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RADIAL LIMITS OF BOUNDED NONPARAMETRIC PRESCRIBED MEAN CURVATURE SURFACES

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Dedicated to the memory of Alan Ross Elcrat

Consider a solution $f \in C^2(\Omega)$ of a prescribed mean curvature equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 2H(x, f) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a domain whose boundary has a corner at $\mathcal{O} = (0, 0) \in \partial\Omega$. If $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite and Ω has a reentrant corner at $\mathcal{O}$, then the (nontangential) radial limits of f at $\mathcal{O}$,

$$Rf(\theta) := \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

are shown to exist, independent of the boundary behavior of f on $\partial\Omega$, and to have a specific type of behavior. If $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite and the trace of f on one side has a limit at $\mathcal{O}$, then the (nontangential) radial limits of f at $\mathcal{O}$ exist, the tangential radial limit of f at $\mathcal{O}$ from one side exists and the radial limits have a specific type of behavior.

1. Introduction and statement of main theorems

Consider the prescribed mean curvature equation

$$Nf = 2H(\,\cdot\,, f) \quad \text{in } \Omega, \tag{1}$$

where Ω is a domain in $\mathbb{R}^2$ whose boundary has a corner at $\mathcal{O} \in \partial\Omega$, $Nf = \nabla\cdot Tf = \operatorname{div}(Tf)$, $Tf = (\nabla f)/\sqrt{1+|\nabla f|^2}$, $H : \Omega\times\mathbb{R} \to \mathbb{R}$ and H satisfies one of the conditions which guarantees that "cusp solutions" (e.g., [Lancaster and Siegel 1996a, §5; 1996b]) do not exist; for example, $H(\boldsymbol{x}, t)$ is strictly increasing in t for each $\boldsymbol{x}$ or is real-analytic (e.g., constant). We will assume $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball in $\mathbb{R}^2$ of radius δ^* about $\mathcal{O}$. Polar coordinates relative to $\mathcal{O}$ will be denoted by r and θ. We assume that $\partial\Omega$ is piecewise smooth and there exists $\alpha \in (0, \pi)$ such that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of

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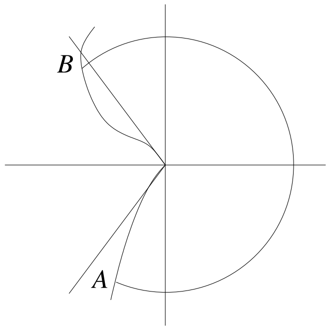

Figure 1. The domain Ω^*.

two arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached (see Figure 1 of [Lancaster and Siegel 1997] or Figure 1).

Suppose

$$\sup_{x\in\Omega} |f(x)| < \infty \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| < \infty. \tag{2}$$

We shall prove

Theorem 1. *Let $f \in C^2(\Omega)$ satisfy (1) and suppose (2) holds and $\alpha \in (\pi/2, \pi)$. Then for each $\theta \in (-\alpha, \alpha)$,*

$$Rf(\theta) := \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta)$$

exists and $Rf(\cdot)$ is a continuous function on $(-\alpha, \alpha)$ which behaves in one of the following ways:

(i) *$Rf : (-\alpha, \alpha) \to \mathbb{R}$ is a constant function (so f has a nontangential limit at $\mathcal{O}$).*

(ii) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on (α_1, α_2).*

(iii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha, \alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1], [\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and either strictly increasing on $(\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2)$ or strictly decreasing on $(\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2)$.*

At a convex corner (i.e., $\alpha \in (0, \pi/2]$), Theorem 1 is not applicable. The additional assumption that the trace of f on one side (e.g., $\partial^-\Omega^*$) has a limit at $\mathcal{O}$ implies the radial limits of f exist.

Theorem 2. *Let $f \in C^2(\Omega) \cap C^0(\Omega \cup \partial^-\Omega^* \setminus \{\mathcal{O}\})$ satisfy (1). Suppose (2) holds and $m = \lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x})$ exists. Then for each $\theta \in (-\alpha, \alpha)$, $Rf(\theta)$ exists and $Rf(\cdot)$ is a continuous function on $[-\alpha, \alpha)$, where $Rf(-\alpha) := m$. If $\alpha \in (0, \pi/2]$,*

Rf can behave as in (i) or (ii) in Theorem 1. If $\alpha \in (\pi/2, \pi)$, Rf can behave as in (i), (ii) or (iii) in Theorem 1.

