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XIAO-LEI LIU

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CURVES WITH PRESCRIBED INTERSECTION WITH BOUNDARY DIVISORS IN MODULI SPACES OF CURVES

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We construct curves in moduli spaces of curves with prescribed intersection with boundary divisors. As applications, we obtain families of curves with maximal slope as well as extremal test curves for the weakly positive cone of the moduli space.

1. Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g over the field of complex numbers $\mathbb{C}$, and $\Delta_0, \Delta_1, \ldots, \Delta_{[g/2]}$ the boundary divisors of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$. Denote by $\mathcal{H}_g$ the moduli space of smooth hyperelliptic curves of genus g; then the restriction of Δ_0 to the closure $\overline{\mathcal{H}}_g$ breaks up into $\Xi_0, \Xi_1, \ldots, \Xi_{[(g-1)/2]}$. The restriction of Δ_i to $\overline{\mathcal{H}}_g$ is often denoted by Θ_i for $i > 0$; see [Harris and Morrison 1998].

A family of curves of genus g over a curve Y is a fibration $f : X \to Y$ whose general fibers are smooth curves of genus g, where X is a smooth projective surface. Let $\omega_{X/Y}$ be the relative dualizing sheaf. If f is nontrivial, the *slope* of f is $\omega_{X/Y}^2 / \deg f_*\omega_{X/Y}$. Let $J_f : Y \to \overline{\mathcal{M}}_g$ denote the moduli map induced by f; see [Tan 2010].

Special curves in moduli spaces play an important rule in the study of birational geometry of moduli spaces: for example, the ample cone, the nef cone and the Mori cone of curves [Gibney 2009]. Before raising our problems, we firstly summarize some interesting properties of curves in moduli spaces with prescribed intersection with boundary divisors. In this paper, we always assume that curves in moduli spaces are complete irreducible and are not contained in boundary divisors.

If $C = J_f(Y) \subset \overline{\mathcal{H}}_g$ is a curve intersecting only Ξ_0 (resp. $\Delta_{[g/2]}$), then the semistable reduction of f has minimal (resp. maximal) slope; see [Liu 2016, Remark 3.9] and [Liu and Tan 2013, Theorem 3.1]. We refer to [Tan 2010] for related discussions. Moreover, if $C = J_f(Y) \subset \overline{\mathcal{M}}_g$ is disjoint from boundary divisors, then the semistable reduction of f is a Kodaira fibration; see [Kodaira 1967].

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On the other hand, curves intersecting exactly one boundary divisor of $\overline{\mathcal{H}}_g$ are the extremal test curves of the weakly positive cone in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. Here the weakly positive cone consists of weakly positive $\mathbb{Q}$-Cartier divisors over $\mathcal{M}_g$; see Section 1 in [Moriwaki 1998] for the definition of weakly positive divisors, and Section 4 of that paper for this result.

Motivated by these properties, we want to study further properties of curves that intersect given boundary divisors. In this paper we discuss the existence of such curves.

Let C be a curve in $\overline{\mathcal{H}}_g$, and $\mathcal{B} = \{\Theta_1, \dots, \Theta_{[g/2]}, \Xi_0, \Xi_1, \dots, \Xi_{[(g-1)/2]}\}$ be the set of boundary divisors of $\overline{\mathcal{H}}_g$. Denote by $\mathcal{B}_C \subset \mathcal{B}$ the set of boundary divisors of $\overline{\mathcal{H}}_g$ that intersect C.

Problem 1.1. *For any nonempty subset $\mathcal{B}' \subseteq \mathcal{B}$, does there exist a curve C in $\overline{\mathcal{H}}_g$ such that the boundary divisors of $\overline{\mathcal{H}}_g$ that intersect C are those in $\mathcal{B}'$, i.e., $\mathcal{B}_C = \mathcal{B}'$?*

Let $\widetilde{\mathcal{M}}_{0,n}$ be the moduli space of stable unordered n-pointed rational curves. Let B_k be the boundary divisor of $\widetilde{\mathcal{M}}_{0,n}$ whose general point parametrizes the union of a k-pointed $\mathbb{P}^1$ and an $(n-k)$-pointed $\mathbb{P}^1$ for $2 \le k \le [n/2]$.

One can regard $\overline{\mathcal{H}}_g$ as the Hurwitz space parametrizing genus g admissible double covers of rational curves. Such a cover uniquely corresponds to a stable $(2g+2)$-pointed rational curve by marking the branch points of the cover. Thus $\overline{\mathcal{H}}_g$ can be further identified as $\widetilde{\mathcal{M}}_{0,2g+2}$. The natural isomorphism $\overline{\mathcal{H}}_g \cong \widetilde{\mathcal{M}}_{0,2g+2}$ induces the identifications $\Xi_i = B_{2i+2}$ and $\Theta_i = B_{2i+1}$. Also denote by $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$ the set of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$. Hence the existence of curves in $\overline{\mathcal{H}}_g$ in Problem 1.1 is the same as that in $\widetilde{\mathcal{M}}_{0,n}$ for $n = 2g+2$. For the sake of completeness, we consider the existence of curves in $\widetilde{\mathcal{M}}_{0,n}$. Precisely, we consider the following problem.

Problem 1.2. *For any nonempty subset $\mathcal{B}' \subseteq \mathcal{B}$, does there exist a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that the boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$ that intersect C are those in $\mathcal{B}'$, i.e., such that $\mathcal{B}_C = \mathcal{B}'$?*

The purpose of this paper is to answer these problems in a number of cases. We have the following uniform solution for small n.

Theorem 1.3. *Assume $n \le 17$. For any nonempty subset $\mathcal{B}'$ of $\mathcal{B}$, there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Unfortunately, our method is invalid for $n = 18$ (see Remark 6.1).

Corollary 1.4. *For $2 \le g \le 7$ and any nonempty $\mathcal{B}' \subset \mathcal{B}$, there exists a curve C in $\overline{\mathcal{H}}_g$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Corollary 1.5. *For $2 \le g \le 7$ and any nonempty $\mathcal{B}' \subset \Delta = \{\Delta_0, \Delta_1, \dots, \Delta_{[g/2]}\}$, there exists a curve C in $\overline{\mathcal{M}}_g$ with $\mathcal{B}_C = \mathcal{B}'$.*

In order to simplify the proof of the above theorem, we use the following two partial solutions of Problem 1.2.

Theorem 1.6. *Let $\mathcal{B}'$ be a subset of the set $\mathcal{B} = \{B_2, \dots, B_{[n/2]}\}$ of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$.*

(1) *For $|\mathcal{B}'| = 1$, if $\mathcal{B}' = \{B_i\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(2) *For $|\mathcal{B}'| = 2$, if $\mathcal{B}' = \{B_i, B_{i+1}\}$ or $\mathcal{B}' = \{B_i, B_{i+2}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(3) *For $|\mathcal{B}'| = 3$, if $\mathcal{B}'$ is one of $\{B_i, B_j, B_{i+j+1}\}$, $\{B_i, B_j, B_{i+j}\}$, $\{B_i, B_j, B_{i+j-1}\}$, or $\{B_i, B_j, B_{i+j-2}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

In particular, we recover the existence of a hyperelliptic family with maximal slope from [Liu and Tan 2013], where the authors construct explicit polynomials to define the family. As another application, we also give an alternative proof of the existence of extremal test curves for the weakly positive cone from Appendix A in [Moriwaki 1998], where the author uses the existence of a concrete linear system. Here we reduce the problem to the existence of rational functions and then give a unitive method. This new method greatly generalizes the former results.

