Pacific Journal of Mathematics

Volume 284 No. 1 September 2016

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.1

BITWIST MANIFOLDS AND TWO-BRIDGE KNOTS

JAMES W. CANNON, WILLIAM J. FLOYD, LEER LAMBERT,
WALTER R. PARRY AND JESSICA S. PURCELL

Though LeeR Lambert spent his life as an actuary and a musician and was a loving father of nine girls and one boy, he had always wanted to earn an advanced degree as a mathematician. With the encouragement of his wife, he earned his Ph.D. in mathematics at the age of 68. Many of the results of this paper appeared in his Ph.D. dissertation at Brigham Young University. At the age of 71, LeeR died of bone cancer. We miss you, LeeR.

We give uniform, explicit, and simple face-pairing descriptions of all the branched cyclic covers of the 3-sphere, branched over two-bridge knots. Our method is to use the bitwisted face-pairing constructions of Cannon, Floyd, and Parry; these examples show that the bitwist construction is often efficient and natural. Finally, we give applications to computations of fundamental groups and homology of these branched cyclic covers.

1. Introduction

Branched cyclic covers of $\mathbb{S}^3$ have played a major role in topology, and continue to appear in a wide variety of contexts. For example, branched cyclic covers of $\mathbb{S}^3$ branched over two-bridge knots have recently appeared in combinatorial work bounding the Matveev complexity of a 3-manifold [Petronio and Vesnin 2009], in algebraic and topological work determining relations between L-spaces, left-orderability, and taut foliations [Gordon and Lidman 2014; Boyer et al. 2013; Hu 2015], and in geometric work giving information on maps of character varieties [Nagasato and Yamaguchi 2012]. They provide a wealth of examples, and a useful collection of manifolds on which to study conjectures. Given their wide applicability, and their continued relevance, it is useful to have many explicit descriptions of these manifolds.

We give a new and elegant construction of the branched cyclic covers of two-bridge knots, using the bitwist construction of [Cannon et al. 2009]. While other presentations of these manifolds are known (see, for example, [Minkus 1982; Mulazzani and Vesnin 2001]), we feel our descriptions have several advantages.

First, they follow from a recipe involving exactly the parameters necessary to describe a two-bridge knot, namely, continued fraction parameters. Our descriptions

---

MSC2010: 57M12, 57M25.
Keywords: bitwist manifolds, two-bridge knots, branched cyclic covers.

apply uniformly to all two-bridge knots, and all branched cyclic covers of $\mathbb{S}^3$ branched over two-bridge knots.

Second, they are obtained from a description of a two-bridge knot using a very straightforward bitwisted face-pairing construction, as in [Cannon et al. 2000; 2002; 2003; 2009]. Bitwisted face-pairings (read "bi-twisted", as in twisted two ways) are known to produce all closed orientable 3-manifolds. The examples of this paper show, in addition, that bitwist constructions are often efficient and natural. While a generic face-pairing will yield a pseudomanifold, which, with probability 1, will not be an actual manifold [Dunfield and Thurston 2006], bitwisted face-pairings avoid this problem. (We will review necessary information on bitwisted face-pairings, so no prior specialized knowledge is required to understand our constructions.)

Third, our description leads to immediate consequences in geometric group theory. We obtain a simple proof of the fact that the fundamental group of the n-fold branched cyclic covering of $\mathbb{S}^3$, branched over a two-bridge knot, has a cyclic presentation. Our description also gives immediate presentations of two well-known families of groups, the Fibonacci and Sieradski groups. These are known to arise as fundamental groups of branched cyclic covers of $\mathbb{S}^3$ branched over the figure-eight and trefoil knots, respectively. These groups have received considerable attention from geometric group theorists; see, for example, [Cavicchioli et al. 1998] for further references, and Section 6 for more history. Our methods recover the fact that the first homology groups of Sieradski manifolds are periodic. We also give a proof that their fundamental groups are distinct using Milnor's characterization of these spaces. We consider orders of abelianizations of Fibonacci groups as well. These orders form an interesting sequence related to the Fibonacci sequence, which we shall see.

1A. ***Bitwisted face-pairing description.*** We will see that the bitwist description of any two-bridge knot is encoded as the image of the north–south axis in a ball labeled as in Figure 1, along with an associated vector of integer multipliers. For the branched cyclic cover, the description is encoded by adding additional longitudinal arcs to the sphere. We now describe the construction briefly, in order to state the main results of the paper. A more detailed description of the construction, with examples, is given in Section 2.

Begin with a finite graph Γ in the 2-sphere $\mathbb{S}^2 = \partial\mathbb{B}^3$ that is the union of the equator e, one longitude NS from the north pole N to the south pole S, and $2k \geq 0$ latitudinal circles, such that Γ is invariant under reflection

$$\epsilon : \mathbb{S}^2 \to \mathbb{S}^2$$

in the equator. Then Γ divides $\mathbb{S}^2$ into $2(k+1)$ faces that are paired by ϵ. This face-pairing is shown in Figure 1.

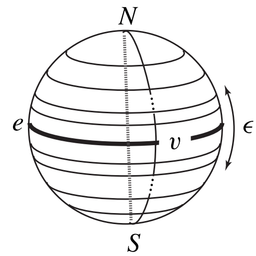

Figure 1. The model face-pairing: a faceted 3-ball with dotted central axis and reflection face-pairing $\epsilon : \partial\mathbb{B}^3 \to \partial\mathbb{B}^3$.

As with any face-pairing, the edges fall into edge cycles. The equator e forms one edge cycle c_0 since the reflection ϵ leaves e invariant. Each other edge of the graph is matched with its reflection to form another edge cycle c_i. We number these edge cycles from 0 through $2k+1$, with even numbers associated with latitudinal edges, as indicated in Figure 2.

Now choose nonzero integer *multipliers*, denoted $m_0, m_1, \ldots, m_{2k}, m_{2k+1}$, for the edge cycles c_i. In the case at hand, restrict the choice of multipliers m_i as follows. Each latitudinal edge cycle c_{2i} is assigned either $+1$ or -1 as multiplier. Each longitudinal edge cycle c_{2i+1} may be assigned any integer multiplier m_{2i+1} whatsoever, including 0. The multiplier $m_{2i+1} = 0$ is usually forbidden, but in this case indicates that the two edges of edge-class c_{2i+1} must be collapsed to a point before the bitwist construction is engaged.

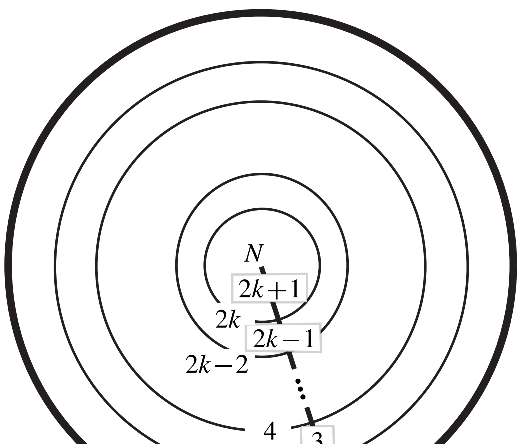

Figure 2. The northern hemisphere, with edge cycles numbered.

Finally, for the general bitwist construction, we obtain a closed manifold $M(\epsilon, m)$ by taking the following quotient. First, subdivide each edge in the edge cycle c_i into $|c_i| \cdot |m_i|$ subedges. Insert an additional edge between each adjacent positive and negative edge, if any. Then twist each subedge by one subedge in a direction indicated by the sign of m_i. Finally, apply the face-pairing map ϵ to glue bitwisted faces. This is the bitwist construction.

In Theorem 4.2, we prove that the bitwist manifold $M(\epsilon, m)$ described above is the 3-sphere $\mathbb{S}^3$. The image of the north–south axis in $\mathbb{S}^3$ is a two-bridge knot. In fact, we prove more. Recall that every two-bridge knot is the closure of a rational tangle. See [Kauffman and Lambropoulou 2002] for an elementary exposition. A rational tangle is determined up to isotopy by a single rational number, which we call the rational number invariant of the tangle. There are two natural ways to close a tangle so that it becomes a knot or link, the numerator closure and the denominator closure. The full statement of Theorem 4.2 is below.

Theorem 4.2. *The bitwist manifold $M(\epsilon, m)$ is the* 3*-sphere $\mathbb{S}^3$. The image of the north–south axis in $\mathbb{S}^3$ is the two-bridge knot which is the numerator closure of the tangle $T(a/b)$ whose rational number invariant a/b is*

$$2 \cdot m_0 + \cfrac{1}{2 \cdot m_1 + \cfrac{1}{\ddots + \cfrac{1}{2 \cdot m_{2k} + \cfrac{1}{2 \cdot m_{2k+1}}}}}.$$

Remark 1.1. The 2's in the continued fraction indicate that the tangle is constructed using only full twists instead of the possible mixture of full and half twists.

Example 1.2. The simplest case, with only equator and longitude, yields the trefoil and figure-eight knots, as we shall see in Theorem 4.1. Simple subdivisions yield their branched cyclic covers, the Sieradski [1986] and Fibonacci [Vesnin and Mednykh 1996] manifolds.

Definition 1.3. We say that the multiplier function m is *normalized* if

(1) $m_{2k+1} \neq 0$, and

(2) if $m_{2i+1} = 0$ for some $i \in \{0, \ldots, k-1\}$, then $m_{2i} = m_{2i+2}$.

With this definition, the previous theorem and well-known results involving two-bridge knots yield the following corollary.

Corollary 4.4. *Every normalized multiplier function yields a nontrivial two-bridge knot. Conversely, every nontrivial two-bridge knot K is realized by either one or two normalized multiplier functions. If K is the numerator closure of the tangle $T(a/b)$, then it has exactly one such realization if and only if $b^2 \equiv 1 \bmod a$.*

Notice that the n-th branched cyclic covering of $\mathbb{S}^3$, branched over K, can be obtained by unwinding the description n times about the unknotted axis that represents K, unwinding the initial face-pairing as in Figure 17. This leads to a new proof of the following result, originally due to Alberto Cavicchioli, Friedrich Hegenbarth and Ana Chi Kim [Cavicchioli et al. 1999a].

Theorem 5.2. *The fundamental group of the n-th branched cyclic covering of $\mathbb{S}^3$, branched over a two-bridge knot K, has a cyclic presentation.*

Problem 1.4. How should one carry out the analogous construction for arbitrary knots?

1B. ***The Fibonacci and Sieradski manifolds.*** Since the knots in the face-pairing description appear as the unknotted axis in $\mathbb{B}^3$, it is easy to unwind $\mathbb{B}^3$ around the axis to obtain face-pairings for the branched cyclic coverings of $\mathbb{S}^3$, branched over the trefoil knot and the figure-eight knot. For the trefoil knot, the n-th branched cyclic cover S_n is called the n-th Sieradski manifold. For the figure-eight knot, the n-th branched cyclic cover F_n is called the n-th Fibonacci manifold. We will prove:

Theorem 5.4. *The fundamental group $\pi_1(F_n)$ is the $2n$-th Fibonacci group with presentation*

$$\langle x_1, \ldots, x_{2n} \mid x_1x_2 = x_3,\ x_2x_3 = x_4,\ \ldots,\ x_{2n-1}x_{2n} = x_1,\ x_{2n}x_1 = x_2\rangle.$$

The fundamental group $\pi_1(S_n)$ is the n-th Sieradski group with presentation

$$\langle y_1, \ldots, y_n \mid y_1 = y_2y_n,\ y_2 = y_3y_1,\ y_3 = y_4y_2,\ \ldots,\ y_n = y_1y_{n-1}\rangle.$$

Remark 1.5. The group presentations are well known once the manifolds are recognized as branched cyclic covers of $\mathbb{S}^3$, branched over the figure-eight knot and the trefoil knot. But these group presentations also follow immediately from the description of the bitwist face-pairings, as we shall see.

The first homology of the Sieradski manifolds has an intriguing periodicity property, which is well known (see, for example, Rolfsen [1976]). In particular, it is periodic of period 6. The following theorem, concerning their fundamental groups, is not as well known; it is difficult to find in the literature. We give a proof using Milnor's characterization of these spaces.

Theorem 5.13. *No two of the Sieradski groups are isomorphic. Hence no two of the branched cyclic covers of $\mathbb{S}^3$, branched over the trefoil knot, are homeomorphic.*

1C. ***Organization.*** In Section 2, we give a more careful description of the bitwisted face-pairing, and work through the description for two examples, which will correspond to the trefoil and figure-eight knots.

In Section 3, we recall many of the results in our previous papers [Cannon et al. 2003; 2009] to make explicit the connections between face-pairings, Heegaard splittings, and surgery descriptions of 3-manifolds. We apply these to the examples of bitwisted face-pairings given here, to give surgery descriptions. We use these descriptions in Section 4 to prove that our constructions yield two-bridge knots. The proofs of the main geometric theorems are given in this section.

In Section 5 we turn to geometric group theory. We prove that our presentations easily lead to well-known results on presentations of fundamental groups. We also give results on Fibonacci and Sieradski groups in this section.

Finally, Section 6 explains some of the history of these problems.

2. Bitwisted face-pairing: trefoil and figure-eight knots

In this section we step through the bitwisted face-pairing description more carefully. We believe it will be most useful to work through a pair of examples. We will see in subsequent sections that these examples lead to Fibonacci and Sieradski manifolds.

As an example, consider the simplest model, shown in Figure 3 (left). The graph has three edges and three vertices, and divides the sphere into two singular "triangles", which are then matched by reflection ϵ in the equator e.

Bitwisted face-pairings require an integer multiplier for each edge cycle. For this simple model there are two edge cycles, namely the singleton $c_0 = \{e\}$ and the pair $c_1 = \{Nv, Sv\}$. We will see that multiplying every multiplier by -1 takes the knot which we construct to its mirror image. So up to taking mirror images, the two simplest choices for multipliers are $m_0 = \pm 1$ for c_0 and $m_1 = 1$ for c_1. The bitwist construction requires that each edge in the cycle c_i be subdivided into $|c_i| \cdot |m_i|$ subedges. When both positive and negative multipliers appear on edges of the same face, we must insert an additional edge, called a *sticker*, between a negative

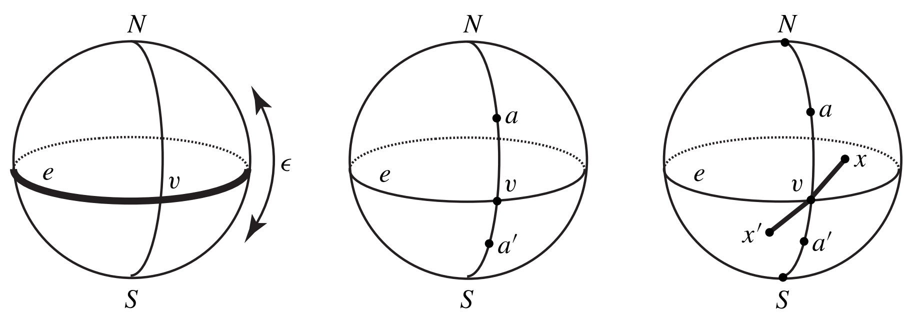

Figure 3. A faceted 3-ball with vertices N, v, and S and edges Nv, Sv, and e (left), subdivisions for M_+ (middle), and subdivisions for M_- (right).

and positive edge in a given, fixed orientation of $\mathbb{S}^2$. We will use the clockwise orientation.

With the facets modified as described in the previous paragraph, we are prepared for the bitwisting. Twist each subedge of each face by one subedge before applying the model map ϵ. Edges with positive multiplier are twisted in the direction of the fixed orientation. Edges with negative multiplier are twisted in the opposite direction. The stickers resolve the twisting conflict between negative and positive subedges. A sticker in the domain of the map splits into two subedges. A sticker in the range of the map absorbs the folding together of two subedges.

We denote by M_+ the face-pairing in which both multipliers are $+1$ and by M_- the face-pairing where one multiplier is $+1$ and the other is -1. The two results are shown in Figure 3 (middle and right).

After this subdivision, the faces can be considered to have five edges for M_+ and seven edges for M_-. Before making the identification of the northern face with the southern face, we rotate the 5-gon one notch (= one edge = one-fifth of a turn, combinatorially) in the direction of the given orientation on $\mathbb{S}^2$ before identification. We rotate the edges of the 7-gon with positive multiplier one notch (= one edge = one-seventh of a turn, combinatorially) in the direction of the orientation before identification. The edges with negative multiplier are twisted one notch in the opposite direction. The stickers absorb the conflict at the joint between positive and negative. Thus the face-pairings ϵ_+ and ϵ_- in terms of the edges forming the boundaries of the faces are given as follows.

For M_+: $$\epsilon_+ : \begin{pmatrix} av & e & va & aN & Na \\ e & va' & a'S & Sa' & a'v \end{pmatrix}.$$

For M_-: $$\epsilon_- : \begin{pmatrix} av & e & vx & xv & va & aN & Na \\ vx' & x'v & e & va' & a'S & Sa' & a'v \end{pmatrix}.$$

The bitwist theorem [Cannon et al. 2009, Theorem 3.1] implies that the resulting identification spaces are closed manifolds, which we denote by F_1 for M_+ and S_1 for M_-. We shall see that both of these manifolds are $\mathbb{S}^3$, and thus topologically uninteresting. But as face-pairings, these identifications are wonderfully interesting because the north–south axis from $\mathbb{B}^3$ becomes the figure-eight knot K_+ in F_1 and becomes the trefoil knot K_- in S_1. We prove this in Theorem 4.1.

3. Pseudo-Heegaard splittings and surgery diagrams

In order to recognize the quotients of $\mathbb{B}^3$ described in Section 1 as the 3-sphere and to recognize the images of the north–south axis as two-bridge knots, we need to make more explicit the connections between face-pairings, Heegaard splittings, and surgery descriptions of 3-manifolds, described in our previous papers [Cannon

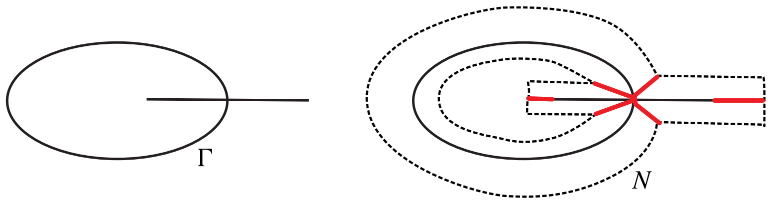

Figure 4. The addition of new red arcs.

et al. 2003; 2009]. We use these connections to transfer knots from the face-pairing description to the surgery descriptions.

3A. ***The pseudo-Heegaard splitting.*** We begin with the following information:

B: a faceted 3-ball which we identify with $\mathbb{B}^3 = [0, 1] \cdot \mathbb{S}^2$ (where $\cdot$ is scalar multiplication).

$\Gamma \subset \partial B = \mathbb{S}^2$: the 1-skeleton of B, a connected, finite graph with at least one edge.

Δ: the dual 1-skeleton, consisting of a cone from the center 0 of B to points of ∂B, one in the interior of each face of B.

N: a regular neighborhood of Γ in ∂B.

$N_\Gamma = \left[\frac{3}{4}, 1\right] \cdot N$: a regular neighborhood of Γ in B.

$N_\Delta = \mathrm{cl}(B - N_\Gamma)$: a regular neighborhood of Δ in B.

Add extra structure to N and N_Γ as follows.

First, from each vertex v of Γ, we extend arcs from v to ∂N, one to each local side of Γ at v so that the interiors of these arcs are mutually disjoint. Label these arcs *red*. Figure 4 shows this for the simplest model described above.

Next, momentarily disregarding both the vertices and edges of Γ, we view the red arcs as subdividing N into quadrilaterals (occasionally singular at the arc ends), every quadrilateral having two sides in ∂N and two sides each of which is the union of two (or one in the singular case) of these red arcs, as on the left of Figure 5.

Every such quadrilateral contains exactly one edge of Γ. We cut these quadrilaterals into half-quadrilaterals by arcs transverse to the corresponding edge of Γ at the middle of that edge. Label these transverse arcs *blue*. For the simplest model, this is shown in Figure 5 (right).

If we cut N along the new red arcs and blue transverse arcs, multiply by the scalar interval $\left[\frac{3}{4}, 1\right]$, and desingularize, we obtain cubes, each containing exactly one vertex of Γ in its boundary. Endow these cubes with a cone structure, coned to its vertex in Γ. See Figure 6.

Finally, we assume that $\epsilon : \partial B \to \partial B$ is an orientation-reversing face-pairing, based on the faceted 3-ball B, that respects all of this structure as much as possible:

Figure 5. The addition of blue transverse arcs.

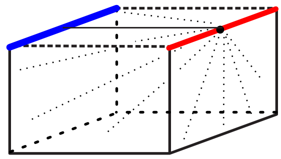

Figure 6. The cone structure.

faces are paired; N is invariant under the pairing; the regions bounded by the new arcs, the transverse arcs, the boundary of N, and Γ are paired by ϵ; and cone structures are preserved.

Definition 3.1. Let C_Γ be the union of the products of the transverse arcs with $\left[\frac{3}{4}, 1\right]$. Let $C_\Delta = N_\Delta \cap (\partial B)$. Define $D_\Gamma = C_\Gamma/\epsilon$, $D_\Delta = C_\Delta/\epsilon$, $H_\Gamma = N_\Gamma/\epsilon$, and $H_\Delta = N_\Delta/\epsilon$, and let $\delta = \partial D_\Delta$ and $\gamma = \partial D_\Gamma$.

The following is essentially contained in [Cannon et al. 2003, Theorem 4.2.1].

Theorem 3.2. *The space H_Δ is a handlebody with one handle for each face pair of B. The set D_Δ is a disjoint union of disks that form a complete set of handle disks for H_Δ; the curves δ form a complete set of handle curves.*

The space H_Γ is a handlebody if and only if $M(\epsilon) = B/\epsilon$ is a 3-manifold. In that case, D_Γ is a disjoint union of disks that form a complete set of handle disks for H_Γ and γ forms a complete set of handle curves. Whether $M(\epsilon)$ is a manifold or not, the disks of D_Γ cut H_Γ into pieces X_i, each containing exactly one vertex v_i of $M(\epsilon)$, and each X_i is a cone $v_i S_i$, where S_i is a closed orientable surface. The space $M(\epsilon)$ is a manifold if and only if each S_i is a 2-sphere. (The cone structure on X_i uses the cone structures of the pieces described above.)

Terminology 3.3. Even when $M(\epsilon) = B/\epsilon$ is not a manifold, we call the disks of D_Γ handle disks for H_Γ and the curves $\gamma = \partial D_\Gamma$ handle curves for H_Γ. We call H_Γ a pseudohandlebody and the pair (H_Γ, H_Δ) a pseudo-Heegaard splitting for $M(\epsilon)$.

All bitwist manifolds based on the face-pairing (B, ϵ) have Heegaard splittings and surgery descriptions that can be based on any unknotted embedding of $H_\Delta = N_\Delta/\epsilon$ in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. The closure of the complement is then also a handlebody, which we shall denote by H. We describe here a particular unknotted embedding

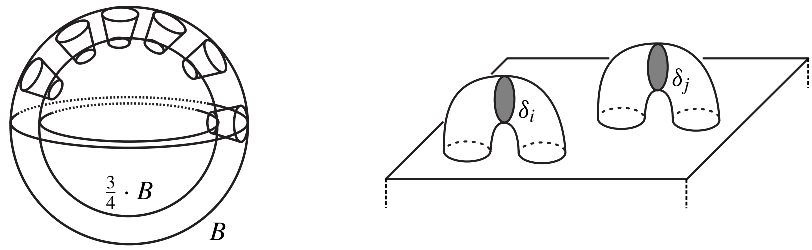

Figure 7. The ball with chimneys N_Δ (left), and the handlebody H_Δ (right).

of H_Δ in $\mathbb{S}^3$, and illustrate with the constructions from Section 1A, especially those of Section 2.

Note that $N_\Delta = \left(\left[0, \frac{3}{4}\right] \cdot \mathbb{S}^2\right) \cup \left(\left[\frac{3}{4}, 1\right] \cdot C_\Delta\right)$, where $\left[0, \frac{3}{4}\right] \cdot \mathbb{S}^2$ is, of course, a 3-ball, and $\left[\frac{3}{4}, 1\right] \cdot C_\Delta$ is a family of chimneys attached to that 3-ball, as in Figure 7 (left).

The space H_Δ is formed by identifying the tops of those chimneys in pairs. We may therefore assume H_Δ is embedded in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ as shown in Figure 7 (right). We identify $\left[0, \frac{3}{4}\right) \cdot \mathbb{S}^2$ with $\mathbb{R}^2 \times (-\infty, 0) \subset \mathbb{R}^3$. The 2-sphere $\left(\frac{3}{4}\right) \cdot \mathbb{S}^2$ minus one point is identified with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. The chimneys with tops identified become handles.

3B. ***Pseudo-Heegaard splittings of our examples.*** For the constructions of Sections 1A and 2, we now determine the curves δ and γ on the handlebody H_Δ.

Begin with the simple face-pairing description of Section 2. The handlebody H_Δ is embedded in $\mathbb{R}^3 \cup \{\infty\}$ as above, with the plane $\mathbb{R}^2 \times \{0\}$ identified with $\left(\frac{3}{4}\right) \cdot \mathbb{S}^3$ minus a point. Sketch the graph $\left(\frac{3}{4}\right) \cdot \Gamma$ on $\mathbb{R}^2 \times \{0\}$, with the vertex $\left(\frac{3}{4}\right) \cdot v$ at ∞, as in Figure 8 (left). There is just one pair of faces, hence just one handle in this case, as shown. Thus D_Δ is a single disk with boundary δ, shown in the left side of the figure.

We need to determine the curves $\gamma = \partial D_\Gamma$. Recall that $D_\Gamma = C_\Gamma / \epsilon$, and the disks C_Γ consist of the union of the products of the blue transverse arcs with $\left[\frac{3}{4}, 1\right]$.

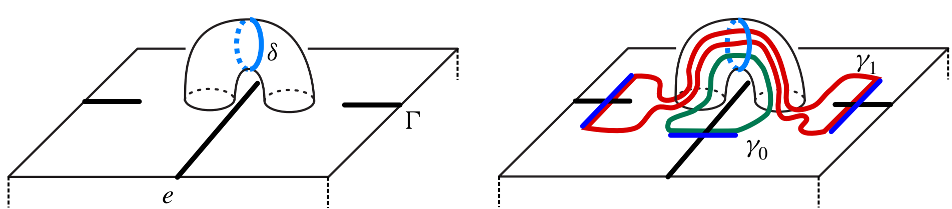

Figure 8. The graph $\left(\frac{3}{4}\right) \cdot \Gamma$ and curve δ for the simple example (left), and the graph with curves γ added in, running partly along blue transverse arcs (right).

Thus curves in γ will contain blue transverse arcs, as well as arcs along the handles of H_Δ, running from the blue transverse arcs to a curve δ_j.

In the case of the simple example, following the action of ϵ, we see that the transverse arc τ_0 dual to the edge e gives a single simple closed curve γ_0 that follows τ_0, then connects the endpoints of τ_0 via an arc that runs over the single handle of H_Δ. The two transverse arcs dual to Nv and Sv are identified by ϵ. Thus endpoints of these arcs are connected by arcs running over the handle. We obtain a simple closed curve γ_1. This is shown in Figure 8 (right).

The general picture, for the construction of Section 1A, follows similarly. We summarize in a lemma.

Lemma 3.4. *Let Γ and ϵ be as in Section 1A, with Γ the union of the equator e, one longitude NS from the north pole N to the south pole S, and $2k \geq 0$ latitudinal circles, such that Γ is invariant under reflection ϵ in the equator. Then the handle curves on H_Δ for this face-pairing are as follows.*

(1) *There are $k+1$ handles of H_Δ, corresponding to the $k+1$ regions in the complement of Γ in the northern hemisphere, each running from the region to its mirror region in the southern hemisphere. These give curves $\delta_0, \dots, \delta_k$ encircling the handles.*

(2) *The transverse arc dual to the edge e gives a curve γ_0 with endpoints connecting to itself over the handle corresponding to the faces on either side of e, which are identified by ϵ.*

(3) *Each latitudinal arc distinct from e, if any, is joined to its mirror over two handles, one for each face on opposite sides of the latitudinal edge. These give curves γ_{2i}, $i = 1, \dots, k$, with index corresponding to the edge label as in Figure 2.*

(4) *Each transverse arc dual to a longitudinal arc is joined to its mirror over a handle corresponding to the region on either side of that arc. These give curves γ_{2i+1}, $i = 0, \dots, k$, with index corresponding to edge label as in Figure 2.*

Curves parallel to those of Lemma 3.4 are illustrated in Figure 9. These curves have been pushed slightly to be disjoint, in a manner described in the next section.

3C. ***The surgery description.*** We assume now that we are given a bitwist construction based on (B, ϵ). We are given the following information:

$c_1, \dots, c_k$: the edge cycles of ϵ.

$m = \{m_1, \dots, m_k\}$: a set of nonzero integer multipliers assigned to these edge cycles.

$\epsilon_m : \partial B \to \partial B$: the associated bitwist face-pairing.

