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A REMARK ON THE NOETHERIAN PROPERTY OF POWER SERIES RINGS

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Let α be a (finite or infinite) cardinal number. An ideal of a ring R is called an α-generated ideal if it can be generated by a set with cardinality at most α. A ring R is called an α-generated ring if every ideal of R is an α-generated ideal. When α is finite, the class of α-generated rings has been studied in literature by scholars such as I. S. Cohen and R. Gilmer. In this paper, the class of α-generated rings when α is infinite (in particular, when $\alpha = \aleph_0$, the smallest infinite cardinal number) is considered. Surprisingly, it is proved that the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$. In other words, if every ideal of $R[\![X]\!]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[\![X]\!]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α, it is proved that R is an α-generated ring if and only if $R[X]$ is an α-generated ring, which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. Let $\mathbb{O}$ be the ring of algebraic integers. Under the continuum hypothesis, we show that $\mathbb{O}[\![X]\!]$ contains an $|\mathbb{O}[\![X]\!]|$-generated (and hence uncountably generated) ideal which is not a β-generated ideal for any cardinal number $\beta < |\mathbb{O}[\![X]\!]|$ and that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different for the power series ring $R[\![X]\!]$.

1. Introduction

In this paper, a ring means a commutative ring with identity. Let R be a ring and let n be a positive integer. An ideal I of R is called an n-generated ideal if I can be generated by a set with cardinality $\leq n$. We call R an n-generated ring if every ideal of R is an n-generated ideal. This class of n-generated rings was first introduced and studied by Cohen [1950]. Principal ideal rings are obviously 1-generated rings. It is well known that Dedekind domains are 2-generated rings. For each

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integer $n \geq 2$, Matson [2009] gave an example of an n-generated ring which is not an $(n-1)$-generated ring. Cohen [1950] proved that if D is an n-generated integral domain, then the Krull dimension of D is at most 1. An easy proof of this result was later given by Gilmer [1973]. He further showed [1972b] that the result is valid for n-generated rings (see also [Sally 1978, Theorem 1.2, p. 51]). By definition, every n-generated ring is a Noetherian ring. However, the converse does not hold [Cohen 1950]. Gilmer [1973] proved a nice result that if R is an n-generated ring, then the integral closure of R is a Noetherian Prüfer ring. He also showed that a Noetherian ring R is an n-generated ring (for some n) if and only if there exists a positive integer m such that R_M is an m-generated ring for each maximal ideal M of R [Gilmer 1972a; 1972b]. For more on n-generated ideals and rings, we refer the readers to [Ameziane Hassani et al. 1996; Cohen 1950; Gilmer 1972b; 1973; Heinzer and Lantz 1983; Matsuda 1979; McLean 1982; Okon et al. 1992; Okon and Vicknair 1992; 1993; Rush 1991; 1992; Sally 1978; Shalev 1986].

According to Cohen and Gilmer's results, the class of n-generated rings is rather small. It is a subclass of Noetherian rings with Krull dimension at most 1. We generalize the definition of n-generated rings in a natural way as follows. Let α be a (finite or infinite) cardinal number (e.g., $\alpha = 1, 2, \dots, \aleph_0, \aleph_1, \dots$). An ideal I of a ring R is called an α-generated ideal if I can be generated by a set with cardinality at most α. R is called an α-generated ring if every ideal of R is an α-generated ideal. In this paper, we mainly deal with the class of α-generated rings when $\alpha = \aleph_0$, the smallest infinite cardinal number. By definition, an $\aleph_0$-generated ring is a ring whose ideals are countably generated. Trivial examples of $\aleph_0$-generated rings are those that have only countably many elements (so that each ideal has itself as a countable generating set). Every Noetherian ring is obviously an $\aleph_0$-generated ring. However, the converse does not hold. Polynomial rings $R[X_1, X_2, \dots, X_n, \dots]$ in countably infinite indeterminates over countable rings R, the ring $\mathbb{O}$ of algebraic integers, the ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials on $\mathbb{Z}$, and 1-dimensional nondiscrete valuation domains are good examples of $\aleph_0$-generated rings that are not Noetherian rings.