The conclusions of these theorems were first obtained in [Lancaster 1985] for minimal surfaces satisfying Dirichlet boundary conditions and then for nonparametric prescribed mean curvature surfaces satisfying Dirichlet [Elcrat and Lancaster 1986; Lancaster 1988] or contact angle [Lancaster and Siegel 1996a] boundary conditions; see also [Jin and Lancaster 1997; Lancaster 1991]. Notice that Theorem 1 applies to a solution of a capillary surface problem whose domain has a reentrant corner even when the contact angle equals 0 and/or π on some (or all) of $\partial\Omega^*$.

Remark. The assumption that Ω has a reentrant corner at $\mathcal{O} \in \partial\Omega$ or that the trace of f from one side of $\partial\Omega$ is continuous at $\mathcal{O}$ is critical here; the nonexistence of radial limits at $(1, 0)$ when $\Omega = B_1(\mathcal{O})$ and the boundary data is symmetric with respect to the horizontal axis is demonistrated in [Lancaster 1989] and in Theorem 3 of [Lancaster and Siegel 1996a]. In [Lancaster 1987], it was conjectured that the existence of radial limits at corners for bounded solutions of Dirichlet problems for the minimal surface equation in $\mathbb{R}^2$, independent of boundary conditions. Although [Lancaster 1989] proved this conjecture false, Theorems 1 and 2 show it is true in many cases.

2. Proof of Theorem 1

Since $f \in C^2(\Omega)$ (and so in $C^0(\Omega)$), we may assume that f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| > \delta\}$ for each $\delta \in (0, \delta^*)$; if this is not true, we may replace Ω with U, $U \subset \Omega$, such that $\partial\Omega \cap \partial U = \{\mathcal{O}\}$ and $\partial U \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+ U$ and $\partial^- U$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached. Set

$$S_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \Omega^*\} \quad \text{and} \quad \Gamma_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \partial\Omega^* \setminus \{\mathcal{O}\}\};$$

the points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled $A \in \partial^-\Omega^*$ and $B \in \partial^+\Omega^*$. From the calculation on page 170 of [Lancaster and Siegel 1996a], we see that the area of S_0^* is finite; let M_0 denote this area. For $\delta \in (0, 1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}}.$$

Let $E = \{(u, v) : u^2 + v^2 < 1\}$. As in [Elcrat and Lancaster 1986; Lancaster and Siegel 1996a], there is a parametric description of the surface S_0^*,

$$Y(u, v) = \big(a(u, v), b(u, v), c(u, v)\big) \in C^2(E : \mathbb{R}^3), \tag{3}$$

which has the following properties:

(a_1) Y is a diffeomorphism of E onto S_0^*.

(a_2) Set $G(u, v) = \big(a(u, v), b(u, v)\big)$, $(u, v) \in E$. Then $G \in C^0(\bar{E} : \mathbb{R}^2)$.

(a_3) Let $\sigma = G^{-1}(\partial\Omega^* \setminus \{\mathcal{O}\})$; then σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^*. We may assume the endpoints of σ are $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ and there exist points $\boldsymbol{a}, \boldsymbol{b} \in \sigma$ such that $G(\boldsymbol{a}) = A$, $G(\boldsymbol{b}) = B$, G maps the (open) arc $\boldsymbol{o}_1\boldsymbol{b}$ onto $\partial^+\Omega$, and G maps the (open) arc $\boldsymbol{o}_2\boldsymbol{a}$ onto $\partial^-\Omega$. (Note that $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ are not assumed to be distinct at this point; Figures 4a and 4b of [Lancaster and Siegel 1997] illustrate this situation.)

(a_4) Y is conformal on E: $Y_u \cdot Y_v = 0$, $Y_u \cdot Y_u = Y_v \cdot Y_v$ on E.

(a_5) $\triangle Y := Y_{uu} + Y_{vv} = H(Y) Y_u \times Y_v$ on E.