$M(\epsilon, m) = M(\epsilon_m) = B/\epsilon_m$: the resulting bitwist manifold.

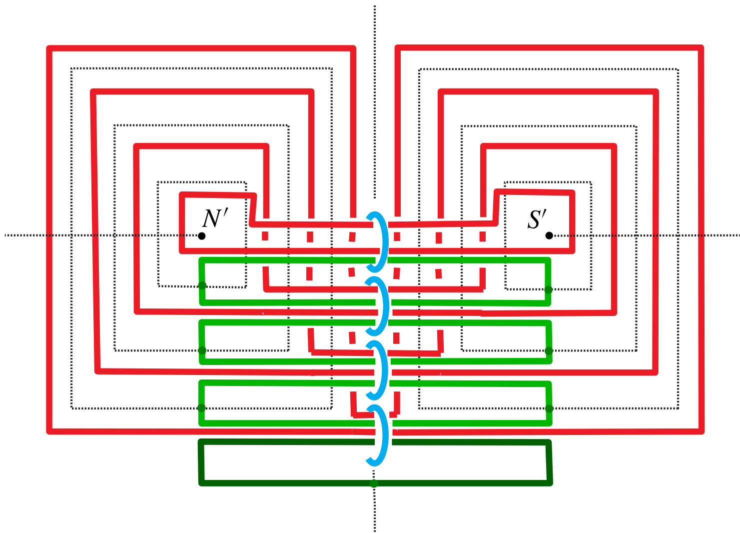

Figure 9. The curves δ and γ in H_Δ. Curves δ are shown in blue, γ_0 is in dark green at the bottom of the diagram, curves of γ corresponding to latitudinal transverse arcs are in green, and curves of γ corresponding to longitudinal transverse arcs are in red.

The set $\delta = \partial D_\Delta$ is a disjoint union of simple closed handle curves $\delta_1, \ldots, \delta_g$ for H_Δ, one for each face pair of ϵ. We first push each δ_i slightly into $\mathbb{R}^3 \smallsetminus H_\Delta$ to a curve δ_i'. We let V_i denote a solid torus neighborhood of δ_i' in $\mathbb{R}^3 \smallsetminus H_\Delta$, remove it, and sew a new solid torus V_i' back in with meridian and longitude reversed (0-surgery on each δ_i'). The curve δ_i now bounds a disk E_i, disjoint from H_Δ, consisting of an annulus from δ_i to $\partial V_i'$ and a meridional disk in V_i'. The result is a new handlebody

$$H' = \left[\mathrm{cl}(\mathbb{S}^3 \smallsetminus H_\Delta) \smallsetminus \bigcup V_i\right] \cup \left[\bigcup V_i'\right]$$

with the same handle curves $\delta_1, \ldots, \delta_g$ as H_Δ and with handle disks $E_1, \ldots, E_g$. The union $H_\Delta \cup H'$ is homeomorphic to $(\mathbb{S}^2 \times \mathbb{S}^1) \# \cdots \# (\mathbb{S}^2 \times \mathbb{S}^1)$.

The set $\gamma = \partial D_\Gamma$ is a disjoint union of simple closed curves $\gamma_1, \ldots, \gamma_k$ on $\partial H'$, one for each edge class of ϵ. We push each γ_j slightly into $\mathrm{int}(H') \smallsetminus \left(\bigcup V_i'\right)$ to a curve γ_j'. On each γ_j' we perform $\mathrm{lk}(\gamma_j, \gamma_j') + (1/m_j)$ surgery. Note from Lemma 3.4 that in our applications, the curves γ_j will be unknotted and the curves γ_j' will have linking number 0 with them.

These surgeries modify H' to form a new handlebody H''. By [Cannon et al. 2009, Theorem 4.3], (H_Δ, H'') is a Heegaard splitting for $M(\epsilon, m)$ (or, because of ambiguities associated with orientations, the manifold $M(\epsilon, -m)$, with $-m = \{-m_1, \ldots, -m_k\}$, which is homeomorphic with $M(\epsilon, m)$).

For our purposes, it is important to see that these surgeries can be realized by an explicit homeomorphism from H' to H'' defined by Dehn–Lickorish moves. To that end, we enclose γ_j' in a solid torus neighborhood U_j that is joined to γ_j by an annulus A_j. We remove U_j and cut the remaining set along A_j. Let A_j' denote one side of the cut. We may parametrize a neighborhood of A_j' by (θ, s, t), where $\theta \in \mathbb{R} \pmod{2\pi}$ is the angle around the circle γ_j, $s \in [0, 1]$ is the depth into H', and $t \in [0, 1]$ is the distance from A_j'. Then one twists this neighborhood of A_j' by the map $(\theta, s, t) \mapsto (\theta + (1 - t) \cdot m_j \cdot 2\pi, s, t)$ before reattaching A_j' to its partner A_j'' to reconstitute A_j. This twisting operation defines a homeomorphism $\phi : \left[H' \setminus \left(\bigcup U_j\right)\right] \to \left[H' \setminus \left(\bigcup U_j\right)\right]$. One then reattaches the solid tori U_j via the homeomorphisms $\phi|_{\partial U_j}$ to form H'', with an extended homeomorphism $\Phi : H' \to H''$. The homeomorphism Φ is the identity except in a small neighborhood of γ. The new handle disks are $\Phi(E_1), \ldots, \Phi(E_g)$.

We apply this to obtain a surgery description for our construction. Recall from Section 1A that our multipliers were chosen to be ± 1 on latitudinal edge cycles, and any integer m_i on longitudinal edges. We record the result in the following lemma.

Lemma 3.5. *Let Γ and ϵ be as in Section 1A, with handle curves as in Lemma 3.4. Then the manifold $M(\epsilon, m)$ has the following surgery description.*

(1) *There are $k + 1$ simple closed curves $\delta_0', \ldots, \delta_k'$, with each δ_j' parallel to δ_j, pushed to the exterior of the handle of H_Δ. Each δ_j' has surgery coefficient* 0*.*

(2) *Each curve of γ corresponding to a latitudinal edge class γ_{2i} appears with surgery coefficient $m_{2i} = \pm 1$, $i = 0, \ldots, k$.*

(3) *Each curve of γ corresponding to longitudinal edge class γ_{2i+1} has surgery coefficient* $1/m_{2i+1}$. *If one of these multipliers is* 0, *so that the edge collapses to a point and disappears as an edge class, we retain the corresponding curve, but with surgery coefficient* $\frac{1}{0} = \infty$. □

The curves are shown in Figure 9.

3D. ***The knot as the image of the north–south axis.*** It is now an easy matter to identify the image of the north–south axis in our bitwist constructions. In particular, we want to recognize this curve in the associated surgery description of the manifold. The portion of the curve in the handlebody H_Δ is obvious. That portion in the handlebody H_Γ is simple, yet not so obvious. We need a criterion that allows us to recognize it.

To that end, suppose that H_Γ is a pseudohandlebody with one vertex x. Recall that $H_\Gamma \setminus D_\Gamma$ has a natural cone structure from x. We say that an arc α in $H_\Gamma \setminus D_\Gamma$ is *boundary parallel* if there is a disk D in $H_\Gamma \setminus D_\Gamma$ such that $(\partial D) \cap (\mathrm{int}(H_\Gamma)) = \mathrm{int}(\alpha)$ and $(\partial D) \cap (\partial H_\Gamma)$ is an arc α'.

Lemma 3.6. *Suppose $a, b \in (\partial H_\Gamma)\smallsetminus D_\Gamma$ with $a \neq b$. Then the arc $\alpha = ax \cup bx$ (using the cone structure) is boundary parallel, and any arc β that has a and b as endpoints and is boundary parallel is, in fact, isotopic to α.*

Proof. The set D_Γ is a disjoint union of handle disks for H_Γ, hence does not separate ∂H_Γ. There is therefore an arc α' from a to b in $(\partial H_\Gamma)\smallsetminus D_\Gamma$. The disk $x\alpha'$, which uses the cone structure, proves that α is boundary parallel. If β is boundary parallel, as certified by disk E and arc β', we may first assume $\mathrm{int}(E) \subset \mathrm{int}(H_\Gamma \smallsetminus D_\Gamma)$, and then we may straighten E so that, near $(\partial H_\Gamma)\smallsetminus D_\Gamma$, E is part of the cone over β'. The arc β may be slid along E near to β', then isotoped along the cone over β' until it coincides with α. □

In our construction, we are mainly interested in a curve of the form $(Ov \cup Ow)/\epsilon_m$, where O is the center of B and v and w are vertices of Γ, all of which are identified by ϵ_m to a single vertex x in H_Γ. The set $(Ov \cup Ow) \cap H_\Delta$ is immediately apparent. However, we must identify $\beta = (v'v \cup w'w)/\epsilon_m$, where $v' = \left(\frac{3}{4}\right) \cdot v$ and $w' = \left(\frac{3}{4}\right) \cdot w$. The images of v and w in H_Γ are the single vertex x of H_Γ, and the image of β is a cone from x in the cone structure on $H_\Gamma \smallsetminus D_\Gamma$. Therefore, by Lemma 3.6, to identify β it suffices to find a boundary parallel arc in H_Γ with endpoints v' and w'.

The vertices $v' = \left(\frac{3}{4}\right) \cdot v$ and $w' = \left(\frac{3}{4}\right) \cdot w$ lie in $\mathbb{R}^2 \times \{0\}$, disjoint from the disks $\left(\frac{3}{4}\right) \cdot D_\Delta$, i.e., the attaching disks of the handles of D_Δ in $\mathbb{R}^2 \times \{0\}$. Hence, there is an arc α' in $(\mathbb{R}^2 \times \{0\})\smallsetminus\left(\frac{3}{4}\right) \cdot D_\Delta$ from v' to w'. Take the product of α' and a small closed interval with left endpoint 0 in $\mathbb{R}^2 \times [0, \infty) \subset \mathbb{R}^3$. We obtain a disk D in the handlebody H that is the closure of $\mathbb{S}^3 \smallsetminus H_\Delta$. This disk exhibits the complementary arc $\alpha \subset \partial D$ as boundary parallel in H. We fix this arc and construct the handlebodies H' and H''. Provided that the annuli and tori used in constructing H' from H are chosen close enough to the curves $\delta = \partial D_\Delta$ to avoid D, the disk D will also certify that α misses the handle disks E_i of H' so that α is boundary parallel in H'. If the annuli A_j and tori U_j are chosen close enough to $\gamma = \partial D_\Gamma$ to avoid α (but not D), then the homeomorphism $\Phi : H' \to H''$ will fix α and will take the disks E_i to handle disks for H'', and the disk $\Phi(D)$ will show that α is boundary parallel in H''. Thus $(Ov' \cup Ow') \cup \alpha$ represents the curve $(Ov \cup Ow)/\epsilon_m$ as desired.

Now we add this axis to our surgery descriptions. For the simplest construction, with equator e and longitudinal arc NS, and handle curves as shown in Figure 8, the surgery description is obtained by pushing δ_0 slightly into H. Let $N' = \left(\frac{3}{4}\right) \cdot N$ and $S' = \left(\frac{3}{4}\right) \cdot S$ on $\left(\frac{3}{4}\right) \cdot \Gamma \subset \mathbb{R}^2 \times \{0\}$. The arc $(ON' \cup OS')$ runs below the plane $\mathbb{R}^2 \times \{0\}$ in H_Δ. To find the arc α, we take an arc α' from N' to S' in $\mathbb{R}^2 \times \{0\}$ disjoint from the handle, and, fixing the endpoints, push this above $\mathbb{R}^2 \times \{0\}$ slightly. By the above discussion, this gives the desired arc of the axis NS. The surgery diagram and the axis are shown for this example in Figure 10.

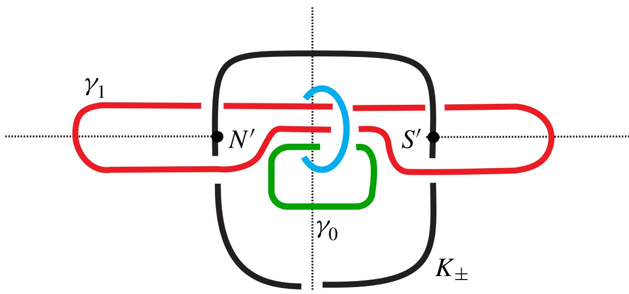

Figure 10. The surgery diagram for $K_\pm$, S_1, and F_1.

4. Two-bridge knots

In this section, we prove that the image of the NS axis in Figure 10 represents the figure-eight knot in $\mathbb{S}^3$ when the surgery coefficient is taken to be $+1$, and the trefoil knot in $\mathbb{S}^3$ when the coefficient is taken to be -1.

More generally, we prove that the NS axis in the general construction represents a two-bridge knot in $\mathbb{S}^3$.

4A. ***Identifying the trefoil and figure-eight.*** We will modify the surgery diagram of Figure 10 by means of Rolfsen twists. We remind the reader of the effect of a Rolfsen twist. We assume we are given an unknotted curve J with surgery coefficient p/q through which pass a number of curves, some of which are surgery curves K_i with surgery coefficients r_i, and some of which may be of interest for some other reasons, such as our knot axis. We perform an n-twist on J. The curves passing through J acquire n full twists as a group. The curve J acquires the new surgery coefficient $p/(q+np)$; in particular, if $p=1$, then a twist of $-q$ will change the coefficient to ∞, and any curve with a surgery coefficient ∞ can be removed from the diagram. Finally, each surgery curve K_i that passes through J acquires the new surgery coefficient $r_i + n \cdot \mathrm{lk}(J, K_i)^2$.

Theorem 4.1. *The surgery description of $M(\epsilon, m)$ for the simple face-pairing of Figure 3 (left) yields the manifold $\mathbb{S}^3$. The image of the north–south axis is the trefoil knot when $m=(-1,1)$ and the figure-eight knot when $m=(1,1)$.*

Proof. We apply Rolfsen twists to our surgery curves in the order γ_1, γ_0, and δ' to change their surgery coefficients, one after the other, to ∞. We trace the effect on the axis $K_{\pm 1}$, and show this in Figure 11.

In detail, we first perform a -1 Rolfsen twist on γ_1. This changes the surgery coefficient on γ_1 to ∞ so that γ_1 can be removed from the diagram. In the process, one negative full twist is added to the axis representing $K_\pm$.

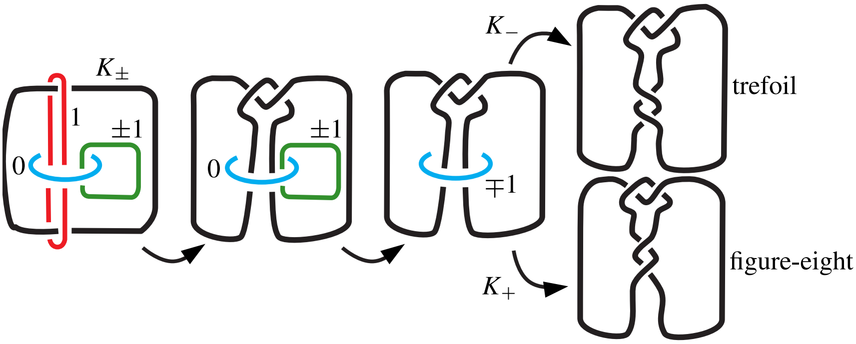

Figure 11. Analyzing S_1, F_1, K_-, and K_+.

We next perform a Rolfsen twist on γ_0 to change its surgery coefficient to ∞ so that it too can be removed from the diagram. If the coefficient on γ_0 was originally 1, this twist must be a -1 twist. If the coefficient on γ_0 was originally -1, this twist must be a $+1$ twist. The coefficient of this twist is added to the 0 coefficient on the δ' curve. The axis is not affected.

Finally, we perform a Rolfsen twist on δ', opposite to its surgery coefficient ∓1 so that its coefficient is changed to ∞. That makes it possible to remove δ' from the diagram. Since the diagram is now empty, we can conclude that the quotient manifold is $\mathbb{S}^3$.

This last twist adds a ±1 full twist to the axis and results in either the trefoil knot for the $(-1, 1)$ multiplier pair or the figure-eight knot for the $(1, 1)$ multiplier pair. $\square$

4B. ***The general case.*** Having analyzed the simplest model face-pairing, we proceed to the general case. Thus we consider the 2-sphere $\mathbb{S}^2 = \partial\mathbb{B}^3$ subdivided by one longitude, the equator e, $k \geq 0$ latitudinal circles in the northern hemisphere, and their reflections in the southern hemisphere. As usual, we pair faces by reflection in the equator. There are $k+1$ face pairs in this model face-pairing.

The general surgery description is given in Lemma 3.5, and illustrated in Figure 9. Section 3D tells us how to recognize the image of the north–south axis in this diagram. It is the union of a boundary parallel arc below the plane $\mathbb{R}^2 \times \{0\}$ from N' to S' and a boundary parallel arc above the plane $\mathbb{R}^2 \times \{0\}$ from N' to S'. Straightening this axis curve and the surgery diagram, we obtain the diagram in Figure 12.

Recall that the integers m_{2i+1} are arbitrary — positive, negative, or zero. The integers m_{2i} are either $+1$ or -1. Note that the surgery curves fall naturally into three families, each with $k+1$ curves: the δ curves, circling the handles with surgery coefficients 0, the latitudinal curves, linking the 0-curves together in a chain and having coefficients $1/m_{2i} = \pm1$, and the longitudinal curves with coefficients

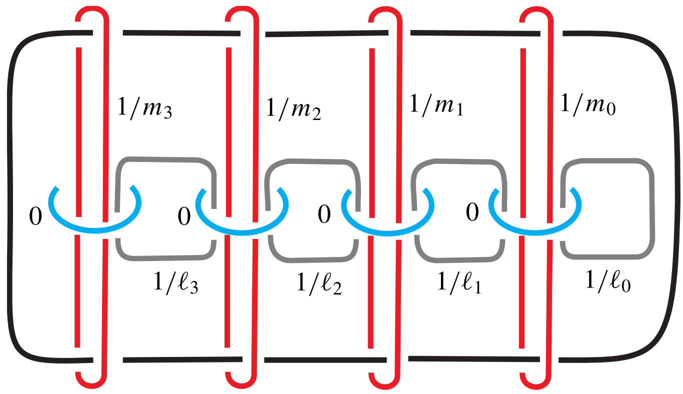

Figure 12. The surgery diagram.

$1/m_{2i+1}$. Each of these curve families has a natural left-to-right order, as in the figure. To simplify notation, we denote the latitudinal curves from left to right by $L_k, L_{k-1}, \dots, L_1, L_0$, and let the corresponding surgery coefficients be denoted $1/\ell_k, 1/\ell_{k-1}, \dots, 1/\ell_i, 1/\ell_0$, respectively (so ℓ_i now replaces notation m_{2i}). We denote the longitudinal curves from left to right by $M_k, M_{k-1}, \dots, M_1, M_0$, and renumber their surgery coefficients to be $1/m_k, 1/m_{k-1}, \dots, 1/m_1, 1/m_0$. We denote the δ curves from left to right by $O_k, O_{k-1}, \dots, O_1, O_0$, with surgery coefficients 0. This decreasing order of subscripts is suggested by the usual inductive description of a rational tangle and the associated continued fraction $[a_0, a_1, \dots, a_n] = a_0 + 1/(a_1 + 1/(a_2 + \cdots + 1/a_n))$, where the coefficient a_n represents the first twist made in the construction and a_0 represents the last twist.

We now prove the following theorem.

Theorem 4.2. *The bitwist manifold $M(\epsilon, m)$ is the 3-sphere $\mathbb{S}^3$. The image of the north–south axis in $\mathbb{S}^3$ is the two-bridge knot which is the numerator closure of the tangle $T(a/b)$ whose rational number invariant a/b is*

$$[2\ell_0, 2m_0, 2\ell_1, 2m_1, \dots, 2\ell_k, 2m_k],$$

or in continued fraction form,

$$2 \cdot \ell_0 + \cfrac{1}{2 \cdot m_0 + \cfrac{1}{2 \cdot \ell_1 + \cfrac{1}{2 \cdot m_1 + \cfrac{1}{\ddots}}}}.$$

Here $\ell_0, \ell_1, \ell_2, \dots$ and $m_0, m_1, m_2, \dots$ are the multipliers of the latitudinal and longitudinal edge cycles, respectively.

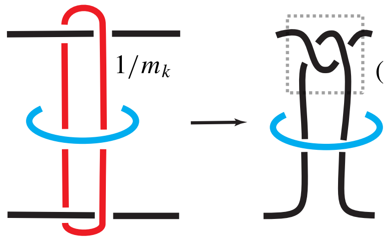

Figure 13. Removing the curve M_k adds $-m_k$ horizontal twists.

Proof. We shall reduce the surgery diagram to the empty diagram by a sequence of Rolfsen twists. This will show that the quotient manifold is $\mathbb{S}^3$. We shall track the development of the axis as we perform those twists and show that, at each stage, the knot is a two-bridge knot. We perform the Rolfsen twists on curves in decreasing order of subscripts in the following order: M_k, L_k, O_k, M_{k-1}, L_{k-1}, O_{k-1}, etc., in order to change surgery coefficients one after the other to ∞. Once a coefficient is ∞, that curve can be removed from the diagram.

There are two cases.

Case 1: If $m_k = 0$, so that $1/m_k = \infty$, we simply remove M_k and the axis is not affected. We may then remove L_k and O_k without affecting the rest of the diagram as follows. First, twist $-\ell_k = \mp 1$ about L_k, to give L_k a surgery coefficient of ∞. This allows us to remove L_k. It also links O_k and O_{k-1} and changes the surgery coefficient on each from 0 to $-\ell_k$, but it does not affect the axis or the other link components. Now twist ℓ_k times about O_k. This allows us to remove O_k, returns the surgery coefficient of O_{k-1} to 0, and leaves the rest of the diagram unchanged. The diagram is now as in Figure 12, only with fewer link components. Thus we repeat the argument with this new link component. By induction, either all $m_j = 0$, all link components can be removed, resulting in $\mathbb{S}^3$ with the unknot as the image of the axis, or eventually we are in case 2.

Case 2: If $m_k \neq 0$, we twist $-m_k$ times about M_k. The coefficient of M_k then becomes ∞ so that M_k can be removed from the surgery diagram. This twists two strands of the axis together as in Figure 13, introducing $-2 \cdot m_k$ half twists into the axis (according to our sign convention). This twist has no effect on the other curves in the diagram.

Note that the axis has formed a rational tangle at the top left of the diagram. To identify the tangle, we will use work of Kauffman and Lambropoulou [2002], with attention to orientation. Our twisting orientation agrees with theirs for horizontal twists, and so at this point, the rational tangle has continued fraction with the single entry $[-2m_k]$.

The proof now proceeds by induction. We will assume that at the j-th step, we have a surgery diagram with image of the axis with the following properties.

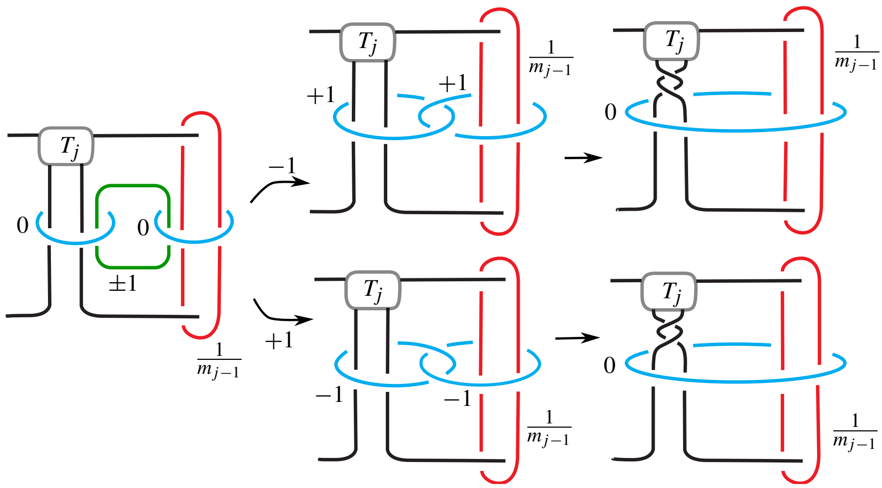

Figure 14. The effect of Rolfsen twists to remove first L_j and then O_j.

(1) In the top-left corner, there is a rational tangle T_j with continued fraction

$$[-2m_j, -2\ell_j, \dots, -2\ell_k, -2m_k].$$

(2) Two strands run from the tangle through the link component O_j.

(3) Link components $M_k, L_k, \dots$ through M_j have been removed.

(4) To the right, the surgery diagram is identical to the original surgery diagram, beginning with link components L_j and running to the right through the components M_0 and L_0. That is, the link components are identical for this portion of the diagram, and the surgery coefficients are also identical.

The next step is to remove link components L_j and O_j. This is shown in Figure 14, for both cases $\ell_j = \pm1$. Carefully, we twist $-\ell_j$ times about L_j. The coefficient of L_j then becomes ∞ so that L_j can be removed from the surgery diagram. That twist adds $-\ell_j$ to the 0 surgery coefficients of O_j and O_{j-1} and links those two curves together with overcrossing having sign equal to $-\ell_j$. This twist has no effect on the axis. Now twist ℓ_j times about O_j. The coefficient of O_j then becomes ∞ so that O_j can be removed from the surgery diagram. The twist returns the surgery coefficient of O_{j-1} back to 0. The twist also adds $2 \cdot \ell_j$ half twists to the two strands of the axis that were running through O_j. Note this yields a new rational tangle, with a vertical twist added to the tangle T_j. Our twisting orientation for vertical twists is opposite that of Kauffman and Lambropoulou [2002], and so the continued fraction of this new tangle becomes $T = [-2\ell_j, -2m_j, \dots, -2\ell_k, -2m_k]$.

We now need to consider M_{j-1}. If $m_{j-1} = 0$, so its surgery coefficient is ∞, we simply remove M_{j-1} from the surgery diagram, and we have completed the

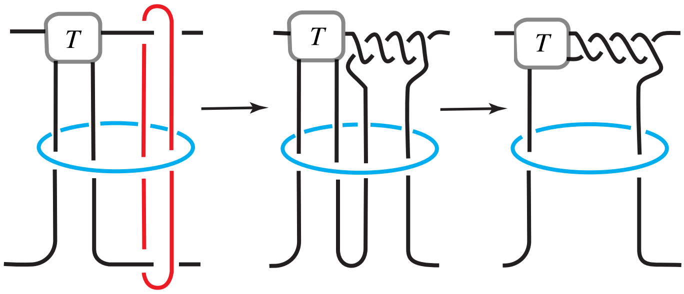

Figure 15. Removing M_{j-1} through twisting.

inductive step. Otherwise, we twist $-m_i$ times about M_i, as in Figure 15, after which four strands of the axis pass through O_{j-1}. However, the central two strands can be isotoped upward through O_{j-1}. This adds $-2m_{j-1}$ horizontal crossings to the tangle T, yielding a tangle T_{j-1}, and completes the inductive step.

After the final step $j = 0$, we have removed all M_j, L_j, O_j from the surgery diagram, yielding $\mathbb{S}^3$, and our axis has become the denominator closure of a rational tangle $T(c/d)$ with continued fraction

$$[-2\ell_0, -2m_0, \ldots, -2\ell_k, -2m_k] = \cfrac{1}{-2\ell_0 + \cfrac{1}{-2m_0 + \cfrac{1}{-2\ell_1 + \cfrac{1}{-2m_1 + \cfrac{1}{\ddots}}}}}.$$

The continued fraction begins with $1/(-2\ell_0 + \cdots)$ instead of $-2\cdot\ell_0 + \cdots$ because ℓ_0 corresponds to a vertical twist. Loosely speaking, horizontal twists correspond to addition and vertical twists correspond to addition and inversion. Hence our knot is the numerator closure of the tangle $T(a/b)$ with $a/b = -d/c$, as in the statement of the theorem. □

Recall from Section 1 that a multiplier function m with values $m_0, \ldots, m_{2k+1}$ is normalized if $m_{2k+1} \neq 0$, and if $m_{2i+1} = 0$ for some $i \in \{0, \ldots, k-1\}$, then $m_{2i} = m_{2i+2}$. The following example helps to motivate this definition.

