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Let α be a (finite or infinite) cardinal number. An ideal of a ring R is called an α -generated ideal if it can be generated by a set with cardinality at most α . A ring R is called an α -generated ring if every ideal of R is an α -generated ideal. When α is finite, the class of α -generated rings has been studied in literature by scholars such as I. S. Cohen and R. Gilmer. In this paper, the class of α -generated rings when α is infinite (in particular, when $\alpha = \aleph_0$, the smallest infinite cardinal number) is considered. Surprisingly, it is proved that the concepts “ \aleph_0 -generated ring” and “Noetherian ring” are the same for the power series ring $R[[X]]$. In other words, if every ideal of $R[[X]]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[[X]]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α , it is proved that R is an α -generated ring if and only if $R[X]$ is an α -generated ring, which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. Let \mathbb{C} be the ring of algebraic integers. Under the continuum hypothesis, we show that $\mathbb{C}[[X]]$ contains an $|\mathbb{C}[[X]]|$ -generated (and hence uncountably generated) ideal which is not a β -generated ideal for any cardinal number $\beta < |\mathbb{C}[[X]]|$ and that the concepts “ \aleph_1 -generated ring” and “ \aleph_0 -generated ring” are different for the power series ring $R[[X]]$.

1. Introduction

In this paper, a ring means a commutative ring with identity. Let R be a ring and let n be a positive integer. An ideal I of R is called an n -generated ideal if I can be generated by a set with cardinality $\leq n$. We call R an n -generated ring if every ideal of R is an n -generated ideal. This class of n -generated rings was first introduced and studied by Cohen [1950]. Principal ideal rings are obviously 1-generated rings. It is well known that Dedekind domains are 2-generated rings. For each

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integer $n \geq 2$, Matson [2009] gave an example of an n -generated ring which is not an $(n-1)$ -generated ring. Cohen [1950] proved that if D is an n -generated integral domain, then the Krull dimension of D is at most 1. An easy proof of this result was later given by Gilmer [1973]. He further showed [1972b] that the result is valid for n -generated rings (see also [Sally 1978, Theorem 1.2, p. 51]). By definition, every n -generated ring is a Noetherian ring. However, the converse does not hold [Cohen 1950]. Gilmer [1973] proved a nice result that if R is an n -generated ring, then the integral closure of R is a Noetherian Prüfer ring. He also showed that a Noetherian ring R is an n -generated ring (for some n) if and only if there exists a positive integer m such that R_M is an m -generated ring for each maximal ideal M of R [Gilmer 1972a; 1972b]. For more on n -generated ideals and rings, we refer the readers to [Ameziane Hassani et al. 1996; Cohen 1950; Gilmer 1972b; 1973; Heinzer and Lantz 1983; Matsuda 1979; McLean 1982; Okon et al. 1992; Okon and Vicknair 1992; 1993; Rush 1991; 1992; Sally 1978; Shalev 1986].

According to Cohen and Gilmer's results, the class of n -generated rings is rather small. It is a subclass of Noetherian rings with Krull dimension at most 1. We generalize the definition of n -generated rings in a natural way as follows. Let α be a (finite or infinite) cardinal number (e.g., $\alpha = 1, 2, \dots, \aleph_0, \aleph_1, \dots$). An ideal I of a ring R is called an α -generated ideal if I can be generated by a set with cardinality at most α . R is called an α -generated ring if every ideal of R is an α -generated ideal. In this paper, we mainly deal with the class of α -generated rings when $\alpha = \aleph_0$, the smallest infinite cardinal number. By definition, an \aleph_0 -generated ring is a ring whose ideals are countably generated. Trivial examples of \aleph_0 -generated rings are those that have only countably many elements (so that each ideal has itself as a countable generating set). Every Noetherian ring is obviously an \aleph_0 -generated ring. However, the converse does not hold. Polynomial rings $R[X_1, X_2, \dots, X_n, \dots]$ in countably infinite indeterminates over countable rings R , the ring \mathbb{C} of algebraic integers, the ring $\text{Int}(\mathbb{Z})$ of integer-valued polynomials on \mathbb{Z} , and 1-dimensional nondiscrete valuation domains are good examples of \aleph_0 -generated rings that are not Noetherian rings.