Here by the (open) arcs $\boldsymbol{o}_1\boldsymbol{b}$ and $\boldsymbol{o}_2\boldsymbol{a}$ we mean the component of $\partial E \setminus \{\boldsymbol{o}_1, \boldsymbol{b}\}$ which does not contain $\boldsymbol{a}$ and the component of $\partial E \setminus \{\boldsymbol{o}_2, \boldsymbol{a}\}$ which does not contain $\boldsymbol{b}$ respectively. Let $\sigma_0 = \partial E \setminus \sigma$.

There are two cases we wish to consider:

(A) $\boldsymbol{o}_1 = \boldsymbol{o}_2$.

(B) $\boldsymbol{o}_1 \neq \boldsymbol{o}_2$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 in [Lancaster and Siegel 1996a]. Let us first assume that (A) holds and set $\boldsymbol{o} = \boldsymbol{o}_1 = \boldsymbol{o}_2$. Let h denote a function on the annulus $\mathcal{A} = \{\boldsymbol{x} : r_1 \le |\boldsymbol{x}| \le r_2\}$ which vanishes on the circle $|\boldsymbol{x}| = r_2$ and whose graph is an unduloid surface with constant mean curvature $-H_0$ which becomes vertical at $|\boldsymbol{x}| = r_1$ and at $|\boldsymbol{x}| = r_2$ (see Figure 2) for suitable $r_1 < r_2$ (e.g., [Lancaster and Siegel 1996a, pp. 170–171]). Let q denote the modulus of continuity of h (i.e., $|h(\boldsymbol{x}_1) - h(\boldsymbol{x}_2)| \le q(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$).

For each $\boldsymbol{p} \in \mathbb{R}^2$ with $|\boldsymbol{p}| = r_1$, set $\mathcal{A}(\boldsymbol{p}) = \{\boldsymbol{x} : r_1 \le |\boldsymbol{x} - \boldsymbol{p}| \le r_2\}$ and define $h_{\boldsymbol{p}} : \mathcal{A}(\boldsymbol{p}) \to \mathbb{R}$ by $h_{\boldsymbol{p}}(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{p})$.

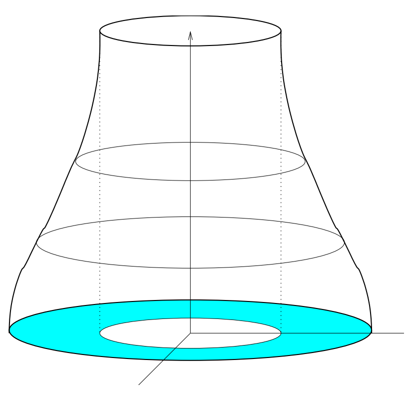

Figure 2. The graph of h over $\mathcal{A}$.

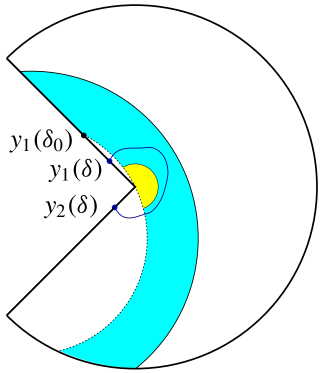

Figure 3. $\Omega^* \cap \mathcal{A}(\boldsymbol{p}_1)$, $C'_{\rho(\delta)}$ (blue curve), $B_{\eta(\delta)}(\mathcal{O}) \cap \mathcal{A}(\boldsymbol{p}_1)$ (yellow).

For $r > 0$, set $B_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| < r\}$, $C_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| = r\}$ and let l_r be the length of the image curve $Y(C_r)$; also let $C'_r = G(C_r)$ and $B'_r = G(B_r)$. From the Courant–Lebesgue lemma (e.g., [Courant 1950, Lemma 3.1]), we see that for each $\delta \in (0, 1)$, there exists a $\rho = \rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength l_ρ of $Y(C_\rho)$ is less than $p(\delta)$. For $\delta > 0$, let $k(\delta) = \inf_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \inf_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$ and $m(\delta) = \sup_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \sup_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$; notice that $m(\delta) - k(\delta) \le l_\rho < p(\delta)$.