Example 4.3. Figure 16 shows an example arising from multipliers

$$\begin{array}{ccccccc} m_6 = 3, & m_5 = 0, & m_4 = 0, & m_3 = 2, & m_2 = -3, & m_1 = 0, & m_0 = 2, \\ \ell_6 = 1, & \ell_5 = -1, & \ell_4 = -1, & \ell_3 = -1, & \ell_2 = 1, & \ell_1 = 1, & \ell_0 = 1. \end{array}$$

In the notation of the previous paragraph, the multiplier function has values $\ell_0, m_0, \ell_1, m_1, \ldots, m_6$. This multiplier function is not normalized since $\ell_5 = -\ell_6$ even though $m_5 = 0$. As a result, the second vertical twist cancels the first one, and so

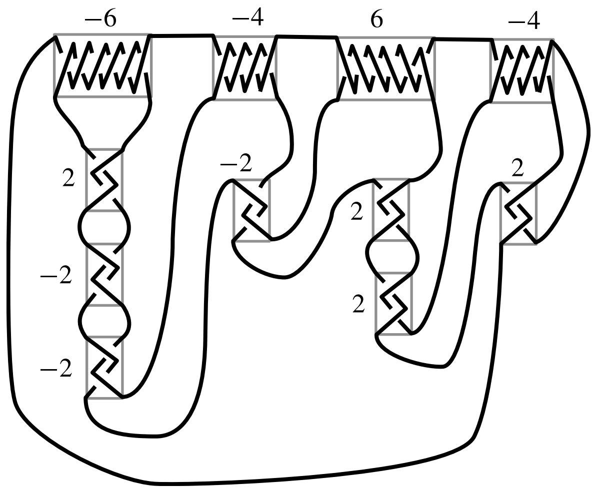

Figure 16. An example.

they can be eliminated. This is consistent with the fact that $x+1/(0+1/y)=x+y$, so that a continued fraction with a term equal to 0 can be simplified. Also notice that if $m_6=0$ instead of $m_6=3$, then the first three vertical twists can be untwisted, and so they can be eliminated. This is consistent with the fact that $x+1/\bigl(y+\frac{1}{0}\bigr)=x$.

Corollary 4.4. *Every normalized multiplier function yields a nontrivial two-bridge knot. Every nontrivial two-bridge knot K is realized by either one or two normalized multiplier functions. Furthermore, if K is the numerator closure of the tangle $T(a/b)$, then it has exactly one such realization if and only if $b^2\equiv 1 \bmod a$.*

Proof. Note that our construction allows us to obtain any two-bridge knot with a rational invariant made only of even integers, by choosing $m_j=0$ appropriately. On the other hand, it is a classical result that any rational number p/q with p odd and q even has a continued fraction expansion of the form $[2a_0,\dots,2a_n]$ with n odd. This result can also be derived by a modification of the Euclidean algorithm. The corollary then follows from Theorem 4.2 and standard results involving two-bridge knots, many of which are contained in [Bleiler and Moriah 1988] and [Kauffman and Lambropoulou 2002]. □

5. Cyclic presentations

Let $M_n(K_m)$ denote the n-fold branched cyclic covering of $\mathbb{S}^3$, branched over the two-bridge knot K_m realized by the multiplier m. It is known (see [Cavicchioli et al. 1999a]) that the fundamental group G_n of $M_n(K_m)$ has a cyclic presentation. We shall show here that the bitwist representation of $M_n(K_m)$ easily leads to the same result.

Definition 5.1. Let $X = \{x_1, \ldots, x_n\}$ be a finite alphabet. Let ϕ denote the cyclic permutation of X that takes each x_i to x_{i+1}, with subscripts taken modulo n. Let $W(X)$ denote a finite word in the letters of X and their inverses. Then the group presentation

$$\langle X \mid W(X), \phi(W(X)), \ldots, \phi^{n-1}(W(X))\rangle$$

is called a *cyclic presentation.*

Theorem 5.2. *The fundamental group of the n-th branched cyclic covering of* $\mathbb{S}^3$*, branched over a two-bridge knot K, has a cyclic presentation.*

Equivalently, the group $G_n = \pi_1(M_n(K_m))$ has a cyclic presentation. Before giving the proof, we recall the algorithm that gives a presentation for the fundamental group of the bitwist manifold $M(\epsilon, m)$. We work with the model faceted 3-ball. We assign a generator $x(f)$ to each face f. We will need to assign a word $W(f, e)$ to each pair (f, e) consisting of a face f and boundary edge e of f, and a word $W(f)$ to each face f.

If f is a face, denote the matching face by f^{-1}. Then $x(f^{-1}) = x(f)^{-1}$. If f is a face and e is a boundary edge of f, then there is a (shortest) finite sequence $(f, e) = (f_1, e_1), (f_2, e_2), \ldots, (f_k, e_k) = (f, e)$ such that $\epsilon(f_i)$ takes e_i onto e_{i+1} and takes f_i onto the face across e_{i+1} from f_{i+1}. We define $W(f, e)$ to be the word $x(f_1) \cdot x(f_2) \cdots x(f_{k-1})$. Finally, if f is a face and $e_1, e_2, \ldots, e_j$ are the edges of f, in order, with assigned multipliers $m_1, m_2, \ldots, m_j$, then we assign f the word

$$W(f) = W(f, e_1)^{m_1} \cdot W(f, e_2)^{m_2} \cdots W(f, e_j)^{m_j}.$$

The next lemma follows from standard results. See also [Cannon et al. 2002, Theorem 4.8].

Lemma 5.3. *The group* $\pi_1(M(\epsilon, m))$ *has presentation*

$$\langle x(f),\ f \text{ a face} \mid W(f),\ f \text{ a face}\rangle$$

Proof of Theorem 5.2. We begin with a model faceted 3-ball and multipliers $\ell_0, m_0, \ldots, \ell_k, m_k$ used to construct $M_1(K_m)$ in Section 4B. We take its n-fold branched cyclic cover branched over the north–south axis. We label the faces of the northern hemisphere $x(i, j)$ as in Figure 17.

We use the same labels $x(i, j)$ as group generators. The corresponding faces and generators for the southern hemisphere are $x(i, j)^{-1}$. We distinguish three types of faces: those bordering on the equator, which are designated as type 0, those touching the poles, which are designated as type 2, and all others, designated type 1. We initially assume that $k > 0$ so that we don't have faces that are both type 0 and type 2. Since edge classes have size 1 or size 2, the words associated with a face-edge pair have length 1 or length 2. Figure 18 shows edges of the three types

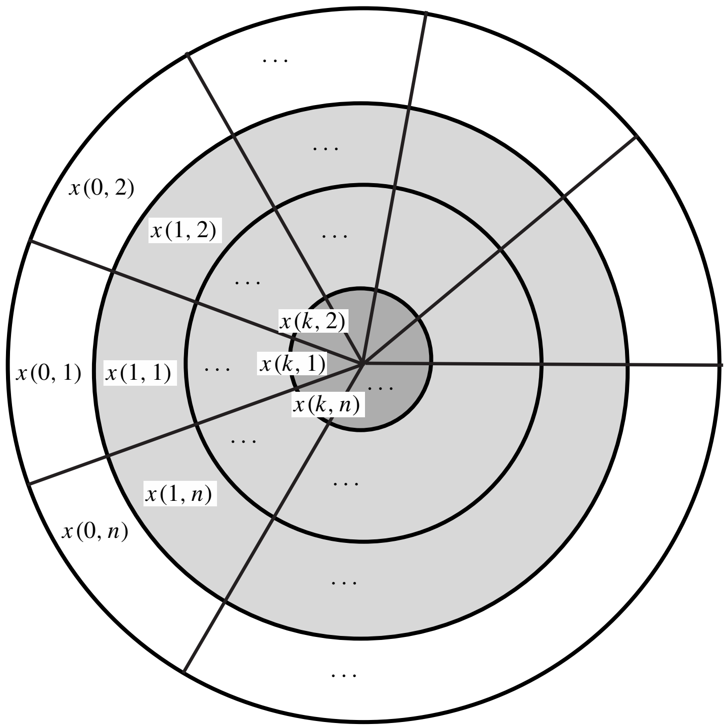

Figure 17. The model for the n-fold branched cyclic cover, with the face generators labeled $x(i, j)$. Faces of type 0 are shaded white, faces of type 1 are shaded light gray, and faces of type 2 are shaded darker gray.

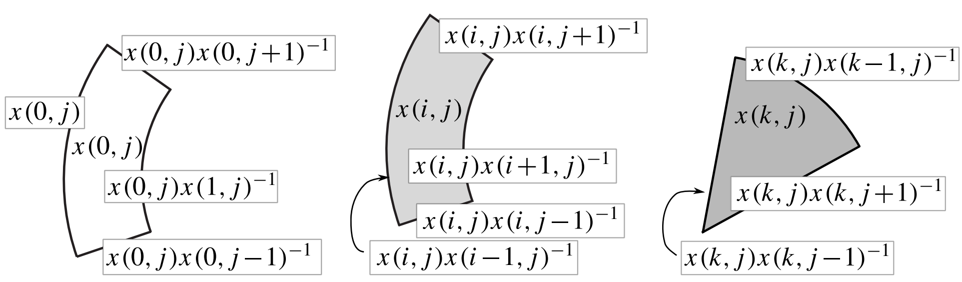

Figure 18. A face of type 0, with face-edge words (left), a face of type 1 (middle), and a face of type 2 (right).

of faces labeled with those face-edge words. These words are then raised to the appropriate powers and multiplied together to give the word associated with the corresponding face. We call these words $R(i, j)$'s since they are the relators of the fundamental group. We have

$$R(0,j) = [x(0,j)]^{\ell_0}[x(0,j)x(0,j+1)^{-1}]^{m_0} \times [x(0,j)x(1,j)^{-1}]^{\ell_1}[x(0,j)x(0,j-1)^{-1}]^{m_0},$$

$$R(i,j) = [x(i,j)x(i-1,j)^{-1}]^{\ell_i}[x(i,j)x(i,j+1)^{-1}]^{m_i} \times [x(i,j)x(i+1,j)^{-1}]^{\ell_{i+1}}[x(i,j)x(i,j-1)^{-1}]^{m_i},$$

$$R(k,j) = [x(k,j)x(k-1,j)^{-1}]^{\ell_k}[x(k,j)x(k,j+1)^{-1}]^{m_k}[x(k,j)x(k,j-1)^{-1}]^{m_k}.$$

We conclude that the fundamental group has a presentation

$$\langle x(i,j) \mid R(i,j),\ i = 0,\dots,k,\ j = 1,\dots,n\rangle.$$

Since each of the multipliers $\ell_0, \ell_1, \dots, \ell_k$ is either $+1$ or -1, the letter $x(1,j)^{\pm 1}$ appears at most once in the relator $R(0,j)$. Similarly, the letter $x(i,j)^{\pm 1}$ appears at most once in the relator $R(i-1,j)$, for $i = 2,\dots,k-1$, and the letter $x(k,j)^{\pm 1}$ appears at most once in the relator $R(k-1,j)$. Hence, these relators may be solved for $x(1,j), x(2,j), \dots, x(k,j)$ iteratively, and then these relators and generators may be removed. The only generators remaining are the generators $x(0,j)$, with $j = 1,\dots,n$; and, with appropriate generator substitutions made, the only remaining relators are the relators $R(k,j)$. The presentation

$$\langle x(0,j) \mid R(k,j),\ j = 1,\dots,n\rangle$$

is clearly a cyclic presentation.

Finally, if $k = 0$, then every face is both type 0 and type 2. In this case the presentation is $\langle x(0,j) \mid R(0,j),\ j = 1,\dots,n\rangle$, which is cyclic. □

5A. ***The Fibonacci and Sieradski manifolds.*** Recall from Section 1 that the n-th branched cyclic cover S_n of the trefoil knot is called the n-th Sieradski manifold. The n-th branched cyclic cover F_n of the figure-eight knot is called the n-th Fibonacci manifold.

We illustrate the above group calculations by proving a well-known theorem.

Theorem 5.4. *The fundamental group $\pi_1(F_n)$ is the $2n$-th Fibonacci group with presentation*

$$\langle x_1,\dots,x_{2n} \mid x_1x_2 = x_3,\ x_2x_3 = x_4,\ \dots,\ x_{2n-1}x_{2n} = x_1,\ x_{2n}x_1 = x_2\rangle.$$

The fundamental group $\pi_1(S_n)$ is the n-th Sieradski group with presentation

$$\langle y_1,\dots,y_n \mid y_1 = y_2y_n,\ y_2 = y_3y_1,\ y_3 = y_4y_2,\ \dots,\ y_n = y_1y_{n-1}\rangle.$$

Proof. The faceted 3-ball that serves as the model for the face-pairings is the same for both manifolds; it is as in Figure 17 with $k = 0$, so without interior latitudinal circles.

For the Fibonacci manifolds, we label the faces of the northern hemisphere as $x(2), x(4), \ldots, x(2n)$. All subscript calculations are modulo $2n$. We obtain the following cyclic presentation for the fundamental group:

$$\langle x(2), x(4), \ldots, x(2n) \mid x(2j) \cdot [x(2j)x(2j+2)^{-1}] \cdot [x(2j)x(2j-2)^{-1}] \rangle,$$

with $j = 1, 2, \ldots, n$. We can then introduce intermediate generators $x(2j-1) = x(2j-2)^{-1} \cdot x(2j)$. The presentation becomes the standard presentation for the $2n$-th Fibonacci group, as desired:

$$\langle x(1), \ldots, x(2n) \mid x(i+2) = x(i) \cdot x(i+1) \rangle.$$

For the Sieradski manifolds, we label the faces of the northern hemisphere as $y(1), y(2), \ldots, y(n)$. Subscript calculations are modulo n. We obtain the following cyclic presentation for the fundamental group:

$$\langle y(1), \ldots, y(n) \mid y(j)^{-1} \cdot [y(j)y(j+1)^{-1}] \cdot [y(j)y(j-1)^{-1}],\ j = 1, \ldots, n \rangle,$$

or, reversing the order of the subscripts so that $x(1) = y(n), \ldots, x(n) = y(1)$,

$$\langle x(1), \ldots, x(n) \mid x(i) = x(i-1) \cdot x(i+1) \rangle,$$

the standard presentation for the n-th Sieradski group. □

5B. *Branched cyclic covers with periodic homology.* In this section we consider first homology groups of our cyclic branched covers of $\mathbb{S}^3$. This is a topic which has received and still receives considerable attention. There are two very different behaviors. The first homology groups of the n-fold cyclic covers M_n of $\mathbb{S}^3$ branched over a knot K are either periodic in n or their orders grow exponentially fast. Specifically, Gordon [1972] proved that when the roots of the Alexander polynomial of K are all roots of unity, then $\mathrm{H}_1(M_n, \mathbb{Z})$ is periodic in n. Riley [1990] and, independently, González-Acuña and Short [1991] proved that if the roots of the Alexander polynomial are not all roots of unity, then the finite values of $\mathrm{H}_1(M_n, \mathbb{Z})$ grow exponentially fast in n. Silver and Williams [2002] extended these results to links and replaced "finite values" with "orders of torsion subgroups". See also [Le 2009; Bergeron and Venkatesh 2013; Brock and Dunfield 2015] for more recent results and conjectures on this topic.

We are particularly fascinated by the first homology of the branched cyclic covers of $\mathbb{S}^3$ branched over the knots that are two-strand braids. These knots are the only two-bridge knots that are not hyperbolic.

The northern hemisphere of the model before bitwisting looks like Figure 17. We construct the n-fold branched cyclic cover of $\mathbb{S}^3$, branched over a knot that is a two-strand braid, by using $k \geq 0$ latitudes and n longitudes in the open northern hemisphere, assigning multipliers -1 to the latitudinal edges, and assigning multipliers $+1$ to all longitudinal edges. We calculate the fundamental group as in the proof of Theorem 5.2 and transform it into a cyclic presentation as explained there. We then abelianize, and let $a_0, a_1, \ldots, a_{2k+2}$ denote the exponent sums of the generators in the defining cyclic word W.

We very briefly indicate by diagram how these integers may be computed. Every relator corresponds to a diagram as follows:

$$R(k, j) \quad \begin{array}{c|ccc} & j-1 & j & j+1 \\ \hline k & 1 & -1 & 1 \\ k-1 & & -1 & \end{array}$$

$$R(i, j) \quad \begin{array}{c|ccc} & j-1 & j & j+1 \\ \hline i+1 & & -1 & \\ i & 1 & 0 & 1 \\ i-1 & & -1 & \end{array}$$

$$R(0, j) \quad \begin{array}{c|ccc} & j-1 & j & j+1 \\ \hline 1 & & -1 & \\ 0 & 1 & 0 & 1 \end{array}$$

We begin with the diagram for $R(k, j)$ and use the diagrams for $R(k-1, j)$, $R(k-2, j), \ldots$ to successively transform the entries in rows $k, k-1, \ldots, 1$ to 0. The defining cyclic word is the final result in row 0.

$$\begin{array}{ccc} 1 & -1 & 1 \\ & -1 & \end{array} \longrightarrow \begin{array}{ccccc} & 0 & 0 & 0 & \\ 1 & -1 & -1+1+1 & -1 & 1 \\ & -1 & 1 & -1 & \end{array}$$

$$\longrightarrow \begin{array}{ccccccc} & & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & 0 & \\ 1 & -1 & -1+1+1 & 1-1+1 & -1+1+1 & -1 & 1 \\ & -1 & 1 & -1 & 1 & -1 & \end{array}$$

$$\longrightarrow \cdots$$

We find that the polynomial $a_0 + a_1 \cdot t + \cdots + a_{2k+2} \cdot t^{2k+2}$ is the cyclotomic polynomial

$$1 - t + t^2 - t^3 + \cdots - t^{2k+1} + t^{2k+2}.$$

(If $2k+3 > n$, then the polynomial folds on itself because powers are to be identified modulo n. However, once $n \geq 2k+3$, there is no folding.)

Remark 5.5. The computation indicated by the diagram is a continued fraction algorithm. For the fundamental group of a general two-bridge knot, the corresponding polynomial may be taken to be the numerator of the continued fraction

$$Q_0 - \cfrac{1}{Q_1 - \cfrac{1}{Q_2 - \cfrac{1}{\ddots - \cfrac{1}{Q_k}}}},$$

where

$$Q_i(t) = m_i t - (\ell_i + \ell_{i+1} + 2m_i) + m_i t^{-1} \quad \text{for } 0 \le i \le k-1$$

and

$$Q_k(t) = m_k t - (\ell_k + 2m_k) + m_k t^{-1}.$$

We shall prove that, for a given knot realized as a two-strand braid, the abelianizations of the fundamental group of the n-fold branched cover are periodic functions of n. However, as a warm up, we use row reduction of the presentation matrix to prove the much easier theorem that no two of the Fibonacci groups $F(n)$ are isomorphic for $n > 1$ since no two of the abelianizations have the same order. Johnson [1976, page 35] poses this problem as an exercise and suggests using the two-variable presentation of the group. We use the n-variable presentation and note that the Fibonacci numbers $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2, \ldots$ appear in a very natural way. In this case we have the behavior of exponential growth of orders.

Theorem 5.6. *Let*

$$F(n) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+1} = x_{i+2} \text{ for all } i \rangle,$$

with subscripts calculated modulo n. For odd n, the order of the abelianization is the sum $f_{n-1} + f_{n+1}$ of two Fibonacci numbers. For even n, the order is $f_{n-1} + f_{n+1} - 2$.

Remark 5.7. Recall that for even n these abelianizations are the first homology groups of the Fibonacci manifolds. This theorem gives successive orders of $1, 1, 4, 5, 11, 16, 29, 45, 76, 121, \ldots$ for the abelianizations of the Fibonacci groups. It is clear from the definition of the Fibonacci numbers that these numbers are strictly increasing after the numbers $1, 1$. These numbers are also known as the *associated Mersenne numbers* [Sloane and Guy 1991]. The sums $f_{n-1} + f_{n+1}$ are also known as Lucas numbers.

Proof. The presentation matrix has the form

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ & & \cdots & & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The absolute value of the determinant of this matrix is the order of the abelianization of the group unless the determinant is 0. In that case, the group is infinite. The goal is to move the entries in the lower-left corner to the right by adding multiples of the upper rows. These operations do not change the determinant.

We use the upper rows in descending order, with each successive row moving the lower-left 2×2 matrix one column to the right. We first trace the evolution of the two entries in the next-to-last row:

$$(-1,0) \to (1,-1) \to (-2,1) \to (3,-2) \to (-5,3) \to (8,-5) \to \cdots .$$

The reader will easily identify the first in the k-th pair as $(-1)^k f_k$, and the second as $(-1)^{k-1} f_{k-1}$. Since the second of these, namely $(1,-1)$, coincides with the first pair in the bottom row, we see that the bottom row evolves just one step ahead of the next-to-last row. Thus after k moves, the 2×2 matrix evolves into the matrix

$$\begin{pmatrix} (-1)^k f_k & (-1)^{k-1} f_{k-1} \\ (-1)^{k+1} f_{k+1} & (-1)^k f_k \end{pmatrix},$$

which has determinant $f_k^2 - f_{k+1} \cdot f_{k-1} = (-1)^{k-1}$. After the appropriate number of moves, this matrix will be added to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

from the lower-right corner to form the very last lower-right-corner matrix

$$\begin{pmatrix} (-1)^k f_k + 1 & (-1)^{k-1} f_{k-1} + 1 \\ (-1)^{k+1} f_{k+1} & (-1)^k f_k + 1 \end{pmatrix}.$$

The matrix then has determinant

$$[f_k^2 + 2 \cdot (-1)^k \cdot f_k + 1] - [f_{k+1} \cdot f_{k-1} + (-1)^{k+1} f_{k+1}] \\ = (-1)^{k+1} + 1 + (-1)^k [f_k + f_{k+2}].$$

The absolute value of this determinant is the order of the abelianization, and since the last value of k is $n-1$, it agrees with the value claimed in the theorem. $\square$

$$
\begin{array}{c}
\overbrace{\hphantom{a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_j \quad\quad}}^{j+1}\ \overbrace{\hphantom{\quad a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_j \quad\quad\quad\quad}}^{k} \\
\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_j & & & & & & \\
 & a_0 & a_1 & a_2 & \cdots & a_j & & & \mathbf{0} & & \\
 & & a_0 & a_1 & a_2 & \cdots & a_j & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & \ddots & & & & \\
a_j & & & \mathbf{0} & & & a_0 & a_1 & a_2 & \cdots & a_j \\
a_{j-1} & a_j & & & & & & a_0 & a_1 & \cdots & a_{j-1} \\
 & \ddots & & & & & & & & \ddots & \\
a_1 & a_2 & \cdots & a_j & & & & & & & a_0
\end{pmatrix}
\end{array}
$$

Figure 19. The relator matrix for $n = j + 1 + k$.

For the moment, we fix two integers $j > 0$ and $k \geq 0$, and let G_n, with $n = j+1+k$, denote an abelian group with generators $x_0, x_1, x_2, \ldots$ such that $x_i = x_{i+n}$ and with relators $a_0 \cdot x_i + a_1 \cdot x_{i+1} + \cdots + a_j \cdot x_{i+j}$ for each i. Then the group has a circulant relator matrix of the form shown in Figure 19. In the following theorem we have the behavior of periodic homology groups.

Theorem 5.8. *Let j, k, and G_n be as immediately above, so that $n = j + 1 + k$. Assume that $p(t) = a_0 + a_1 \cdot t + \cdots + a_j \cdot t^j$ is a cyclotomic polynomial, by which we mean that there is a polynomial $q(t) = b_0 + b_1 \cdot t + \cdots + b_\ell \cdot t^\ell$ such that $p(t) \cdot q(t) = 1 - t^{j+\ell}$. Then the groups G_n and $G_{n+j+\ell}$ are isomorphic.*

Proof. We manipulate the relator matrix for $G_{n+j+\ell}$ using integral row and column operations. See Figure 19. We use the rows at the top of the matrix to remove entries from the triangle at the lower-left corner of the matrix.

Let x be such an entry in row R_a. Let R_b denote the row whose initial entry on the diagonal is above x. Subtract from row R_a the expression

$$x \cdot [b_0 \cdot R_b + b_1 \cdot R_{b+1} + \cdots + b_\ell \cdot R_{b+\ell}].$$

The effect is to move entry x to the right $j + \ell$ places. Similarly, we move all entries in the lower-left triangle $j + \ell$ places to the right. Because $a_0 = \pm 1$, we may use column operations to make every entry to the right of the first $j + \ell$ a_0's equal to 0. The lower-right $n \times n$ block of the resulting matrix is the relator matrix for G_n. The theorem follows. $\square$

Remark 5.9. The same calculation can be carried out if the polynomial is any integer multiple $\alpha \cdot p(t)$ of a cyclotomic polynomial $p(t)$, except that the diagonal entries above the periodic box all become α's. Thus the abelianization has a periodic component together with an increasing direct sum of $\mathbb{Z}_\alpha$'s. It can be shown that these are the only polynomials with these periodicity properties.

Corollary 5.10. *If K is a knot that is a two-strand braid and M_n is the n-fold cyclic branched cover of $\mathbb{S}^3$ over K, then the homology groups $H_1(M_n)$ are periodic in n.*

Remark 5.11. Lambert [2010] explicitly calculated all of the homology groups of the branched cyclic covers of $\mathbb{S}^3$, branched over knots that are two-strand braids. These are the only two-bridge knots that are not hyperbolic. His tables give an explicit picture of the periodicity we have just proved. Rolfsen [1976] notes that the period for the trefoil is 6. We shall also see that as follows.

Proof of Corollary 5.10. It suffices to find the appropriate polynomials $q(t)$, and thereby determine the period. If $p(t) = 1 - t + t^2$, as for the trefoil, then the appropriate $q(t)$ of smallest degree is $q(t) = 1 + t - t^3 - t^4$ so that the period is $2+4=6$. With five half twists, $p(t)=1-t+t^2-t^3+t^4$ and $q(t)=1+t-t^5-t^6$ and the period is $4+6=10$. Each added pair of half twists in the braid adds two terms to $p(t)$, multiplies the negative entries of $q(t)$ by t^2, and increases the period by 4. □

Remark 5.12. By [Gordon 1972], the homology groups $H_1(M_n)$ of the cyclic branched covers M_n of the complement of a knot K are periodic with period dividing m if and only if the first Alexander invariant (the quotient of the first two Alexander polynomials) of K is a divisor of the polynomial $t^m - 1$. Furthermore, if the first Alexander invariant is a divisor of $t^m - 1$ and n is a positive integer, then $H_1(M_n) = H_1(M_{(m,n)})$, where (m, n) is the greatest common divisor of m and n. Since the first Alexander invariant of the trefoil knot is $1-t+t^2$, which divides t^6-1, Gordon's theorem shows that the first homology groups of the cyclic branched covers of the trefoil knot are periodic with period 6 and $H_1(S_{6j+2}) = H_1(S_{6j+4})$ for all j.

We use the calculation of the period of the trefoil in establishing the next theorem.

Theorem 5.13. *No two of the Sieradski groups are isomorphic. Hence no two of the branched cyclic covers of $\mathbb{S}^3$, branched over the trefoil knot, are homeomorphic.*

Proof. Milnor [1975] defines the Brieskorn manifold $M(p, q, r)$ to be the orientable closed 3-manifold obtained by intersecting the complex algebraic surface given by $z_1^p + z_2^q + z_3^r = 0$ with the unit sphere given by $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Here p, q, r should be integers at least 2. Theorem 2.1 of [Cavicchioli et al. 1998], by Cavicchioli, Hegenbarth, and A. C. Kim, states that S_n is the Brieskorn manifold $M(2, 3, n)$. This follows from the fact that S_n is the n-fold cyclic branched cover of $\mathbb{S}^3$ branched over the trefoil knot, which is the torus knot of type $(2, 3)$, and

Lemma 1.1 of [Milnor 1975], which states that the Brieskorn manifold $M(p, q, r)$ is the r-fold cyclic branched cover of $\mathbb{S}^3$ branched over a torus link of type (p, q).

The first few n-fold cyclic covers of $\mathbb{S}^3$ branched over the right-hand trefoil knot are discussed in Section 10D of Rolfsen's book [1976], which begins on page 304. Here are the results.

- $n = 1$: The manifold S_1 is the 3-sphere $\mathbb{S}^3$, and so $G_1 = 1$.
- $n = 2$: The manifold S_2 is the lens space $L(3, 1)$, so $G_2 \cong \mathbb{Z}/3\mathbb{Z}$.
- $n = 3$: The manifold S_3 is the spherical 3-manifold with fundamental group G_3 the quaternion group of order 8. It appears in Example 7.2 of [Cannon et al. 2002]. This group might be called the binary Klein 4-group.
- $n = 4$: The manifold S_4 is the spherical 3-manifold with fundamental group G_4 the binary tetrahedral group.
- $n = 5$: The manifold S_5 is the spherical 3-manifold with fundamental group G_5 the binary icosahedral group. In other words, this is the Poincaré homology sphere.
- $n = 6$: The manifold S_6 is the Heisenberg manifold. Here

$$G_6 \cong \langle x, y : [x, [x, y]] = [y, [x, y]] = 1\rangle.$$

Milnor [1975] proves that $M(2, 3, n)$, which we know is homeomorphic to S_n, is an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-manifold for $n \geq 7$. It follows that $G_1, \ldots, G_6$ are distinct and that they are not $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ manifold groups. Because of this and Milnor's result that S_n is an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-manifold for $n \geq 7$, to prove that the groups G_n are distinct, it suffices to prove that the groups G_n are distinct for $n \geq 7$.

As stated on page 304 of [Rolfsen 1976], for every positive integer n the first homology group $H_1(S_n)$ is $\mathbb{Z} \oplus \mathbb{Z}$, 0, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when $n \equiv 0, \pm 1, \pm 2$, or $3 \bmod 6$. So to prove that Sieradski groups G_m and G_n are distinct, we may assume that $m \equiv \pm n \bmod 6$.