Even though the class of $\aleph_0$-generated rings is strictly larger than the class of Noetherian rings, we show that, when restricted to power series rings, they are actually the same. In other words, the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$ (Theorem 13). This means if every ideal of $R[\![X]\!]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[\![X]\!]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α, we prove that R is an α-generated ring if and only if $R[X]$ is an α-generated ring (Theorem 22), which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We show under the continuum hypothesis that (1) $\mathbb{O}[\![X]\!]$ contains an $|\mathbb{O}[\![X]\!]|$-generated (and hence uncountably generated) ideal which is not a β-generated ideal for any cardinal number $\beta < |\mathbb{O}[\![X]\!]|$ (Corollary 16) and that (2) $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring that is not an $\aleph_0$-generated ring (Corollary 17). In fact, these two results hold if $\mathbb{O}$ is replaced by any non-Noetherian countable ring R. As a consequence, it is shown that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different (while the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same) for the power series ring $R[\![X]\!]$.

2. Some examples of α-generated rings

For each integer $n \geq 2$, Matson [2009] gave an example of an n-generated ring which is not an $(n-1)$-generated ring. For an infinite cardinal number α, we give an example of an α-generated ring that is not a β-generated ring for any cardinal number $\beta < \alpha$.

Proposition 1. *For an infinite cardinal number α, there exists an α-generated ring that is not a β-generated ring for any cardinal number $\beta < \alpha$.*

Proof. Let R be any ring with cardinality $< \alpha$ and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over R, where Λ is a set of cardinality α. Then the polynomial ring $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is clearly an α-generated ring since it has cardinality α. We now show that the ideal J of $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ generated by $\{X_\lambda\}_{\lambda\in\Lambda}$ is not a β-generated ideal for any cardinal number $\beta < \alpha$. Let β be any cardinal number such that $\beta < \alpha$. If J is a β-generated ideal, then $J = (\{f_\mu\})$ for some $f_\mu \in J$ such that $|\{f_\mu\}| \leq \beta$. Since each f_μ involves only finitely many indeterminates, $(\{X_\lambda\}_{\lambda\in\Lambda}) = J = (\{f_\mu\}) \subseteq (\{X_\lambda\}_{\lambda\in\Gamma})$ for some subset Γ of Λ such that $|\Gamma| < |\Lambda| = \alpha$, a contradiction. □

In the next section, we are going to prove that the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$. We however note that these two concepts are different in general. We give here some examples of (finite-dimensional or infinite-dimensional) $\aleph_0$-generated rings that are not Noetherian rings.

Example 2. Let $R_1 := R[X_1, X_2, \ldots, X_n, \ldots]$ be the polynomial ring in countably infinite indeterminates over a countable ring R. Then R_1 is an $\aleph_0$-generated ring since it is countable. It is easy to see that the ideal of R_1 generated by $X_1, X_2, \ldots, X_n, \ldots$ is not a finitely generated ideal and hence R_1 is not a Noetherian ring.

In Example 2, the (Krull) dimension of R_1 is infinite. We now give examples of finite-dimensional $\aleph_0$-generated rings that are not Noetherian rings.

Example 3. Let $\mathbb{O}$ be the ring of algebraic integers (an algebraic integer is a complex number that is integral over $\mathbb{Z}$). It is well known that $\mathbb{O}$ is a 1-dimensional non-Noetherian Bézout domain (for example, see p. 72 of [Kaplansky 1974]). However, since $\mathbb{O}$ is countable, it is an $\aleph_0$-generated ring.

Example 4. Let $\mathrm{Int}(\mathbb{Z})$ be the ring of integer-valued polynomials on $\mathbb{Z}$, i.e.,

$$\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

By [Cahen and Chabert 1997, Proposition V.2.7], $\mathrm{Int}(\mathbb{Z})$ is a 2-dimensional non-Noetherian domain. However, since $\mathrm{Int}(\mathbb{Z})$ is countable, it is an $\aleph_0$-generated ring.

So far, the given examples of $\aleph_0$-generated rings are all countable. We finally show that there do exist uncountable $\aleph_0$-generated rings that are not Noetherian rings (see Example 6).