For each $\delta \in (0, 1)$ with $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, there are two points in $C_{\rho(\delta)} \cap \partial E$; we denote these points as $\boldsymbol{e}_1(\delta) \in \boldsymbol{ob}$ and $\boldsymbol{e}_2(\delta) \in \boldsymbol{oa}$ and set $\boldsymbol{y}_1(\delta) = G(\boldsymbol{e}_1(\delta))$ and $\boldsymbol{y}_2(\delta) = G(\boldsymbol{e}_2(\delta))$. Notice that $C'_{\rho(\delta)}$ is a curve in $\bar{\Omega}$ which joins $\boldsymbol{y}_1 \in \partial^+\Omega^*$ and $\boldsymbol{y}_2 \in \partial^-\Omega^*$ and $\partial\Omega \cap C'_{\rho(\delta)} \setminus \{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \varnothing$; therefore there exists $\eta = \eta(\delta) > 0$ such that $B_{\eta(\delta)}(\mathcal{O}) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < \eta(\delta)\} \subset B'_{\rho(\delta)}$ (see Figure 3).

Fix $\delta_0 \in (0, \delta^*)$ with $\sqrt{\delta_0} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$. Let $\boldsymbol{p}_1 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_1| = r_1$ and $|\boldsymbol{p}_1 - \boldsymbol{y}_1(\delta_0)| = r_1$ such that $\boldsymbol{p}_1$ lies below (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_1(\delta_0)$. Let $\boldsymbol{p}_2 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_2| = r_1$ and $|\boldsymbol{p}_2 - \boldsymbol{y}_2(\delta_0)| = r_1$ such that $\boldsymbol{p}_2$ lies above (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_2(\delta_0)$. Set $\Omega_0 = \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_1| > r_1\} \cup \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_2| > r_1\}$ (see Figure 4).

Claim. *f is uniformly continuous on Ω_0.*

Proof. Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) < \frac{1}{4}\epsilon$ and $p(\delta) < r_2 - r_1$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b_j^\pm : \mathcal{A}(\boldsymbol{p}_j) \to \mathbb{R}$ by

$$b_j^\pm(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h_{\boldsymbol{p}_j}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{A}(\boldsymbol{p}_j)$$

for $j \in \{1, 2\}$. Notice that

$$b_j^-(\boldsymbol{x}) < f(\boldsymbol{x}) < b_j^+(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_j), \quad j \in \{1, 2\}.$$

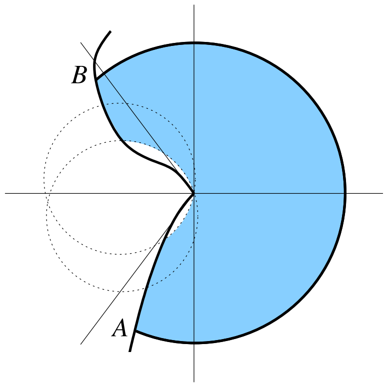

Figure 4. Ω_0.

If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfy $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$, then there exist $\boldsymbol{x}_3 \in \mathcal{A}(\boldsymbol{p}_1) \cap \mathcal{A}(\boldsymbol{p}_2)$ with $|\boldsymbol{x}_3| < \eta(\delta)$ such that $|\boldsymbol{x}_1 - \boldsymbol{x}_3| < \eta(\delta)$ and $|\boldsymbol{x}_2 - \boldsymbol{x}_3| < \eta(\delta)$ and so

(4) $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| \le |f(\boldsymbol{x}_1) - f(\boldsymbol{x}_3)| + |f(\boldsymbol{x}_1) - f(\boldsymbol{x}_3)| < 4p(\delta) + 4q(p(\delta)) < \epsilon.$

Since f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| \ge \frac{1}{2}\eta(\delta)\}$, there exists a $\lambda > 0$ such that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega^*$ satisfy $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \ge \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \lambda$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Now set $d = d(\epsilon) = \min\{\lambda, \frac{1}{2}\eta(\delta)\}$. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \le \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1| < \frac{1}{2}\eta(\delta)$, then $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$; hence $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$ by (4). Next, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \le \lambda$, $|\boldsymbol{x}_1| \ge \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_2| \ge \frac{1}{2}\eta(\delta)$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Therefore, for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon)$, we have $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. □