Even though the class of \aleph_0 -generated rings is strictly larger than the class of Noetherian rings, we show that, when restricted to power series rings, they are actually the same. In other words, the concepts “ \aleph_0 -generated ring” and “Noetherian ring” are the same for the power series ring $R[[X]]$ (Theorem 13). This means if every ideal of $R[[X]]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[[X]]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α , we prove that R is an α -generated ring if and only if $R[X]$ is an α -generated ring (Theorem 22), which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We show under

the continuum hypothesis that (1) $\mathbb{C}[[X]]$ contains an $|\mathbb{C}[[X]]|$ -generated (and hence uncountably generated) ideal which is not a β -generated ideal for any cardinal number $\beta < |\mathbb{C}[[X]]|$ (Corollary 16) and that (2) $\mathbb{C}[[X]]$ is an \aleph_1 -generated ring that is not an \aleph_0 -generated ring (Corollary 17). In fact, these two results hold if \mathbb{C} is replaced by any non-Noetherian countable ring R . As a consequence, it is shown that the concepts “ \aleph_1 -generated ring” and “ \aleph_0 -generated ring” are different (while the concepts “ \aleph_0 -generated ring” and “Noetherian ring” are the same) for the power series ring $R[[X]]$.

2. Some examples of α -generated rings

For each integer $n \geq 2$, Matson [2009] gave an example of an n -generated ring which is not an $(n-1)$ -generated ring. For an infinite cardinal number α , we give an example of an α -generated ring that is not a β -generated ring for any cardinal number $\beta < \alpha$.

Proposition 1. *For an infinite cardinal number α , there exists an α -generated ring that is not a β -generated ring for any cardinal number $\beta < \alpha$.*

Proof. Let R be any ring with cardinality $< \alpha$ and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over R , where Λ is a set of cardinality α . Then the polynomial ring $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ is clearly an α -generated ring since it has cardinality α . We now show that the ideal J of $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ generated by $\{X_\lambda\}_{\lambda \in \Lambda}$ is not a β -generated ideal for any cardinal number $\beta < \alpha$. Let β be any cardinal number such that $\beta < \alpha$. If J is a β -generated ideal, then $J = (\{f_\mu\})$ for some $f_\mu \in J$ such that $|\{f_\mu\}| \leq \beta$. Since each f_μ involves only finitely many indeterminates, $(\{X_\lambda\}_{\lambda \in \Lambda}) = J = (\{f_\mu\}) \subseteq (\{X_\lambda\}_{\lambda \in \Gamma})$ for some subset Γ of Λ such that $|\Gamma| < |\Lambda| = \alpha$, a contradiction. \square

In the next section, we are going to prove that the concepts “ \aleph_0 -generated ring” and “Noetherian ring” are the same for the power series ring $R[[X]]$. We however note that these two concepts are different in general. We give here some examples of (finite-dimensional or infinite-dimensional) \aleph_0 -generated rings that are not Noetherian rings.

Example 2. Let $R_1 := R[X_1, X_2, \dots, X_n, \dots]$ be the polynomial ring in countably infinite indeterminates over a countable ring R . Then R_1 is an \aleph_0 -generated ring since it is countable. It is easy to see that the ideal of R_1 generated by $X_1, X_2, \dots, X_n, \dots$ is not a finitely generated ideal and hence R_1 is not a Noetherian ring.

In Example 2, the (Krull) dimension of R_1 is infinite. We now give examples of finite-dimensional \aleph_0 -generated rings that are not Noetherian rings.

Example 3. Let \mathbb{O} be the ring of algebraic integers (an algebraic integer is a complex number that is integral over \mathbb{Z}). It is well known that \mathbb{O} is a 1-dimensional

non-Noetherian Bézout domain (for example, see p. 72 of [Kaplansky 1974]). However, since \mathbb{C} is countable, it is an \aleph_0 -generated ring.

Example 4. Let $\text{Int}(\mathbb{Z})$ be the ring of integer-valued polynomials on \mathbb{Z} , i.e.,

$$\text{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

By [Cahen and Chabert 1997, Proposition V.2.7], $\text{Int}(\mathbb{Z})$ is a 2-dimensional non-Noetherian domain. However, since $\text{Int}(\mathbb{Z})$ is countable, it is an \aleph_0 -generated ring.