Proposition 5. *If V is a 1-dimensional nondiscrete valuation domain, then V is an $\aleph_0$-generated ring that is not a Noetherian ring.*

Proof. Since V is a 1-dimensional valuation domain, its value group is (isomorphic to) a subgroup of $\mathbb{R}$, the (additive) group of real numbers. This implies that every ideal of V is countably generated, i.e., V is an $\aleph_0$-generated ring. Since V is nondiscrete, it is not a Noetherian ring. □

Example 6. Let K be a field and let V be the valuation ring of the field $K(X;\mathbb{R})$, which is the quotient field of the group ring $K[X;\mathbb{R}]$ of $\mathbb{R}$ over K, associated with the valuation v defined by

$$v\left(\sum_{i=0}^{n} a_{r_i} X^{r_i}\right) := \min\{r_i \mid i = 0, 1, \dots, n\}.$$

Then V is a 1-dimensional nondiscrete valuation domain with value group $\mathbb{R}$. Hence, V is an $\aleph_0$-generated ring that is not a Noetherian ring by Proposition 5. Since $\mathbb{R}$ is uncountable, so is V.

3. Power series rings over α-generated rings

In this section, we prove that the power series ring $R[[X]]$ is an $\aleph_0$-generated ring if and only if $R[[X]]$ is a Noetherian ring (and hence) if and only if R is a Noetherian ring. In order to prove this result, we only need to show that if R is a non-Noetherian ring, then the power series ring $R[[X]]$ is not an $\aleph_0$-generated ring since, if R is a Noetherian ring, then $R[[X]]$ is also a Noetherian ring (for example, see [Kaplansky 1974, Theorem 71]) and hence an $\aleph_0$-generated ring.

Suppose that R is a non-Noetherian ring. Our task is to construct an ideal J of $R[[X]]$ that cannot be generated by any countable subset of J. The desired uncountable generating set for J is indexed by the following special uncountable set, which is called a fathomless set.

3.1. ***Fathomless sets.*** Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and let $\mathcal{U}$ be the set of all subsets U of $\mathbb{N}$ such that $U = \{n, n+1, \dots\}$ for some $n \in \mathbb{N}$. For two strictly increasing sequences $s = \{s_n\}$ and $t = \{t_n\}$ of positive integers, we set $s \gg t$ (we also write $t \ll s$) if for each positive integer k there is a set $U \in \mathcal{U}$ (depending on k) such that $s_n > kt_n$ for each $n \in U$, i.e., $s_n > kt_n$ for all large n.

Let $\mathcal{S}$ be the collection of all $\mathcal{A}$ having the following properties:

(1) $\mathcal{A}$ is a nonempty collection of strictly increasing sequences $s = \{s_n\}$ of positive integers.

(2) If $s \in \mathcal{A}$ then $s \gg b$, where b is the sequence defined by $b_n := n$ for all n.

(3) If $s, t \in \mathcal{A}$ and $s \neq t$, then $s \gg t$ or $t \gg s$.

If u is the sequence defined by $u_n := b_n^2$ for each n, then it is easy to see that u is a strictly increasing sequence of positive integers and that $u \gg b$. It follows that the set $\mathcal{S}$ is nonempty. We order $\mathcal{S}$ by set-theoretic inclusion. By Zorn's lemma, there exists a maximal element in $\mathcal{S}$. Let $\mathcal{A}$ be a maximal element in $\mathcal{S}$. This choice of $\mathcal{A}$ will be fixed through the rest of the article. For $s, t \in \mathcal{A}$, we define $s \underline{\ll} t$ if and only if $s = t$ or $s \ll t$. Then $(\mathcal{A}, \underline{\ll})$ becomes a totally ordered set.

Definition 7. A totally ordered set $(\mathcal{Y}, \underline{\ll})$ is called a *fathomless set* if, for every nonempty countable subset $\mathcal{C}$ of $\mathcal{Y}$, there exists an element $y \in \mathcal{Y}$ such that $y \ll \mathcal{C}$, i.e., $y \ll c$ for all $c \in \mathcal{C}$.

The following theorem tells us that the set $(\mathcal{A}, \underline{\ll})$ is a fathomless set (for the proof, see [Kang et al. 2013, Theorem 5]).

Theorem 8. *The set $(\mathcal{A}, \underline{\ll})$ is a fathomless set.*

Remark 9. By definition, every fathomless set is an uncountable set. Hence, the set $(\mathcal{A}, \underline{\ll})$ is uncountable.