If $\{(r\cos(\theta^-(\delta_0)), r\sin(\theta^-(\delta_0))) : r \ge 0\}$ is the tangent ray to $\partial\mathcal{A}(\boldsymbol{p}_2)$ at $\mathcal{O}$, $\{(r\cos(\theta^+(\delta_0)), r\sin(\theta^+(\delta_0))) : r \ge 0\}$ is the tangent ray to $\partial\mathcal{A}(\boldsymbol{p}_1)$ at $\mathcal{O}$ and $\theta^-(\delta_0), \theta^+(\delta_0) \in (-\alpha, \alpha)$, then it follows from the claim that $f \in C^0(\overline{\Omega}_0)$, the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ exist for $\theta \in [\theta^-(\delta_0), \theta^+(\delta_0)]$ and the radial limits are identical (i.e., $Rf(\theta) = f(\mathcal{O})$ for all $\theta \in [\theta^-(\delta_0), \theta^+(\delta_0)]$.) Since

(5) $$\lim_{\delta_0 \downarrow 0} \theta^-(\delta_0) = -\alpha \quad \text{and} \quad \lim_{\delta_0 \downarrow 0} \theta^+(\delta_0) = \alpha,$$

Theorem 1 is proven in this case.

Next assume that (B) holds. For $r > 0$ and $j \in \{1, 2\}$, set $B_r^j = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}_j| < r\}$, $C_r^j = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}_j| = r\}$, and let l_r^j be the length of the image curve $Y(C_r^j)$; also let $C_r^{j,\prime} = G(C_r^j)$ and $B_r^{j,\prime} = G(B_r^j)$. From the Courant–Lebesgue lemma, we see that for each $\delta \in (0, 1)$ and $j \in \{1, 2\}$, there exists a $\rho_j = \rho_j(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength $l_{j,\rho}$ of $Y(C_{\rho_j}^j)$ is less than $p(\delta)$.

We will only consider $\delta \le \delta_0$, where δ_0 is small enough that the endpoints of $C_{\rho_j(\delta)}^j$ lie on $\sigma_0 \cup \sigma_N^j$ for $j \in \{1, 2\}$ and $C_{\sqrt{\delta_0}}^1 \cap C_{\sqrt{\delta_0}}^2 = \varnothing$, where $\sigma_N^1 = \boldsymbol{o}_1\boldsymbol{b}$ and

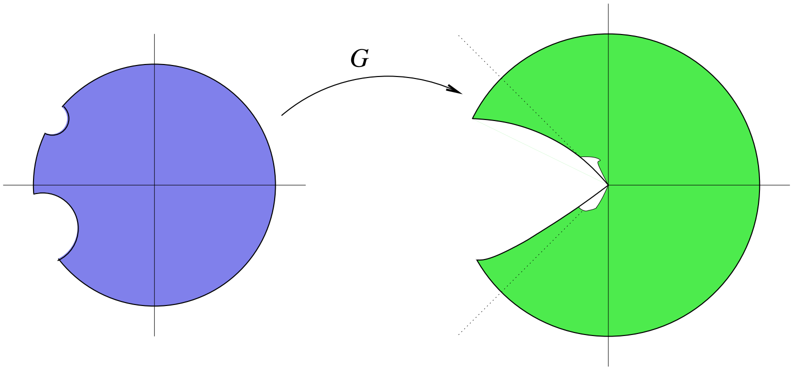

Figure 5. $E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}})$ and Ω_1.

$\sigma_N^2 = \boldsymbol{o}_2\boldsymbol{a}$. For each $\delta \in (0, \delta_0)$, the fact that $l_{j,\ \rho_j(\delta)}$ is finite for $j \in \{1, 2\}$ implies that

$$\lim_{C^{j,\prime}_{\rho_j(\delta)} \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x}) \quad \text{exists for } j \in \{1, 2\}.$$

If we set $\Omega_1 = G(E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}}))$ and define $\phi : \partial\Omega_1 \to \mathbb{R}$ by $\phi = f$, then ϕ has (at worst) a jump discontinuity at $\mathcal{O}$. If we consider ϕ to be the Dirichlet data for the boundary value problem

$$\operatorname{div}(Th) = 2H(\cdot, f) \quad \text{in } \Omega_1, \tag{6}$$

$$h = \phi \quad \text{on } \partial\Omega_1 \setminus \{\mathcal{O}\}, \tag{7}$$