So far, the given examples of \aleph_0 -generated rings are all countable. We finally show that there do exist uncountable \aleph_0 -generated rings that are not Noetherian rings (see Example 6).

Proposition 5. *If V is a 1-dimensional nondiscrete valuation domain, then V is an \aleph_0 -generated ring that is not a Noetherian ring.*

Proof. Since V is a 1-dimensional valuation domain, its value group is (isomorphic to) a subgroup of \mathbb{R} , the (additive) group of real numbers. This implies that every ideal of V is countably generated, i.e., V is an \aleph_0 -generated ring. Since V is nondiscrete, it is not a Noetherian ring. \square

Example 6. Let K be a field and let V be the valuation ring of the field $K(X; \mathbb{R})$, which is the quotient field of the group ring $K[X; \mathbb{R}]$ of \mathbb{R} over K , associated with the valuation v defined by

$$v\left(\sum_{i=0}^n a_{r_i} X^{r_i}\right) := \min\{r_i \mid i = 0, 1, \dots, n\}.$$

Then V is a 1-dimensional nondiscrete valuation domain with value group \mathbb{R} . Hence, V is an \aleph_0 -generated ring that is not a Noetherian ring by Proposition 5. Since \mathbb{R} is uncountable, so is V .

3. Power series rings over α -generated rings

In this section, we prove that the power series ring $R[[X]]$ is an \aleph_0 -generated ring if and only if $R[[X]]$ is a Noetherian ring (and hence) if and only if R is a Noetherian ring. In order to prove this result, we only need to show that if R is a non-Noetherian ring, then the power series ring $R[[X]]$ is not an \aleph_0 -generated ring since, if R is a Noetherian ring, then $R[[X]]$ is also a Noetherian ring (for example, see [Kaplansky 1974, Theorem 71]) and hence an \aleph_0 -generated ring.

Suppose that R is a non-Noetherian ring. Our task is to construct an ideal J of $R[[X]]$ that cannot be generated by any countable subset of J . The desired uncountable generating set for J is indexed by the following special uncountable set, which is called a fathomless set.

3.1. Fathomless sets. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and let \mathcal{U} be the set of all subsets U of \mathbb{N} such that $U = \{n, n + 1, \dots\}$ for some $n \in \mathbb{N}$. For two strictly increasing sequences $s = \{s_n\}$ and $t = \{t_n\}$ of positive integers, we set $s \gg t$ (we also write $t \ll s$) if for each positive integer k there is a set $U \in \mathcal{U}$ (depending on k) such that $s_n > kt_n$ for each $n \in U$, i.e., $s_n > kt_n$ for all large n .

Let \mathcal{S} be the collection of all \mathcal{A} having the following properties:

- (1) \mathcal{A} is a nonempty collection of strictly increasing sequences $s = \{s_n\}$ of positive integers.
- (2) If $s \in \mathcal{A}$ then $s \gg b$, where b is the sequence defined by $b_n := n$ for all n .
- (3) If $s, t \in \mathcal{A}$ and $s \neq t$, then $s \gg t$ or $t \gg s$.

If u is the sequence defined by $u_n := b_n^2$ for each n , then it is easy to see that u is a strictly increasing sequence of positive integers and that $u \gg b$. It follows that the set \mathcal{S} is nonempty. We order \mathcal{S} by set-theoretic inclusion. By Zorn’s lemma, there exists a maximal element in \mathcal{S} . Let \mathcal{A} be a maximal element in \mathcal{S} . This choice of \mathcal{A} will be fixed through the rest of the article. For $s, t \in \mathcal{A}$, we define $s \ll t$ if and only if $s = t$ or $s \ll t$. Then (\mathcal{A}, \ll) becomes a totally ordered set.

Definition 7. A totally ordered set (\mathcal{Y}, \ll) is called a *fathomless set* if, for every nonempty countable subset \mathcal{C} of \mathcal{Y} , there exists an element $y \in \mathcal{Y}$ such that $y \ll \mathcal{C}$, i.e., $y \ll c$ for all $c \in \mathcal{C}$.

The following theorem tells us that the set (\mathcal{A}, \ll) is a fathomless set (for the proof, see [Kang et al. 2013, Theorem 5]).