Theorem 1.7. *Let $\mathcal{B}'$ be a subset of the set $\mathcal{B} = \{B_2, \dots, B_{[n/2]}\}$ of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$.*

(1) *If $\mathcal{B}' = \{B_2, B_{i_1}, \dots, B_{i_k}\}$ and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(2) *If $\mathcal{B}' = \{B_2, B_{k+1}, B_{k+2}, \dots, B_{[n/2]}\}$ and $2 \le k \le [n/2]$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(3) *If $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 3$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Now we explain the main idea of the proofs. In order to construct curves in moduli spaces intersecting the given boundary divisors, we use the following three different methods:

(i) We regard a rational function ϕ of degree n as a 1-dimensional family of unordered n marked points in $\mathbb{P}^1$ in Section 3. Only the critical points of ϕ may correspond to points in boundary divisors; see the correspondence (3-1). Then we just need to construct rational functions with the desired critical points. The existence of such rational functions is from the main Theorem of [Scherbak 2002].

(ii) The graph G_ϕ of ϕ is a smooth rational curve in $\mathbb{P}^1 \times E$, where $E \cong \mathbb{P}^1$. Let p be the second projection and R the reducible curve consisting of G_ϕ and certain sections of p. Then the restriction $p|_R$ is a branched cover over E, and this cover induces different 1-dimensional families of marked points in $\mathbb{P}^1$. The correspondence (4-1) gives a relation between points in boundary divisors with ramification points of $p|_R$, i.e., between singular points of R and ramification points of $p|_{G_\phi}$. So we only need to construct curves R with suitable ramification points in Section 4. This method generalizes method (i), and both methods are effective for many cases.

On the other hand, if n is large, $\mathcal{B}'$ may have so many elements that there is no desired rational function, rendering the above two methods invalid; see Remark 6.1. For this difficulty, we have the following method for special $\mathcal{B}'$.

(iii) Taking the intersection of general very ample divisors of Hassett's weighted moduli spaces, we can obtain curves intersecting all the boundary divisors of these spaces. The proper transforms of these curves in $\widetilde{\mathcal{M}}_{0,n}$ by the reduction morphism are also the curves we need; see Section 5.

2. Existence of rational functions

Let $\phi(x) = f(x)/g(x)$ be a *rational function of degree n*; i.e., $f(x)$ and $g(x)$ are polynomials without common roots and $n = \max\{\deg f(x), \deg g(x)\}$. A point $z \in \mathbb{C}$ is a *critical point of multiplicity m* if z is a root of the Wronskian $W(x)$ of $f(x)$ and $g(x)$ of multiplicity m, where

$$W(x) = g^2(x)\phi'(x) = f'(x)g(x) - f(x)g'(x).$$

Let all the finite critical points of $\phi(x)$ be denoted $z_1, \dots, z_{l-1}$ with multiplicities $m_1, \dots, m_{l-1}$, respectively. According to the Riemann–Hurwitz formula, $2n-2 \geq m_1 + \cdots + m_{l-1}$. Let z_l be the point at infinity; then the difference

$$m_l = 2n - 2 - (m_1 + \cdots + m_{l-1}) \tag{2-1}$$

is *the multiplicity of $\phi(x)$ at infinity*. If $z_1, \dots, z_{l-1}$ are in general position and $1 \leq m_i \leq n-1$ for each $1 \leq i \leq l$, then we say that $\phi(x)$ is a rational function *of type* $(n; m_1, \dots, m_l)$.

Note that, up to the point at infinity, the types of rational functions here are the same as in [Scherbak 2002].

If $\phi_1(x)$ and $\phi_2(x)$ are two rational functions satisfying

$$\phi_2(x) = \frac{a\phi_1(x) + b}{c\phi_1(x) + d}, \quad ad - bc \neq 0,$$

then $\phi_1(x)$ and $\phi_2(x)$ have the same type, and we say that $\phi_1(x)$ and $\phi_2(x)$ are in the same class of rational functions.

Theorem 2.1 [Scherbak 2002]. *Let $l \geq 3$ and $n \geq 2$ be integers, and also let $1 \leq m_i \leq n-1$ for $1 \leq i \leq l$ be integers satisfying (2-1). Then the number $\#(n; m_1, \dots, m_l)$ of classes of rational functions of type $(n; m_1, \dots, m_l)$ is*

$$\#(n; m_1, \dots, m_l) = \sum_{q=1}^{l-1} (-1)^{l-1-q} \sum_{1 \leq i_1 < \cdots < i_q \leq l-1} \binom{m_{i_1} + \cdots + m_{i_q} + q - n - 1}{l-3}, \tag{2-2}$$

and any nonempty class can be represented by the ratio of polynomials without multiple roots.

Proof. Let $\boldsymbol{m} = (m_1, \dots, m_{l-1})$; then $\#(n; m_1, \dots, m_l)$ is equal to $\#(n, l-1; \boldsymbol{m})$ in the main theorem in [Scherbak 2002], from which the result directly follows. □

As usual, we set $\binom{a}{b} = 0$ for $a < b$.

Corollary 2.2.

(1) *There exists a rational function $\phi(x)$ of type $(n; m_1, \dots, m_{l-1}, m_l = n-1)$.*

(2) *There exists a rational function $\phi(x)$ of type $(n; m_1, m_2, m_3)$.*

Proof. (1) Since $m_l = n-1$, we have $m_1 + \cdots + m_{l-1} = n-1$, and the right side of (2-2) is

$$\binom{m_1 + \cdots + m_{l-1} + l - 1 - n - 1}{l-3} = \binom{l-3}{l-3} = 1.$$

So the desired rational function exists by Theorem 2.1.

(2) If $m_i = n-1$ for some i, then the existence is from (1). We may assume that $m_i \leq n-2$ for $i = 1, 2, 3$. Then the right side of (2-2) is

$$\binom{m_1 + m_2 + 2 - n - 1}{0} - \binom{m_1 - n}{0} - \binom{m_2 - n}{0} = \binom{m_1 + m_2 - n + 1}{0} = 1. \quad □$$

3. Curves from rational functions

Let $\phi(x)$ be a rational function of degree n; then it induces a degree n branched cover $D \cong \mathbb{P}^1 \to E \cong \mathbb{P}^1$. Varying a point $t \in E$, the union of the n preimage points also varies in D; hence it provides a 1-dimensional family E of unordered n points in $\mathbb{P}^1$. So we obtain a curve in $\widetilde{\mathcal{M}}_{0,n}$. When t hits a branch point, suppose that over t there is a ramification point z_i locally of type $y = \phi(x) = x^{m_i+1}$, i.e., a critical point of $\phi(x)$ with multiplicity m_i. Making a degree $m_i + 1$ base change, we then get an ordinary singularity of degree $m_i + 1$. Blowing up the singularity separates the $m_i + 1$ sheets, and thus t corresponds to a point in the boundary component B_{m_i+1} (or B_{n-1-m_i} if $m_i \geq [n/2]$, but not $m_i = n-1$ or $n-2$, which correspond to a point in the interior of $\widetilde{\mathcal{M}}_{0,n}$). Hence, if $\phi(x)$ is of type $(n; m_1, \dots, m_l)$ and C is its corresponding curve in $\widetilde{\mathcal{M}}_{0,n}$, then

$$\mathcal{B}_C = \{B_{m_i+1} : 1 \leq m_i < [n/2]\} \cup \{B_{n-1-m_i} : [n/2] \leq m_i \leq n-3\}. \tag{3-1}$$