then $h = f$ is the unique solution of this boundary value problem and so we may parametrize the graph of f over Ω_1 in isothermal coordinates as above and the arguments in [Elcrat and Lancaster 1986; Lancaster 1988; Lancaster and Siegel 1996a] can be used to show that c is uniformly continuous on Ω_1 and so extends to be continuous on $\overline{\Omega}_1$. That is, let $k : E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}}) \to E$ be a conformal map. From the works just cited we see that $c \circ k^{-1} \in C^0(\overline{E})$ and so $c \in C^0(\overline{E \setminus (B^1_{\rho_1(\delta)} \cup B^2_{\rho_2(\delta)})})$. Since

$$\bigcup_{\delta \in (0,1)} (E \setminus (B^1_{\rho_1(\delta)} \cup B^2_{\rho_2(\delta)})) = E,$$

we see $c \in C^0(\overline{E} \setminus \{\boldsymbol{o}_1, \boldsymbol{o}_2\})$.

As at the end of Step 1 of the proof of Theorem 1 of [Lancaster and Siegel 1996a], we define $X : B \to \mathbb{R}^3$ by $X = Y \circ g$ and $K : B \to \mathbb{R}^2$ by $K = G \circ g$, where $B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1,\ v > 0\}$ and $g : \overline{B} \to \overline{E}$ is an indirectly conformal (or anticonformal) map from $\overline{B}$ onto $\overline{E}$ such that $g(1, 0) = \boldsymbol{o}_1$, $g(-1, 0) = \boldsymbol{o}_2$ and $g(u, 0) \in \boldsymbol{o}_1\boldsymbol{o}_2$ for each $u \in [-1, 1]$. Notice that $K(u, 0) = \mathcal{O}$ for $u \in [-1, 1]$ (see Figure 6). Set

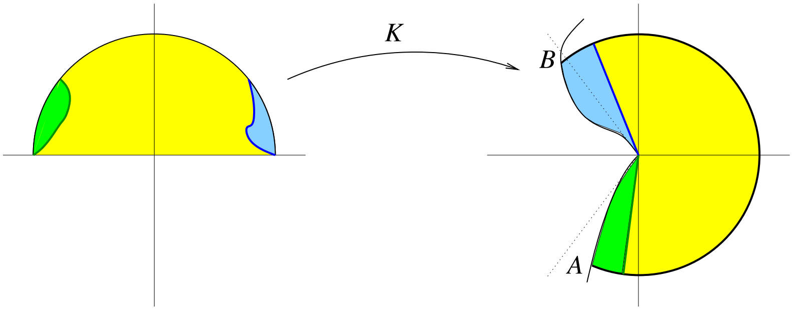

Figure 6. $L(\alpha_2)$, $K^{-1}(L(\alpha_2))$ (blue curves); $L(\alpha_1)$, $K^{-1}(L(\alpha_1))$ (green curves).

$x = a \circ g$, $y = b \circ g$ and $z = c \circ g$, so that $X(u, v) = (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in B$. Now, from Step 2 of the proof of Theorem 1 of [Lancaster and Siegel 1996a],

$$X \in C^0\big(\overline{B} \setminus \{(\pm 1, 0)\} : \mathbb{R}^3\big) \cap C^{1,\iota}\big(B \cup \{(u, 0) : -1 < u < 1\} : \mathbb{R}^3\big)$$

for some $\iota \in (0, 1)$ and $X(u, 0) = (0, 0, z(u, 0))$ cannot be constant on any non-degenerate interval in $(-1, 1)$. Define $\Theta(u) = \arg(x_v(u, 0) + i y_v(u, 0))$. From equation (12) of [Lancaster and Siegel 1996a], we see that

$$\alpha_1 = \lim_{u \downarrow -1} \Theta(u) \quad \text{and} \quad \alpha_2 = \lim_{u \uparrow 1} \Theta(u);$$

here $\alpha_1 < \alpha_2$. As in Steps 2–5 of the proof of Theorem 1 of [Lancaster and Siegel 1996a], we see that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$,

$$\overline{G^{-1}(L(\alpha_2))} \cap \partial E = \{\boldsymbol{o}_1\} \quad \big(\text{and } \overline{K^{-1}(L(\alpha_2))} \cap \partial B = \{(1, 0)\}\big) \quad \text{if } \alpha_2 < \alpha,$$
$$\overline{G^{-1}(L(\alpha_1))} \cap \partial E = \{\boldsymbol{o}_2\} \quad \big(\text{and } \overline{K^{-1}(L(\alpha_1))} \cap \partial B = \{(-1, 0)\}\big) \quad \text{if } \alpha_1 > -\alpha,$$

where $L(\theta) = \big\{(r \cos\theta, r \sin\theta) \in \Omega : 0 < r < \delta^*\big\}$, and one of the following cases holds:

(a) Rf is strictly increasing or strictly decreasing on (α_1, α_2).