Theorem 8. *The set (\mathcal{A}, \ll) is a fathomless set.*

Remark 9. By definition, every fathomless set is an uncountable set. Hence, the set (\mathcal{A}, \ll) is uncountable.

3.2. Power series rings over \aleph_0 -generated rings. Using the above fathomless set (\mathcal{A}, \ll) , we construct generators for an ideal J of $R[[X]]$ that is not countably generated. The following observation is useful:

Proposition 10. *For a ring R , the following are equivalent:*

- (1) R is a non-Noetherian ring.
- (2) There exists a sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R such that

$$a_m \notin (a_0, a_1, \dots, a_{m-1})$$

for each $m \geq 1$.

Proof. (1) \implies (2) Suppose that R is a non-Noetherian ring. Then there exists an ideal I of R such that I is not finitely generated. We will find the desired sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R by using induction as follows. Choose an

element $a_0 \in I$. Since $(a_0) \subsetneq I$, there exists an element $a_1 \in I \setminus (a_0)$. Suppose that there exist elements a_0, a_1, \dots, a_{m-1} in I ($m \geq 2$) such that $a_i \notin (a_0, a_1, \dots, a_{i-1})$ for each $1 \leq i \leq m - 1$. Since $(a_0, a_1, \dots, a_{m-1}) \subsetneq I$, there exists an element $a_m \in I \setminus (a_0, a_1, \dots, a_{m-1})$.

(2) \implies (1) The ideal $(a_0, a_1, \dots, a_m, \dots)$ of R is not finitely generated. \square

Let R be a non-Noetherian ring. By Proposition 10, there exists a sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R such that

$$a_m \notin (a_0, a_1, \dots, a_{m-1})$$

for each $m \geq 1$. For each integer $m \geq 0$, we let $I_m := (a_0, a_1, \dots, a_m)$. Then $a_m \notin I_{m-1}$ for each $m \geq 1$. For each sequence $s = \{s_n\} \in \mathcal{A}$, we define

$$f_s := a_0 + a_1 X^{s_1} + a_2 X^{s_2} + \dots + a_n X^{s_n} + \dots \in R[[X]].$$

We let J be the ideal of $R[[X]]$ generated by all f_s with $s \in \mathcal{A}$.

Remark 11. The generators f_s of J are constructed by stretching out the coefficients of the power series $\sum_{n=0}^\infty a_n X^n$ so that its coefficient at X^{s_n} (s_n is much greater than n for all large n since $s \gg b$) is still a_n . This property will play a crucial role in showing that J is not a countably generated ideal.

An ideal I of a ring R is called an uncountably generated ideal if it is not a countably generated ideal. If R is a non-Noetherian ring, then so is the power series ring $R[[X]]$. Hence, $R[[X]]$ has some ideals that are not finitely generated. These ideals can be either countably generated or uncountably generated. However, we show that $R[[X]]$ has at least one uncountably generated ideal if R is a non-Noetherian ring.

Theorem 12. *If R is a non-Noetherian ring, then the power series ring $R[[X]]$ has at least one uncountably generated ideal.*

Proof. It suffices to show that the ideal J constructed above is not a countably generated ideal. Suppose on the contrary that J is countably generated. Then there exists a countable subset \mathcal{B} of \mathcal{A} such that J is generated by $\{f_s \mid s \in \mathcal{B}\}$. Since \mathcal{A} is a fathomless set, there exists a sequence $v \in \mathcal{A}$ such that $v \ll \mathcal{B}$. Since $f_v \in J$, f_v is a finite sum of elements of the form $h(s)f_s$,

$$(1) \quad f_v = \sum_s h(s)f_s,$$

where $h(s) \in R[[X]]$ and $s \in \mathcal{B}$. Since $v \ll \mathcal{B}$, by taking a finite intersection of members of \mathcal{U} , we can find a set $U \in \mathcal{U}$ such that $v_m < s_m$ for each $m \in U$ and for each s appearing in the finite sum (1). Choose any number $m \in U$. Since $v_m < s_m$, the coefficient of f_s at X^j belongs to I_{m-1} for all $j \leq v_m$. It follows that the

coefficient of $h(s)f_s$ at X^{v_m} belongs to I_{m-1} . This holds for every s appearing in the finite sum (1). Therefore, the coefficient of $\sum_s h(s)f_s$ at X^{v_m} belongs to I_{m-1} . This is a contradiction since the coefficient of $f_v = \sum_s h(s)f_s$ at X^{v_m} is a_m and $a_m \notin I_{m-1}$. \square

We can now obtain the main result of the paper.