Theorem 3.1. *Let $\mathcal{B}' = \{B_{i_1}, \dots, B_{i_k}\}$ be a subset of $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$ with $(i_1 - 1) + \cdots + (i_k - 1) = n - 1$; then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$. Moreover,*

(1) *if $\mathcal{B}' = \{B_2, B_{i_1}, \dots, B_{i_k}\}$, and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$;*

(2) *if $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$, and $n - 1 - (i_1 - 1) - \cdots - (i_k - 1) = 2j \ge 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $(i_1 - 1) + \cdots + (i_k - 1) = n - 1$, set $i_{k+1} = n$; then there exists a rational function $\phi(x)$ of type $(n; i_1 - 1, \dots, i_k - 1, i_{k+1} - 1)$ by Corollary 2.2(1). Thus the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi(x)$ satisfies $\mathcal{B}_C = \{B_{i_1}, \dots, B_{i_k}\}$ by (3-1).

(1) Let $h = n - 1 - ((i_1 - 1) + \cdots + (i_k - 1)) \ge 1$ and $i_{k+1} = \cdots = i_{k+h} = 2$; then $(i_1 - 1) + \cdots + (i_{k+h} - 1) = n - 1$, and hence there exists a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \{B_2, B_{i_1}, \dots, B_{i_k}\}$.

(2) If we take $i_{k+1} = \cdots = i_{k+j} = 3$, then $(i_1 - 1) + \cdots + (i_{k+j} - 1) = n - 1$, and there exists a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \{B_3, B_{i_1}, \dots, B_{i_k}\}$. □

Theorem 3.2. *Let $\mathcal{B}'$ be a subset of $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$. If $\mathcal{B}'$ is one of $\{B_i, B_{i+1}\}$, $\{B_i, B_{i+2}\}$, or $\{B_i, B_j, B_{i+j+1}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $\mathcal{B}' = \{B_i, B_{i+1}\}$, then there exists a rational function $\phi_1(x)$ of type $(n; i, n-1-i, n-1)$ by Corollary 2.2(2). Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_1(x)$ satisfies $\mathcal{B}_C = \{B_i, B_{i+1}\}$ by (3-1).

By the same reasoning, for $\mathcal{B}' = \{B_i, B_{i+2}\}$, there exists a rational function $\phi_2(x)$ of type $(n; i+1, n-1-i, n-2)$, and the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_2(x)$ satisfies $\mathcal{B}_C = \{B_i, B_{i+2}\}$.

Similarly, for $\mathcal{B}' = \{B_i, B_j, B_{i+j+1}\}$, there exists a rational function $\phi_3(x)$ of type $(n; n-1-i, n-1-j, i+j)$, and the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_3(x)$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j+1}\}$. □

4. Curves from rational functions and sections

Let $\phi(x)$ be a rational function of type $(n - s; m_1, \dots, m_l)$, where $s < n$ is a nonnegative integer. Let G_ϕ be its graph in $\mathbb{P}^1 \times E$, where $E \cong \mathbb{P}^1$; then G_ϕ is a smooth rational curve. We now consider the reducible curve

$$R_{\phi,\Gamma_1,\dots,\Gamma_s} = G_\phi + \Gamma_1 + \cdots + \Gamma_s,$$

where the Γ_i are sections of the second projection

$$p : \mathbb{P}^1 \times E \to E, \quad p((x, t)) = t.$$

So the restriction of p to $R_{\phi,\Gamma_1,\dots,\Gamma_s}$

$$p : R_{\phi,\Gamma_1,\dots,\Gamma_s} \to E$$

is a cover of degree n. Similarly as in Section 3, varying a point $t \in E$, we obtain a 1-dimensional family of unordered n marked points in $\mathbb{P}^1$. Hence we get a curve C in $\widetilde{\mathcal{M}}_{0,n}$. Note that this construction is the same as that in Section 3 when $s = 0$.

Let $S_1 = (\Gamma_1 + \cdots + \Gamma_s) \cap G_\phi$, and let $S_2 = \{z_1, \dots, z_l\}$ be the set of all the critical points of $\phi(x)$, including the point at infinity. Here we identify the critical point z_i of ϕ with its image $(z_i, \phi(z_i))$ in G_ϕ.

If $z \in S_2 \cap S_1$, then the local equation of p at z is $x(x^{m+1} + t) = 0$, where m is the multiplicity of z. Making a degree $m + 1$ base change

$$t \mapsto u^{m+1},$$

we get an ordinary singularity of degree $m + 2$. Blowing it up, we then see that $p(z) \in E$ corresponds to a point in the boundary component B_{m+2} (or B_{n-2-m} if $[n/2] - 1 \le m \le n - 4$, but not if $m = n - 3$ or $n - 2$, corresponding to a point in the interior of $\widetilde{\mathcal{M}}_{0,n}$).

If $z \in S_2 - S_1$, then the point in $\widetilde{\mathcal{M}}_{0,n}$ corresponding to $p(z)$ has been considered in Section 3. Hence we have that

$$\begin{aligned}(4\text{-}1)\quad \mathcal{B}_C = \{B_2 : z \in S_1 \backslash S_2\} &\cup \left\{B_{m_i+1} : 1 \le m_i < \left[\tfrac{1}{2}n\right], z_i \in S_2 \backslash S_1\right\} \\ &\cup \left\{B_{n-1-m_i} : \left[\tfrac{1}{2}n\right] \le m_i \le n-3, z_i \in S_2 \backslash S_1\right\} \\ &\cup \left\{B_{m_i+2} : 1 \le m_i < \left[\tfrac{1}{2}n\right] - 1, z_i \in S_2 \cap S_1\right\} \\ &\cup \left\{B_{n-2-m_i} : \left[\tfrac{1}{2}n\right] - 1 \le m_i \le n-4, z_i \in S_2 \cap S_1\right\}.\end{aligned}$$

Since we have considered the case that $\mathcal{B}'$ contains B_2 in Theorem 3.1(1), we now consider the case where each section passes through a critical point of $\phi(x)$, that is, where $S_1 \subset S_2$.