3.2. ***Power series rings over $\aleph_0$-generated rings.*** Using the above fathomless set $(\mathcal{A}, \underline{\ll})$, we construct generators for an ideal J of $R[\![X]\!]$ that is not countably generated. The following observation is useful:

Proposition 10. *For a ring R, the following are equivalent*:

(1) *R is a non-Noetherian ring.*

(2) *There exists a sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R such that*

$$a_m \notin (a_0, a_1, \dots, a_{m-1})$$

for each $m \geq 1$.

Proof. (1) $\Longrightarrow$ (2) Suppose that R is a non-Noetherian ring. Then there exists an ideal I of R such that I is not finitely generated. We will find the desired sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R by using induction as follows. Choose an element $a_0 \in I$. Since $(a_0) \subsetneq I$, there exists an element $a_1 \in I \setminus (a_0)$. Suppose that there exist elements $a_0, a_1, \ldots, a_{m-1}$ in I ($m \geq 2$) such that $a_i \notin (a_0, a_1, \ldots, a_{i-1})$ for each $1 \leq i \leq m-1$. Since $(a_0, a_1, \ldots, a_{m-1}) \subsetneq I$, there exists an element $a_m \in I \setminus (a_0, a_1, \ldots, a_{m-1})$.

(2) $\Longrightarrow$ (1) The ideal $(a_0, a_1, \ldots, a_m, \ldots)$ of R is not finitely generated. □

Let R be a non-Noetherian ring. By Proposition 10, there exists a sequence $a_0, a_1, \ldots, a_m, \ldots$ of elements in R such that

$$a_m \notin (a_0, a_1, \ldots, a_{m-1})$$

for each $m \geq 1$. For each integer $m \geq 0$, we let $I_m := (a_0, a_1, \ldots, a_m)$. Then $a_m \notin I_{m-1}$ for each $m \geq 1$. For each sequence $s = \{s_n\} \in \mathscr{A}$, we define

$$f_s := a_0 + a_1 X^{s_1} + a_2 X^{s_2} + \cdots + a_n X^{s_n} + \cdots \in R[\![X]\!].$$

We let J be the ideal of $R[\![X]\!]$ generated by all f_s with $s \in \mathscr{A}$.

Remark 11. The generators f_s of J are constructed by stretching out the coefficients of the power series $\sum_{n=0}^{\infty} a_n X^n$ so that its coefficient at X^{s_n} (s_n is much greater than n for all large n since $s \gg b$) is still a_n. This property will play a crucial role in showing that J is not a countably generated ideal.

An ideal I of a ring R is called an uncountably generated ideal if it is not a countably generated ideal. If R is a non-Noetherian ring, then so is the power series ring $R[\![X]\!]$. Hence, $R[\![X]\!]$ has some ideals that are not finitely generated. These ideals can be either countably generated or uncountably generated. However, we show that $R[\![X]\!]$ has at least one uncountably generated ideal if R is a non-Noetherian ring.

Theorem 12. *If R is a non-Noetherian ring, then the power series ring $R[\![X]\!]$ has at least one uncountably generated ideal.*

Proof. It suffices to show that the ideal J constructed above is not a countably generated ideal. Suppose on the contrary that J is countably generated. Then there exists a countable subset $\mathscr{B}$ of $\mathscr{A}$ such that J is generated by $\{f_s \mid s \in \mathscr{B}\}$. Since $\mathscr{A}$ is a fathomless set, there exists a sequence $v \in \mathscr{A}$ such that $v \ll \mathscr{B}$. Since $f_v \in J$, f_v is a finite sum of elements of the form $h(s) f_s$,

$$f_v = \sum_s h(s) f_s, \tag{1}$$

where $h(s) \in R[\![X]\!]$ and $s \in \mathscr{B}$. Since $v \ll \mathscr{B}$, by taking a finite intersection of members of $\mathscr{U}$, we can find a set $U \in \mathscr{U}$ such that $v_m < s_m$ for each $m \in U$ and for each s appearing in the finite sum (1). Choose any number $m \in U$. Since $v_m < s_m$, the coefficient of f_s at X^j belongs to I_{m-1} for all $j \leq v_m$. It follows that the coefficient of $h(s)f_s$ at X^{v_m} belongs to I_{m-1}. This holds for every s appearing in the finite sum (1). Therefore, the coefficient of $\sum_s h(s)f_s$ at X^{v_m} belongs to I_{m-1}. This is a contradiction since the coefficient of $f_v = \sum_s h(s)f_s$ at X^{v_m} is a_m and $a_m \notin I_{m-1}$. □

We can now obtain the main result of the paper.