(b) There exist α_L, α_R so that $\alpha_1 < \alpha_L < \alpha_R < \alpha_2$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $[\alpha_L, \alpha_R]$ and either increasing on $(\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2)$ or decreasing on $(\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2)$.

If $\alpha_2 = \alpha$ and $\alpha_1 = -\alpha$, then Theorem 1 is proven. Otherwise, suppose $\alpha_2 < \alpha$ and fix $\delta_0 \in (0, \delta^*)$ and Ω_0 (see Figure 4) as before in case (i).

Claim. *Suppose $\alpha_2 < \alpha$. Then f is uniformly continuous on Ω_0^+, where*

$$\Omega_0^+ := \big\{(r \cos\theta, r \sin\theta) \in \Omega_0 : 0 < r < \delta^*,\ \alpha_2 < \theta < \pi\big\}.$$

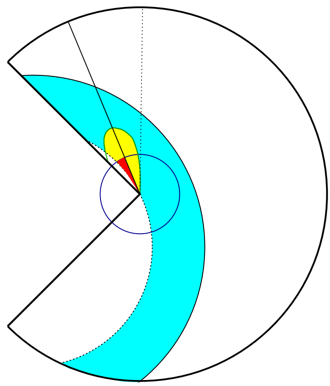

Figure 7. $\Omega^* \cap \mathcal{A}(\boldsymbol{p}_1)$ (blue, yellow and red regions), $\partial B_{\eta(\delta)}(\mathcal{O})$ (blue circle).

Proof. Suppose $\alpha - \alpha_2 < \pi$ (see the blue region in Figure 6). Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) < \frac{1}{4}\epsilon$ and $p(\delta) < r_2 - r_1$. Let $C_r = \{(u, v) \in \overline{B} : |(u, v) - (1, 0)| = r\}$ and let l_r be the arclength of the image curve $X(C_r)$. The Courant–Lebesgue lemma implies that for each $\delta \in (0, 1)$, there exists a $\rho(\delta) \in (\delta, \sqrt{\delta})$ such that $l_{\rho(\delta)} < p(\delta)$. Denote the endpoints of $C_{\rho(\delta)}$ as $(u_1(\delta), v_1(\delta))$ and $(u_2(\delta), 0)$, where $(u_1(\delta))^2 + (v_1(\delta))^2 = 1$, $v_1(\delta) > 0$ and $u_2(\delta) \in (-1, 1)$. Notice $\Theta(u_2(\delta)) < \alpha_2$; let us assume that δ is small enough that $\alpha - \Theta(u_2(\delta)) < \pi$.

Now $X(C_{\rho(\delta)})$ is a curve whose tangent ray at $\mathcal{O}$ exists and has direction $\theta = \Theta(u_2(\delta))$ and $\partial\Omega \cap X\big(C_{\rho(\delta)} \setminus \{(u_1(\delta), v_1(\delta)), (u_2(\delta), 0)\}\big) = \varnothing$; hence there exists $\eta = \eta(\delta) > 0$ such that $\{\boldsymbol{x} \in \Omega_0^+ : |\boldsymbol{x}| < \eta(\delta)\}$ (the red region in Figure 7) is a subset of $\Omega_0 \cap X\big(\{(u, v) \in \overline{B} : |(u, v) - (1, 0)| < \rho(\delta)\}\big)$ (the yellow region plus the red region in Figure 7). From (4) and the arguments in the proof of the claim in case (i), we see that f is uniformly continuous on Ω_0^+.