Theorem 4.1. *Let $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ be a subset of $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$ with $n - 1 - ((i_1 - 1) + \cdots + (i_k - 1)) = 2j + 1 \ge 3$; then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. Set $i_{k+1} = \cdots = i_{k+j} = 3$, then

$$(i_1 - 1) + \cdots + (i_k - 1) + (3 - 1)j + ((n - 1) - 1) = 2((n - 1) - 1).$$

Hence there exists a rational function $\phi(x)$ of type $(n-1; i_1-1, \dots, i_{k+j}-1, n-2)$ by Corollary 2.2(1). Let Γ be the section passing through the critical point z_{k+j+1}, where the multiplicity of z_{k+j+1} of $\phi(x)$ is $n - 2$. Thus the reducible curve $R_{\phi,\Gamma} = G_\phi + \Gamma$ induces a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ by (4-1). □

Corollary 4.2. *If $\mathcal{B}' = \{B_3, B_{i_1}, \ldots, B_{i_k}\}$, and $n-3-((i_1-1)+\cdots+(i_k-1)) \geq 0$, then there exists a curve C such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $n-3-((i_1-1)+\cdots+(i_k-1))$ is even, then it follows directly from Theorem 3.1(2); if $n-3-((i_1-1)+\cdots+(i_k-1))$ is odd, then it is from Theorem 4.1. □

Theorem 4.3. *Let $\mathcal{B}'$ be a subset of $\mathcal{B} = \{B_2, B_3, \ldots, B_{[n/2]}\}$. If $\mathcal{B}'$ is one of $\{B_i\}$, $\{B_i, B_j, B_{i+j-2}\}$, $\{B_i, B_j, B_{i+j}\}$, or $\{B_i, B_j, B_{i+j-1}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. Let $\phi_1(x)$ be a rational function of type $(n-1; i-1, n-i-1, n-2)$ and Γ_1 the section passing through the critical point z_3 of $\phi_1(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by R_{ϕ_1,Γ_1} satisfies $\mathcal{B}_C = \{B_i\}$ by (4-1).

Let $\phi_2(x)$ be a rational function of type $(n-3; n-i-2, n-j-2, i+j-4)$ and Γ_{2i} the section passing through the critical point z_i $(i = 1, 2, 3)$ of $\phi_2(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by $R_{\phi_2,\Gamma_{21},\Gamma_{22},\Gamma_{23}}$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j-2}\}$ by (4-1).

Let $\phi_3(x)$ be a rational function of type $(n-2; n-2-i, n-2-j, i+j-2)$ and Γ_{3i} the section passing through the critical point z_i $(i = 1, 2)$ of $\phi_3(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by $R_{\phi_3,\Gamma_{31},\Gamma_{32}}$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j-1}\}$ by (4-1).

Let $\phi_4(x)$ be a rational function of type $(n-1; n-1-i, n-1-j, i+j-2)$ and Γ_4 the section passing through the critical point z_3 of $\phi_4(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by R_{ϕ_4,Γ_4} satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j}\}$ by (4-1). □

Proof of Theorem 1.6. It follows directly from Theorem 3.2 and Theorem 4.3. □

5. Curves from birational geometry of moduli spaces

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of stable n-pointed rational curves. The space $\overline{\mathcal{M}}_{0,n}$ has a natural S_n action by reordering the marked points. Let $\rho : \overline{\mathcal{M}}_{0,n} \to \widetilde{\mathcal{M}}_{0,n}$ be the finite quotient morphism via S_n.

A *weight datum* $\mathcal{A} = (a_1, \ldots, a_n)$ is a sequence of rational numbers such that $0 < a_i \leq 1$. Given a weight datum $\mathcal{A}$ satisfying $2g-2+\sum_{i=1}^n a_i > 0$, an n-pointed curve $(C; p_1, \ldots, p_n)$ of genus g is *$\mathcal{A}$-stable* if

(1) C has, at worst, ordinary double points as singularities, and $p_1, \ldots, p_n$ are smooth points of C;

(2) $\omega_C\left(\sum_{i=1}^n a_i p_i\right)$ is ample;

(3) $\mathrm{mult}_x\left(\sum_{i=1}^n a_i p_i\right) \leq 1$ for any $x \in C$.

For any weight datum $\mathcal{A}$ such that $2g-2+\sum_{i=1}^n a_i > 0$, there exists a projective coarse moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$ [Hassett 2003, Theorem 2.1] of weighted n-pointed $\mathcal{A}$-stable curves. Note that $\overline{\mathcal{M}}_{g,\mathcal{A}} = \overline{\mathcal{M}}_{g,n}$ when $a_1 = \cdots = a_n = 1$.

Let $\mathscr{A}_1 = (a_1, \dots, a_n)$ and $\mathscr{A}_2 = (b_1, \dots, b_n)$ be two weight data and suppose that $a_i \geq b_i$ for all $i = 1, 2, \dots, n$. Then there exists a *reduction morphism* [Hassett 2003, Theorem 4.1]

$$\varphi_{\mathscr{A}_1,\mathscr{A}_2} : \overline{\mathscr{M}}_{0,\mathscr{A}_1} \to \overline{\mathscr{M}}_{0,\mathscr{A}_2}.$$

If $(C, p_1, \dots, p_n) \in \overline{\mathscr{M}}_{0,\mathscr{A}_1}$, then $\varphi_{\mathscr{A}_1,\mathscr{A}_2}(C, p_1, \dots, p_n)$ is obtained by collapsing components of C on which $\omega_C + \sum b_i p_i$ fails to be ample.

Theorem 5.1. *Let $\mathscr{B}' = \{B_2, B_{k+1}, B_{k+2}, \dots, B_{[n/2]}\}$ for $2 \leq k \leq [n/2]$; then there exists a curve $C \subset \widetilde{\mathscr{M}}_{0,n}$ such that $\mathscr{B}_C = \mathscr{B}'$.*

Proof. If $k = [n/2]$, then $\mathscr{B}' = \{B_2\}$, and existence was proved in Theorem 3.1(1).

If $k = 2$, then $\mathscr{B}' = \mathscr{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$. Taking the intersection of $n - 4$ general very ample divisors of $\widetilde{\mathscr{M}}_{0,n}$, we obtain a curve C in $\widetilde{\mathscr{M}}_{0,n}$ that intersects all the boundary divisors; that is, $\mathscr{B}_C = \mathscr{B}$.

In the following, we assume that $3 \leq k \leq [n/2] - 1$. Let $\mathscr{A}(k) = \{1/k, \dots, 1/k\}$ be the symmetric weight datum that assigns $1/k$ to each marked point. Then $\sum_{i=1}^n a_i = n \cdot (1/k) > 2$. Let $\widetilde{\mathscr{M}}_{0,\mathscr{A}(k)}$ be the quotient of the weighted moduli space $\overline{\mathscr{M}}_{0,\mathscr{A}(k)}$ via the natural S_n action. Denote by $f_k : \widetilde{\mathscr{M}}_{0,n} \to \widetilde{\mathscr{M}}_{0,\mathscr{A}(k)}$ the corresponding reduction morphism. Since three points on $\mathbb{P}^1$ (the two marked points and the attaching node) have no nontrivial moduli, f_k does not contract B_2. The degree of the total weight on a rational tail with j $(3 \leq j \leq k)$ marked points is $(1/k) \cdot j + 1 \leq 2 = -\deg K_{\mathbb{P}^1}$, where the $+1$ comes from the attaching node. Hence it violates the stability condition. So f_k contracts the boundary divisors B_j for $3 \leq j \leq k$. Furthermore, we see that f_k contracts only these boundary divisors. Now taking the intersection of $n - 4$ general very ample divisors, we obtain a curve C in $\widetilde{\mathscr{M}}_{0,\mathscr{A}(k)}$ that is disjoint with $f_k(B_j)$ for $3 \leq j \leq k$ and that intersects all the other (not contracted) boundary divisors $B_2, B_{k+1}, \dots, B_{[n/2]}$ in the loci where f_k is an isomorphism. Then the proper transform C' of C in $\overline{\mathscr{M}}_{0,n}$ satisfies $\mathscr{B}_{C'} = \mathscr{B}' = \{B_2, B_{k+1}, \dots, B_{[n/2]}\}$. □