For the rest of this section suppose that $n \geq 7$. Milnor [1975] (see the bottom of page 213 and Lemma 3.1) proved that G_n is isomorphic to the commutator subgroup of the centrally extended triangle group

$$\Gamma(2, 3, n) = \langle \gamma_1, \gamma_2, \gamma_3 : \gamma_1^2 = \gamma_2^3 = \gamma_3^n = \gamma_1\gamma_2\gamma_3\rangle.$$

Let $\Delta(2, 3, n) = \langle \delta_1, \delta_2, \delta_3 : \delta_1^2 = \delta_2^3 = \delta_3^n = \delta_1\delta_2\delta_3 = 1\rangle$, a homomorphic image of $\Gamma(2, 3, n)$. The group $\Delta(2, 3, n)$ is the group of orientation-preserving elements of the $(2, 3, n)$-triangle group. Let $\Delta'(2, 3, n)$ denote the commutator subgroup of $\Delta(2, 3, n)$. We see that the quotient group $\Delta(2, 3, n)/\Delta'(2, 3, n)$ is isomorphic to the group generated by the elements $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in $\mathbb{Z}^3$ with relations corresponding to a matrix which row reduces as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & n \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & n \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & n \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \\ 0 & 0 & n \end{bmatrix}.$$

So $\Delta(2,3,n)/\Delta'(2,3,n)$ is a cyclic group of order $k = \mathrm{GCD}(6,n)$. This computation also shows that $\delta_1 \in \Delta'(2,3,n)$ if and only if $n \not\equiv 0 \bmod 2$, that $\delta_2 \in \Delta'(2,3,n)$ if and only if $n \not\equiv 0 \bmod 3$, and that δ_3^k is the smallest power of δ_3 in $\Delta'(2,3,n)$. In particular δ_3^k is a nontrivial elliptic element of $\Delta'(2,3,n)$. Every element of $\Delta'(2,3,n)$ which commutes with δ_3^k must fix the fixed point of δ_3^k. It easily follows that the center of $\Delta'(2,3,n)$ is trivial, and in the same way that the center of $\Delta(2,3,n)$ is trivial.

Since the kernel of the homomorphism from $\Gamma(2,3,n)$ to $\Delta(2,3,n)$ is generated by the central element $\gamma_1\gamma_2\gamma_3$ and the center of $\Delta(2,3,n)$ is trivial, it follows that the kernel of this homomorphism is the center of $\Gamma(2,3,n)$. So $\Gamma(2,3,n)$ modulo its center is isomorphic to $\Delta(2,3,n)$. Similarly, G_n modulo its center is isomorphic to $\Delta'(2,3,n)$.

Now suppose that $n \equiv \pm 1 \bmod 6$. Then G_n modulo its center is isomorphic to $\Delta'(2,3,n) = \Delta(2,3,n)$. The largest order of a torsion element in $\Delta(2,3,n)$ is n. So G_m and G_n are distinct if $m \equiv n \equiv \pm 1 \bmod 6$. Next suppose that $n \equiv \pm 2 \bmod 6$. In this case the largest order of a torsion element in $\Delta'(2,3,n)$ is $n/2$. So G_m and G_n are distinct if $m \equiv n \equiv \pm 2 \bmod 6$. The same argument is valid if $n \equiv 3 \bmod 6$. Finally suppose that $n \equiv 0 \bmod 6$. In this case neither δ_1 nor δ_2 are in $\Delta'(2,3,n)$. In this case every torsion element in $\Delta'(2,3,n)$ is conjugate to a power of δ_3^6, which has order $n/6$. Again G_m and G_n are distinct if $m \equiv n \equiv 0 \bmod 6$. □

6. History

There is a large literature concerning the Fibonacci groups, the Sieradski groups, their generalizations, cyclic presentations of groups, the relationship between cyclic presentations and branched cyclic covers of manifolds, two-bridge knots, and their generalizations. We are incapable of digesting, let alone giving an adequate summary of, this work. We plead forgiveness for having omitted important and beautiful work and for misrepresenting work that we have not adequately studied.

6A. ***The Fibonacci groups.*** John Conway told the first-named author of this paper that he created the Fibonacci group $F(5)$, with presentation

$$\langle x_1, \dots, x_5 \mid x_1x_2 = x_3,\ x_2x_3 = x_4,\ x_3x_4 = x_5,\ x_4x_5 = x_1,\ x_5x_1 = x_2 \rangle,$$

and asked that his graduate students calculate its structure as an exercise to demonstrate that it is not easy to read the structure of a group from a group presentation. For example, our straightforward coset enumeration program creates four layers and more than 200 vertices before the coset graph collapses to its final 11 elements. Conway [1965] presented the calculation as a problem. The definition was immediately generalized to give the group $F(n)$. Coset enumeration showed that $F(n)$ is finite for $n < 6$ and for $n = 7$. The Cayley graph for group $F(6)$ can be constructed systematically and recognized as a 3-dimensional infinite Euclidean group. Roger Lyndon proved, using small cancellation theory, that $F(n)$ is infinite if $n \geq 11$ (unpublished). A. M. Brunner [1974] proved that $F(8)$ and $F(10)$ are infinite. George Havas, J. S. Richardson, and Leon S. Sterling [Havas et al. 1979] showed that $F(9)$ has a quotient of order $152 \cdot 5^{18}$, and, finally, M. F. Newman [1990] proved that $F(9)$ is infinite. Derek F. Holt [1995] later reported a proof by computer that $F(9)$ is automatic, from which it could be seen directly from the word-acceptor that the generators have infinite order.

At the International Congress in Helsinki (1978), Bill Thurston was advertising the problem (eventually solved by Misha Gromov) of proving that a group of polynomial growth has a nilpotent subgroup of finite index. The first-named author brought up the example of $F(6)$ as such a group. Thurston immediately recognized the group as a branched cyclic cover of $\mathbb{S}^3$, branched over the figure-eight knot. And before our dinner of reindeer steaks was over, Thurston had conjectured that the even-numbered Fibonacci groups were probably also branched cyclic covers of $\mathbb{S}^3$, branched over the figure-eight knot. This conjecture was verified by H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia [Hilden et al. 1992] and by H. Helling, A. C. Kim, and J. L. Mennicke [Helling et al. 1998]. C. Maclachlan [1995] proved that, for odd n, the group $F(n)$ is not a fundamental group of a hyperbolic 3-orbifold of finite volume.

6B. ***Sieradski manifolds.*** The Sieradski manifolds have a similar rich history, but not one we know as well. They were introduced by A. Sieradski [1986], who used the same faceted 3-ball that we employ, though his face-pairings were different. Richard M. Thomas [1991] showed that the Sieradski groups, which he calls $G(n)$, are infinite if and only if $n \geq 6$ and that $G(6)$ is metabelian. Cavicchioli, Hegenbarth, and A. C. Kim [Cavicchioli et al. 1998] showed that the Sieradski manifolds are branched over the trefoil knot.

6C. ***Cyclic presentations.*** Cyclic presentations are particularly interesting because of their connections with branched cyclic coverings of 3-manifolds. Fundamental results about cyclic presentations appear in the book *Presentations of groups* by D. L. Johnson [1976, Chapter 16]. Arye Juhász [2007] considered the question of when cyclically presented groups are finite. Andrzej Szczepański and Andrei Y.

Vesnin [2000] asked which cyclically presented groups can be groups of hyperbolic 3-orbifolds of finite volume and which cannot. Cavicchioli and Fulvia Spaggiari [2006] showed that nonisomorphic cyclically presented groups can have the same polynomial.

6D. ***Dunwoody manifolds.*** M. J. Dunwoody [1995] managed to enumerate, with parameters, a large class of 3-manifolds admitting Heegaard splittings with cyclic symmetry. The fundamental groups were all cyclically presented. He observed that the polynomials associated with the cyclic presentations were Alexander polynomials of knots and asked whether the spaces were in fact branched cyclic covers of $\mathbb{S}^3$, branched over knots or links. Cavicchioli, Hegenbarth, and A. C. Kim [Cavicchioli et al. 1999a] showed that the Dunwoody manifolds included branched covers with singularities that were torus knots of a specific type. L. Grasselli and Michele Mulazzani [2001] showed that Dunwoody manifolds are cyclic coverings of lens spaces branched over (1, 1)-knots. Cavicchioli, Beatrice Ruini, and Spaggiari [Cavicchioli et al. 2001] proved Dunwoody's conjecture that the Dunwoody manifolds are n-fold cyclic coverings branched over knots or links. Soo Hwan Kim and Yangkok Kim [2004] determined the Dunwoody parameters explicitly for a family of cyclically presented groups that are the n-fold cyclic coverings branched over certain torus knots and certain two-bridge knots. Nurullah Ankaralioglu and Huseyin Aydin [2008] identified certain of the Dunwoody parameters with generalized Sieradski groups.

6E. ***Two-bridge knots.*** The first general presentation about the branched cyclic coverings of the two-bridge knots seems to be that of Jerome Minkus [1982]. A very nice presentation appears in [Cavicchioli et al. 1999b], where cyclic presentations are developed that correspond to cyclically symmetric Heegaard decompositions. In that paper, Cavicchioli, Ruini, and Spaggiari showed that the polynomial of the presentation is the Alexander polynomial. They use the very clever and efficient RR-system descriptions of the Heegaard decompositions. They pass from the Heegaard decompositions to face-pairings and determine many of the geometric structures. Mulazzani and Vesnin [2001] exhibited the many ways cyclic branched coverings can be viewed: polyhedral, Heegaard, Dehn surgery, colored graph constructions.

In addition to these very general presentations, there are a number of concrete special cases in the literature [Bleiler and Moriah 1988; Kim et al. 1998; Kim 2000; Kim and Kim 2003; 2004; Jeong 2006; Jeong and Wang 2008; Grasselli and Mulazzani 2009; Telloni 2010].

Significant progress has been made beyond the two-bridge knots. Maclachlan and A. Reid [1997] and Vesnin and A. C. Kim [1998] considered 2-fold branched covers over certain 3-braids. Alexander Mednykh and Vesnin [1995] considered 2-fold branched covers over Turk's head links.

Alessia Cattabriga and Mulazzani [Mulazzani 2003; Cattabriga and Mulazzani 2003] developed strongly cyclic branched coverings with cyclic presentations over the class of (1, 1) knots, which includes all of the two-bridge knots as well as many knots in lens spaces. P. Cristofori, Mulazzani, and Vesnin [Cristofori et al. 2007] described strongly cyclic branched coverings of knots via $(g, 1)$-decompositions. Every knot admits such a description.

Acknowledgements

We thank the referees of this and an earlier version of this paper for numerous helpful comments. Purcell is partially supported by NSF grant DMS-1252687 and by ARC grant DP160103085.

References

[Ankaralioglu and Aydin 2008] N. Ankaralioglu and H. Aydin, "Some Dunwoody parameters and cyclic presentations", *Gen. Math.* **16**:2 (2008), 85–93. MR 2439228 Zbl 1240.57001

[Bergeron and Venkatesh 2013] N. Bergeron and A. Venkatesh, "The asymptotic growth of torsion homology for arithmetic groups", *J. Inst. Math. Jussieu* **12**:2 (2013), 391–447. MR 3028790 Zbl 1266.22013

[Bleiler and Moriah 1988] S. A. Bleiler and Y. Moriah, "Heegaard splittings and branched coverings of B^3", *Math. Ann.* **281**:4 (1988), 531–544. MR 958258 Zbl 0627.57007

[Boyer et al. 2013] S. Boyer, C. M. Gordon, and L. Watson, "On L-spaces and left-orderable fundamental groups", *Math. Ann.* **356**:4 (2013), 1213–1245. MR 3072799 Zbl 1279.57008

[Brock and Dunfield 2015] J. F. Brock and N. M. Dunfield, "Injectivity radii of hyperbolic integer homology 3-spheres", *Geom. Topol.* **19**:1 (2015), 497–523. MR 3318758 Zbl 1312.57022

[Brunner 1974] A. M. Brunner, "The determination of Fibonacci groups", *Bull. Austral. Math. Soc.* **11** (1974), 11–14. MR 0349849 Zbl 0282.20025

[Cannon et al. 2000] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Introduction to twisted face-pairings", *Math. Res. Lett.* **7**:4 (2000), 477–491. MR 1783626 Zbl 0958.57021

[Cannon et al. 2002] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Twisted face-pairing 3-manifolds", *Trans. Amer. Math. Soc.* **354**:6 (2002), 2369–2397. MR 1885657 Zbl 0986.57015

[Cannon et al. 2003] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Heegaard diagrams and surgery descriptions for twisted face-pairing 3-manifolds", *Algebr. Geom. Topol.* **3** (2003), 235–285. MR 1997321 Zbl 1025.57026

[Cannon et al. 2009] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Bitwist 3-manifolds", *Algebr. Geom. Topol.* **9**:1 (2009), 187–220. MR 2482073 Zbl 1179.57030

[Cattabriga and Mulazzani 2003] A. Cattabriga and M. Mulazzani, "Strongly-cyclic branched coverings of (1, 1)-knots and cyclic presentations of groups", *Math. Proc. Cambridge Philos. Soc.* **135**:1 (2003), 137–146. MR 1990837 Zbl 1050.57003

[Cavicchioli and Spaggiari 2006] A. Cavicchioli and F. Spaggiari, "Certain cyclically presented groups with the same polynomial", *Comm. Algebra* **34**:8 (2006), 2733–2744. MR 2250565 Zbl 1104.20031

[Cavicchioli et al. 1998] A. Cavicchioli, F. Hegenbarth, and A. C. Kim, "A geometric study of Sieradski groups", *Algebra Colloq.* **5**:2 (1998), 203–217. MR 1682984 Zbl 0902.57023

[Cavicchioli et al. 1999a] A. Cavicchioli, F. Hegenbarth, and A. C. Kim, "On cyclic branched coverings of torus knots", *J. Geom.* **64**:1-2 (1999), 55–66. MR 1675959 Zbl 0919.57004

[Cavicchioli et al. 1999b] A. Cavicchioli, B. Ruini, and F. Spaggiari, "Cyclic branched coverings of 2-bridge knots", *Rev. Mat. Complut.* **12**:2 (1999), 383–416. MR 1740466 Zbl 0952.57001

[Cavicchioli et al. 2001] A. Cavicchioli, B. Ruini, and F. Spaggiari, "On a conjecture of M. J. Dunwoody", *Algebra Colloq.* **8**:2 (2001), 169–218. MR 1838517 Zbl 0994.57004

[Conway 1965] J. H. Conway, "Problems and solutions: advanced problems: 5327", *Amer. Math. Monthly* **72**:8 (1965), 915. MR 1533441

[Cristofori et al. 2007] P. Cristofori, M. Mulazzani, and A. Y. Vesnin, "Strongly-cyclic branched coverings of knots via $(g, 1)$-decompositions", *Acta Math. Hungar.* **116**:1-2 (2007), 163–176. MR 2341048 Zbl 1164.57001

[Dunfield and Thurston 2006] N. M. Dunfield and W. P. Thurston, "Finite covers of random 3-manifolds", *Invent. Math.* **166**:3 (2006), 457–521. MR 2257389 Zbl 1111.57013

[Dunwoody 1995] M. J. Dunwoody, "Cyclic presentations and 3-manifolds", pp. 47–55 in *Groups — Korea '94* (Pusan, 1994), edited by A. C. Kim and D. L. Johnson, De Gruyter, Berlin, 1995. MR 1476948 Zbl 0871.20026

[González-Acuña and Short 1991] F. González-Acuña and H. Short, "Cyclic branched coverings of knots and homology spheres", *Rev. Mat. Univ. Complut. Madrid* **4**:1 (1991), 97–120. MR 1142552 Zbl 0756.57001

[Gordon 1972] C. M. Gordon, "Knots whose branched cyclic coverings have periodic homology", *Trans. Amer. Math. Soc.* **168** (1972), 357–370. MR 0295327 Zbl 0238.55001

[Gordon and Lidman 2014] C. M. Gordon and T. Lidman, "Taut foliations, left-orderability, and cyclic branched covers", *Acta Math. Vietnam.* **39**:4 (2014), 599–635. MR 3292587 Zbl 1310.57023

[Grasselli and Mulazzani 2001] L. Grasselli and M. Mulazzani, "Genus one 1-bridge knots and Dunwoody manifolds", *Forum Math.* **13**:3 (2001), 379–397. MR 1831091 Zbl 0963.57002

[Grasselli and Mulazzani 2009] L. Grasselli and M. Mulazzani, "Многообразия Зейферта и (1, 1)-узлы", *Sibirsk. Mat. Zh.* **50**:1 (2009), 28–39. Translated as "Seifert manifolds and (1, 1)-knots" in *Siberian Math. J.* **50**:1 (2009), 22–31. MR 2502871 Zbl 1224.57004

[Havas et al. 1979] G. Havas, J. S. Richardson, and L. S. Sterling, "The last of the Fibonacci groups", *Proc. Roy. Soc. Edinburgh Sect. A* **83**:3-4 (1979), 199–203. MR 549854 Zbl 0416.20026

[Helling et al. 1998] H. Helling, A. C. Kim, and J. L. Mennicke, "A geometric study of Fibonacci groups", *J. Lie Theory* **8**:1 (1998), 1–23. MR 1616794 Zbl 0896.20026

[Hilden et al. 1992] H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia, "On the Borromean orbifolds: geometry and arithmetic", pp. 133–167 in *Topology '90* (Columbus, OH, 1990), edited by B. Apanasov et al., Ohio State University Mathematics Research Institute Publications **1**, De Gruyter, Berlin, 1992. MR 1184408 Zbl 0787.57001

[Holt 1995] D. F. Holt, "An alternative proof that the Fibonacci group $F(2, 9)$ is infinite", *Experiment. Math.* **4**:2 (1995), 97–100. MR 1377412 Zbl 0853.20019

[Hu 2015] Y. Hu, "Left-orderability and cyclic branched coverings", *Algebr. Geom. Topol.* **15**:1 (2015), 399–413. MR 3325741 Zbl 1312.57001

[Jeong 2006] K.-W. Jeong, "On the cyclic branched coverings of the 2-bridge knot $b(17, 4)$", *Algebra Colloq.* **13**:1 (2006), 173–180. MR 2188485 Zbl 1093.57001

[Jeong and Wang 2008] K.-W. Jeong and M.-O. Wang, “Notes on more Fibonacci groups”, *Algebra Colloq.* **15**:4 (2008), 699–706. MR 2452002 Zbl 1156.57012

[Johnson 1976] D. L. Johnson, *Presentations of groups*, London Mathematical Society Lecture Note Series **22**, Cambridge University Press, 1976. 2nd ed. published in London Mathematical Society Student Texts **15**, Cambridge University Press, 1997. MR 0396763 Zbl 0324.20040

[Juhász 2007] A. Juhász, “On a Freiheitssatz for cyclic presentations”, *Int. J. Algebra Comput.* **17**:5-6 (2007), 1049–1053. MR 2355683 Zbl 1182.20036

[Kauffman and Lambropoulou 2002] L. H. Kauffman and S. Lambropoulou, “Classifying and applying rational knots and rational tangles”, pp. 223–259 in *Physical knots: knotting, linking, and folding geometric objects in* $\mathbb{R}^3$ (Las Vegas, NV, 2001), edited by J. A. Calvo et al., Contemporary Mathematics **304**, American Mathematical Society, Providence, RI, 2002. MR 1953344 Zbl 1014.57009

[Kim 2000] Y. Kim, “About some infinite family of 2-bridge knots and 3-manifolds”, *Int. J. Math. Math. Sci.* **24**:2 (2000), 95–108. MR 1775047 Zbl 0963.57003

[Kim and Kim 2003] A. C. Kim and Y. Kim, “A polyhedral description of 3-manifolds”, pp. 157–162 in *Advances in algebra* (Hong Kong, 2002), edited by K. P. Shum et al., World Scientific, River Edge, NJ, 2003. MR 2088439 Zbl 1034.57003

[Kim and Kim 2004] S. H. Kim and Y. Kim, “Torus knots and 3-manifolds”, *J. Knot Theory Ramifications* **13**:8 (2004), 1103–1119. MR 2108650 Zbl 1065.57002

[Kim et al. 1998] G. Kim, Y. Kim, and A. Y. Vesnin, “The knot 5_2 and cyclically presented groups”, *J. Korean Math. Soc.* **35**:4 (1998), 961–980. MR 1666482 Zbl 0916.57015

[Lambert 2010] L. Lambert, *A toolkit for the construction and understanding of* 3*-manifolds*, Ph.D. thesis, Brigham Young University, Provo, UT, 2010, Available at http://scholarsarchive.byu.edu/etd/2188. MR 2801765

[Le 2009] T. Le, “Hyperbolic volume, Mahler measure, and homology growth”, lecture slides, 2009, Available at http://www.math.columbia.edu/~volconf09/notes/leconf.pdf.

[Maclachlan 1995] C. Maclachlan, “Generalisations of Fibonacci numbers, groups and manifolds”, pp. 233–238 in *Combinatorial and geometric group theory* (Edinburgh, 1993), edited by A. J. Duncan et al., London Mathematical Society Lecture Note Series **204**, Cambridge University Press, 1995. MR 1320285 Zbl 0851.20026

[Maclachlan and Reid 1997] C. Maclachlan and A. W. Reid, “Generalised Fibonacci manifolds”, *Transform. Groups* **2**:2 (1997), 165–182. MR 1451362 Zbl 0890.57023

[Mednykh and Vesnin 1995] A. D. Mednykh and A. Y. Vesnin, “On the Fibonacci groups, the Turk's head links and hyperbolic 3-manifolds”, pp. 231–239 in *Groups — Korea '94* (Pusan, 1994), edited by A. C. Kim and D. L. Johnson, De Gruyter, Berlin, 1995. MR 1476964 Zbl 0871.57001

[Milnor 1975] J. Milnor, “On the 3-dimensional Brieskorn manifolds $M(p,q,r)$”, pp. 175–225 in *Knots, groups, and* 3*-manifolds: papers dedicated to the memory of R. H. Fox*, edited by L. P. Neuwirth, Annals of Mathematics Studies **84**, Princeton University Press, 1975. MR 0418127 Zbl 0305.57003

[Minkus 1982] J. Minkus, *The branched cyclic coverings of* 2 *bridge knots and links*, Memoirs of the American Mathematical Society **35**:255, American Mathematical Society, Providence, RI, 1982. MR 643587 Zbl 0491.57005

[Mulazzani 2003] M. Mulazzani, “Cyclic presentations of groups and cyclic branched coverings of $(1,1)$-knots”, *Bull. Korean Math. Soc.* **40**:1 (2003), 101–108. MR 1958228 Zbl 1037.57002

[Mulazzani and Vesnin 2001] M. Mulazzani and A. Y. Vesnin, "The many faces of cyclic branched coverings of 2-bridge knots and links", *Atti Sem. Mat. Fis. Univ. Modena* **49**:supplement (2001), 177–215. MR 1881097 Zbl 1221.57009 arXiv math/0106164

[Nagasato and Yamaguchi 2012] F. Nagasato and Y. Yamaguchi, "On the geometry of the slice of trace-free $SL_2(\mathbb{C})$-characters of a knot group", *Math. Ann.* **354**:3 (2012), 967–1002. MR 2983076 Zbl 1270.57045

[Newman 1990] M. F. Newman, "Proving a group infinite", *Arch. Math.* (*Basel*) **54**:3 (1990), 209–211. MR 1037607 Zbl 0662.20023

[Petronio and Vesnin 2009] C. Petronio and A. Y. Vesnin, "Two-sided bounds for the complexity of cyclic branched coverings of two-bridge links", *Osaka J. Math.* **46**:4 (2009), 1077–1095. MR 2604922 Zbl 1191.57012

[Riley 1990] R. Riley, "Growth of order of homology of cyclic branched covers of knots", *Bull. London Math. Soc.* **22**:3 (1990), 287–297. MR 1041145 Zbl 0727.57002

[Rolfsen 1976] D. Rolfsen, *Knots and links*, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, CA, 1976. Revised edition by AMS Chelsea, Providence, RI, 2003. MR 0515288 Zbl 0339.55004

[Sieradski 1986] A. J. Sieradski, "Combinatorial squashings, 3-manifolds, and the third homology of groups", *Invent. Math.* **84**:1 (1986), 121–139. MR 830041 Zbl 0604.57001

[Silver and Williams 2002] D. S. Silver and S. G. Williams, "Mahler measure, links and homology growth", *Topology* **41**:5 (2002), 979–991. MR 1923995 Zbl 1024.57007

[Sloane and Guy 1991] N. J. A. Sloane and R. K. Guy, "Associated Mersenne numbers", pp. A001350 in *The online encyclopedia of integer sequences*, 1991.

[Szczepański and Vesnin 2000] A. Szczepański and A. Y. Vesnin, "On generalized Fibonacci groups with an odd number of generators", *Comm. Algebra* **28**:2 (2000), 959–965. MR 1736775 Zbl 0951.20023

[Telloni 2010] A. I. Telloni, "Combinatorics of a class of groups with cyclic presentation", *Discrete Math.* **310**:22 (2010), 3072–3079. MR 2684075 Zbl 1231.05293

[Thomas 1991] R. M. Thomas, "On a question of Kim concerning certain group presentations", *Bull. Korean Math. Soc.* **28**:2 (1991), 219–224. MR 1127741 Zbl 0752.20013

[Vesnin and Kim 1998] A. Y. Vesnin and A. C. Kim, "Дробные группы Фибоначчи и многообразия", *Sibirsk. Mat. Zh.* **39**:4 (1998), 765–775. Translated as "Fractional Fibonacci groups and manifolds" in *Siberian Math. J.* **39**:4 (1998), 655–664. MR 1654144 Zbl 0917.20032

[Vesnin and Mednykh 1996] A. Y. Vesnin and A. D. Mednykh, "Многообразия Фибоначчи как двулистные накрытия над трехмерной сферой и гипотеза Мейергофа–Ноймана", *Sibirsk. Mat. Zh.* **37**:3 (1996), 534–542. Translated as "Fibonacci manifolds as two-fold coverings of the three-dimensional sphere and the Meyerhoff–Neumann conjecture" in *Siberian Math. J.* **37**:3 (1996), 461–467. MR 1434698 Zbl 0882.57011

Received May 26, 2015. Revised January 28, 2016.

JAMES W. CANNON
DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
279 TMCB
PROVO, UT 84602
UNITED STATES
jmcnnn@gmail.com

WILLIAM J. FLOYD
DEPARTMENT OF MATHEMATICS
VIRGINIA TECH
BLACKSBURG, VA 24061
UNITED STATES
floyd@math.vt.edu

LEER LAMBERT
DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
PROVO, UT 84602
UNITED STATES
leer.lambert@gmail.com

WALTER R. PARRY
DEPARTMENT OF MATHEMATICS
EASTERN MICHIGAN UNIVERSITY
YPSILANTI, MI 48197
UNITED STATES
wparry@emich.edu

JESSICA S. PURCELL
SCHOOL OF MATHEMATICAL SCIENCES
MONASH UNIVERSITY
9 RAINFOREST WALK, ROOM 401
CLAYTON, VIC 3800
AUSTRALIA
jessica.purcell@monash.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.41

RECOGNIZING RIGHT-ANGLED COXETER GROUPS USING INVOLUTIONS

CHARLES CUNNINGHAM, ANDY EISENBERG,
ADAM PIGGOTT AND KIM RUANE

We consider the question of determining whether or not a given group (especially one generated by involutions) is a right-angled Coxeter group. We describe a group invariant, the *involution graph*, and we characterize the involution graphs of right-angled Coxeter groups. We use this characterization to describe a process for constructing candidate right-angled Coxeter presentations for a given group or proving that one cannot exist. We apply this process to a number of examples. Our new results imply several known results as corollaries. In particular, we provide an elementary proof of rigidity of the defining graph for a right-angled Coxeter group, and we recover an existing result stating that if Γ satisfies a particular graph condition (called *no SILs*), then $\mathrm{Aut}^0(W_\Gamma)$ is a right-angled Coxeter group.

1. Introduction

Given a finite simple graph Γ, the *right-angled Coxeter group defined by* Γ is the group $W = W_\Gamma$ generated by the vertices of Γ. The relations of W_Γ declare that the generators all have order 2, and adjacent vertices commute with each other. Right-angled Coxeter groups (commonly abbreviated RACG) have a rich combinatorial and geometric history [Davis 2008]. The particular presentation specified by Γ is called a *right-angled Coxeter system.* When encountering a group generated by involutions, a natural question is to ask whether or not this group might be a right-angled Coxeter group, and if so, how to identify the preferred presentation.

MSC2010: primary 20F55, 20F65; secondary 05C75.
Keywords: Coxeter group, involutions, graph theory, automorphisms.

The main objective of this paper is the development of a recognition procedure that successfully answers this question for certain families of groups. Although the procedure may be applied more generally, our applications focus primarily on two classes of examples. Given a right-angled Coxeter group W_Γ, we consider

(1) extensions of W_Γ by subgroups of $\mathrm{Out}^0(W_\Gamma)$, and

(2) subgroups of W_Γ generated by chosen sets of involutions.