Theorem 13. *For a ring R, the following are equivalent*:

(1) *$R[\![X]\!]$ is an $\aleph_0$-generated ring.*

(2) *$R[\![X]\!]$ is a Noetherian ring.*

(3) *R is a Noetherian ring.*

Proof. We only need to prove that (1) implies (3). However, this follows from Theorem 12. □

Since a ring R is a Noetherian ring if and only if every countably generated ideal is finitely generated, we have the following corollary.

Corollary 14. *For a ring R, the following are equivalent*:

(1) *Every ideal of $R[\![X]\!]$ is countably generated.*

(2) *Every countably generated ideal of $R[\![X]\!]$ is finitely generated.*

Corollary 15. *If R is an $\aleph_0$-generated ring, then the power series ring $R[\![X]\!]$ is not necessarily an $\aleph_0$-generated ring.*

Proof. Let R be any $\aleph_0$-generated ring which is not a Noetherian ring (see Proposition 5 and Examples 2, 3, and 4). Then $R[\![X]\!]$ is not an $\aleph_0$-generated ring by Theorem 13. □

Corollary 16. *Let $\mathbb{O}$ be the ring of algebraic integers and let $\alpha = |\mathbb{O}[\![X]\!]|$. Then, under the continuum hypothesis, $\mathbb{O}[\![X]\!]$ contains an α-generated ideal that is not a β-generated ideal for any cardinal number $\beta < \alpha$.*

Proof. We have $\alpha = |\mathbb{O}[\![X]\!]| = 2^{\aleph_0} = \aleph_1$ under the continuum hypothesis. By Theorem 12, $\mathbb{O}[\![X]\!]$ has an uncountably generated ideal J. Obviously, J is an α-generated ideal since $|J| \le |\mathbb{O}[\![X]\!]| = \alpha$. But, since J is not countably generated, it is not a β-generated ideal for any cardinal number $\beta < \aleph_1 = \alpha$. □

The following corollary shows that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different for the power series ring $R[\![X]\!]$.

Corollary 17. *Under the continuum hypothesis, $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring but not an $\aleph_0$-generated ring.*

Proof. Since $|\mathbb{O}[\![X]\!]| = \aleph_1$, $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring. The fact that $\mathbb{O}[\![X]\!]$ is not an $\aleph_0$-generated ring follows from Corollary 16. □

Remark 18. Corollaries 16 and 17 hold for any non-Noetherian countable ring (see Examples 2, 3, and 4).

In the rest of this section, we consider power series rings over n-generated rings, where n is a positive integer. We first note that, for the power series ring $R[\![X]\!]$ to be an n-generated ring, it is necessary that the ring R be zero-dimensional.

Proposition 19. *If the power series ring $R[\![X]\!]$ is an n-generated ring for some positive integer n, then* $\dim R = 0$.

Proof. If $R[\![X]\!]$ is an n-generated ring, then $\dim R[\![X]\!] \leq 1$. It is easy to see that $\dim R+1 \leq \dim R[\![X]\!]$. Thus, $\dim R+1 \leq \dim R[\![X]\!] \leq 1$ and hence $\dim R = 0$. □

Proposition 20. *Suppose that D is an integral domain. Then the following are equivalent*:

(1) *$D[\![X]\!]$ is an n-generated ring for some positive integer n.*

(2) *$D[\![X]\!]$ is a* 1*-generated ring.*

(3) *D is a field.*

Proof. (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) These are obvious.

(1) $\Longrightarrow$ (3) By Proposition 19, $\dim D = 0$. Thus, D is a field by the assumption that D is an integral domain. □

Remark 21. By Proposition 20, if D is an (n-generated) integral domain which is not a field, then the power series ring $D[\![X]\!]$ is never an m-generated ring for any positive integer m. In particular, $D[\![X]\!]$ has an ideal that is not an $(n+1)$-generated ideal despite the fact that all prime ideals of $D[\![X]\!]$ are $(n+1)$-generated ideals (see [Kaplansky 1974, Theorem 70]).