Proof of Theorem 1.7. It follows from Theorems 3.1 and 5.1 and Corollary 4.2. □

6. Proof of Theorem 1.3

Let us first introduce some notation. Below, $\{i_1, \dots, i_k\}$ stands for $\{B_{i_1}, \dots, B_{i_k}\}$. If $\mathscr{B}' = \{i_1, \dots, i_k\}$ is one of the sets in Theorem 1.6, we call $\mathscr{B}'$ *of type* $T_{i_1,\dots,i_k}$. For example, if $\mathscr{B}' = \{2, 3, 5\}$, we call $\mathscr{B}'$ of type $T_{2,3,5}$. If $\mathscr{B}' = \{2, i_1, \dots, i_k\}$ is one of the sets in Theorem 1.7(1)–(2), we call $\mathscr{B}'$ *of type* $T_{2,*}$. If $\mathscr{B}' = \{3, i_1, \dots, i_k\}$ is one of the sets in Theorem 1.7(3), we call $\mathscr{B}'$ *of type* $T_{3,*}$.

Proof of Theorem 1.3. For $n \leq 11$, it is easy to check that the result follows directly from Theorem 1.6 and Theorem 1.7. In the following we always assume that $12 \leq n \leq 17$. Then we have the following three cases.

Case 1. If $\mathscr{B}'$ is one of the sets in Theorems 1.6 and 1.7, we are done.

Case 2. If $n = 14$ and $\mathscr{B}' = \{4, 7\}$, then there exists a rational function $\phi_1(x)$ of type (13; 3, 3, 6, 12) by Corollary 2.2(1). Denote by Γ_1 the section passing through the critical point z_4 of $\phi_1(x)$, where the multiplicity of z_4 is 12. Then the curve C_1 in $\widetilde{\mathscr{M}}_{0,14}$ corresponding to R_{ϕ_1,Γ_1} satisfies $\mathscr{B}_{C_1} = \{4, 7\}$ by (4-1).

If $n = 17$ and $\mathscr{B}' = \{4, 7\}$, then there exists a rational function $\phi_2(x)$ of type (16; 3, 6, 6, 15) by Corollary 2.2(1). Denote by $\Gamma_2(x)$ the section passing through the critical point z_4 of $\phi_2(x)$, where the multiplicity of z_4 is 15. Then the curve C_2 in $\widetilde{\mathscr{M}}_{0,17}$ corresponding to R_{ϕ_2,Γ_2} satisfies $\mathscr{B}_{C_2} = \{4, 7\}$ by (4-1).

Case 3. We discuss the remaining $\mathscr{B}'$ case by case using the method in Section 3.

Let $\mathscr{B}' = \{i_1, \dots, i_k\}$ be a set of boundary divisors of $\widetilde{\mathscr{M}}_{0,n}$. If there is a sequence $(m_1, \dots, m_l)$ satisfying

(i) $m_1 + \cdots + m_l = 2n - 2$,

(ii) for any $1 \le t \le l$, $m_t \in \{i_1 - 1, \dots, i_k - 1, n-1-i_1, \dots, n-1-i_k, n-1, n-2\}$,

(iii) for any $1 \le j \le k$, there is $1 \le t_j \le l$ such that $i_j = m_{t_j} + 1$ or $n - 1 - m_{t_j}$,

(iv) the right side of (2-2) is positive,

then there exists a rational function of type $(n; m_1, \dots, m_l)$ by Theorem 2.1. Thus we get a curve C in $\widetilde{\mathscr{M}}_{0,n}$ such that $\mathscr{B}_C = \mathscr{B}'$ by (3-1). So we finish the proof of the theorem by giving a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) for each n and $\mathscr{B}'$; see Table 1–Table 10.

We now show the meaning of these tables. If $\mathscr{B}'$ is in Case 1, we give its type in the tables. If $\mathscr{B}'$ is in Case 3, we give a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) in the tables. For example, if $\mathscr{B}' = \{2, 3\}$, then $(m_1, \dots, m_l)$ is $T_{2,*}$ in Table 1; by

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{3, 4}	$T_{3,*}$	{3, 5}	$T_{3,*}$
{3, 6}	$T_{3,*}$	{4, 5}	$T_{4,5}$	{4, 6}	$T_{4,6}$
{5, 6}	$T_{5,6}$				
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 4, 5}	$T_{2,*}$	{2, 4, 6}	$T_{2,*}$	{2, 5, 6}	$T_{2,*}$
{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$	{3, 5, 6}	$T_{3,5,6}$
{4, 5, 6}	(4, 5, 6, 7)				
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 5, 6}	(2, 5, 6, 9)
{2, 4, 5, 6}	$T_{2,*}$	{3, 4, 5, 6}	(3, 5, 6, 8)		

Table 1. $\widetilde{\mathscr{M}}_{0,12}$.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
$\{2, 3\}$	$T_{2,*}$	$\{2, 4\}$	$T_{2,*}$	$\{2, 5\}$	$T_{2,*}$
$\{2, 6\}$	$T_{2,*}$	$\{3, 4\}$	$T_{3,*}$	$\{3, 5\}$	$T_{3,*}$
$\{3, 6\}$	$T_{3,*}$	$\{4, 5\}$	$T_{4,5}$	$\{4, 6\}$	$T_{4,6}$
$\{5, 6\}$	$T_{5,6}$				
$\{2, 3, 4\}$	$T_{2,*}$	$\{2, 3, 5\}$	$T_{2,*}$	$\{2, 3, 6\}$	$T_{2,*}$
$\{2, 4, 5\}$	$T_{2,*}$	$\{2, 4, 6\}$	$T_{2,*}$	$\{2, 5, 6\}$	$T_{2,*}$
$\{3, 4, 5\}$	$T_{3,4,5}$	$\{3, 4, 6\}$	$T_{3,4,6}$	$\{3, 5, 6\}$	$T_{3,5,6}$
$\{4, 5, 6\}$	$(3, 6, 7, 8)$				
$\{2, 3, 4, 5\}$	$T_{2,*}$	$\{2, 3, 4, 6\}$	$T_{2,*}$	$\{2, 3, 5, 6\}$	$T_{2,*}$
$\{2, 4, 5, 6\}$	$T_{2,*}$	$\{3, 4, 5, 6\}$	$(2, 3, 4, 6, 9)$		

Table 2. $\widetilde{\mathcal{M}}_{0,13}$.

this we mean that there is a curve C in $\widetilde{\mathcal{M}}_{0,12}$ of type $T_{2,*}$ with $\mathcal{B}_C = \{B_2, B_3\}$. If $\mathcal{B}' = \{4, 5, 6\}$, then $(m_1, \dots, m_l) = (4, 5, 6, 7)$ in Table 1; by this we mean that there exists a curve C in $\widetilde{\mathcal{M}}_{0,12}$ with $\mathcal{B}_C = \{4, 5, 6\}$, where C is induced by a rational function of type $(12; 4, 5, 6, 7)$.