(Recall that $\mathrm{Aut}^0(W_\Gamma)$ consists of the automorphisms of W_Γ which map each generator to a conjugate of itself, and $\mathrm{Out}^0(W_\Gamma)$ is the quotient $\mathrm{Aut}^0(W_\Gamma)/\operatorname{Inn}(W_\Gamma)$.) In each of these cases, we give examples of groups which are right-angled Coxeter and examples which are not. For those cases which are right-angled Coxeter, our procedure produces the preferred presentations. We show:

Theorem 1.1 (p. 57). *Suppose $\chi_1, \dots, \chi_k$ are pairwise commuting partial conjugations of the right-angled Coxeter group W_Γ such that whenever χ_i and χ_j have the same acting letter, their domains don't intersect. Then $G = W \rtimes \langle \chi_1, \dots, \chi_k \rangle$ is a right-angled Coxeter group. Further, writing $S_i \subseteq \{\chi_1, \dots, \chi_k\}$ for the set comprising those partial conjugations with acting letter a_i, we have*

$$\Big\{a_1 \prod_{\chi_i \in S_1} \chi_i, \dots, a_n \prod_{\chi_i \in S_n} \chi_i\Big\} \cup \{\chi_1, \dots, \chi_k\}$$

is a Coxeter generating set for G.

If a group G has only 2-torsion, and G is not a right-angled Coxeter group, then G is not a Coxeter group. So our procedure may in fact enable one to show that a given group is not a Coxeter group. Cunningham [2015] has used some of the methods described here to show that $\mathrm{Out}^0(W_n)$ for $n \geq 4$ is not a Coxeter group. (W_n is the *universal Coxeter group* whose defining graph has n vertices and no edges.)

Given a group G, the *involution graph Δ_G of G* is the group invariant defined as follows: the vertices in Δ_G correspond to the conjugacy classes of involutions in G; vertices are adjacent when there exist commuting representatives of the corresponding conjugacy classes. In general, this invariant is unwieldy. It may be infinite, and even when it's finite, it may be impossible to construct. Nevertheless, for certain classes of groups the invariant promises insights. Like any invariant, it can allow us to distinguish between groups. It also carries information on the automorphism group of G. Since an automorphism must permute conjugacy classes of involutions and must preserve commuting relations, $\mathrm{Aut}(G)$ acts naturally on Δ_G. The kernel of this action is therefore a natural normal subgroup of $\mathrm{Aut}(G)$, and has finite index in $\mathrm{Aut}(G)$ when Δ_G is finite.

The involution graph for a right-angled Coxeter group W_Γ is easily constructed directly from Γ: the vertices in Δ_W correspond to cliques in Γ; vertices are adjacent when the union of the corresponding cliques is also a clique. When constructed in this manner, we denote the graph Γ_K and call it the *clique graph for* Γ. Tits [1988] proved that the kernel of the action $\mathrm{Aut}(W) \circlearrowright \Delta_W$ has a natural complement, which is therefore a finite subgroup of $\mathrm{Aut}(\Delta_W)$. Thus the involution graphs of right-angled Coxeter groups are significantly more tractable than the involution graphs of arbitrary groups, and may be more convenient for certain purposes than the defining graph Γ. Aaron Meyers, in his undergraduate thesis under the supervision of Piggott, began to explore some properties of clique graphs and how to recover their base graphs. (As this work is unpublished, new proofs are given in the following sections.)

The reader may compare our use of the clique graph and involution graph to the use of the *clique graph*, *extension graph*, and *commutation graph* in [Kim and Koberda 2013] in the context of right-angled Artin groups. Our use of the term and notation for the clique graph comes from that reference. In addition, Kim and Koberda define the *extension graph* Γ^e *of* Γ and the *commutation graph* of a subset $S \subset A(\Gamma)$ of elements of the right-angled Artin group. The vertices of Γ^e are the words in the right-angled Artin group $A(\Gamma)$ which are conjugate to a vertex of Γ, and two such vertices are connected by an edge if they commute with one another. More generally, the commutation graph of S has vertices given by the elements of S, and two of these are connected by an edge if they commute with each other.

It is straightforward to define the extension and commutation graphs in the context of right-angled Coxeter groups. Note that the vertices of Γ^e are the individual group elements, not conjugacy classes, so that Γ^e is infinite whereas Δ_{W_Γ} is finite. Moreover, Γ^e does not contain words that are only conjugate to a product of pairwise commuting generators, so it is not the case that Δ_{W_Γ} is a quotient graph of Γ^e. [Kim and Koberda 2013, Theorem 1.3] states that, given graphs Λ and Γ, if Λ is contained in Γ^e, then $A(\Lambda) \le A(\Gamma)$. The analogous statement about right-angled Coxeter groups is certainly false, and a counterexample is provided by

$$D_\infty = W_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

The defining graph Γ consists of two vertices with no edges. The extension graph Γ^e has countably many vertices and no edges, but D_∞ cannot contain subgroups which are free products of more than two copies of $\mathbb{Z}/2\mathbb{Z}$. If we replace the extension graph with the involution graph Δ_{W_Γ} in [Kim and Koberda 2013], the claim would still be false: Δ_{W_Γ} contains cliques which are larger than any clique in Γ.

Finally, we note that the involution graph Δ_G of a group which is not a right-angled Coxeter group may not be a commutation graph on any subset $\{g_1, \dots, g_n\}$ of elements. A priori, it could be the case that there is no single collection of elements, one from each conjugacy class, which simultaneously exhibit all commuting and noncommuting relationships dictated by the involution graph. (When W_Γ is a right-angled Coxeter group, Δ_{W_Γ} is the commutation graph on the set of products of pairwise commuting generators.) It may be that the techniques of [Kim and Koberda 2013] could be adapted to the case of right-angled Coxeter groups, but as the current paper focuses on the recognition problem, we have not considered questions of embeddability.

In Section 2, we summarize our recognition procedure, which attempts to construct right-angled Coxeter presentations for a given group. This procedure relies on many facts about clique graphs and involution graphs which, for clarity of exposition, are only stated in that section. Detailed proofs have been relegated to Section 4 at the end of the paper. Section 2 contains all necessary definitions and results to understand the applications in Section 3.

In Section 3, we apply our procedure to several first examples of potential right-angled Coxeter groups. Section 3A collects examples of families of groups which are right-angled Coxeter. Γ is said to contain a *separating intersection of links* (*SIL*) if, for some pair of vertices v and w with $d(v, w) \geq 2$, there is a connected component of $\Gamma \setminus (\mathrm{Lk}(v) \cap \mathrm{Lk}(w))$ which contains neither v nor w. Otherwise, we say Γ *contains no SILs*. Section 3A also gives a new, shortened proof of [Charney et al. 2010, Theorem 3.6]: that $\mathrm{Aut}^0(W_\Gamma)$ is right-angled Coxeter if Γ contains no SILs. Section 3B shows several examples of groups which we prove cannot be right-angled Coxeter. This includes, in particular, an iterated extension

$$\overbrace{\underbrace{(W_\Gamma \rtimes \mathbb{Z}/2\mathbb{Z})}_{H} \rtimes \mathbb{Z}/2\mathbb{Z}}^{G}$$

in which H is not right-angled Coxeter, but G is. We also note that $\mathrm{Aut}^0(W_3)$ is not right-angled Coxeter, answering a motivating question for the authors.

Section 3C states some results that essentially identify features of a given graph Λ which indicate that W_Λ has a semidirect product decomposition $W_\Lambda = W_\Gamma \rtimes H$, where $H \leq \mathrm{Out}^0(W_\Gamma)$. The results of this section follow from those in Section 3A quite easily, and the semidirect product decompositions are certainly not unique.

Section 4 presents detailed proofs for many facts stated without proof in Section 2. In this section, we present a characterization of those finite graphs which arise as clique graphs (i.e., a characterization of those graphs which arise as the involution graphs of right-angled Coxeter groups). We present a collapsing procedure to recover Γ from Γ_K, and we establish the correctness of our recognition procedure for constructing right-angled Coxeter presentations.

Finally, in Section 5 we give many follow-up questions which may be approachable using our recognition procedure. These include the question of characterizing those subgroups $H \leq \mathrm{Out}^0(W_\Gamma)$ such that $W_\Gamma \rtimes H$ is again right-angled Coxeter, and determining when the involution graph of a subgroup $H \leq G$ can be calculated easily from the involution graph of G.

2. A summary of the recognition algorithm

In this section, we present the definitions and basic properties of the clique graph, star poset, and involution graph constructions. We state one of our main theorems characterizing those finite graphs which arise as clique graphs, and we describe a procedure which recovers a graph Γ from its clique graph Γ_K. Finally, we prove several algebraic results about right-angled Coxeter groups which allow us to modify this procedure to seek right-angled Coxeter presentations of a given group. Many of the proofs of this section are elementary or nongeometric in nature, so they have been pushed to Section 4 at the end of the paper, where the interested reader will find all of the details. In this section, we present only the definitions and statements of results necessary to understand the applications in Section 3.

A finite simple graph $\Gamma = (V, E)$ is an ordered pair of finite sets. We require that V, the set of *vertices*, is nonempty and E, the set of *edges*, consists of 2-element subsets of V. We say $a, b \in V$ are *adjacent* if $\{a, b\} \in E$. All graphs we consider in this paper will be undirected and have finitely many vertices, no loops, and no parallel edges. We will use the notation

$$\mathrm{Lk}(v) = \{w \in V \mid \{v, w\} \in E\}$$

for the *link of* v and

$$\mathrm{St}(v) = \mathrm{Lk}(v) \cup \{v\}$$

for the *star of* v.

Definition 2.1. Let Γ be a graph. A *clique in* Γ is a nonempty subset of pairwise adjacent vertices. The *clique graph of* Γ is the graph $\Gamma_K = (V_K, E_K)$ whose vertices correspond to the cliques of Γ. Two vertices of Γ_K are adjacent if the union of the corresponding cliques in Γ is also a clique. Figure 1 depicts an example.

The relation $v \sim w$ when $\mathrm{St}(v) = \mathrm{St}(w)$ is an equivalence relation on $V(\Gamma)$. Write $[v]$ for the equivalence class of v. Declaring that $[v] \le [w]$ if $\mathrm{St}(v) \subseteq \mathrm{St}(w)$ we define a partial ordering, and we write $\mathcal{P}(\Gamma)$ for the poset of star-equivalence classes of vertices in Γ.

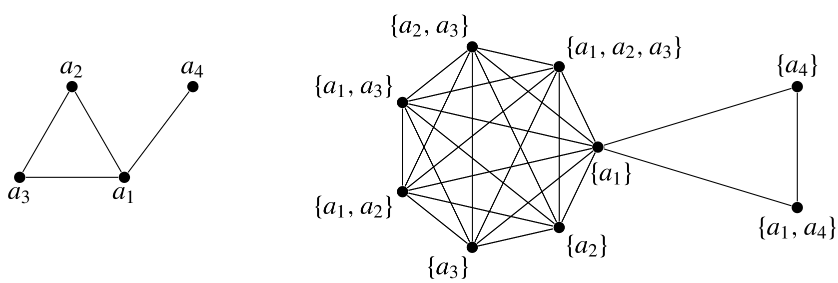

Figure 1. A graph Γ (left) and its corresponding clique graph Γ_K (right).

Throughout this paper, we will write $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ for the maximal cliques of Γ. If $I \subset \{1, 2, \ldots, r\}$, then

$$\Gamma_I = \bigcap_{i \in I} \Gamma_i$$

is the corresponding intersection of maximal cliques.

Definition 2.2. A vertex $v \in \Gamma$ is called *minimal* if it is contained in a unique maximal clique. Given $J \subset \{1, 2, \ldots, r\}$, we say v is *J-minimal* if there is no $J' \supset J$ such that $\Gamma_{J'} \subsetneq \Gamma_J$ and $v \in \Gamma_{J'}$.

Theorem 2.3 (p. 64). *Let Γ' be a graph. There exists a graph Γ such that $\Gamma' = \Gamma_K$ if and only if the following three conditions are satisfied*:

(1) *Maximal clique condition* (*MCC*): *For all I, there exists some k_I such that*

$$|\Gamma'_I| = 2^{k_I} - 1.$$

(2) *Minimal vertex condition* (*MVC*): *Each nonempty intersection Γ'_J contains some J-minimal vertex v_J.*

(3) *Inclusion-exclusion condition* (*IEC*): *For each J*,

$$\sum_{I \supsetneq J} (-1)^{|I \setminus J|+1} k_I \leq k_J.$$

Moreover, if Γ' is a clique graph, then the graph Γ such that $\Gamma' = \Gamma_K$ is unique. The following procedure, which we call the *collapsing procedure*, recovers Γ from Γ'. We may write $\Gamma = C(\Gamma')$.

Theorem 2.4 (p. 68). *Let Γ' be a graph which satisfies the MCC*, *MVC*, *and IEC. Then there is a unique* (*up to isomorphism*) *graph Γ such that Γ' is isomorphic to Γ_K. Moreover*, *the following collapsing procedure produces the graph Γ if it exists.*

(1) *Initially*, *let $V = \{\,\}$.*

(2) *Let $[w] \in \mathcal{P}(\Gamma')$ be a class such that every class $[v]$ with $[w] < [v]$ has already been considered. Write*

$$S_w = \bigcup_{[v] \geq [w]} [v].$$

Then there is some k such that $|S_w| = 2^k - 1$. Let k' be the number of vertices of S_w which are already contained in V. Choose $k - k'$ vertices of $[w]$ to add to the vertex set V.

(3) *Repeat the previous step until all classes of $\mathcal{P}(\Gamma')$ have been considered.*

(4) *Return the graph $C(\Gamma')$ which is the induced subgraph of Γ' on the vertex set V.*

The set S_w forms a clique in Γ' which is an intersection of maximal cliques, so its size has the desired form by the MCC. The details can be found in Section 4A.

Definition 2.5. Let G be a (finitely generated) group. The *involution graph* Δ_G *of* G is defined as follows. The vertices are the conjugacy classes of involutions in G. Two vertices $[x]$ and $[y]$ are connected by an edge if there exist representatives gxg^{-1} and hyh^{-1} that commute with each other.

We make a few remarks. The particular conjugates which witness commutativity are chosen for each edge individually. A system of representatives of each conjugacy class which act as witnesses for every edge simultaneously is called a *full system of representatives*. Such a system need not exist in general, but a right-angled Coxeter group will always have a full system of representatives.

We have also said earlier that all graphs we consider do not have loops, although the involution graph as defined here may contain a loop if an involution commutes with a conjugate of itself. This may happen in general, but it will never happen in a right-angled Coxeter group. So, if the involution graph of a given G contains a loop, we may immediately conclude that G is not a right-angled Coxeter group.

Lemma 2.6. *Let Γ be a graph. Then $\Delta_{W_\Gamma} = \Gamma_K$.*

Proof. It is a well-known fact about right-angled Coxeter groups that the only nontrivial torsion elements have order 2, and that any involution is conjugate to some product of pairwise commuting generators. The set of products of pairwise commuting generators forms a full system of representatives for the involution graph (this follows essentially from the deletion condition), and two such products commute if and only if all the generators involved in each product pairwise commute, i.e., if the collection of all these generators forms a clique in Γ. □

We recover the rigidity of right-angled Coxeter groups as an immediate consequence. This was originally proven in [Green 1990] (for a more general class of groups), and many other proofs have been presented for different classes of groups containing right-angled Coxeter groups as a subclass; see, for example, [Droms 1987; Laurence 1995; Radcliffe 2003].

Corollary 2.7. *The defining graph of a right-angled Coxeter group W_Γ is unique up to isomorphism.*

Proof. The involution graph is an algebraic invariant (it does not depend on the chosen right-angled Coxeter presentation). By the previous lemma, the involution graph Δ_{W_Γ} is a clique graph, and by Theorem 2.3 the collapsed graph $C(\Delta_{W_\Gamma})$ is unique (up to isomorphism). □

At this point, we can essentially describe our recognition procedure for seeking a right-angled Coxeter presentation for a given group G. First, we form the involution graph Δ_G. If this is not a clique graph, then G is not a right-angled Coxeter group.

If it is, then we must find a full system of representatives for the vertices. If such a system does not exist, then G is not a right-angled Coxeter group. If we find a full system of representatives, then the collapsing procedure will produce a labeled graph $\Gamma = C(\Delta_G)$, which gives a map $W_\Gamma \to G$ by sending the generators of W_Γ to the labels of the corresponding vertices. If we can show the candidate map is an isomorphism, then G is a right-angled Coxeter group, and the labels of Γ form a right-angled Coxeter generating set. (On the other hand, if the candidate map is not an isomorphism, we cannot conclude that G is not a right-angled Coxeter group. We may have simply chosen the wrong full system of representatives for Δ_G.)

We must address one subtlety in this procedure. In Theorem 2.4, we chose vertices from $[w]$ to add to the vertex set V arbitrarily. It only mattered that we had the right number of vertices from each intersection of maximal cliques. In the algebraic setting, this is not sufficient, as the following simple example shows.

Example 2.8. Let Γ be a triangle with vertices a, b, c. Then $\Gamma_K = \Delta_{W_\Gamma}$ is a clique of size 7 with the labels a, b, c, ab, ac, bc, abc. In the star poset $\mathcal{P}(\Gamma_K)$, all vertices are equivalent, so there is only one $[w]$ to consider. The collapsing procedure says to choose 3 vertices from this class at random. If we choose, for example, the vertices a, b, c, then the collapsing procedure recovers Γ. If we choose a, ab, abc, then we find a new right-angled Coxeter presentation for W_Γ. However, if we pick a, b, ab, then we don't get a right-angled Coxeter presentation (because there is an additional relation between these vertices).

Essentially, at this step in the collapsing procedure we are choosing which vertices of the involution graph represent generators and which represent products of generators. There are (generally) many different ways that we can make this choice, but we have to make use of some algebraic information to avoid choosing products as if they were generators. The following results are certainly of independent interest, but we will, in particular, use them to make intelligent choices during the collapsing procedure.

Since we wish to avoid choosing vertices whose labels have a nontrivial product relation, it would certainly help if we could solve the word problem in G. However, depending on how G is presented, such a solution may or may not be evident (if it even exists). For this reason, we pass to the abelianization G^{ab}, in which there is a solution to the word problem. If G is a right-angled Coxeter group, then

$$G^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^n,$$

and a product relation among involutions in G must also occur in G^{ab}.

From this point forward, for $g \in G$, we will write $\bar{g}$ for the image of g in the abelianization. An important fact about right-angled Coxeter groups is that the abelianization is injective on conjugacy classes of involutions.

Proposition 2.9. *Let W_Γ be a right-angled Coxeter group. Let $x, y \in W_\Gamma$ such that $x^2 = y^2 = 1$. Then $\bar{x} = \bar{y}$ in W_Γ^{ab} if and only if x and y are conjugate in W_Γ.*

Proof. The "if" direction is trivial. Now, suppose x and y are not conjugate in W_Γ. Since x, y are involutions, there are pairwise commuting generators $a_1, a_2, \ldots, a_k$, pairwise commuting generators $b_1, b_2, \ldots, b_\ell$, and words g, h such that

$$x = ga_1a_2 \cdots a_k g^{-1} \quad \text{and} \quad y = hb_1b_2 \cdots b_\ell h^{-1}.$$

Without loss of generality, since x and y are not conjugate, there is a b_j that does not appear among the a_i. But since it is a generator, there is a $\mathbb{Z}/2\mathbb{Z}$ direct factor in W_Γ^{ab} corresponding to that $\overline{b_j}$. Therefore, $\bar{y}$ will have a 1 in this factor and $\bar{x}$ will have a 0. Thus, $\bar{x} \neq \bar{y}$ in W_Γ^{ab}. □

Corollary 2.10. *For a right-angled Coxeter group W_Γ, if H is a subgroup generated by distinct, commuting involutions, then $H \cong H^{\mathrm{ab}}$ injects into W_Γ^{ab}.*

Proof. H is a finite subgroup of W_Γ and so is conjugate to a special subgroup H'. Each element of H' is a distinct product of commuting generators from W_Γ and so each gets sent to a distinct element of W_Γ^{ab}. Thus, no two elements of H' can be conjugate in W_Γ and so neither can any two elements of H. By Proposition 2.9, H injects into W_Γ^{ab}. □

Proposition 2.11 (p. 72). *If W_Γ is a right-angled Coxeter group, then in step 2 of the collapsing procedure in Theorem 2.4, we can choose the $k - k'$ involutions of W_Γ so that the chosen elements do not exhibit a nontrivial product relation.*

This proposition, which is proved in Section 4B, makes use of the available algebraic information to amend our collapsing procedure and avoid nontrivial product relations. We can make further use of the available algebraic information to improve upon the procedure. In general, we have no particular method (or hope of finding a method) to construct Δ_G for an arbitrary G. Each of the following steps seem to be generally insurmountable:

(1) Identify all involutions in G.

(2) Separate all involutions into their conjugacy classes.

(3) Determine the presence or lack of each edge in Δ_G (i.e., find a pair of commuting representatives or prove that none exist).

(4) Find a full system of representatives.

(5) Identify a full system of representatives so that the candidate maps are isomorphisms.

For a right-angled Coxeter system, it happens that all of these steps are not just possible, but straightforward.

Proposition 2.12 (p. 74). *If W_Γ is a right-angled Coxeter group, then two conjugacy classes of involutions $[x]$ and $[y]$ are connected by an edge in Δ_{W_Γ} if and only if there exists another class $[z]$ such that $\bar{z} = \overline{xy}$ in the abelianization.*

If we are given a group G, supposing we can identify the conjugacy classes of involutions (i.e., the vertices of Δ_G), we can identify hypothetical edges and nonedges by looking for such $\bar{z}$ in G^{ab}. If G is a right-angled Coxeter group, then this will produce the correct involution graph, and the remainder of the procedure will (hopefully, if we pick a good full system of representatives) identify a right-angled Coxeter presentation. On the other hand, if this not-quite involution graph of G is not a clique graph, we can be certain that G is not a right-angled Coxeter group. At no point do we directly need to check that we have calculated the true involution graph of G. We summarize this discussion with the following amended collapsing procedure. For details (including a full description of how to do these calculations in the abelianization), refer to Section 4B.

Theorem 2.13 (p. 74). *Suppose G is a group whose only torsion elements all have order 2, so that $G^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some n. If the following procedure returns* TRUE, *then G is a right-angled Coxeter group (and the procedure indicates a right-angled Coxeter presentation). If the procedure returns* FALSE, *then G is not a right-angled Coxeter group.*

(1) *Determine all conjugacy classes of involutions in G, and let these be the vertices of a graph Γ'. If there are not finitely many, return* FALSE.

(2) *Apply Proposition 2.12 to construct the edges of Γ'.*

(3) *If Γ' is not a clique graph, return* FALSE.

(4) *Find a full system of representatives for the vertices of Γ'. If no such system exists, return* FALSE.

(5) *Collapse as in Theorem 2.4, using Proposition 2.11 to ensure that nontrivial product relations are avoided. Write $C(\Gamma')$ for the resulting graph.*

(6) *Let Γ be a graph isomorphic to $C(\Gamma')$ with generic vertex labels $a_1, \ldots, a_n$. Let $\varphi : W_\Gamma \to G$ be the map which sends the generators of W_Γ to the word given by the corresponding labels of vertices in $C(\Gamma')$. If φ is an isomorphism, return* TRUE.

(7) *Otherwise, return* UNKNOWN.

3. Applications and results

In this section, we apply the recognition procedure from Section 2 to seek out right-angled Coxeter presentations for certain families of groups. We focus in particular on

(1) semidirect products of a given right-angled Coxeter group W_Γ by certain subgroups of $\mathrm{Out}^0(W_\Gamma)$, and

(2) subgroups of a given W_Γ generated by chosen subsets of involutions.

In particular, we note that the families of groups that we consider are already generated by involutions, have no torsion of order other than 2, and are usually given by presentations which are nearly right-angled Coxeter.

If D is a union of connected components of $\Gamma \setminus \mathrm{St}(a_i)$ for some i, then the automorphism of W_Γ determined by

$$\chi_{i,D}(a_j) = \begin{cases} a_i a_j a_i, & a_j \in D, \\ a_j, & \text{otherwise}, \end{cases}$$

is called the *partial conjugation with acting letter a_i and domain D*. (Note that this terminology is not entirely consistent in the literature. Other papers have reserved partial conjugation for the case in which D is a single connected component [Gutierrez et al. 2012; Charney et al. 2010], while Laurence [1995] used the term *locally inner automorphism* before the term partial conjugation became common. We have preferred here to allow for multiple connected components in the domain of a partial conjugation, and we would propose the term *elementary partial conjugation* for the case in which D consists of a single connected component.) The partial conjugations generate $\mathrm{Out}^0(W_\Gamma)$.

In Section 3A, we present families of groups which our procedure shows to be right-angled Coxeter. One example is worked out in full detail to demonstrate the procedure. For the remaining results, we simply state the resulting right-angled Coxeter group and the isomorphism determined by our procedure. The reader is left to verify the details. Most of these results are about split extensions of a given W_Γ by a finite subgroup of $\mathrm{Out}^0(W_\Gamma)$ generated by (pairwise commuting) partial conjugations.

In Section 3B, we present families of groups which our procedure shows cannot be right-angled Coxeter. Again, one example is worked out in full detail. We note one example which is of particular interest: we find a group W_Γ with two elements $x, y \in \mathrm{Out}^0(W_\Gamma)$ such that $G = W_\Gamma \rtimes \langle x, y\rangle$ is a right-angled Coxeter group, but $H = W_\Gamma \rtimes \langle xy\rangle$ is not. In particular, we can realize G as the iterated semidirect product

$$G = (W_\Gamma \rtimes \langle xy\rangle) \rtimes \langle x\rangle,$$

where each extension has degree 2. So this gives, to our knowledge, the first example in which the existence of a right-angled Coxeter presentation is lost and then recovered by semidirect product extensions.

Finally, in Section 3C, we note that many of our examples of right-angled Coxeter families arise as semidirect products. By analyzing the properties of the defining graphs of the groups arising from these semidirect products, we can identify semidirect product decompositions in many cases. Such decompositions are generally not unique, and we cannot at the moment provide an exhaustive list of graph features of Γ which indicate a semidirect product decomposition of W_Γ.

3A. *Groups which are right-angled Coxeter.*

Example 3.1. We begin with an explicit example in which we demonstrate the recognition procedure in detail. Consider the defining graph in Figure 2.

Write $x = \chi_{1,\{2\}}$ for the partial conjugation with acting letter a_1 and domain $\{a_2\}$. We consider the group $G = W_\Gamma \rtimes \langle x \rangle$, which has the presentation

$$G = \big\langle a_1, a_2, a_3, a_4, x \;\big|\; a_i^2 = x^2 = 1, [a_1, a_4] = [a_2, a_4] = [a_3, a_4] = 1, \\ [a_1, x] = [a_3, x] = [a_4, x] = 1, xa_2x = a_1a_2a_1 \big\rangle.$$

This is not quite a right-angled Coxeter presentation, so we apply our procedure to see if we can find one.

First, we compute G^{ab} (removing any relations that become trivial and understanding that group presentations with additive notation are assumed to be abelian):

$$\begin{aligned} G^{\mathrm{ab}} &= \langle \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \bar{x} \mid 2\overline{a_i} = 2\bar{x} = 0 \rangle \\ &\cong \langle \overline{a_1} \rangle \times \langle \overline{a_2} \rangle \times \langle \overline{a_3} \rangle \times \langle \overline{a_4} \rangle \times \langle \bar{x} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^5. \end{aligned}$$

The relation matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

is already in Smith normal form, and so our canonical abelianization map $G \to G^{\mathrm{ab}}$ is given by $g \mapsto \bar{g}$.

We now want to list all conjugacy classes of involutions in G. The classes of involutions in W_Γ are evident by inspection of Γ: a_i for each i, and a_ja_4 for each $1 \le j \le 3$. The new generator x is also an involution, and the products of x with the other generators that commute with it give new involutions: xa_1, xa_3, xa_4. There are two remaining conjugacy classes of involutions, namely xa_1a_2 and $xa_1a_2a_4$.

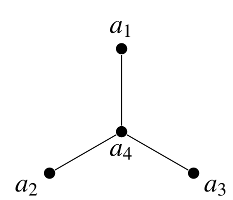

Figure 2. The defining graph Γ.

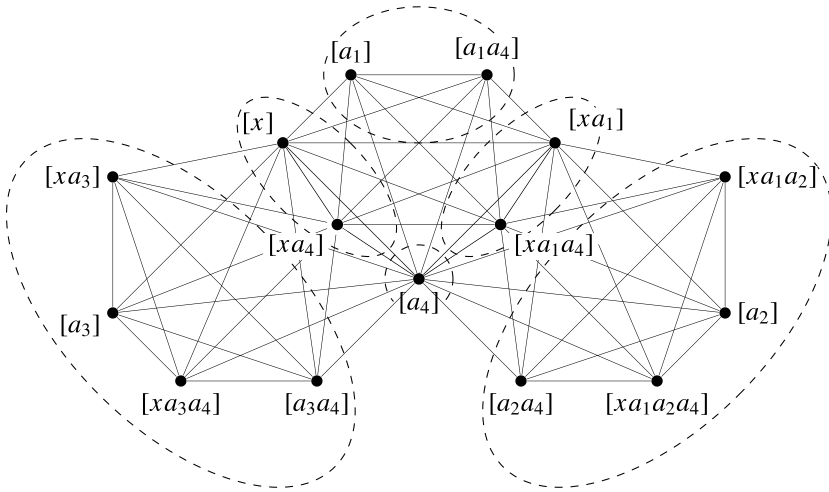

Figure 3. The involution graph Δ_G.