4. Polynomial rings over α-generated rings

In this section, we prove that for any infinite cardinal number α, a ring R is an α-generated ring if and only if the polynomial ring $R[X]$ is an α-generated ring, which is an analogue of the Hilbert basis theorem, which states that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We also note that the result fails if α is a finite cardinal number.

Theorem 22. *For any infinite cardinal number α, a ring R is an α-generated ring if and only if the polynomial ring $R[X]$ is an α-generated ring.*

Proof. We follow the standard proof of the Hilbert basis theorem. Any homomorphic image of an α-generated ring is obviously an α-generated ring. Hence, if $R[X]$ is an α-generated ring, then so is R.

For the converse, suppose that R is an α-generated ring and let J be an ideal of $R[X]$. We show that J is an α-generated ideal. For each $n \geq 0$, let I_n be the set of $r \in R$ such that $r = 0$ or r is the leading coefficient of a polynomial in J of degree n. For each $n \geq 0$, since R is an α-generated ring, I_n is an α-generated ideal. Let

$$\{r_{n\lambda} \mid \lambda \in \Lambda_n\}$$

be a generating set of I_n with cardinality at most α. For each $r_{n\lambda}$, let $f_{n\lambda} \in J$ be a polynomial of degree n with leading coefficient $r_{n\lambda}$. We will show that the ideal J of $R[X]$ is generated by the set

$$F := \{f_{n\lambda} \mid n \geq 0, \lambda \in \Lambda_n\},$$

which also has cardinality at most α. Denote by (F) the ideal of $R[X]$ generated by F. Since $F \subseteq J$, we have $(F) \subseteq J$. Conversely, the polynomials of degree 0 in J are precisely the elements of I_0 and hence are contained in (F). Proceeding by induction, assume that (F) contains all polynomials of J of degree less than k and let $g \in J$ have degree k and leading coefficient r. Then

$$r = \sum_{i=1}^{m} s_i r_{k\lambda_i}$$

for some $s_i \in R$ and $\lambda_i \in \Lambda_k$. The polynomial $\sum_{i=1}^{m} s_i f_{k\lambda_i}$ also has degree k and leading coefficient r. Hence,

$$g - \sum_{i=1}^{m} s_i f_{k\lambda_i} \in J$$

has degree at most $k - 1$. By the induction hypothesis,

$$g - \sum_{i=1}^{m} s_i f_{k\lambda_i} \in (F)$$

and hence $g \in (F)$. Therefore, $J = (F)$ is an α-generated ideal. □

Proposition 23. *If the polynomial ring $R[X]$ is an n-generated ring for some positive integer n, then* $\dim R = 0$.

Proof. As in the proof of Proposition 19, we have $\dim R + 1 \leq \dim R[X] \leq 1$ and hence $\dim R = 0$. □

As in the power series ring case, we can prove the following:

Proposition 24. *Suppose that D is an integral domain. Then the following are equivalent*:

(1) *$D[X]$ is an n-generated ring for some positive integer n.*

(2) *$D[X]$ is a 1-generated ring.*

(3) *D is a field.*

Remark 25. By Proposition 24, if D is an (n-generated) integral domain which is not a field, then the polynomial ring $D[X]$ is never an m-generated ring for any positive integer m. In particular, Theorem 22 always fails for any finite cardinal number α.

Remarks 26. (1) For rings with zero-divisors, the polynomial ring $R[X]$ may not be a 2-generated ring even if the ring R is a 1-generated ring. For example, let $R = V/(a^3)$, where V is a 1-dimensional discrete valuation domain (or equivalently, a local principal ideal domain) with maximal ideal $M = (a)$. Then R is a 1-generated ring. However, if $\overline{M}$ denotes the maximal ideal $M/(a^3)$ of R, then $\overline{M}^2 \neq 0$. By [Matsuda 1979, (5.7)], $R[X]$ is not a 2-generated ring.

(2) More generally, for any integer $n \geq 2$, let $R = V/(a^{n+1})$, where V is the same as above. Then R is a 1-generated ring. However, by [Matsuda 1979, (5.13)], the polynomial ring $R[X]$ is not an n-generated ring.

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