Note that only the cases of $\mathcal{B}'$ with $|\mathcal{B}'| = 1$ and $|\mathcal{B}'| = [n/2] - 1$ are not contained in these tables, since the theorem for such $\mathcal{B}'$ holds true for any n; see Theorem 4.3(1) and Theorem 5.1 for $k = 2$. □

Remark 6.1. Assume that $n = 18$ and $\mathcal{B}' = \{B_3, B_4, \dots, B_9\}$. Suppose that there exists a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) for $\mathcal{B}'$. From (ii) and (iii), we know that

$$m_1 + \cdots + m_l \geq \sum_{i=3}^{9} (i - 1) = 35 > 2n - 2 = 34,$$

which contradicts (i). Hence the method in Section 3 is invalid for $\widetilde{\mathcal{M}}_{0,18}$. Similarly, we know that the method in Section 4 is also invalid.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
$\{2, 3\}$	$T_{2,*}$	$\{2, 4\}$	$T_{2,*}$	$\{2, 5\}$	$T_{2,*}$
$\{2, 6\}$	$T_{2,*}$	$\{2, 7\}$	$T_{2,*}$	$\{3, 4\}$	$T_{3,*}$
$\{3, 5\}$	$T_{3,*}$	$\{3, 6\}$	$T_{3,*}$	$\{3, 7\}$	$T_{3,*}$
$\{4, 5\}$	$T_{4,5}$	$\{4, 6\}$	$T_{4,6}$	$\{4, 7\}$	see Case 2
$\{5, 6\}$	$T_{5,6}$	$\{5, 7\}$	$T_{5,7}$	$\{6, 7\}$	$T_{6,7}$

Table 3. $\widetilde{\mathcal{M}}_{0,14}$ for $|\mathcal{B}'| = 2$.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 3, 7}	$T_{2,*}$	{2, 4, 5}	$T_{2,*}$	{2, 4, 6}	$T_{2,*}$
{2, 4, 7}	$T_{2,*}$	{2, 5, 6}	$T_{2,*}$	{2, 5, 7}	$T_{2,*}$
{2, 6, 7}	$T_{2,*}$	{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$
{3, 4, 7}	$T_{3,4,7}$	{3, 5, 6}	$T_{3,5,6}$	{3, 5, 7}	$T_{3,5,7}$
{3, 6, 7}	$T_{3,6,7}$	{4, 5, 6}	(4, 5, 8, 9)	{4, 5, 7}	$T_{4,5,7}$
{4, 6, 7}	(3, 3, 5, 6, 9)	{5, 6, 7}	(5, 6, 7, 8)		
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$	{2, 3, 6, 7}	(1, 2, 6, 7, 10)
{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	(1, 4, 6, 6, 9)	{2, 4, 6, 7}	(1, 5, 5, 6, 9)
{2, 5, 6, 7}	$T_{2,*}$	{3, 4, 5, 6}	(3, 5, 8, 10)	{3, 4, 5, 7}	(3, 3, 4, 6, 10)
{3, 4, 6, 7}	(3, 6, 7, 10)	{3, 5, 6, 7}	(2, 5, 5, 6, 8)	{4, 5, 6, 7}	(4, 6, 7, 9)

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4, 5, 6}	(1, 2, 3, 4, 5, 11)	{2, 3, 4, 5, 7}	(1, 2, 3, 4, 6, 10)
{2, 3, 4, 6, 7}	(1, 2, 3, 5, 6, 9)	{2, 3, 5, 6, 7}	(1, 2, 4, 5, 6, 8)
{2, 4, 5, 6, 7}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 3, 4, 5, 6, 6)

Table 4. $\widetilde{\mathcal{M}}_{0,14}$ for $3 \le |\mathcal{B}'| \le 5$.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{2, 7}	$T_{2,*}$	{3, 4}	$T_{3,*}$
{3, 5}	$T_{3,*}$	{3, 6}	$T_{3,*}$	{3, 7}	$T_{3,*}$
{4, 5}	$T_{4,5}$	{4, 6}	$T_{4,6}$	{4, 7}	(6, 6, 6, 10)
{5, 6}	$T_{5,6}$	{5, 7}	$T_{5,7}$	{6, 7}	$T_{6,7}$
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 3, 7}	$T_{2,*}$	{2, 4, 5}	$T_{2,*}$	{2, 4, 6}	$T_{2,*}$
{2, 4, 7}	$T_{2,*}$	{2, 5, 6}	$T_{2,*}$	{2, 5, 7}	$T_{2,*}$
{2, 6, 7}	$T_{2,*}$	{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$
{3, 4, 7}	$T_{3,4,7}$	{3, 5, 6}	$T_{3,5,6}$	{3, 5, 7}	$T_{3,5,7}$
{3, 6, 7}	$T_{3,6,7}$	{4, 5, 6}	(4, 5, 9, 10)	{4, 5, 7}	$T_{4,5,7}$
{4, 6, 7}	(3, 7, 8, 10)	{5, 6, 7}	(4, 7, 8, 9)		
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$	{2, 3, 6, 7}	$T_{2,*}$
{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	$T_{2,*}$	{2, 4, 6, 7}	(1, 3, 5, 7, 12)
{2, 5, 6, 7}	$T_{2,*}$	{3, 4, 5, 6}	$T_{3,*}$	{3, 4, 5, 7}	(2, 7, 9, 10)
{3, 4, 6, 7}	(3, 6, 8, 11)	{3, 5, 6, 7}	(2, 4, 5, 6, 11)	{4, 5, 6, 7}	(3, 4, 5, 6, 10)

Table 5. $\widetilde{\mathcal{M}}_{0,15}$ for $2 \le |\mathcal{B}'| \le 4$.

$\mathscr{B}'$	$(m_1,\dots,m_l)$	$\mathscr{B}'$	$(m_1,\dots,m_l)$
{2, 3, 4, 5, 6}	(1, 2, 3, 4, 8, 10)	{2, 3, 4, 5, 7}	(2, 3, 4, 7, 12)
{2, 3, 4, 6, 7}	(2, 3, 5, 6, 12)	{2, 3, 5, 6, 7}	(1, 4, 5, 7, 11)
{2, 4, 5, 6, 7}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 4, 5, 7, 10)

Table 6. $\widetilde{\mathcal{M}}_{0,15}$ for $|\mathscr{B}'| = 5$.