These are all of the conjugacy classes of involutions in G. We could try to prove this directly, but it will also end up following from the fact that our procedure in this case does in fact construct an explicit isomorphism with a right-angled Coxeter group. Thus, we can omit the details.

We claim that the graph in Figure 3 is the involution graph Δ_G. The given system of representatives is a full system, and the commuting relations are straightforward to check. (If they weren't as straightforward, we could easily construct the edge relations given by Proposition 2.12.)

The brackets in the involution graph represent conjugacy classes. Since we now have a full system of representatives, we may stop writing these brackets. For the remainder of the calculation, brackets around a vertex label will denote its star-equivalence class. Before calculating the star poset structure, we observe that this graph clearly satisfies the MCC and MVC, and the IEC is straightforward to verify.

The equivalence classes in the star poset are the following (identified by the dashed ellipses in Figure 3):

$$[a_1] = \{a_1, a_1a_4\}, \qquad [a_2] = \{a_2, a_2a_4, xa_1a_2, xa_1a_2a_4\},$$
$$[a_3] = \{a_3, a_3a_4, xa_3, xa_3a_4\}, \qquad [a_4] = \{a_4\},$$
$$[x] = \{x, xa_4\}, \qquad [xa_1] = \{xa_1, xa_1a_4\}.$$

The Hasse diagram for this poset is depicted in Figure 4.

The element $[a_4]$ is maximal in the poset structure and contains a single element. We add a_4 to V. Next, we consider $[x]$ (or $[xa_1]$; the order in which we consider these classes is irrelevant). The clique above $[x]$ has size 3, so 2 of its vertices

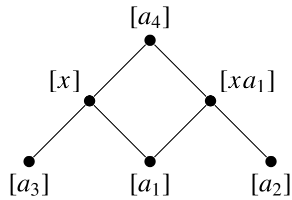

Figure 4. The Hasse diagram for the poset $\mathcal{P}(\Delta_G)$.

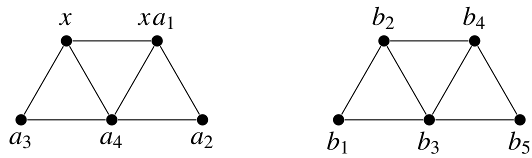

Figure 5. The collapsed graph Λ (left) and an isomorphic graph with generic labels (right).

must be added to V. We have already added 1, so we must pick one more from $[x]$. Examining the abelianization, $\langle \overline{a_4}, \bar{x}\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ and either of $\bar{x}$ or $\overline{xa_4}$ will extend $\overline{a_4}$ into a basis. So we choose to add x to V. Similarly, we consider $[xa_1]$ and add xa_1 to V.

The remaining three classes are all minimal. Suppose we take $[a_2]$ next. The clique above $[a_2]$ has size 7, so we must choose 3 elements from it. We have already chosen 2, so we need to choose 1 more. Checking the abelianization again, we see that any choice of the 4 elements in $[a_2]$ will extend to a basis, and so we add a_2 to V. Similarly, from $[a_3]$, we add a_3 to V.

Finally, we consider $[a_1]$. The clique above $[a_1]$ has size 7, and we have already chosen 3 of these vertices, so we choose no more. This leaves us with $V = \{a_2, a_3, a_4, x, xa_1\}$. We take the induced subgraph Λ of Δ_G on these vertices; see Figure 5.

We now have a candidate map $\varphi : W_\Lambda \to G$. It is straightforward to check that the map ψ below is the inverse, and that φ and ψ are isomorphisms:

$$\begin{aligned} \varphi:&\quad b_1 \mapsto a_3, \quad b_2 \mapsto x, \quad b_3 \mapsto a_4, \quad b_4 \mapsto xa_1, \quad b_5 \mapsto a_2,\\ \psi:&\quad a_1 \mapsto b_2b_4, \quad a_2 \mapsto b_5, \quad a_3 \mapsto b_1, \quad a_4 \mapsto b_3, \quad x \mapsto b_2. \end{aligned}$$

Thus, G is a right-angled Coxeter group, completing the example.

In this example, we were extending a right-angled Coxeter group by a single partial conjugation. It turns out that this will always yield a right-angled Coxeter group, and in fact we can say much more.

Lemma 3.2. *Suppose W_Γ is a right-angled Coxeter group. If $\alpha_1, \dots, \alpha_k$ are partial conjugations of W with the same acting letter and pairwise disjoint domains, then $G = W \rtimes \langle \alpha_1, \dots, \alpha_k\rangle$ is a right-angled Coxeter group.*

Proof. Without loss of generality, we may assume each α_j has acting letter a_1. Let D_i denote the domain of α_i for each $1 \le i \le k$. Now G is generated by the elements

$$\{a_1, \dots, a_n, \alpha_1, \dots, \alpha_k\}$$

with the relations

$$\begin{array}{lrl}
\text{(R1)} & a_i^2 = 1, & \text{for } 1 \le i \le n, \\
\text{(R2)} & [a_i, a_j] = 1, & \text{for } \{a_i, a_j\} \in E(\Gamma), \\
\text{(R3)} & \alpha_i^2 = 1, & \text{for } 1 \le i \le k, \\
\text{(R4)} & [\alpha_i, \alpha_j] = 1, & \text{for } 1 \le i < j \le k, \\
\text{(R5)} & [\alpha_i, a_j] = 1, & \text{for } a_j \notin D_j, \\
\text{(R6)} & \alpha_i a_j \alpha_i = a_1 a_j a_1, & \text{for } a_j \in D_i.
\end{array}$$

Let H be the group generated by

$$\{b_1, \dots, b_n, \beta_1, \dots, \beta_k\}$$

with the relations

$$\begin{array}{lrl}
\text{(S1)} & b_i^2 = 1, & \text{for } 1 \le i \le n, \\
\text{(S2)} & [b_i, b_j] = 1, & \text{for } \{a_i, a_j\} \in E(\Gamma), \\
\text{(S3)} & \beta_i^2 = 1, & \text{for } 1 \le i \le k, \\
\text{(S4)} & [\beta_i, \beta_j] = 1, & \text{for } 1 \le i < j \le k, \\
\text{(S5)} & [\beta_i, b_j] = 1, & \text{for } a_j \notin D_i, \\
\text{(S6)} & [b_1, b_i] = 1, & \text{for } 2 \le i \le n \text{ and } a_i \in D_1 \cup \cdots \cup D_k.
\end{array}$$

We note that the given presentation for H is a right-angled Coxeter presentation. We define maps

$$\begin{aligned}
\hat{\varphi} : \{a_1, \dots, a_n, \alpha_1, \dots, \alpha_k\} &\to \{b_1, \dots, b_n, \beta_1, \dots, \beta_k\}, \\
a_1 &\mapsto b_1 \beta_1 \cdots \beta_k, \\
\alpha_i &\mapsto \beta_i \quad (1 \le i \le k), \\
a_i &\mapsto b_i \quad (2 \le i \le n), \\
\hat{\psi} : \{b_1, \dots, b_n, \beta_1, \dots, \beta_k\} &\to \{a_1, \dots, a_n, \alpha_1, \dots, \alpha_k\}, \\
b_1 &\mapsto a_1 \alpha_1 \cdots \alpha_k, \\
\beta_i &\mapsto \alpha_i \quad (1 \le i \le k), \\
b_i &\mapsto a_i \quad (2 \le i \le n).
\end{aligned}$$

It is straightforward to check that $\hat{\varphi}$ and $\hat{\psi}$ preserve the relations (R1)–(R6) and (S1)–(S6), respectively, so they induce homomorphisms $\varphi : G \to H$ and $\psi : H \to G$.

(Note that the preservation of the relation (S6) uses the assumption that the domains D_i are pairwise disjoint.) Finally, it is straightforward to see that φ and ψ are inverses to each other, hence G and H are isomorphic. That is, G is a right-angled Coxeter group. □

Suppose $H \leq \mathrm{Out}^0(W_\Gamma)$ is generated by partial conjugations $\chi_1, \ldots, \chi_k$. Having shown that the semidirect product extension of W_Γ by any single partial conjugation is again right-angled Coxeter, we might hope to show that $W_\Gamma \rtimes H$ is right-angled Coxeter by observing that this is isomorphic to taking the iterated semidirect products, each by a single χ_i:

$$W_\Gamma \rtimes H = \big(\cdots((W_\Gamma \rtimes \langle\chi_1\rangle) \rtimes \langle\chi_2\rangle) \rtimes \cdots \rtimes \langle\chi_k\rangle\big).$$

However, there is a subtlety that ruins this argument, namely, that χ_2 will extend to some automorphism of $W_\Gamma \rtimes \langle\chi_1\rangle$, but not necessarily to a partial conjugation. We cannot extend inductively, since we cannot ensure that we are always extending by single partial conjugations. The following lemma and theorem identify certain cases in which this inductive argument works.

Lemma 3.3. *Suppose W, Γ, a_1, $\alpha_1, \ldots, \alpha_k$, H, and G are as in the lemma and proof above. Let γ be a partial conjugation of W with acting letter $a_2 \neq a_1$ and such that γ commutes with each of the automorphisms $\alpha_1, \ldots, \alpha_k$. Then γ acts on G as a partial conjugation.*

Proof. Without loss of generality we may assume γ has acting letter a_2 and domain D. Recall that $a_2 = b_2$. To show that γ acts on G as a partial conjugation we consider the result of conjugation by γ on each of the generators $b_1, \ldots, b_n, \beta_1, \ldots, \beta_k$. Firstly we note: $\gamma\beta_i\gamma = \beta_i$ for $1 \leq i \leq k$; $\gamma b_i \gamma = b_i$ for $1 \leq i \leq n$ and $a_i \notin D$; $\gamma b_i \gamma = b_2 b_i b_2$ for $2 \leq i \leq n$ and $a_i \in D$. If $a_1 \notin D$, then $\gamma b_1 \gamma = \gamma a_1 \gamma = b_1$. Suppose $a_1 \in D$. Since γ commutes pairwise with $\alpha_1, \ldots, \alpha_k$, we have $a_2 \notin D_1 \cup \cdots \cup D_k$. We compute

$$\begin{aligned} \gamma b_1 \gamma &= \gamma a_1 \alpha_1 \cdots \alpha_k \gamma \\ &= \gamma a_1 \gamma \alpha_1 \cdots \alpha_k \\ &= a_2 a_1 a_2 \alpha_1 \cdots \alpha_k \\ &= a_2 a_1 \alpha_1 \cdots \alpha_k a_2 \\ &= b_2 b_1 b_2. \end{aligned}$$

Since γ is an automorphism of G, and γ takes each generator to either itself or the conjugate of itself by b_2, we may conclude that γ is a partial conjugation of G.

Write $\varphi : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$ for the map $\varphi(a_i) = b_i$. From the calculations above, the domain of γ acting on G is $\varphi(D)$. □

Theorem 1.1. *Suppose $\chi_1, \dots, \chi_k$ are pairwise commuting partial conjugations of the right-angled Coxeter group W_Γ such that whenever χ_i and χ_j have the same acting letter, their domains don't intersect. Then $G = W \rtimes \langle \chi_1, \dots, \chi_k \rangle$ is a right-angled Coxeter group. Further, writing $S_i \subseteq \{\chi_1, \dots, \chi_k\}$ for the set comprising those partial conjugations with acting letter a_i,*

$$\left\{a_1 \prod_{\chi_i \in S_1} \chi_i, \dots, a_n \prod_{\chi_i \in S_n} \chi_i\right\} \cup \{\chi_1, \dots, \chi_k\}$$

is a Coxeter generating set for G.

Proof. The proof is by induction, applying the lemmas above at each step. Let $\alpha_1, \dots, \alpha_{k_1}$ be those χ_i with acting letter 1. By assumption, they have pairwise disjoint domains. By Lemma 3.2, $W_\Gamma \rtimes \langle \alpha_1, \dots, \alpha_{k_1} \rangle$ is a RACG.

Moreover, by Lemma 3.3, the remaining χ_i still act like partial conjugations, and their domains do not intersect, since they didn't before the extension. Now take $\beta_1, \dots, \beta_{k_2}$ among the remaining χ_i to be those which have acting letter 2, and extend by $\langle \beta_1, \dots, \beta_{k_2} \rangle$.

Continuing inductively, we extend at the i-th step by all remaining partial conjugations with acting letter i. The result follows. □

In [Gutierrez et al. 2012], the authors investigate the automorphism groups of graph products of cyclic groups. In the case that W is a right-angled Coxeter group, the authors recover a result of Tits [1988] which shows $\mathrm{Aut}(W) = \mathrm{Aut}^0(W) \rtimes \mathrm{Aut}^1(W)$ with $\mathrm{Aut}^1(W)$ finite. Thus $\mathrm{Aut}^0(W)$ (sometimes denoted $\mathrm{Aut}^{\mathrm{PC}}(W)$), which is the subgroup of $\mathrm{Aut}(W)$ generated by all partial conjugations of W, is a finite index subgroup of $\mathrm{Aut}(W)$. They also show that $\mathrm{Aut}^0(W)$ splits as $\mathrm{Inn}(W) \rtimes \mathrm{Out}^0(W)$. Finally, they give the following condition on Γ, called *no SILs*, which characterizes exactly when $\mathrm{Out}^0(W)$ is finite and is thus isomorphic to $\mathbb{Z}_2^n$.

Definition 3.4. A graph Γ has a *separating intersection of links* (*SIL*) if, for some vertices v and w with $d(v, w) \geq 2$, there is a component of $\Gamma \setminus (\mathrm{Lk}(v) \cap \mathrm{Lk}(w))$ which contains neither v nor w. Otherwise, Γ is said to have *no SILs*.

$\mathrm{Inn}(W_\Gamma)$ is known to be a right-angled Coxeter group. In the case that Γ has no SILs, $\mathrm{Aut}^0(W_\Gamma)$ is a finite extension of $\mathrm{Inn}(W_\Gamma)$. In [Charney et al. 2010], it is shown that $\mathrm{Aut}^0(W_\Gamma)$ is again a right-angled Coxeter group in that case. We arrive at this same result as a direct application of the previous corollary.

Corollary 3.5. *If Γ contains no SILs, then $\mathrm{Aut}^0(W)$ is a right-angled Coxeter group and thus $\mathrm{Aut}(W)$ contains a right-angled Coxeter group as a subgroup of finite index.*

Proof. Without loss of generality we may assume W has trivial center. Suppose Γ contains no SILs. Then

$$\mathrm{Aut}^0(W) = \mathrm{Inn}(W) \rtimes \mathrm{Out}^0(W) \cong W \rtimes \mathrm{Out}^0(W),$$

and $\mathrm{Out}^0(W)$ is generated by pairwise commuting partial conjugations which satisfy the condition in the corollary above. □

In general, one should not expect $\mathrm{Aut}(W)$ to be right-angled Coxeter. The elements of $\mathrm{Aut}^1(W)$ include graph symmetries, which could then introduce torsion elements of order other than 2. One should not generally expect that $\mathrm{Aut}^0(W)$ is a right-angled Coxeter group, but one might see the no SILs result as suggesting that we restrict our attention to extensions of right-angled Coxeter groups by finite subgroups of $\mathrm{Out}^0(W)$ (although Example 3.8 in the following section demonstrates that even this restriction is not sufficient).

3B. ***Groups which are not right-angled Coxeter.***

Example 3.6. As in the previous section, we begin with an explicitly worked out example. Let G denote the group presented as

$$G = \langle a, b, c, x, y \mid a^2,\ b^2,\ c^2,\ x^2,\ y^2,\ xax = a,\ xbx = b,\ xcx = aca,\\ yay = a,\ yby = b,\ ycy = bcb\rangle.$$

Let $W = \langle a, b, c\rangle$ and $H = \langle x, y\rangle$. Then $W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, $H \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, and $G \cong W \rtimes H$, where x and y act as a pair of noncommuting partial conjugations.

To construct Δ_G, we must understand the involutions in G. Since $G = W \rtimes H$, each $g \in G$ may be written uniquely in the form $g = wh$, where $w \in W$ and $h \in H$. Further, $g^2 = whwh = whw(h^{-1}h)h = ww^{h^{-1}}h^2$. Since every element in G can be uniquely written as a product of an element of G and an element of H, if g is an involution, then h is an involution and $w^{h^{-1}} = w^h = w^{-1}$. Because H is a right-angled Coxeter group (in fact, D_∞), every nontrivial involution in H is conjugate to either x or y; it follows that, up to conjugation, we may suppose g has one of the forms

(1) w such that $w^2 = 1$,

(2) wx such that $w^x = w^{-1}$, or

(3) wy such that $w^y = w^{-1}$.

Every element of the first type is conjugate to either a, b, or c. Now we'll try to list elements of the second type (elements of the third type will be analogous).

Suppose $g = wx$ with $w^x = w^{-1}$. We further suppose that, within the collection of words of this form in the conjugacy class of g, we choose the shortest possible w. The element w can be written uniquely in the form $u_0bu_1b \cdots u_{m-1}bu_m$, where

$m \geq 0$, each u_i is a geodesic word in $\{a, c\}^*$, and only u_0 and u_m may be trivial. Then $w^x = u_0^x b u_1^x b \cdots u_{m-1}^x b u_m^x = w^{-1}$ implies that $u_0^x = u_m^{-1}$, $u_1^x = u_{m-1}^{-1}$, and so on. We now consider a few subcases.

If $m > 0$ and u_0 is not trivial, then

$$\begin{aligned} u_0^{-1}(wx)u_0 &= u_0^{-1}(u_0 b u_1 b \cdots u_{m-1} b u_m x)u_0 \\ &= b u_1 b \cdots u_{m-1} b u_m u_0^x x \\ &= b u_1 b \cdots u_{m-1} b x. \end{aligned}$$

This contradicts the minimality of the length of w, so either $m = 0$ or u_0 is trivial. If u_0 is trivial and $m > 1$, then w begins and ends with b, so $|b(wx)b| < |wx|$. Again, this contradicts minimality, hence either $m = 0$ or $w = b$.

If $m = 0$, then $w = u_0 \in \langle a, c\rangle$ is geodesic and so is an alternating string of a and c. If $|w| > 1$ and $|w|$ is odd, then w begins and ends with the same letter. If w begins and ends with a, then $|awxa| = |awax| < |wx|$; if w begins and ends with c then w^x begins and ends with a; hence $w^x \neq w^{-1}$. In either case, we have a contradiction, so $|w| = 1$, in which case $w = a$ or $w = c$, or else $|w|$ is even. If $w = (ac)^n$ and $n > 1$, then $|aca(wx)aca| < |wx|$; if $w = (ca)^n$ and $n > 1$, then $|cwxc| < |wx|$. In both cases, we have a contradiction. Our only case left is $m = 0$, $n = 1$, which corresponds to $w = ac$ or $w = ca$. Therefore, our only nontrivial possibilities for w are $w = b, a, c, ac, ca$.

Note that $a(cax)a = acx$, so these cases fall into the same conjugacy classes. In summary, we have that each involution of the form wx is conjugate to exactly one of the elements x, ax, bx, acx. (We observe that the final option cx is not, in fact, an involution. In this case, $w = c$, and $w^x \neq w^{-1}$.) We also observe that none of these involutions are conjugate to each other since they all map to distinct elements in G^{ab}.

Similarly, each involution of the form wy is conjugate to exactly one of the elements y, ay, by, bcy. Therefore, the following is the complete list of conjugacy classes in G, and hence serves as the list of vertex labels in Δ_G:

$$[a],\ [b],\ [c],\ [x],\ [ax],\ [bx],\ [acx],\ [y],\ [ay],\ [by],\ [bcy].$$

We now consider pairs of distinct conjugacy classes, to see whether or not they should be adjacent in Δ_G. By Proposition 2.12, we can just check the product relations among the images of the involutions in G^{ab}. We omit the actual calculation and show the resulting involution graph in Figure 6.

Now Δ_G is not a clique graph, since, for example, the IEC fails. (The reader can check this directly for the maximal cliques labeled Γ_3 and Γ_4 in the figure.)

Example 3.7. $\mathrm{Aut}^0(W_3)$ is not a right-angled Coxeter group. The details are very similar to the previous example (we extend by one further partial conjugation), and are omitted here. The involution graph is shown in Figure 7.

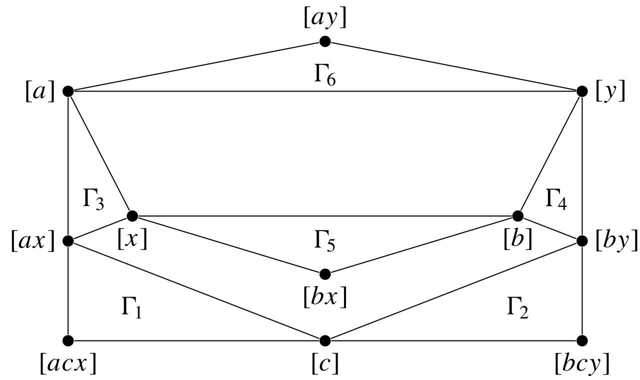

Figure 6. An involution graph which cannot be a clique graph. The labeled triangles Γ_i are the maximal cliques.

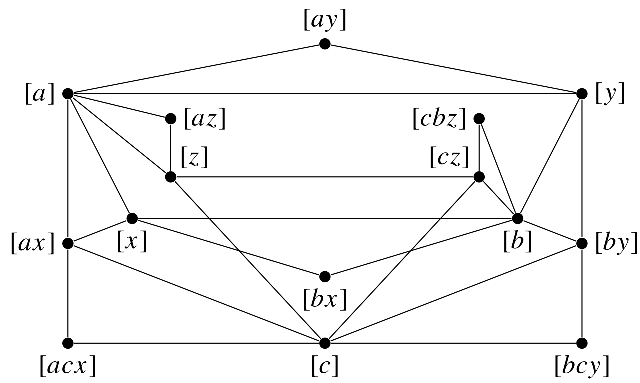

Figure 7. The involution graph for $\mathrm{Aut}^0(W_3)$.

Here we must give the following warning. The proof above relies on finding a portion of the involution graph which we know should not appear in any clique graph. In the example, it is the "triangle of triangles" configuration (see Example 4.2). This should not occur in the involution graph of a right-angled Coxeter group, essentially because it means that all three vertices of the central triangle must be generators (whereas, by construction of the involution graph in the case of right-angled Coxeter groups, we should expect two of the vertices to be generators and the third to be their product).

However, we must point out that, strictly speaking, there is no such thing as a "poison pill" subgraph — a subgraph which, by its presence, prevents the given graph from being a clique graph. Indeed, if Γ is any graph, then Γ is an induced subgraph of Γ_K. In this way, any finite graph may appear as an induced subgraph in some clique graph (even the "triangle of triangles"). In the example above, it

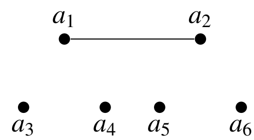

Figure 8. The defining graph Γ.

is important that we know the central triangles Γ_3 and Γ_4 to be not just induced subgraphs, but also maximal cliques.

In all of the previous results, we have only considered split extensions by subgroups $H \le \mathrm{Out}^0(W_\Gamma)$ which were generated by partial conjugations. In particular, if the partial conjugations commuted pairwise, then H was finite and the extension $G = W_\Gamma \rtimes H$ was right-angled Coxeter. On the other hand, in the example above, the partial conjugations did not commute, thus H was infinite and G was not right-angled Coxeter. One might wonder whether the existence of a right-angled Coxeter presentation for the extension G depends only on the finiteness of H. The following example answers this question in the negative.

Example 3.8. Let Γ be the graph shown in Figure 8. Let x be the partial conjugation with acting letter a_1 and domain $\{a_3, a_4\}$, and let y be the partial conjugation with acting letter a_2 and domain $\{a_3, a_5\}$. Since a_1 and a_2 commute, so do x and y. Now write $z = xy$ for the product, which is also an involution. It follows from Theorem 1.1 that $G = W_\Gamma \rtimes \langle x, y\rangle$ is a right-angled Coxeter group. Consider the subgroup $H = W_\Gamma \rtimes \langle z\rangle \le G$. The defining graph for G and the involution graph for H are shown in Figure 9.

The reader could verify Δ_H in two ways — first, by directly calculating the involutions and checking their commuting relations; and second, using the defining graph of G to calculate Δ_G, and then picking out the subset of vertices in Δ_G which are labeled by elements in the subgroup H. (Note that this latter method

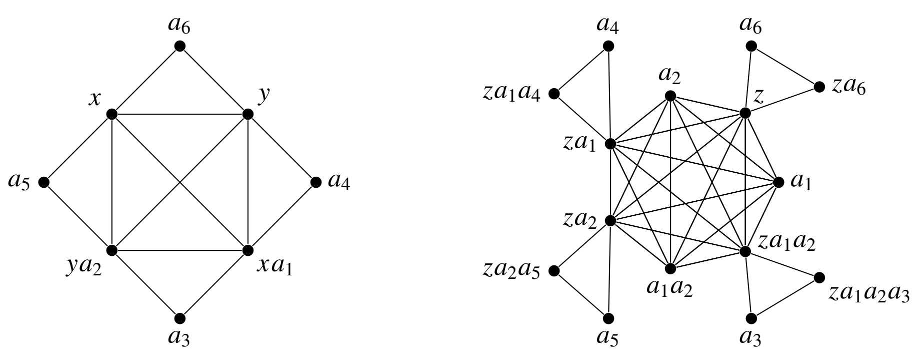

Figure 9. The defining graph of G (left) and the involution graph of H (right).

of constructing the involution graph of a subgroup will not work in general. It works for the current example because G is a right-angled Coxeter group and H is normal.)

We can realize G as the iterated semidirect product

$$G = (W_\Gamma \rtimes \langle z \rangle) \rtimes \langle x \rangle = H \rtimes \langle x \rangle.$$

This gives an example of a right-angled Coxeter group W_Γ with a degree-2 split extension H which is not right-angled Coxeter. Moreover, taking a further degree-2 extension G, we recover the right-angled Coxeter property.

3C. ***Semidirect product decompositions.*** Here we present some results which are unrelated to the problem of recognizing right-angled Coxeter groups. These results fall naturally out of the applications in Section 3A, and they generally address our ability to recognize semidirect product decompositions of W_Γ by identifying features of Γ.

To give the basic idea of how to generate these results, we give the following alternate description of Lemma 3.2. Suppose $a_1, \dots, a_n$ are the vertices of Γ and $\alpha_1, \dots, \alpha_k$ are partial conjugations as in the lemma. We will suppose that a_1 is the acting letter and D_i is the domain of α_i. The lemma says that the group $G = W_\Gamma \rtimes \langle \alpha_1, \dots, \alpha_k \rangle$ is a right-angled Coxeter group, and the proof of the lemma gives the right-angled Coxeter generating set. We can directly construct the defining graph Λ for G from Γ as follows:

(1) Add k new vertices labeled $\alpha_1, \dots, \alpha_k$, all connected to one another and to a_1.

(2) Connect each α_i to every a_j where $a_j \notin D_i$.

(3) Relabel a_1 as $a_1\alpha_1\alpha_2 \cdots \alpha_k$ and connect this to each vertex in $D_1 \cup D_2 \cup \cdots \cup D_k$.

The vertices $a_1, \alpha_1, \dots, \alpha_k$ form a clique of size $k+1$, and the union of the stars of these vertices cover all of Λ. The restriction in Lemma 3.2 that the domains be pairwise disjoint implies the following: we can distinguish D_i as those elements in $\mathrm{St}(a_1) \setminus \mathrm{St}(\alpha_i)$ which are contained in $\mathrm{St}(\alpha_j)$ for every $j \neq i$. The following corollary is immediate from this description.

Corollary 3.9. *Suppose Λ contains $k+1$ vertices $a_1, \alpha_1, \dots, \alpha_k$ satisfying: the following properties:*

(1) *$a_1, \alpha_1, \dots, \alpha_k$ form a clique.*

(2) $\mathrm{St}(a_1) \cup \bigcup_i \mathrm{St}(\alpha_i) = \Lambda$.