If $\alpha - \alpha_2 \geq \pi$, we argue as in the proof of the claim in case (i) and see that f is uniformly continuous on Ω_0^+. □

Thus $f \in C^0(\overline{\Omega}_0^+)$; hence (5) implies that $Rf(\theta) = \lim_{\tau \uparrow \alpha_2} Rf(\tau)$ for all $\theta \in [\alpha_2, \alpha)$. Suppose $\alpha_1 > -\alpha$. Then, as above, f is uniformly continuous on

$$\Omega_0^- := \{(r\cos\theta, r\sin\theta) \in \Omega_0 : 0 < r < \delta^*, -\pi < \theta < \alpha_1\}$$

and $f \in C^0(\overline{\Omega}_0^-)$; hence (5) implies

$$Rf(\theta) = \lim_{\tau \downarrow \alpha_1} Rf(\tau) \quad \text{for all } \theta \in (-\alpha, \alpha_1].$$

Thus Theorem 1 is proven. □

3. Proof of Theorem 2

The parametric representation (3) with properties $(a_1)-(a_5)$ continues to be valid and either case (A) or case (B) holds true.

Suppose case (A) holds. Let q_1 denote the modulus of continuity of the trace of f on the (closed) set $\partial^-\Omega^*$ (i.e., $|f(\boldsymbol{x}_1)-f(\boldsymbol{x}_2)| \le q_1(|\boldsymbol{x}_1-\boldsymbol{x}_2|)$ if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \partial^-\Omega^*$). Fix $\delta_0 \in (0, \delta^*)$ with $\sqrt{\delta_0} < \min\{|\boldsymbol{o}-\boldsymbol{a}|, |\boldsymbol{o}-\boldsymbol{b}|\}$. Let $\boldsymbol{p}_1 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_1| = r_1$ and $|\boldsymbol{p}_1 - \boldsymbol{y}_1(\delta_0)| = r_1$ such that $\boldsymbol{p}_1$ lies above (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_1(\delta_0)$. Set $\Omega_0 = \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_1| > r_1\}$.

Claim. *f is uniformly continuous on Ω_0.*

Proof. Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) + q_1(p(\delta)) < \frac{1}{2}\epsilon$ and $p(\delta) < r_2 - r_1$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b_1^{\pm} : \mathcal{A}(\boldsymbol{p}_1) \to \mathbb{R}$ by

$$b_1^{\pm}(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h_{\boldsymbol{p}_1}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{A}(\boldsymbol{p}_1).$$

Notice that

$$b_1^{-}(\boldsymbol{x}) < f(\boldsymbol{x}) < b_1^{+}(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_1).$$

Now there exists $\eta = \eta(\delta) > 0$ such that $\{\boldsymbol{x} \in \Omega_0 : |\boldsymbol{x}| < \eta(\delta)\}$ (the red regions in Figure 8) is a subset of $B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_1)$ (the yellow regions plus the red regions in Figure 8). Thus, for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfying $|\boldsymbol{x}_1| < \eta(\delta)$, $|\boldsymbol{x}_2| < \eta(\delta)$, we have

$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < 2p(\delta) + 2q(p(\delta)) + 2q_1(p(\delta)) < \epsilon.$$

The remainder of the proof of the claim follows as before.

The proof of Theorem 2 in this case now follows the proof of Theorem 1 in the same case.

If case (B) holds, then the proof of Theorem 2 is essentially the same as the proof of Theorem 1; the only significant difference is that $z \in C^0(\overline{B} \setminus \{(1,0)\})$ (and $c \in C^0(\overline{E} \setminus \{\boldsymbol{o}_1\})$) and hence $Rf(\theta)$ exists for $\theta \in [-\alpha, \alpha)$. □

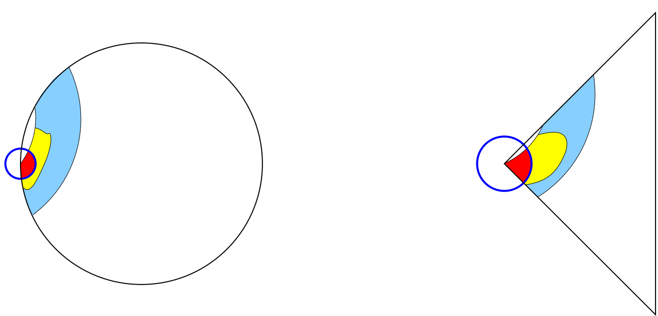

Figure 8. $\Omega_0 \cap \mathcal{A}(\boldsymbol{p}_1)$ (blue, yellow and red regions), $\partial B_{\eta(\delta)}(\mathcal{O})$ (blue circle).

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