Theorem 13. *For a ring R , the following are equivalent:*

- (1) $R[[X]]$ is an \aleph_0 -generated ring.
- (2) $R[[X]]$ is a Noetherian ring.
- (3) R is a Noetherian ring.

Proof. We only need to prove that (1) implies (3). However, this follows from Theorem 12. \square

Since a ring R is a Noetherian ring if and only if every countably generated ideal is finitely generated, we have the following corollary.

Corollary 14. *For a ring R , the following are equivalent:*

- (1) Every ideal of $R[[X]]$ is countably generated.
- (2) Every countably generated ideal of $R[[X]]$ is finitely generated.

Corollary 15. *If R is an \aleph_0 -generated ring, then the power series ring $R[[X]]$ is not necessarily an \aleph_0 -generated ring.*

Proof. Let R be any \aleph_0 -generated ring which is not a Noetherian ring (see Proposition 5 and Examples 2, 3, and 4). Then $R[[X]]$ is not an \aleph_0 -generated ring by Theorem 13. \square

Corollary 16. *Let \mathbb{C} be the ring of algebraic integers and let $\alpha = |\mathbb{C}[[X]]|$. Then, under the continuum hypothesis, $\mathbb{C}[[X]]$ contains an α -generated ideal that is not a β -generated ideal for any cardinal number $\beta < \alpha$.*

Proof. We have $\alpha = |\mathbb{C}[[X]]| = 2^{\aleph_0} = \aleph_1$ under the continuum hypothesis. By Theorem 12, $\mathbb{C}[[X]]$ has an uncountably generated ideal J . Obviously, J is an α -generated ideal since $|J| \leq |\mathbb{C}[[X]]| = \alpha$. But, since J is not countably generated, it is not a β -generated ideal for any cardinal number $\beta < \aleph_1 = \alpha$. \square

The following corollary shows that the concepts “ \aleph_1 -generated ring” and “ \aleph_0 -generated ring” are different for the power series ring $R[[X]]$.

Corollary 17. *Under the continuum hypothesis, $\mathbb{C}[[X]]$ is an \aleph_1 -generated ring but not an \aleph_0 -generated ring.*

Proof. Since $|\mathbb{C}[[X]]| = \aleph_1$, $\mathbb{C}[[X]]$ is an \aleph_1 -generated ring. The fact that $\mathbb{C}[[X]]$ is not an \aleph_0 -generated ring follows from Corollary 16. \square

Remark 18. Corollaries 16 and 17 hold for any non-Noetherian countable ring (see Examples 2, 3, and 4).

In the rest of this section, we consider power series rings over n -generated rings, where n is a positive integer. We first note that, for the power series ring $R[[X]]$ to be an n -generated ring, it is necessary that the ring R be zero-dimensional.

Proposition 19. *If the power series ring $R[[X]]$ is an n -generated ring for some positive integer n , then $\dim R = 0$.*

Proof. If $R[[X]]$ is an n -generated ring, then $\dim R[[X]] \leq 1$. It is easy to see that $\dim R + 1 \leq \dim R[[X]]$. Thus, $\dim R + 1 \leq \dim R[[X]] \leq 1$ and hence $\dim R = 0$. \square

Proposition 20. *Suppose that D is an integral domain. Then the following are equivalent:*

- (1) $D[[X]]$ is an n -generated ring for some positive integer n .
- (2) $D[[X]]$ is a 1-generated ring.
- (3) D is a field.

Proof. (3) \implies (2) \implies (1) These are obvious.

(1) \implies (3) By Proposition 19, $\dim D = 0$. Thus, D is a field by the assumption that D is an integral domain. \square

Remark 21. By Proposition 20, if D is an (n -generated) integral domain which is not a field, then the power series ring $D[[X]]$ is never an m -generated ring for any positive integer m . In particular, $D[[X]]$ has an ideal that is not an $(n+1)$ -generated ideal despite the fact that all prime ideals of $D[[X]]$ are $(n+1)$ -generated ideals (see [Kaplansky 1974, Theorem 70]).