(3) *the sets $D_i = (\mathrm{St}(a_1) \setminus \mathrm{St}(\alpha_i)) \cap \bigcap_{j \neq i} \mathrm{St}(\alpha_j)$ are all nonempty.*

Define Γ to be the graph obtained from Λ by removing the vertices $\alpha_1, \dots, \alpha_k$ and any edge from a_1 to any D_i. Then W_Λ can be realized as the semidirect product

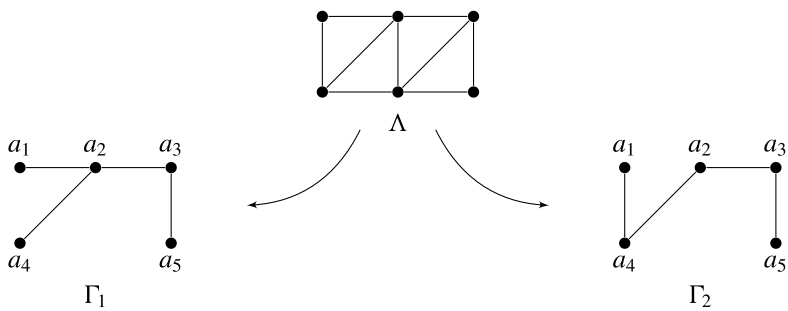

Figure 10. $W_\Lambda = W_{\Gamma_1} \rtimes \langle x \rangle = W_{\Gamma_2} \rtimes \langle y \rangle$, where x, y act like the partial conjugations $x = \chi_{4,\{1\}}$ and $y = \chi_{2,\{1\}}$ on Γ_1, Γ_2, respectively.

$W_\Gamma \rtimes H$, *where* $H \leq \mathrm{Out}^0(W_\Gamma)$ *is generated by the partial conjugations with acting letter* a_1 *and domains* D_i.

Theorem 1.1 yields an analogous corollary, since in each case they tell how to build the defining graph of the extension from the original defining graph, and the process is always reversible. It is not uniquely reversible. A given right-angled Coxeter group will, in general, have many semidirect product decompositions. As an example, consider the decompositions shown in Figure 10.

4. Details

In this section we explore the properties of the clique graph, the star poset, and the involution graph introduced in Section 2. We present detailed proofs of these properties, including proofs establishing claims made in that section and the correctness of our collapsing algorithms.

4A. ***The clique graph and the star poset.*** Recall that, given a graph Γ, we write Γ_I for the intersections of maximal cliques in Γ. We begin by establishing a correspondence between the *maximal clique structure* of a graph Γ and its clique graph Γ_K. By *maximal clique structure*, we mean that there is a bijection between the maximal cliques of Γ and those of Γ_K, which respects intersections.

Proposition 4.1. *Suppose* Γ *is a finite graph with maximal cliques* $\Gamma_1, \ldots, \Gamma_r$. *For any subset* $I \subseteq \{1, 2, \ldots, r\}$, *write*

$$\Gamma_I = \bigcap_{i \in I} \Gamma_i.$$

Similarly, write $\Gamma_{K,1}, \ldots, \Gamma_{K,s}$ *for the maximal cliques of* Γ_K, *and write* $\Gamma_{K,I}$ *for the intersections of maximal cliques. Then, possibly after reindexing*:

(1) $r = s$.

(2) *Each* $\Gamma_{K,J}$ *contains at least one* J*-minimal vertex* (*namely*, *the vertex labeled by the clique* Γ_J).

(3) $\Gamma_{K,i} = (\Gamma_i)_K$ (*that is*, $(\Gamma_i)_K$ *naturally injects as a labeled graph into* Γ_K, *and the image is precisely* $\Gamma_{K,i}$).

(4) $\Gamma_{K,I} = (\Gamma_I)_K$.

(5) *If* Γ_I *is a clique of size* k, *then* $\Gamma_{K,I}$ *is a clique of size* $2^k - 1$.

Proof. (1) For each maximal clique Γ_i in Γ, there is a corresponding vertex v_i in Γ_K. This vertex is adjacent only to vertices representing subsets of Γ_i since Γ_i is maximal, and so v_i is contained in the unique maximal clique $\mathrm{St}(v_i)$ in Γ_K. In particular, since each v_i, v_j can be in the same maximal clique of Γ_K, we have $r \leq s$.

Conversely, each vertex of the maximal clique $\Gamma_{K,i}$ is labeled by some clique of vertices in Γ. Since $\Gamma_{K,i}$ forms a clique, the collection of all vertices of Γ which appear in the labels of vertices of $\Gamma_{K,i}$ must form a clique Λ in Γ. It is clear that Λ is maximal, since $\Gamma_{K,i}$ is. Thus $\Lambda = \Gamma_j$ for some j. That is, $s \leq r$, establishing (1). The description we have just given of the cliques in Γ_K also establishes the correspondence in (3), and therefore in (4).

As noted in the claim, the clique Γ_J forms a vertex of Γ_K. It is straightforward to see that this vertex is J-minimal in $\Gamma_{K,J}$, establishing (2).

Finally, if Γ_I is a clique of size k, then every nonempty subset of vertices induces a clique, and so corresponds to a vertex in $\Gamma_{K,I}$. There are $2^k - 1$ of these subsets, which correspond to $2^k - 1$ vertices in $\Gamma_{K,I}$. □

Let Γ be a finite graph with maximal cliques $\Gamma_1, \ldots, \Gamma_r$. As before, write Γ_I for the intersections of the maximal cliques, and suppose $|\Gamma_I| = k_I$. Then

$$\sum_{I \supsetneq J} (-1)^{|I \setminus J|+1} k_I \leq k_J.$$

This is a direct application of the inclusion-exclusion principle, since the left hand side of the inequality counts the number of vertices in $\Gamma_J \cap \bigcup_{i \notin J} \Gamma_i$ (while the right-hand side is, by definition, the total number of vertices in Γ_J). We have therefore established that any clique graph must satisfy the MCC, MVC, and IEC. This gives one direction of the characterization theorem:

Theorem 2.3. *Let* Γ' *be a graph. There exists a graph* Γ *such that* $\Gamma' = \Gamma_K$ *if and only if the following three conditions are satisfied*:

(1) *Maximal clique condition* (*MCC*): *For all* I, *there exists some* k_I *such that*

$$|\Gamma'_I| = 2^{k_I} - 1.$$

(2) *Minimal vertex condition* (*MVC*): *Each nonempty intersection* Γ'_J *contains some* J*-minimal vertex* v_J.

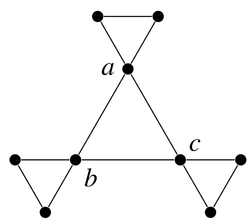

Figure 11. The triangle $\{a, b, c\}$ forms a maximal clique which fails the IEC. This was essentially the feature of Example 3.6 which prevented the group there from being right-angled Coxeter.

(3) *Inclusion-exclusion condition* (*IEC*): *For each* J,

$$\sum_{I \supsetneq J} (-1)^{|I \setminus J|+1} k_I \leq k_J.$$

If we are faced with some graph which we do not know to be a clique graph, we can check directly that the intersections of maximal cliques have sizes of the form $n_I = 2^{k_I} - 1$, and we can check directly that the system of integers k_I satisfies the inclusion-exclusion inequalities. Thus, determining whether a graph arises as a clique graph is reduced to checking a system of integer inequalities (once we establish the other direction of the theorem).

Example 4.2. Consider the graph in Figure 11. In this graph, all intersections of maximal cliques have sizes of the form $2^k - 1$, but the IEC fails. So the graph cannot arise as a clique graph.

We will establish the converse of Theorem 2.3 by proving that, for any graph which satisfies the MCC, MVC, and IEC, the proposed collapsing procedure of Theorem 2.4 produces the desired output. In order to evaluate the collapsing procedure, we must explore some properties of the star poset $\mathcal{P}(\Gamma)$.

Lemma 4.3. *Let* $[v] \in \mathcal{P}(\Gamma)$. *Then the vertices*

$$S = \bigcup_{[v] \leq [w]} [w]$$

form a clique in Γ. *If this clique is maximal*, *then* $[v]$ *is minimal in* $\mathcal{P}(\Gamma)$.

Proof. If $w, w' \in S$ are any vertices, then $w \in \mathrm{St}(v) \subseteq \mathrm{St}(w')$, so w and w' are adjacent. Thus S forms a clique.

We now suppose $[v]$ is not minimal. Then there is some $[w] < [v]$. In particular, $w \notin S$, but $w \in \mathrm{St}(w) \subseteq \mathrm{St}(s)$ for any $s \in S$, hence w is a vertex outside of S adjacent to all of S. Thus S is not maximal. $\square$

Definition 4.4. For $[v] \in \mathcal{P}(\Gamma)$, we call the clique S defined in the lemma the *clique above* $[v]$. We will use the notation S_v if we need to keep track of the vertex v.

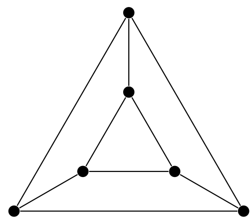

Figure 12. It is easy to check that each vertex is its own star-equivalence class, and that these equivalence classes are pairwise not comparable. In particular, each $[v]$ is minimal, and each $S_v = \{v\}$ is not a maximal clique.

The converse of the lemma (i.e., that minimality of $[v]$ implies maximality of S_v) is false in general. A simple example is given in Figure 12. However, we claim that the converse does hold for those Γ which are clique graphs. Namely:

Proposition 4.5. *Suppose Γ satisfies the MVC. Then $[v]$ is a minimal element of $\mathcal{P}(\Gamma)$ if and only if v is a minimal vertex of Γ. In this case, S_v is the unique maximal clique containing v.*

Proof. Suppose v is a minimal vertex of Γ. Then $\mathrm{St}(v)$ is the unique maximal clique containing v. Since S_v is a clique containing v, it is clear that $S_v \subseteq \mathrm{St}(v)$. Conversely, if $x \in \mathrm{St}(v)$, then $\mathrm{St}(v) \subseteq \mathrm{St}(x)$, hence $[v] \leq [x]$ and $x \in S_v$. Thus $\mathrm{St}(v) = S_v$ is maximal. By the previous lemma, since S_v is maximal, $[v]$ is minimal.

Conversely, suppose v is not minimal. Then v is contained in the intersection of two distinct maximal cliques, Γ_1 and Γ_2. Since Γ_i are maximal cliques, they contain minimal vertices w_i. By the above argument, $[w_i] \leq [v]$, and this must be a strict inequality since, e.g., $w_2 \in \mathrm{St}(v) \setminus \mathrm{St}(w_1)$. Thus $[v]$ is not minimal. □

Proposition 4.6. *For any finite graph Γ and $[v] \in \mathcal{P}(\Gamma)$, S_v is an intersection of maximal cliques.*

Proof. Let $\Gamma_1, \dots, \Gamma_k$ be all the maximal cliques of Γ containing S_v. It is clear that $S_v \subseteq \bigcap \Gamma_i$.

Conversely, let $v' \in \bigcap \Gamma_i$ and suppose $v' \notin S_v$. Since $\mathrm{St}(v) \not\subseteq \mathrm{St}(v')$, there is some $x \in \mathrm{St}(v)$ which is not in $\mathrm{St}(v')$. In particular, since $\bigcup \Gamma_i \subseteq \mathrm{St}(v')$, we must have $x \notin \Gamma_i$ for any i. By construction of S_v, we must have $x \in \mathrm{St}(w)$ for each $w \in S_v$. Now $S_v \cup \{x\}$ forms a clique which contains S_v and is not equal to Γ_i for any i, contradicting our assumption that the list of Γ_i contained all maximal cliques containing S_v. So there can exist no such v', hence $S_v = \bigcap \Gamma_i$, proving the claim. □

We observe that the previous two propositions say the following in the case of clique graphs (which must satisfy the MVC):

Corollary 4.7. *Suppose Γ_K is a clique graph.*

(1) $[v]$ *is minimal in* $\mathcal{P}(\Gamma_K)$ *if and only if* v *is minimal in* Γ_K.

(2) *If* $[v]$ *is nonminimal, then* S_v *is the intersection of maximal cliques* (*and therefore has size of the form* $2^k - 1$). *In this case,*

$$S_v = \bigcap_{\substack{[w]\text{ minimal} \\ [w]\leq[v]}} S_w.$$

This shows that the star poset also records information about the intersections of maximal cliques: any clique above $[v]$ is such an intersection. Finally, we prove the converse.

Proposition 4.8. *Suppose* Γ_K *is a clique graph. Then any intersection of maximal cliques is equal to* S_v *for some* v.

Proof. Since Γ_K is a clique graph, it satisfies the MVC. Let $\Gamma_{K,J}$ be any intersection of maximal cliques, and let $v \in \Gamma_{K,J}$ be a J-minimal vertex. Without loss of generality, let J be the maximal index set without changing the intersection. In particular, J is precisely the index set of all maximal cliques containing v, so that $\mathrm{St}(v) = \bigcup_{j\in J} \Gamma_{K,j}$.

We claim that $S_v = \Gamma_{K,J}$. Let $u \in S_v$. By definition of S_v, $[v] \leq [u]$, so

$$\mathrm{St}(u) \supset \mathrm{St}(v) = \bigcup_{j\in J} \Gamma_{K,j}.$$

That is, u is adjacent to every vertex in $\Gamma_{K,j}$, for each $j \in J$. Since each $\Gamma_{K,j}$ is a maximal clique, this shows $u \in \Gamma_{K,j}$ for each $j \in J$. That is, $u \in \Gamma_{K,J}$.

Conversely, let $w \in \Gamma_{K,J}$. Then w is adjacent to all vertices in $\Gamma_{K,j}$ for $j \in J$, thus $\bigcup_{j\in J} \Gamma_{K,j} \subseteq \mathrm{St}(w)$. That is, $\mathrm{St}(v) \subseteq \mathrm{St}(w)$, so $[v] \leq [w]$. By its definition, S_v contains w. □

We note that the previous proof gives a nice description of the elements of each star-equivalence class.

Corollary 4.9. *Suppose* Γ_K *is a clique graph. Then any* $[v] \in \mathcal{P}(\Gamma_K)$ *consists precisely of the* J*-minimal vertices of* Γ_K, *where* J *is the largest index set such that* $v \in \Gamma_J$.

Proof. Clearly, all J-minimal vertices for the same index set J must have the same star (namely, $\bigcup_{j\in J} \Gamma_{K,j}$). Conversely, suppose v is J-minimal and $[v] = [w]$ for some w. Then $w \in \Gamma_j$ for each $j \in J$, and $w \notin \Gamma_i$ for any $i \notin J$. (Otherwise, all of Γ_i would be in $\mathrm{St}(w)$, which we have assumed to be equal to $\mathrm{St}(v)$, a contradiction.) Therefore, w is J-minimal. □

These results establish that, for a clique graph, the cliques above vertices are precisely the intersections of maximal cliques, and every intersection of maximal cliques is the clique above some vertex. (This is not, in general, a bijective correspondence. As remarked earlier, it may be that $\Gamma_{K,J} = \Gamma_{K,J'} = S_v$, where $J \neq J'$.) In our collapsing algorithm to recover Γ from Γ_K, we begin at the top of the poset (this is the deepest intersection of maximal cliques) and work downwards. The previous proposition ensures that the algorithm examines every intersection of maximal cliques as it traverses every element in the poset structure.

We now wish to prove the correctness of our collapsing procedure, which also establishes the other direction of Theorem 2.3. Recall the procedure:

Theorem 2.4. *Let Γ' be a graph which satisfies the MCC, MVC, and IEC. Then there is a unique (up to isomorphism) graph Γ such that Γ' is isomorphic to Γ_K. Moreover, the following collapsing procedure produces the graph Γ if it exists.*

(1) *Initially, let $V = \{\}$.*

(2) *Let $[w] \in \mathcal{P}(\Gamma')$ be a class such that every class $[v]$ with $[w] < [v]$ has already been considered. Write*
$$S_w = \bigcup_{[v] \geq [w]} [v].$$
Then there is some k such that $|S_w| = 2^k - 1$. Let k' be the number of vertices of S_w which are already contained in V. Choose $k - k'$ vertices of $[w]$ to add to the vertex set V.

(3) *Repeat the previous step until all classes of $\mathcal{P}(\Gamma')$ have been considered.*

(4) *Return the graph $C(\Gamma')$ which is the induced subgraph of Γ' on the vertex set V.*

We first must address a subtlety, namely, that we can carry out the choice in step 2 of the algorithm.

Proposition 4.10. *In step 2 of the collapsing procedure, $0 \leq k - k' \leq |[w]|$. So we are able to choose an appropriate number of vertices from $[w]$ to add to V.*

Proof. The clique S_w is some intersection of maximal cliques Γ'_J by Proposition 4.5. From this clique, we have already chosen k' vertices, and every vertex among those already chosen comes from a larger poset element, which is therefore a strictly smaller intersection of maximal cliques. By the IEC, the number of elements we could have chosen is at most $k_J = k$, hence $k' \leq k$.

Now $S_w = \left(\bigcup_{[w]<[v]} S_v\right) \cup [w]$. Because
$$|S_w| = 2^k - 1, \qquad \left|\bigcup_{[w]<[v]} S_v\right| \leq 2^{k'} - 1, \quad \text{and} \quad [w] \leq 2^{|[w]|},$$
we have $2^k - 1 \leq 2^{k'} - 1 + 2^{|[w]|}$. Therefore, $2^k \leq 2^{k'} + 2^{|[w]|}$. But $2^x + 2^y \leq 2^{x+y}$ for all pairs of positive integers x, y. Thus $2^k \leq 2^{k'+|[w]|}$, and $k \leq k' + |[w]|$. $\square$

We also see that step 2 does not tell us explicitly which vertices of $[w]$ to add to V. We claim this choice does not matter:

Proposition 4.11. *Given Γ', if the procedure above does not return* FALSE, *then the isomorphism type of the graph Γ does not depend on the choices made in step* 2 *of the collapsing procedure.*

Proof. Without loss of generality, we will suppose our choices differ by a single vertex. Suppose we are about to consider $[v]$ and have constructed the set V thus far. Let $v_1, \ldots, v_{k+1} \in [v]$, where $k > 0$ is the number of vertices from $[v]$ which we must add to V. Let

$$V_1 = V \cup \{v_1, \ldots, v_k\},$$
$$V_2 = V \cup \{v_1, \ldots, v_{k-1}, v_{k+1}\}.$$

We observe that we can make all future choices the same (since we haven't changed the number of vertices we must pick from $[w]$ for any $[w] \leq [v]$), so that we create two final graphs Γ_1 and Γ_2 whose vertex sets differ only by switching v_k and v_{k+1}.

We now claim that the resulting graphs Γ_1 and Γ_2 are isomorphic. By the previous observation, the vertex sets of Γ_1 and Γ_2 differ only by switching v_k and v_{k+1}. So we can define a map $\varphi : \Gamma_1 \to \Gamma_2$ which sends each vertex other than v_k to itself, and which sends v_k to v_{k+1}. We claim that φ defines a graph isomorphism. Clearly any adjacency relation not involving v_k is preserved under φ. Suppose w is a vertex of Γ_1 adjacent to v_k. Then $w \in \mathrm{St}(v_k) = \mathrm{St}(v_{k+1})$, so w is adjacent to v_{k+1}. Thus φ is a graph homomorphism. By the same argument, the analogous map $\psi : \Gamma_2 \to \Gamma_1$ is also a graph homomorphism, and the two maps are clearly inverses. Hence Γ_1 is isomorphic to Γ_2. The full result follows by induction. □

This shows that the isomorphism type of an output graph $C(\Gamma')$ is determined. However, a priori it could be the case that there are two graphs Γ, Λ such that Γ_K and Λ_K are isomorphic, but the collapsing procedure applied to Γ_K always outputs the isomorphism type Γ. The following proposition says that the maximal clique structure of Γ' determines the maximal clique structure of the output $C(\Gamma')$. The theorem following the proposition establishes that the maximal clique structure (including information about the sizes of all intersections of maximal cliques) determines a graph up to isomorphism. By Proposition 4.1, any graph whose clique graph is Γ' will have the same clique graph structure, and will therefore be isomorphic. These results together show that the collapsing procedure outputs the unique graph Γ up to isomorphism so that $\Gamma_K = \Gamma'$.

Proposition 4.12. *Let Γ' be a finite graph satisfying the MCC, MVC, and IEC. In particular, this implies there is a system of integers k_I such that $|\Gamma'_I| = 2^{k_I} - 1$. Let $C(\Gamma') = \Gamma$. Then the maximal cliques of Γ correspond to the maximal cliques of Γ', and $|\Gamma_I| = k_I$ for all I.*

Proof. By assumption, each Γ_I' contains an I-minimal vertex v_I'. We have $|\Gamma_I'| = 2^{k_I} - 1$, and the algorithm chooses exactly k_I vertices from $S_{v_I'}$. Corollary 4.7 implies that the maximal cliques in Γ have sizes of the form k_i, and Proposition 4.8 ensures that we have $|\Gamma_I| = k_I$ for all intersections of maximal cliques (since all intersections Γ_I' occur as the clique above some element in the poset). □

We have shown now that, if the algorithm returns any graph, then it returns a graph with a certain number of maximal cliques, and the intersections of the maximal cliques have certain sizes. We now establish that a finite graph is determined up to isomorphism by the sizes of the intersections of maximal cliques.

Theorem 4.13. *Let Γ, Λ be finite graphs. Suppose both graphs have r maximal cliques which may be indexed in such a way that, for all index sets $I \subset \{1, 2, \ldots, r\}$, $|\Gamma_I| = |\Lambda_I|$. That is, all intersections of maximal cliques have the same sizes in each graph. Then there is an isomorphism $\varphi : \Gamma \to \Lambda$ which maps Γ_i to Λ_i for each i.*

Proof. We first claim that the poset structures $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Lambda)$ are the same, and the corresponding equivalence classes have the same sizes. For each $v \in \Gamma$, let J_v be the maximal index set such that $v \in \Gamma_{J_v}$. Then $\mathrm{St}(v) = \bigcup_{j \in J_v} \Gamma_j$. The equivalence class of v consists of the J_v-minimal vertices of Γ by Corollary 4.9. By assumption, $|\Gamma_{J_v}| = |\Lambda_{J_v}|$. Moreover, the number of vertices which are in some further intersection is given by the inclusion-exclusion formula:

$$\sum_{J \supsetneq J_v} (-1)^{|J \setminus J_v|+1} |\Gamma_J| = \sum_{J \supsetneq J_v} (-1)^{|J \setminus J_v|+1} |\Lambda_J|.$$

That is, the number of J_v-minimal vertices in Γ and in Λ is the same. Since this is for any v, the sizes of star-equivalence classes of vertices in Γ and Λ are equal for every class. Each equivalence class is represented by some index set J (although not every index set represents a class).

An equivalence class represented by J is smaller in the poset structure than another represented by J' if and only if $J \subseteq J'$. Since this holds in both Γ and Λ, it follows that the poset structures are equivalent.

Now, we build a map $\varphi : \Gamma \to \Lambda$ by piecing together (arbitrary) bijections between each pair of corresponding equivalence classes. We observe that, by construction,

$$\varphi([v]) = [\varphi(v)].$$

We also observe that Γ_i is mapped to Λ_i for each i. Let $v \in \Gamma_i$, so that $i \in J_v$. By construction, $\varphi(v) \in \Lambda_{J_v}$, which is an intersection of maximal cliques including Λ_i. That is, $\varphi(v) \in \Lambda_i$. It follows that φ maps Γ_I to Λ_I for each I.

We must show that φ preserves adjacency. Suppose $v, w \in \Gamma$ are adjacent. Then the edge $\{v, w\}$ extends to some maximal clique Γ_i. Now φ maps Γ_i to Λ_i, so $\varphi(v)$ and $\varphi(w)$ are still adjacent. □

This completes the proofs of Theorem 2.4 and Theorem 2.3.

4B. ***Calculations in the abelianization.*** We now discuss the modifications to the collapsing procedure to make use of algebraic information. Recall from the discussion in Section 2 that, given a group G, we first form the involution graph Δ_G and try to find a full system of representatives (i.e., a labeling of the vertices of Δ_G which exhibit all commuting relations simultaneously). If Δ_G is a clique graph, the collapsing procedure will give a graph $\Gamma = C(\Delta_G)$ such that $\Gamma_K = \Delta_G$. Moreover, Γ will carry the labels of the vertices chosen during the collapsing, so that the choice of which vertices to keep and which to omit is essentially the choice of which elements of G will be the generators in a (hypothetical) right-angled Coxeter presentation. For this reason, we must take care when choosing our generator vertices to avoid choosing group elements which have a nontrivial product relation. We will now demonstrate a method of passing to the abelianization G^{ab} to determine product relations using straightforward calculations.

Suppose we are given a finitely presented group

$$G = \langle s_1, \dots, s_m \mid r_1, \dots, r_k \rangle.$$

Recall that, for $g \in G$, we write $\bar{g}$ for the image of g in the abelianization. A presentation for G^{ab} is given by

$$G^{\mathrm{ab}} \cong \langle \overline{s_1}, \overline{s_2}, \dots, \overline{s_m} \mid \overline{r_1}, \overline{r_2}, \dots, \overline{r_k}, [\overline{s_i}, \overline{s_j}] \text{ for } 1 \le i, j \le m \rangle.$$

Writing the group operation additively in G^{ab}, we can write the relations as linear combinations of the generators with integer coefficients:

$$\overline{r_i} = a_{i,1}\overline{s_1} + a_{i,2}\overline{s_2} + \cdots + a_{i,m}\overline{s_m}.$$

The coefficients $(a_{i,j})$ form a $k \times m$ matrix R, called the *relations matrix for* G^{ab}.

We briefly recall the *Smith normal form*. Given the $k \times m$ integer matrix R, there exist $k \times k$ and $m \times m$ invertible matrices P, Q and a diagonal matrix S such that $R = PSQ$, and the diagonal elements of S are $\alpha_1, \dots, \alpha_r, 0, \dots, 0$ such that $\alpha_i \mid \alpha_{i+1}$. The diagonal matrix S is called the *Smith normal form of* R.

Interpreting S as the relation matrix for a presentation, we have that G^{ab} is in a canonical form as a direct product of cyclic groups. Normal forms are immediate and computations in G^{ab} are much easier. Moreover, we now have an effective quotient map from $G \to G^{\mathrm{ab}}$ in this canonical form. Namely, for any $g \in G$ with $g = \prod s_j$, we have $\bar{g} = \sum \overline{s_j} = \sum_{i=1}^{m} b_i \overline{s_i}$. The vector-matrix product $(b_1\, b_2\, \cdots\, b_m)Q$ will give the coefficients of $\bar{g}$ in the Smith normal form presentation of G^{ab}. This makes product relations easy to compute.

We now apply this method to show that, in step 2 of the collapsing procedure, we can avoid nontrivial product relations.

Proposition 2.11. *If W_Γ is a right-angled Coxeter group, then in step 2 of the collapsing procedure in Theorem 2.4, we can choose the $k-k'$ involutions of W_Γ so that the chosen elements do not exhibit a nontrivial product relation.*

Proof. In step 2 of our collapsing procedure, we consider an equivalence class $[w]$ of Δ_{W_Γ} and the clique above it, S_w, where $|S_w| = 2^k - 1$ for some k. If (W_Γ, S) is a right-angled Coxeter system for W_Γ and the labels are distinct, pairwise commuting involutions, then $H = S_w \cup \{e\}$ is a finite subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$: the elements of S_w are all involutions which pairwise commute. Any product g of these elements is an involution and commutes with all other elements of S_w (so it is connected to all of S_w). Moreover, any h which commutes with all of S_w commutes with any product of elements in S_w (namely g), and so g is contained in any maximal clique containing all of S_w. Since S_w is an intersection of maximal cliques and g is in all of these cliques, g lies in S_w. So H is a subgroup.

By Corollary 2.10, this subgroup projects injectively as a vector subspace into W_Γ^{ab}. Inductively, we assume that there exists a choice of a right-angled Coxeter system (W_Γ, S) such that $\overline{V}$ is a set of *standard basis elements* for $(W_\Gamma, S)^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^k$, i.e., each element has only one nonzero component in the representation for the abelianization given by our choice of right-angled Coxeter system (W_Γ, S). (The base case is $\overline{V} = \varnothing$ and any choice of (W_Γ, S).)

It follows that $\overline{V \cap S_w}$ is a linearly independent set in the $\mathbb{Z}/2\mathbb{Z}$-vector space W_Γ^{ab}. We can then choose $k-k'$ labels in $S_w - V$ to extend this linearly independent set to a basis $\overline{B}$ of $\langle \overline{S_w} \rangle$. (It's possible that $k-k'=0$.) Since H projects injectively, choosing a basis for $\langle \overline{S_w} \rangle$ is the same as choosing a basis for $\langle S_w \rangle$. We need to show that $\overline{V} \cup \overline{B}$ is linearly independent as well.

To clarify, we are now keeping track of *two* different representations of the abelianization. W_Γ^{ab} is the form calculated from the Smith normal form in step 0 of the procedure, and $(W_\Gamma, S)^{\mathrm{ab}}$ is the form wherein each element of $\overline{V}$ is a standard basis element. We will show that such a form must exist if W_Γ is a right-angled Coxeter group, but it will not be directly computable during the procedure itself. The existence of this form will be used to show that *any* choice of $\overline{B}$ during our procedure will result in no nontrivial product relations.

Since H is a finite subgroup of (W_Γ, S), it is conjugate to a special subgroup: $gHg^{-1} = \langle a_1, a_2, \dots, a_k \rangle$ for $\{a_1, a_2, \dots, a_k\} \subseteq S$. Consider $b \in B \subseteq S_w$. Then, reordering the vertices of S if necessary, $gbg^{-1} = a_1 a_2 \cdots a_m$ in (W_Γ, S). By the deletion condition of right-angled Coxeter groups (see, for example, [Davis 2008]), a product $c_1 c_2 \cdots c_\ell$ of distinct commuting generators of (W_Γ, S) commutes with $a_1 a_2 \cdots a_m$ if and only if c_j commutes with a_i for each i, j. In particular, $[b] = [a_1 a_2 \cdots a_m] \le [a_i]$ for each $1 \le i \le m$.

