$ such that $\lambda(A(t))$ is isometric to $\lambda(K)$ and $A(1)$ is diagonal with u_+ entries equal to -1 and u_- entries equal to 1. Borodzik and Friedl [2014] have already shown that, for every hermitian matrix $A(t)$ representing $\lambda(K)$ such that $A(1)$ is diagonal with u_+ -1's and u_- $+1$'s, there exists an algebraic unknotting sequence for K consisting of u_+ positive and u_- negative crossing changes. Theorem 3.1 follows.

4. Untwisting number does not equal unknotting number

Although the algebraic versions of tu and u are equal, $tu \neq u$ in general. We use a result of Miyazawa [1998] to give our first example of a knot K with $tu(K) = 1$ but $u(K) > 1$.

Theorem 4.1. *Let K be the knot resulting from blowing down the $+1$-framed unknot $U \subset S^3 \setminus V$ in Figure 3. Then $tu(K) = 1$ but $u(K) > 1$.*

From this point forward, we will denote the signature of any knot K by $\sigma(K)$. In order to analyze the unknotting number of K, we will use the following theorem:

Theorem 4.2 [Miyazawa 1998]. *If $u(K) = 1$ and $\sigma(K) = \pm 2$, then*

$$V_K^{(1)}(-1) \equiv 24\, a_4(K) - \tfrac{1}{8}\sigma(K)(\det K + 1)(\det K + 5) \pmod{48},$$

where $V_K^{(1)}$ denotes the first derivative of the Jones polynomial of K and a_4 is the coefficient of z^4 in the Conway polynomial $\nabla_K(z) = \sum_{n=0}^{\infty} a_{2n}(K)z^{2n}$.

We compute using the Mathematica package KnotTheory (http://katlas.org/wiki/The_Mathematica_Package_KnotTheory) that $\sigma(K) = 2$; therefore Theorem 4.2 applies. We also compute using the KnotTheory package that the Jones polynomial $V_K(q)$ for our knot K is

$$V_K(q) = q - q^2 + 2q^3 - q^4 + q^6 - q^7 + q^8 - q^9 - q^{12} + q^{13};$$

hence $V_K^{(1)}(-1) = 8$. The Conway polynomial of K is computed to be

$$\nabla_K(z) = \sum_{n=0}^{\infty} a_{2n}(K)z^{2n} = 1 + z^2.$$

(hence $a_4 = 0$), and the determinant of K is 3. In our case, the right-hand side of the congruence in Theorem 4.2 becomes

$$0 - \tfrac{1}{4}(4)(8) = -8$$

and $8 \not\equiv -8 \pmod{48}$. Hence K cannot have unknotting number one, although it was constructed to have untwisting number one. Note that this also shows Miyazawa's Jones polynomial criterion does not extend to untwisting-number-one knots.

5. Arbitrarily large gaps between unknotting and untwisting numbers

5A. *Arbitrarily large gaps between u and tu$_p$.* Now that we have shown that there exists a knot K with $tu(K) < u(K)$, it is natural to ask how large the difference $u(K) - tu(K)$ can be. Recall that the (p, q)-cable of a knot K is denoted $K_{p,q}$; we denote the (p, q)-torus knot by $U_{p,q}$, the (p, q)-cable of the unknot. The knots we will be working with are (p, q)-cables of knots K with $u(K) = 1$ and $\tau(K) > 0$, where $p, q > 0$.

To get a lower bound on $u(K_{p,q})$ for such knots, we compute $\tau(K_{p,q})$ for all p, q. For cables of alternating (or more generally, "homologically thin") knots such as the trefoil, Petkova [2013] gives a formula for computing τ. However, since we will later compute τ for cables of nonalternating knots, we use a more general method of computing $\tau(K_{p,q})$ using the ϵ-invariant $\epsilon(K) \in \{-1, 0, 1\}$ introduced by Hom:

Theorem 5.1 [Hom 2014]. *Let $K \subset S^3$.*

(1) *If $\epsilon(K) = 1$, then $\tau(K_{p,q}) = p\tau(K) + \tfrac{1}{2}(p-1)(q-1)$.*

(2) *If $\epsilon(K) = -1$, then $\tau(K_{p,q}) = p\tau(K) + \tfrac{1}{2}(p-1)(q+1)$.*

(3) *If $\epsilon(K) = 0$, then $\tau(K) = 0$ and*

$$\tau(K_{p,q}) = \tau(U_{p,q}) = \begin{cases} \tfrac{1}{2}(p-1)(q+1), & q < 0, \\ \tfrac{1}{2}(p-1)(q-1), & q > 0. \end{cases}$$

Theorem 5.2 [Ozsváth and Szabó 2003]. *For the (p, q)-torus knot $U_{p,q}$ with $p, q > 0$, τ equals the 3-sphere genus of $U_{p,q}$, denoted $g(U_{p,q})$:*

$$\tau(U_{p,q}) = g(U_{p,q}) = \tfrac{1}{2}(p-1)(q-1).$$

Proposition 5.3 [Hom 2014]. *Let $K \subset S^3$ be a knot. If $|\tau(K)| = g(K)$, then $\epsilon(K) = \operatorname{sgn} \tau(K)$.*

Theorem 5.4. *Let K be a knot in S^3 with unknotting number one. If $\tau(K) > 0$ and $p, q > 0$, then*

$$u(K_{p,q}) - tu_p(K_{p,q}) \geq p - 1.$$

In particular, $tu_p(K_{p,1}) = 1$, while $u(K_{p,1}) \geq p$.