4. Polynomial rings over α -generated rings

In this section, we prove that for any infinite cardinal number α , a ring R is an α -generated ring if and only if the polynomial ring $R[X]$ is an α -generated ring, which is an analogue of the Hilbert basis theorem, which states that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We also note that the result fails if α is a finite cardinal number.

Theorem 22. *For any infinite cardinal number α , a ring R is an α -generated ring if and only if the polynomial ring $R[X]$ is an α -generated ring.*

Proof. We follow the standard proof of the Hilbert basis theorem. Any homomorphic image of an α -generated ring is obviously an α -generated ring. Hence, if $R[X]$ is an α -generated ring, then so is R .

For the converse, suppose that R is an α -generated ring and let J be an ideal of $R[X]$. We show that J is an α -generated ideal. For each $n \geq 0$, let I_n be the

set of $r \in R$ such that $r = 0$ or r is the leading coefficient of a polynomial in J of degree n . For each $n \geq 0$, since R is an α -generated ring, I_n is an α -generated ideal. Let

$$\{r_{n\lambda} \mid \lambda \in \Lambda_n\}$$

be a generating set of I_n with cardinality at most α . For each $r_{n\lambda}$, let $f_{n\lambda} \in J$ be a polynomial of degree n with leading coefficient $r_{n\lambda}$. We will show that the ideal J of $R[X]$ is generated by the set

$$F := \{f_{n\lambda} \mid n \geq 0, \lambda \in \Lambda_n\},$$

which also has cardinality at most α . Denote by (F) the ideal of $R[X]$ generated by F . Since $F \subseteq J$, we have $(F) \subseteq J$. Conversely, the polynomials of degree 0 in J are precisely the elements of I_0 and hence are contained in (F) . Proceeding by induction, assume that (F) contains all polynomials of J of degree less than k and let $g \in J$ have degree k and leading coefficient r . Then

$$r = \sum_{i=1}^m s_i r_{k\lambda_i}$$

for some $s_i \in R$ and $\lambda_i \in \Lambda_k$. The polynomial $\sum_{i=1}^m s_i f_{k\lambda_i}$ also has degree k and leading coefficient r . Hence,

$$g - \sum_{i=1}^m s_i f_{k\lambda_i} \in J$$

has degree at most $k - 1$. By the induction hypothesis,

$$g - \sum_{i=1}^m s_i f_{k\lambda_i} \in (F)$$

and hence $g \in (F)$. Therefore, $J = (F)$ is an α -generated ideal. □

Proposition 23. *If the polynomial ring $R[X]$ is an n -generated ring for some positive integer n , then $\dim R = 0$.*

Proof. As in the proof of Proposition 19, we have $\dim R + 1 \leq \dim R[X] \leq 1$ and hence $\dim R = 0$. □

As in the power series ring case, we can prove the following:

Proposition 24. *Suppose that D is an integral domain. Then the following are equivalent:*

- (1) $D[X]$ is an n -generated ring for some positive integer n .
- (2) $D[X]$ is a 1-generated ring.
- (3) D is a field.

Remark 25. By Proposition 24, if D is an (n -generated) integral domain which is not a field, then the polynomial ring $D[X]$ is never an m -generated ring for any positive integer m . In particular, Theorem 22 always fails for any finite cardinal number α .

Remarks 26. (1) For rings with zero-divisors, the polynomial ring $R[X]$ may not be a 2-generated ring even if the ring R is a 1-generated ring. For example, let $R = V/(a^3)$, where V is a 1-dimensional discrete valuation domain (or equivalently, a local principal ideal domain) with maximal ideal $M = (a)$. Then R is a 1-generated ring. However, if \bar{M} denotes the maximal ideal $M/(a^3)$ of R , then $\bar{M}^2 \neq 0$. By [Matsuda 1979, (5.7)], $R[X]$ is not a 2-generated ring.

(2) More generally, for any integer $n \geq 2$, let $R = V/(a^{n+1})$, where V is the same as above. Then R is a 1-generated ring. However, by [Matsuda 1979, (5.13)], the polynomial ring $R[X]$ is not an n -generated ring.

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
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