$\mathscr{B}'$	$(m_1,\dots,m_l)$	$\mathscr{B}'$	$(m_1,\dots,m_l)$	$\mathscr{B}'$	$(m_1,\dots,m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{2, 7}	$T_{2,*}$	{2, 8}	$T_{2,*}$
{3, 4}	$T_{3,*}$	{3, 5}	$T_{3,*}$	{3, 6}	$T_{3,*}$
{3, 7}	$T_{3,*}$	{3, 8}	$T_{3,*}$	{4, 5}	$T_{4,5}$
{4, 6}	$T_{4,6}$	{4, 7}	(8, 11, 11)	{4, 8}	(3, 3, 3, 7, 7, 7)
{5, 6}	$T_{5,6}$	{5, 7}	$T_{5,7}$	{5, 8}	(4, 4, 4, 4, 7, 7)
{6, 7}	$T_{6,7}$	{6, 8}	$T_{6,8}$	{7, 8}	$T_{7,8}$
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 3, 7}	$T_{2,*}$	{2, 3, 8}	$T_{2,*}$	{2, 4, 5}	$T_{2,*}$
{2, 4, 6}	$T_{2,*}$	{2, 4, 7}	$T_{2,*}$	{2, 4, 8}	$T_{2,*}$
{2, 5, 6}	$T_{2,*}$	{2, 5, 7}	$T_{2,*}$	{2, 5, 8}	$T_{2,*}$
{2, 6, 7}	$T_{2,*}$	{2, 6, 8}	$T_{2,*}$	{2, 7, 8}	$T_{2,*}$
{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$	{3, 4, 7}	$T_{3,4,7}$
{3, 4, 8}	$T_{3,4,8}$	{3, 5, 6}	$T_{3,5,6}$	{3, 5, 7}	$T_{3,5,7}$
{3, 5, 8}	$T_{3,5,8}$	{3, 6, 7}	$T_{3,6,7}$	{3, 6, 8}	$T_{3,6,8}$
{3, 7, 8}	$T_{3,7,8}$	{4, 5, 6}	(9, 10, 11)	{4, 5, 7}	$T_{4,5,7}$
{4, 5, 8}	$T_{4,5,8}$	{4, 6, 7}	(5, 6, 8, 11)	{4, 6, 8}	$T_{4,6,8}$
{4, 7, 8}	(6, 6, 7, 11)	{5, 6, 7}	(5, 6, 9, 10)	{5, 6, 8}	(4, 7, 9, 10)
{5, 7, 8}	(6, 7, 7, 10)	{6, 7, 8}	(6, 7, 8, 9)		
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 4, 8}	$T_{2,*}$	{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$
{2, 3, 5, 8}	$T_{2,*}$	{2, 3, 6, 7}	$T_{2,*}$	{2, 3, 6, 8}	$T_{2,*}$
{2, 3, 7, 8}	(2, 7, 8, 13)	{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	$T_{2,*}$
{2, 4, 5, 8}	$T_{2,*}$	{2, 4, 6, 7}	$T_{2,*}$	{2, 4, 6, 8}	(1, 3, 5, 5, 7, 9)
{2, 4, 7, 8}	(1, 3, 6, 7, 13)	{2, 5, 6, 7}	(1, 4, 6, 9, 10)	{2, 5, 6, 8}	(1, 4, 5, 7, 13)
{2, 5, 7, 8}	(4, 6, 7, 13)	{2, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6}	(2, 3, 4, 9, 12)
{3, 4, 5, 7}	(2, 2, 3, 4, 8, 11)	{3, 4, 5, 8}	(2, 7, 10, 11)	{3, 4, 6, 7}	(2, 8, 9, 11)
{3, 4, 6, 8}	(3, 3, 5, 7, 12)	{3, 4, 7, 8}	(2, 2, 7, 8, 11)	{3, 5, 6, 7}	(2, 2, 4, 5, 8, 9)
{3, 5, 6, 8}	(2, 2, 7, 9, 10)	{3, 5, 7, 8}	(2, 2, 4, 7, 7, 8)	{3, 6, 7, 8}	(5, 6, 7, 12)
{4, 5, 6, 7}	(4, 6, 9, 11)	{4, 5, 6, 8}	(3, 4, 5, 7, 11)	{4, 5, 7, 8}	(4, 7, 8, 11)
{4, 6, 7, 8}	(3, 5, 6, 7, 9)	{5, 6, 7, 8}	(5, 7, 8, 10)		

Table 7. $\widetilde{\mathcal{M}}_{0,16}$ for $2 \le |\mathscr{B}'| \le 4$.

$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$
{2, 3, 4, 5, 6}	(1, 2, 3, 5, 9, 10)	{2, 3, 4, 5, 7}	(1, 2, 3, 6, 8, 10)
{2, 3, 4, 5, 8}	(1, 2, 3, 4, 7, 13)	{2, 3, 4, 6, 7}	(1, 2, 3, 5, 6, 13)
{2, 3, 4, 6, 8}	(2, 3, 5, 7, 13)	{2, 3, 4, 7, 8}	(1, 1, 3, 6, 7, 12)
{2, 3, 5, 6, 7}	(1, 2, 4, 5, 6, 12)	{2, 3, 5, 6, 8}	(1, 1, 4, 5, 7, 12)
{2, 3, 5, 7, 8}	(1, 4, 6, 7, 12)	{2, 3, 6, 7, 8}	(1, 2, 5, 6, 7, 9)
{2, 4, 5, 6, 7}	(1, 3, 4, 5, 6, 11)	{2, 4, 5, 6, 8}	(1, 3, 4, 5, 7, 10)
{2, 4, 5, 7, 8}	(1, 3, 4, 6, 7, 9)	{2, 4, 6, 7, 8}	(1, 3, 5, 6, 7, 8)
{2, 5, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 3, 4, 5, 6, 10)
{3, 4, 5, 6, 8}	(2, 3, 4, 5, 7, 9)	{3, 4, 5, 7, 8}	(2, 2, 4, 5, 6, 11)
{3, 4, 6, 7, 8}	(2, 3, 5, 6, 7, 7)	{3, 5, 6, 7, 8}	(2, 5, 6, 7, 10)
{4, 5, 6, 7, 8}	(3, 4, 5, 5, 6, 7)		
{2, 3, 4, 5, 6, 7}	(1, 2, 3, 4, 5, 6, 9)	{2, 3, 4, 5, 6, 8}	(1, 2, 4, 5, 7, 11)
{2, 3, 4, 5, 7, 8}	(1, 1, 2, 3, 6, 7, 10)	{2, 3, 4, 6, 7, 8}	(1, 2, 3, 7, 8, 9)
{2, 3, 5, 6, 7, 8}	(1, 2, 4, 5, 5, 6, 7)	{2, 4, 5, 6, 7, 8}	$T_{2,*}$
{3, 4, 5, 6, 7, 8}	(2, 3, 3, 4, 5, 6, 7)		

Table 8. $\widetilde{\mathcal{M}}_{0,16}$ for $5 \le |\mathcal{B}'| \le 6$.