Proof. Let V be the unknot that results from performing the unknotting crossing change on K. Consider a generalized crossing change diagram for V together with the ± 1-framed surgery curve U that transforms V back into K. Then take the (p, q)-cable $V_{p,q}$ of V in this diagram, leaving U alone. The resulting $V_{p,q}$ is the (p, q)-torus knot before performing the ± 1-surgery, but the image of V under ± 1-surgery on U is K; hence the image of $V_{p,q}$ under the ± 1-surgery on U is $K_{p,q}$. Therefore, blowing down the surgery curve U (through which $V_{p,q}$ passes $2p$ times) results in a diagram for $K_{p,q}$ in S^3. Since $K_{p,q}$ and $V_{p,q}$ differ by a single twist,

$$tu_p(K_{p,q}) \leq tu_p(V_{p,q}) + 1.$$

Since

$$tu_p(V_{p,q}) \leq u(V_{p,q}) = \tfrac{1}{2}(p-1)(q-1),$$

we get that

$$tu_p(K_{p,q}) \leq \tfrac{1}{2}(p-1)(q-1) + 1.$$

In particular, this inequality shows that $tu_p(K_{p,1}) = 1$. If $\tau(K) > 0$, then necessarily $\epsilon(K) \neq 0$ by (3) of Theorem 5.1, so that $\epsilon(K) = \pm 1$. In this case,

$$\tau(K_{p,q}) = p\tau(K) + \tfrac{1}{2}(p-1)(q \mp 1),$$

and thus

$$u(K_{p,q}) \geq |\tau(K_{p,q})| = p\tau(K) + \tfrac{1}{2}(p-1)(q \mp 1) \geq p + \tfrac{1}{2}(p-1)(q \mp 1).$$

When $q = 1$, we get that $u(K_{p,1}) \geq p$. Combining our estimates,

$$
\begin{aligned}
u(K_{p,q}) - tu_p(K_{p,q}) &\geq \left(p + \tfrac{1}{2}(p-1)(q \mp 1)\right) - \left(1 + \tfrac{1}{2}(p-1)(q-1)\right) \\
&\geq \left(p + \tfrac{1}{2}(p-1)(q-1)\right) - \left(1 + \tfrac{1}{2}(p-1)(q-1)\right) \\
&\geq p - 1. \qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

5B. *Arbitrarily large gaps between u and tu_q.* The above examples $\{K_{p,1}\}$ show that for every p there exists a knot $K_{p,1}$ with $u(K_{p,1}) \geq p$, even though $tu_p(K_{p,1}) = 1$. However, in order to untwist any such $K_{p,1}$, we must twist at least $2p$ strands at once. A natural follow-up question is whether there exists a knot K with $u(K) \geq p$ that can be untwisted by a single $\pm q$-generalized crossing change, where $q < p$. More generally, we may ask whether, for any fixed q, there is a family of knots which give us arbitrarily large gaps between u and tu_q. We answer this question in the affirmative.

Theorem 5.5. *Let K be a knot with $u(K) = 1$ and $\tau(K) > 0$, and let $J_p^q := \#^p K_{q,1}$. For any $p > 0$ and $q > 1$, we have $tu_q(J_p^q) \leq p$ and $u(J_p^q) - tu_q(J_p^q) \geq p$.*

Proof. First, we note that for any knot K, $J_p^q = \#^p K_{q,1}$ can be unknotted by performing p generalized crossing changes on at most $2q$ strands each, one generalized

crossing change to unknot each copy of $K_{q,1}$. Therefore, $tu_q(J_p^q) \leq p$. Since τ is additive under connected sum,

$$\tau(J_p^q) = p \cdot \tau(K_{q,1}) \geq pq,$$

and hence $u(J_p^q) \geq pq$ for all p. Therefore,

$$u(J_p^q) - tu_q(J_p^q) \geq pq - p = p(q-1) \geq p, \qquad \square$$

Note. In the case where K has $\sigma(K) = \pm 2$, e.g., when K is a right-handed trefoil knot, we can do better by computing tu_q precisely. We use the fact that $\frac{1}{2}|\sigma(K)|$ is a lower bound for $tu_q(K)$ for any q. First, recall that the *Tristram–Levine signature function* of a knot K, $\sigma_\omega(K)$, is equal to the signature of the matrix $(1-\omega)V + (1-\bar{\omega})V^T$, where $\omega \in \mathbb{C}$ has norm 1 and V is a Seifert matrix for K. Note that

$$\sigma_{-1}(K) = \sigma(2(V + V^T)) = \sigma(V + V^T) = \sigma(K).$$

We use Litherland's formula [1979] for Tristram–Levine signatures of cable knots to compute that

$$\sigma_{-1}(K_{p,q}) = \sigma_{(-1)^p}(K) + \sigma_{-1}(U_{p,q})$$

and, since $\sigma_1 \equiv 0$, while $\sigma_{-1} = \sigma$,

$$\sigma(K_{q,1}) = \begin{cases} \sigma(K) + \sigma(U_{q,1}) = \sigma(K), & q \text{ odd}, \\ \sigma(U_{q,1}) = 0, & q \text{ even}, \end{cases}$$

since the $(q, 1)$-torus knot is the unknot for any q. Now, since the knot signature is additive over connected sum,

$$\sigma(J_p^q) = p\sigma(K_{q,1}) = \begin{cases} \sigma(K) \cdot p = \pm 2p, & q \text{ odd}, \\ 0, & q \text{ even}, \end{cases}$$

and therefore, when p is odd,

$$tu_q(J_p^q) \geq \tfrac{1}{2}|\sigma(\kappa_p^q)| = p.$$

Since we already know $tu_q(J_p^q) \leq p$, in fact we must have $tu_q(J_p^q) = p$ for odd $p \geq 1$.

5C. *Arbitrarily large gaps between u and tu_q for topologically slice knots.* Consider the diagram of an unknot $U(K)$ in Figure 6, where K is any knot with $\tau(K) > 0$. Let $p \geq 2$ be an integer.

We take the $(q, 1)$-cable of $U(K)$, which is still an unknot. Then, we perform a -1-twist on the $+1$-framed unknot, obtaining a knot S^q. Clearly, $tu_q(S^q) = 1$.

Furthermore, S^q is the $(q, 1)$-cable of the knot $D_+(K, 0)$, the untwisted Whitehead double of K. This is because $U(K)$ represents $D_+(K, 0)$ in the manifold obtained from the $+1$-surgery, and the cabling operation converts this knot into the
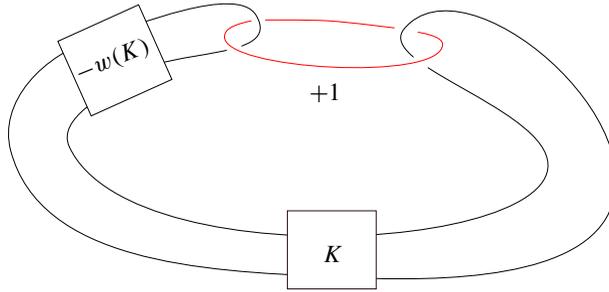
Figure 6. The knot $U(K)$ (an unknot), together with a $+1$-surgery curve.

$(q, 1)$-cable of $D_+(K, 0)$. Since untwisted Whitehead doubles are topologically (but not necessarily smoothly) slice [Freedman and Quinn 1990], $D_+(K, 0)$ is topologically concordant to the unknot. It is well known that, if K is concordant to J, then $K_{m,n}$ is concordant to $J_{m,n}$ for all integers m, n. Hence $S_{q,1}$ is also topologically concordant to the unknot $U_{q,1}$, and therefore S_p is topologically slice for all p.

Now, define $S_p^q := \#^p D_+(K, 0)$. Connected sums of topologically slice knots are topologically slice; hence S_p^q is topologically slice. Moreover, as above, we have that $tu_q(S_p^q) \le p \cdot tu_q(S^q) = p$.

We will now get a lower bound on $u(S_p^q)$ and thus show that $u(S_p^q) - tu_q(S_p^q)$ can be arbitrarily large. The Ozsváth–Szabó τ invariant gives such a lower bound. Thus, we need to compute $\tau(S_p^q)$ for all p, q.

We show that $\epsilon(D_+(K, 0)) = 1$ and hence, applying Theorem 5.1, that

$$\tau(S^q) = q\tau(D_+(K, 0)).$$

We first compute $\tau(D_+(K, 0))$.

Theorem 5.6 [Hedden 2007]. *Let $D_+(K, t)$ denote the positive t-twisted Whitehead double of a knot K. Then*

$$\tau(D_+(K, t)) = \begin{cases} 1, & t < 2\tau(K), \\ 0, & otherwise. \end{cases}$$

Since $\tau(K) > 0$ in our case, $t = 0 < 2 \le 2\tau(K)$, and so $\tau(D_+(K, 0)) = 1$. Furthermore, as is the case with any Whitehead double, $g(D_+(K, 0)) = 1$, so $|\tau(D_+(K, 0))| = 1 = g(D_+(K, 0))$ and, by Proposition 5.3,

$$\epsilon(D_+(K, 0)) = \operatorname{sgn} \tau(D_+(K, 0)) = +1.$$

We then apply Theorem 5.1 to S^q to get that

$$\tau(S^q) = q\tau(D_+(K, 0)).$$

Since $\tau(D_+(K, 0)) = 1$, we have that $\tau(S^q) = q$ and, hence, $\tau(S_p^q) = pq$. Thus, $u(S_p^q) \geq pq$. Therefore,

$$u(S_p) - tu_q(S_p) \geq pq - p = p(q - 1) \geq p,$$

as desired.

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