$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{2, 7}	$T_{2,*}$	{2, 8}	$T_{2,*}$
{3, 4}	$T_{3,*}$	{3, 5}	$T_{3,*}$	{3, 6}	$T_{3,*}$
{3, 7}	$T_{3,*}$	{3, 8}	$T_{3,*}$	{4, 5}	$T_{4,5}$
{4, 6}	$T_{4,6}$	{4, 7}	see Case 2	{4, 8}	(3, 3, 7, 7, 12)
{5, 6}	$T_{5,6}$	{5, 7}	$T_{5,7}$	{5, 8}	(7, 7, 7, 11)
{6, 7}	$T_{6,7}$	{6, 8}	$T_{6,8}$	{7, 8}	$T_{7,8}$
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 3, 7}	$T_{2,*}$	{2, 3, 8}	$T_{2,*}$	{2, 4, 5}	$T_{2,*}$
{2, 4, 6}	$T_{2,*}$	{2, 4, 7}	$T_{2,*}$	{2, 4, 8}	$T_{2,*}$
{2, 5, 6}	$T_{2,*}$	{2, 5, 7}	$T_{2,*}$	{2, 5, 8}	$T_{2,*}$
{2, 6, 7}	$T_{2,*}$	{2, 6, 8}	$T_{2,*}$	{2, 7, 8}	$T_{2,*}$
{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$	{3, 4, 7}	$T_{3,4,7}$
{3, 4, 8}	$T_{3,4,8}$	{3, 5, 6}	$T_{3,5,6}$	{3, 5, 7}	$T_{3,5,7}$
{3, 5, 8}	$T_{3,5,8}$	{3, 6, 7}	$T_{3,6,7}$	{3, 6, 8}	$T_{3,6,8}$
{3, 7, 8}	$T_{3,7,8}$	{4, 5, 6}	(4, 5, 11, 12)	{4, 5, 7}	$T_{4,5,7}$
{4, 5, 8}	$T_{4,5,8}$	{4, 6, 7}	(5, 6, 9, 12)	{4, 6, 8}	$T_{4,6,8}$
{4, 7, 8}	(6, 6, 8, 12)	{5, 6, 7}	(5, 6, 10, 11)	{5, 6, 8}	(4, 7, 10, 11)
{5, 7, 8}	(6, 7, 8, 11)	{6, 7, 8}	(6, 7, 9, 10)		

Table 9. $\widetilde{\mathcal{M}}_{0,17}$ for $2 \le |\mathcal{B}'| \le 3$.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 4, 8}	$T_{2,*}$	{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$
{2, 3, 5, 8}	$T_{2,*}$	{2, 3, 6, 7}	$T_{2,*}$	{2, 3, 6, 8}	$T_{2,*}$
{2, 3, 7, 8}	$T_{2,*}$	{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	$T_{2,*}$
{2, 4, 5, 8}	$T_{2,*}$	{2, 4, 6, 7}	$T_{2,*}$	{2, 4, 6, 8}	$T_{2,*}$
{2, 4, 7, 8}	(1, 3, 7, 9, 12)	{2, 5, 6, 7}	$T_{2,*}$	{2, 5, 6, 8}	(1, 5, 7, 8, 11)
{2, 5, 7, 8}	(4, 6, 8, 14)	{2, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6}	$T_{3,*}$
{3, 4, 5, 7}	$T_{3,*}$	{3, 4, 5, 8}	$T_{3,*}$	{3, 4, 6, 7}	$T_{3,*}$
{3, 4, 6, 8}	(2, 8, 10, 12)	{3, 4, 7, 8}	(3, 7, 9, 13)	{3, 5, 6, 7}	(2, 9, 10, 11)
{3, 5, 6, 8}	(2, 4, 5, 8, 13)	{3, 5, 7, 8}	(2, 4, 6, 7, 13)	{3, 6, 7, 8}	(5, 6, 8, 13)
{4, 5, 6, 7}	(4, 6, 10, 12)	{4, 5, 6, 8}	(3, 4, 5, 8, 12)	{4, 5, 7, 8}	(4, 7, 9, 12)
{4, 6, 7, 8}	(3, 6, 6, 7, 10)	{5, 6, 7, 8}	(5, 7, 9, 11)		

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4, 5, 6}	$T_{2,*}$	{2, 3, 4, 5, 7}	$T_{2,*}$
{2, 3, 4, 5, 8}	(1, 2, 3, 4, 8, 14)	{2, 3, 4, 6, 7}	(2, 2, 3, 5, 6, 14)
{2, 3, 4, 6, 8}	(2, 3, 5, 8, 14)	{2, 3, 4, 7, 8}	(2, 3, 6, 7, 14)
{2, 3, 5, 6, 7}	(1, 4, 5, 9, 13)	{2, 3, 5, 6, 8}	(2, 4, 5, 7, 14)
{2, 3, 5, 7, 8}	(1, 4, 6, 8, 13)	{2, 3, 6, 7, 8}	(1, 5, 6, 7, 13)
{2, 4, 5, 6, 7}	(3, 4, 5, 6, 14)	{2, 4, 5, 6, 8}	(1, 3, 7, 10, 11)
{2, 4, 5, 7, 8}	(1, 3, 8, 9, 11)	{2, 4, 6, 7, 8}	(1, 5, 6, 8, 12)
{2, 5, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 4, 5, 9, 12)
{3, 4, 5, 6, 8}	(3, 4, 5, 7, 13)	{3, 4, 5, 7, 8}	(2, 3, 7, 9, 11)
{3, 4, 6, 7, 8}	(2, 5, 6, 7, 12)	{3, 5, 6, 7, 8}	(2, 5, 6, 8, 11)
{4, 5, 6, 7, 8}	(3, 5, 6, 7, 11)		
{2, 3, 4, 5, 6, 7}	(1, 3, 4, 5, 6, 13)	{2, 3, 4, 5, 6, 8}	(1, 2, 4, 5, 8, 12)
{2, 3, 4, 5, 7, 8}	(1, 2, 3, 4, 6, 7, 9)	{2, 3, 4, 6, 7, 8}	(1, 2, 3, 7, 9, 10)
{2, 3, 5, 6, 7, 8}	(1, 2, 5, 6, 7, 11)	{2, 4, 5, 6, 7, 8}	$T_{2,*}$
{3, 4, 5, 6, 7, 8}	(2, 3, 4, 6, 7, 10)		

Table 10. $\widetilde{\mathscr{M}}_{0,17}$ for $4 \le |\mathscr{B}'| \le 6$.

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References

[Gibney 2009] A. Gibney, "Numerical criteria for divisors on $\overline{M}_g$ to be ample", *Compos. Math.* **145**:5 (2009), 1227–1248. MR Zbl

[Harris and Morrison 1998] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics **187**, Springer, New York, 1998. MR Zbl

[Hassett 2003] B. Hassett, "Moduli spaces of weighted pointed stable curves", *Adv. Math.* **173**:2 (2003), 316–352. MR Zbl

[Kodaira 1967] K. Kodaira, "A certain type of irregular algebraic surfaces", *J. Analyse Math.* **19** (1967), 207–215. MR Zbl

[Liu 2016] X.-L. Liu, "Modular invariants and singularity indices of hyperelliptic fibrations", preprint, 2016. To appear in *Chin. Ann. Math.* (*B*). arXiv

[Liu and Tan 2013] X.-L. Liu and S.-L. Tan, "Families of hyperelliptic curves with maximal slopes", *Sci. China Math.* **56**:9 (2013), 1743–1750. MR Zbl

[Moriwaki 1998] A. Moriwaki, "Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves", *J. Amer. Math. Soc.* **11**:3 (1998), 569–600. MR Zbl

[Scherbak 2002] I. Scherbak, "Rational functions with prescribed critical points", *Geom. Funct. Anal.* **12**:6 (2002), 1365–1380. MR Zbl

[Tan 2010] S.-L. Tan, "Chern numbers of a singular fiber, modular invariants and isotrivial families of curves", *Acta Math. Vietnam.* **35**:1 (2010), 159–172. MR Zbl

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XIAO-LEI LIU
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCE
55, EAST ZHONGGUANCUN ROAD
HAIDIAN
BEIJING, 100190
CHINA
xlliu1124@amss.ac.cn

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