Suppose that $[b] \lneq [a_i]$ for each i. Then the procedure has already considered

$[a_i]$, and a subset of $\overline{V}$ is a basis for $\langle\overline{S_{a_i}}\rangle$, which contains $\overline{a_i}$. But by our inductive hypothesis, $\overline{V}$ is a set of standard basis elements relative to $(W_\Gamma, S)^{\mathrm{ab}}$; moreover $\overline{a_i}$ is also a standard basis element since $a_i \in S$. So the only way that $\overline{a_i} \in \langle\overline{V}\rangle$ is if $\overline{a_i} \in \overline{V}$ (and so by injectivity $g^{-1}a_i g \in V$). Thus, $b = g^{-1}a_1a_2\cdots a_m g \in \langle V\rangle$, and so b would *not* be chosen by the procedure to linearly extend $\overline{V}$.

Therefore, there must be some i such that $[b] = [a_i]$. By reordering the vertices of S if necessary, $[b] = [a_1]$. But then $gbg^{-1} = a_1a_2\cdots a_m$ and a_1 are involutions that commute with exactly the same involutions, and so

$$\varphi : W_\Gamma \to W_\Gamma, \quad \varphi(a_j) = \begin{cases} a_1a_2\cdots a_m & \text{if } j = 1, \\ a_j & \text{otherwise}, \end{cases}$$

is an involutive automorphism (in fact a transvection) of (W_Γ, S).

Now, $(W_\Gamma, \varphi(S))$ is also a right-angled Coxeter system for W_Γ with the exact same generators except for swapping a_1 and the product $a_1a_2\cdots a_m$. The set $\overline{V}$ consists of standard basis elements *not* including $\overline{a_1}$ and so is unchanged under the induced map $\bar{\varphi} : (W_\Gamma, S)^{\mathrm{ab}} \to (W_\Gamma, \varphi(S))^{\mathrm{ab}}$. Alternatively, $\varphi(b) = \varphi(g)^{-1}a_1\varphi(g)$ and so $\overline{\varphi(b)} = \overline{a_1}$. So if we let $(W_\Gamma, S') = (W_\Gamma, \varphi(S))$ be our new right-angled Coxeter system and let $V' = V \cup \{b\}$ be our new subset of labels from our chosen full set of representatives of Δ_{W_Γ}, then the inductive hypothesis is still satisfied. In particular, in our Smith normal form W_Γ^{ab}, the set $\overline{V'}$ is still linearly independent.

For each $b \in B$, we can perform this procedure in succession, making sure that for each b we choose different a_i such that $[b] = [a_i]$. If at any point this were not possible, it would mean that there was some $b_n = g^{-1}a_1a_2\cdots a_m g$ (in the updated system (W_Γ, S') with V') such that each a_j either satisfies

(1) $[b_n] \lneq [a_j]$, in which case $\overline{a_j} \in \overline{V'}$ from a previous step in the procedure, or

(2) $\overline{b_l} = \overline{a_j}$ for some $l < n$, in which case $\overline{a_j} \in \overline{V'}$ from a previous element of the basis.

In either case, since all of the $\overline{a_j}$ lie in $\overline{S_w}$, this would give a linear dependence in $\overline{S_w}$ among $\overline{B}$, which contradicts its choice as a basis.

Thus, by induction on both elements of the poset, and then within each class on the elements of each chosen basis, it will always be the case that $\overline{V}$ will consist of elementary basis elements in $(W_\Gamma, S)^{\mathrm{ab}}$ for some choice of system (W_Γ, S). Since every generator a_i of S is in S_{a_i}, we have $\overline{a_i} \in \langle\overline{V} \cap \overline{S_{a_i}}\rangle$, but since $\overline{V}$ are all elementary basis vectors, it must be that $\overline{a_i} \in \overline{V}$. Thus, at the end of the procedure, $\overline{V}$ will always be the full standard basis for some system $(W_\Gamma, S)^{\mathrm{ab}}$, and in particular, $\overline{V}$ will always be a basis of W_Γ^{ab}.

Any nontrivial product relation among the elements of V would induce a linear dependence among their images in W_Γ^{ab}. But since $\overline{V}$ is a basis, this can never happen. $\square$

Finally, we prove the proposition that allows us to hypothetically build edges in the involution graph of a given group by doing calculations in the abelianization:

Proposition 2.12. *If W_Γ is a right-angled Coxeter group, then two conjugacy classes of involutions $[x]$ and $[y]$ are connected by an edge in Δ_{W_Γ} if and only if there exists another class $[z]$ such that $\bar{z} = \overline{xy}$ in the abelianization.*

Proof. Let (W_Γ, S) be a right-angled Coxeter system for W_Γ, and let $[x]$ and $[y]$ be conjugacy classes of involutions. Since x and y are involutions in a right-angled Coxeter group, they are each conjugate to a product of commuting generators. So there exist $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in S$ and $g, h \in W_\Gamma$ such that

$$gxg^{-1} = a_1a_2\cdots a_n \quad \text{and} \quad hyh^{-1} = b_1b_2\cdots b_m,$$

where all of the a_i pairwise commute, and all of the b_j pairwise commute. Consider the product

$$w = a_1a_2\cdots a_nb_1b_2\cdots b_m = c_1c_2\cdots c_k,$$

where the c_ℓ are the generators that appear among either the a_i or the b_j but not both. (The ones that appear in both cancel with each other since they can be brought to the front or back of their respective words.) In the abelianization W_Γ^{ab}, we have $\bar{x} = \overline{a_1a_2\cdots a_n}$, $\bar{y} = \overline{b_1b_2\cdots b_m}$, and $\overline{w} = \overline{c_1c_2\cdots c_k}$.

Now suppose that $[x]$ and $[y]$ are connected by an edge in Δ_{W_Γ}. That means that some conjugates of x and y commute. This implies that the product z of those conjugates is an involution. But then $\bar{z} = \bar{x}\bar{y}$ in W_Γ^{ab}.

Conversely, suppose that there exists an involution z such that $\bar{z} = \bar{x}\bar{y}$. Since z is an involution, it must be conjugate to a product of distinct, commuting generators, each of which is mapped to its corresponding generator of W_Γ^{ab} and so can be recovered directly from $\bar{z}$. Thus, these generators must be exactly the c_l, and so they each pairwise commute. In particular, w is an involution, and gxg^{-1} and hyh^{-1} commute. Thus, $[x]$ and $[y]$ should be connected by an edge in Δ_{W_Γ}. ☐

We have now established the correctness of our right-angled Coxeter recognition procedure:

Theorem 2.13. *Suppose G is a group whose only torsion elements all have order* 2, *so that $G^{\text{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some n. If the following procedure returns* TRUE, *then G is a right-angled Coxeter group* (*and the procedure indicates a right-angled Coxeter presentation*). *If it returns* FALSE, *then G is not a right-angled Coxeter group.*

(1) *Determine all conjugacy classes of involutions in G, and let these be the vertices of a graph Γ'. If there are not finitely many, return* FALSE.

(2) *Apply Proposition 2.12 to construct the edges of Γ'.*

(3) *If Γ' is not a clique graph, return* FALSE.

(4) *Find a full system of representatives for the vertices of* Γ'. *If no such system exists*, *return* FALSE.

(5) *Collapse as in Theorem 2.4*, *using Proposition 2.11 to ensure that nontrivial product relations are avoided. Write* $C(\Gamma')$ *for the resulting graph.*

(6) *Let* Γ *be a graph isomorphic to* $C(\Gamma')$ *with generic vertex labels* $a_1, \ldots, a_n$. *Let* $\varphi : W_\Gamma \to G$ *be the map which sends the generators of* W_Γ *to the word given by the corresponding labels of vertices in* $C(\Gamma')$. *If* φ *is an isomorphism*, *return* TRUE.

(7) *Otherwise*, *return* UNKNOWN.

5. Further research

While we have used our decision procedure to successfully establish both positive and negative identification of right-angled Coxeter presentations among extensions of right-angled Coxeter groups, much work remains to be done. One might hope to eventually characterize all subgroups $H \leq \mathrm{Out}^0(W_\Gamma)$ (or $H \leq \mathrm{Aut}(W_\Gamma)$) such that $W_\Gamma \rtimes H$ is right-angled Coxeter. We note that subgroups $H \leq \mathrm{Out}^0(W_\Gamma)$ are not necessarily generated by partial conjugations (they may be generated by products of partial conjugations). Even if we only considered those H generated by partial conjugations, we could not extend Lemma 3.2 by induction. If x, y are two commuting partial conjugations of W_Γ, then

$$W_\Gamma \rtimes \langle x, y\rangle \cong (W_\Gamma \rtimes \langle x\rangle) \rtimes \langle y\rangle;$$

however, y may not act on $W_\Gamma \rtimes \langle x\rangle$ as a partial conjugation (it will generally act as a product of partial conjugations). Theorem 1.1 extends the lemma by induction, but we have many more examples of right-angled Coxeter extensions which are not covered by this theorem. More work is required for a complete characterization.

As in Section 3C, following a characterization of extensions $W_\Gamma \rtimes H$ which are right-angled Coxeter, we would also gain insight into semidirect product decompositions of right-angled Coxeter groups. Given a graph Λ, we could hope to obtain a complete list of graph features which identify W_Λ as $W_\Gamma \rtimes H$, where $H \leq \mathrm{Out}^0(W_\Gamma)$. (We observe that this would not identify all semidirect product decompositions of right-angled Coxeter groups. There are certainly decompositions which are not of this form.)

We strongly suspect that, whenever $H \leq \mathrm{Out}^0(W_\Gamma)$ is isomorphic to D_∞, the product $W_\Gamma \rtimes H$ is not right-angled Coxeter. The first example of Section 3B is of this form. Much of the argument in that example rests on using a normal form to establish that the given list of classes of involutions is complete. A general proof would require substantially more work to prove that we can accurately build the involution graph in the general case.

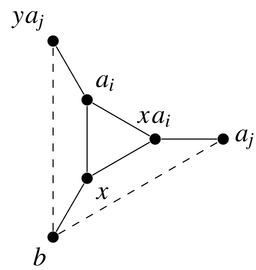

Figure 13. Dashed lines represent edges that may be present in some cases.

In particular, in the case of universal right-angled Coxeter groups (those whose defining graphs have no edges), the outer automorphism groups act on a contractible simplicial complex called McCullough–Miller space [Piggott 2012]. This space is analogous to Culler–Vogtmann outer space for the case of free groups [Culler and Vogtmann 1986], and we can use the action to classify all conjugacy classes of involutions in the outer automorphism groups. An analogous structure does not currently exist for the outer automorphism group of a general right-angled Coxeter group, and such a theory would need to be developed in order to construct the involution graph and confirm our conjecture.

Nevertheless, we can provide the following heuristic about what ought to go wrong in such an extension. Consider, for simplicity, a D_∞ generated by two noncommuting partial conjugations. If $x = \chi_{i,D}$ and $y = \chi_{j,E}$ are the partial conjugations, let b be any vertex other than a_j which is outside $\mathrm{St}(a_i) \cup D$. Then Figure 13 shows part of the involution graph of the extension.

In the figure, the edge from b to ya_j will be present if $b \in E$; the edge from b to a_j will be present if $b \in \mathrm{St}(a_j)$. The figure as drawn so far cannot be a clique graph, because the central triangle is a maximal clique which does not satisfy the IEC. But even if other vertices were present which could turn the central triangle into a 7-clique (or larger) so that the condition would be satisfied, the collapsing procedure would need to choose all three vertices x, a_i, xa_i, which are not linearly independent in the abelianization. However, this only establishes that the given pattern of labeling vertices in the involution graph — a pattern which has produced full systems of labels in all other examples so far — does not give an isomorphism to a right-angled Coxeter group in this case. We have not sufficiently established that the extension could not have any right-angled Coxeter presentation.

Acknowledgements

This work was partially supported by a grant from the Simons Foundation (#317466 to Adam Piggott). The authors would also like to thank Mauricio Gutierrez and the anonymous referee for a careful reading and helpful comments and suggestions.

References

[Charney et al. 2010] R. Charney, K. Ruane, N. Stambaugh, and A. Vijayan, "The automorphism group of a graph product with no SIL", *Illinois J. Math.* **54**:1 (2010), 249–262. MR 2012f:20106 Zbl 1243.20047

[Culler and Vogtmann 1986] M. Culler and K. Vogtmann, "Moduli of graphs and automorphisms of free groups", *Invent. Math.* **84**:1 (1986), 91–119. MR 87f:20048 Zbl 0589.20022

[Cunningham 2015] C. Cunningham, *Automorphisms of right-angled Coxeter groups*, Ph.D. thesis, Tufts University, May 2015.

[Davis 2008] M. W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series **32**, Princeton University Press, Princeton, NJ, 2008. MR 2008k:20091 Zbl 1142.20020

[Droms 1987] C. Droms, "Isomorphisms of graph groups", *Proc. Amer. Math. Soc.* **100**:3 (1987), 407–408. MR 88e:20033 Zbl 0619.20015

[Green 1990] E. Green, *Graph products of groups*, Ph.D. thesis, University of Leeds, 1990.

[Gutierrez et al. 2012] M. Gutierrez, A. Piggott, and K. Ruane, "On the automorphisms of a graph product of abelian groups", *Groups Geom. Dyn.* **6**:1 (2012), 125–153. MR 2888948 Zbl 1242.20041

[Kim and Koberda 2013] S.-h. Kim and T. Koberda, "Embedability between right-angled Artin groups", *Geom. Topol.* **17**:1 (2013), 493–530. MR 3039768

[Laurence 1995] M. R. Laurence, "A generating set for the automorphism group of a graph group", *J. London Math. Soc.* (2) **52**:2 (1995), 318–334. MR 96k:20068 Zbl 0836.20036

[Piggott 2012] A. Piggott, "The symmetries of McCullough–Miller space", *Algebra Discrete Math.* **14**:2 (2012), 239–266. MR 3099973 Zbl 1288.20033

[Radcliffe 2003] D. G. Radcliffe, "Rigidity of graph products of groups", *Algebr. Geom. Topol.* **3** (2003), 1079–1088. MR 2004h:20056 Zbl 1053.20025

[Tits 1988] J. Tits, "Sur le groupe des automorphismes de certains groupes de Coxeter", *J. Algebra* **113**:2 (1988), 346–357. MR 89b:20077

Received July 20, 2015. Revised December 9, 2015.

CHARLES CUNNINGHAM
DEPARTMENT OF MATHEMATICS
BOWDOIN COLLEGE
BRUNSWICK, ME 04011
UNITED STATES
ccunning@bowdoin.edu

ADAM PIGGOTT
DEPARTMENT OF MATHEMATICS
BUCKNELL UNIVERSITY
LEWISBURG, PA 17837
UNITED STATES
ap030@bucknell.edu

ANDY EISENBERG
DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER, OK 74078
UNITED STATES
andy.eisenberg@okstate.edu

KIM RUANE
DEPARTMENT OF MATHEMATICS
TUFTS UNIVERSITY
MEDFORD, MA 02155
UNITED STATES
kim.ruane@tufts.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.79

ON YAMABE-TYPE PROBLEMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

MARCO GHIMENTI, ANNA MARIA MICHELETTI AND ANGELA PISTOIA

Let (M, g) be an n-dimensional compact Riemannian manifold with boundary. We consider the Yamabe-type problem

$$\begin{cases} -\Delta_g u + au = 0 & \text{on } M, \\ \partial_\nu u + \frac{n-2}{2} bu = (n-2)u^{n/(n-2)\pm\varepsilon} & \text{on } \partial M, \end{cases}$$

where $a \in C^1(M)$, $b \in C^1(\partial M)$, ν is the outward pointing unit normal to ∂M, $\Delta_g u := \mathrm{div}_g \nabla_g u$, and ε is a small positive parameter. We build solutions which blow up at a point of the boundary as ε goes to zero. The blowing-up behavior is ruled by the function $b - H_g$, where H_g is the boundary mean curvature.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ with a boundary ∂M which is the union of a finite number of smooth closed compact submanifolds embedded in M.

A well-known problem in differential geometry is whether (M, g) is necessarily conformally equivalent to a manifold of constant scalar curvature whose boundary is minimal. When the boundary is empty this is called the Yamabe problem (see Yamabe [1960]), which has been completely solved by Aubin [1976], Schoen [1984] and Trudinger [1968]. Cherrier [1984] and Escobar [1992a; 1992b] studied the problem in the context of manifolds with boundary and gave an affirmative solution to the question in almost every case. The remaining cases were studied by Marques [2005; 2007], by Almaraz [2010] and by Brendle and Chen [2014].

Once the problem is solvable, a natural question about compactness of the full set of solutions arises. Concerning the Yamabe problem, it was first raised by Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold is not conformally equivalent to the standard sphere $\mathbb{S}^n$, since the group of

The research that lead to the present paper was partially supported by the group GNAMPA of Istituto Nazionale di Alta Matematica (INdAM).
MSC2010: 35J20, 58J05.
Keywords: Yamabe problem, blowing-up solutions, compactness.

conformal transformations of the round sphere is not compact itself. The problem of compactness has been widely studied in recent years and has been completely solved by Brendle [2008], Brendle and Marques [2009] and Khuri, Marques and Schoen [Khuri et al. 2009].

In the presence of a boundary, a necessary condition is that M is not conformally equivalent to the standard ball $\mathbb{B}^n$. The problem when the boundary of the manifold is not empty has been studied by V. Felli and M. Ould Ahmedou [2003; 2005], Han and Li [1999] and Almaraz [2011a; 2011b]. In particular, Almaraz studied the compactness property in the case of scalar-flat metrics. Indeed, the zero scalar curvature case is particularly interesting because it leads one to study a linear equation in the interior with a critical Neumann-type nonlinear boundary condition

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)} R_g u = 0 & \text{on } M,\ u > 0 \text{ in } M, \\ \partial_\nu u + \dfrac{n-2}{2} H_g u = (n-2) u^{n/(n-2)} & \text{on } \partial M, \end{cases} \tag{1-1}$$

where ν is the outward pointing unit normal to ∂M, R_g is the scalar curvature of M with respect to g, and H_g is the boundary mean curvature with respect to g.

We note that in this case compactness of solutions is equivalent to establish a priori estimates for solutions to equation (1-1). Almaraz [2011b] proved that compactness holds for a generic metric g. On the other hand, in [Almaraz 2011a] it was proved that if the dimension of the manifold is $n \geq 25$, compactness does not hold because it is possible to build blowing-up solutions to (1-1) for a suitable metric g. We point out that the problem of compactness in dimension $n \leq 24$ is still not completely understood.

An interesting issue, closely related to the compactness property, is the stability problem. One can ask whether or not the compactness property is preserved under perturbations of the equation, which is equivalent to having or not having uniform a priori estimates for solutions of the perturbed problem. Let us consider the more general problem

$$\begin{cases} -\Delta_g u + a(x) u = 0 & \text{in } M,\ u > 0 \text{ in } M, \\ \partial_\nu u + b(x) u = (n-2) u^{n/(n-2)} & \text{on } \partial M. \end{cases} \tag{1-2}$$

We say that the problem (1-2) is stable if for any sequences of C^1 functions $a_\varepsilon : M \to \mathbb{R}$ and $b_\varepsilon : \partial M \to \mathbb{R}$ converging in C^1 to functions $a : M \to \mathbb{R}$ and $b : \partial M \to \mathbb{R}$, for any sequence of exponents $p_\varepsilon := n/(n-2) \pm \varepsilon$ converging to the critical one $n/(n-2)$ and for any sequence of associated solutions u_ε bounded in $H^1(M)$ of the perturbed problems

$$\begin{cases} -\Delta_g u + a_\varepsilon(x) u = 0 & \text{in } M,\ u_\varepsilon > 0 \text{ in } M, \\ \partial_\nu u + \dfrac{n-2}{2} b_\varepsilon(x) u = (n-2) u_\varepsilon^{n/(n-2) \pm \varepsilon} & \text{on } \partial M, \end{cases} \tag{1-3}$$

there is a subsequence u_{ε_k} which converges in C^2 to a solution to the limit problem (1-2). The stability of the Yamabe problem has been introduced and studied by Druet [2003; 2004] and by Druet and Hebey [2005a; 2005b]. Recently, Esposito, Pistoia and Vetois [Esposito et al. 2014], Micheletti, Pistoia and Vetois [Micheletti et al. 2009] and Esposito and Pistoia [2014] proved that a priori estimates fail for perturbations of the linear potential or of the exponent.

In this paper, we investigate the question of stability of the problem (1-2). It is clear that it is not stable if it is possible to build solutions u_ε to perturbed problems (1-3) which blow up at one or more points of the manifold as the parameter ε goes to zero. Here, we show that the behavior of the sequence u_ε is dictated by the difference

$$\varphi(q) = b(q) - H_g(q) \quad \text{for } q \in \partial M. \tag{1-4}$$

More precisely, we consider the problem

$$\begin{cases} -\Delta_g u + a(x)u = 0 & \text{on } M,\ u > 0 \text{ in } M, \\ \dfrac{\partial}{\partial \nu}u + \dfrac{n-2}{2}b(x)u = (n-2)u^{n/(n-2)\pm\varepsilon} & \text{on } \partial M. \end{cases} \tag{1-5}$$

We assume that $a \in C^1(M)$ and $b \in C^1(\partial M)$ are such that the linear operator $\mathcal{L}u := -\Delta_g u + au$ with Neumann boundary condition $\mathcal{B}u := \partial_\nu u + \frac{1}{2}(n-2)bu$ is coercive; namely, there exists a constant $c > 0$ such that

$$\int_M \big(|\nabla_g u|^2 + a(x)u^2\big)\, d\mu_g + \frac{n-2}{2}\int_{\partial M} b(x)u^2\, d\sigma \ge c\|u\|^2_{H^1(M)}. \tag{1-6}$$

Here $\varepsilon > 0$ is a small parameter, $\Delta_g u := \operatorname{div}_g \nabla_g u$, and the space $H^1(M)$ is the closure of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H^1} = \left(\int_M \big(|\nabla_g u|^2 + u^2\big)\, d\mu_g\right)^{1/2}.$$

The problem (1-5) turns out to be either slightly subcritical or slightly supercritical if the exponent in the nonlinearity is either $n/(n-2)-\varepsilon$ or $n/(n-2)+\varepsilon$, respectively. Let us state our main result.

Theorem 1. *Assume* (1-6) *and $n \ge 7$.*

(i) *If $q_0 \in \partial M$ is a strict local minimum point of the function φ defined in* (1-4) *with $\varphi(q_0) > 0$, then provided $\varepsilon > 0$ is small enough, there exists a solution u_ε of* (1-5) *in the slightly subcritical case such that u_ε blows up at a boundary point when $\varepsilon \to 0^+$.*

(ii) *If $q_0 \in \partial M$ is a strict local maximum point of the function φ defined in* (1-4) *with $\varphi(q_0) < 0$, then provided $\varepsilon < 0$ is small enough, there exists a solution u_ε of* (1-5) *in the slightly supercritical case such that u_ε blows up at a boundary point when $\varepsilon \to 0^+$.*

We say that u_ε blows up at a point q_0 of the boundary if there exists a family of points $q_\varepsilon \in \partial M$ such that $q_\varepsilon \to q_0$ as $\varepsilon \to 0$ and, for any neighborhood $U \subset M$ of q_0, we have that $\sup_{q\in U} u_\varepsilon(q) \to +\infty$ as $\varepsilon \to 0$.

Our result does not concern the stability of the geometric Yamabe problem (1-1). Indeed, the function φ in (1-4) turns out to be identically zero. It would be interesting to discover the function which rules the behavior of blowing-up sequences in this case. We expect that it depends on the trace-free second fundamental form as it is suggested by Almaraz [2011b], where a compactness result in the subcritical case is established.

The case of low dimension also remains open, where we expect that the function φ in (1-4) should be replaced by a function which depends on the Weyl tensor of the boundary, as suggested by Escobar [1992a; 1992b].

The proof of our result relies on a very well known Ljapunov–Schmidt procedure. In Section 2 we set up the problem, and in Section 3 we reduce the problem to a finite dimensional one, which is then studied in Section 4.

2. Setting of the problem

Let us rewrite problem (1-5) in a more convenient way.

First of all, assumption (1-6) allows us to endow the Hilbert space $H := H^1(M)$ with the scalar product

$$\langle\langle u, v\rangle\rangle_H := \int_M (\nabla_g u \nabla_g v + a(x)uv)\, d\mu_g + \frac{n-2}{2}\int_{\partial M} b(x)uv\, d\sigma$$

and the induced norm $\|u\|_H^2 := \langle\langle u, u\rangle\rangle_H$. We define the exponent

$$s_\varepsilon = \begin{cases} \dfrac{2(n-1)}{n-2} & \text{in the subcritical case,} \\ \dfrac{2(n-1)}{n-2} + n\varepsilon & \text{in the supercritical case,} \end{cases}$$

and the Banach space $\mathcal{H} := H^1(M) \cap L^{s_\varepsilon}(\partial M)$ endowed with the norm $\|u\|_{\mathcal{H}} = \|u\|_H + |u|_{L^{s_\varepsilon}(\partial M)}$.

Notice that in the subcritical case $\mathcal{H}$ is identical to the Hilbert space H.

By trace theorems, we have the inclusion $W^{1,\tau}(M) \subset L^t(\partial M)$ for any t and τ satisfying $t \le \tau(n-1)/(n-\tau)$.

We consider $i : H^1(M) \to L^{2(n-1)/(n-2)}(\partial M)$ and its adjoint with respect to $\langle\langle \cdot, \cdot \rangle\rangle_H$, namely

$$i^* : L^{2(n-1)/n}(\partial M) \to H^1(M)$$

defined by

$$\langle\langle \varphi, i^*(g)\rangle\rangle_H = \int_{\partial M} \varphi g\, d\sigma \quad \text{for all } \varphi \in H^1,$$

so that $u = i^*(g)$ is the weak solution of the problem

$$\begin{cases} -\Delta_g u + a(x)u = 0 & \text{on } M, \\ \dfrac{\partial}{\partial \nu}u + \dfrac{n-2}{2}b(x)u = g & \text{on } \partial M. \end{cases} \tag{2-1}$$

We recall that by [Nittka 2011], if $u \in H^1(M)$ is a solution of (2-1), then for $2n/(n+2) \le q \le n/2$ and $r > 0$ we have

$$\|u\|_{L^{(n-1)q/(n-2q)}(\partial M)} = \|i^*(g)\|_{L^{(n-1)q/(n-2q)}(\partial M)} \le \|g\|_{L^{(n-1)q/(n-q)+r}(\partial M)}. \tag{2-2}$$

By this result, we can choose q and r such that

$$\frac{(n-1)q}{n-2q} = \frac{2(n-1)}{n-2} + n\varepsilon \quad \text{and} \quad \frac{(n-1)q}{n-q} + r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}, \tag{2-3}$$

that is,

$$q = \frac{2n + n^2\left(\frac{n-2}{n-1}\right)\varepsilon}{n+2+2n\left(\frac{n-2}{n-1}\right)\varepsilon} \quad \text{and} \quad r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon} - \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\left(\frac{n}{n-1}\right)\varepsilon}.$$

So, if $u \in L^{2(n-1)/(n-2)+n\varepsilon}(\partial M)$, then

$$|u|^{\frac{n}{n-2}+\varepsilon} \in L^{\frac{2(n-1)+n(n-2)\varepsilon}{n+\varepsilon(n-2)}}(\partial M)$$

and, in light of (2-2), also $i^*(|u|^{n/(n-2)+\varepsilon}) \in L^{2(n-1)/(n-2)+n\varepsilon}(\partial M)$.

Finally, we rewrite problem (1-5) — both in the subcritical and the supercritical case — as

$$u = i^*(f_\varepsilon(u)), \quad u \in \mathcal{H}, \tag{2-4}$$

where the nonlinearity $f_\varepsilon(u)$ is defined as $f_\varepsilon(u) := (n-2)(u^+)^{n/(n-2)+\varepsilon}$ in the supercritical case or $f_\varepsilon(u) := (n-2)(u^+)^{n/(n-2)-\varepsilon}$ in the subcritical case. Here $u^+(x) := \max\{0, u(x)\}$. By assumption (1-6), a solution to problem (2-4) is strictly positive and actually is a solution to problem (1-5). Therefore, we are led to build solutions to problem (2-4) which blow-up at a boundary point as ε goes to zero.

The main ingredient to cook up our solutions are the standard bubbles

$$U_{\delta,\xi}(x,t) := \frac{\delta^{(n-2)/2}}{((\delta+t)^2+|x-\xi|^2)^{(n-2)/2}}, \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}_+,\ \delta > 0,\ \xi \in \mathbb{R}^{n-1},$$

which are all the solutions to the limit problem

$$\begin{cases} -\Delta U = 0 & \text{on } \mathbb{R}^{n-1} \times \mathbb{R}_+, \\ \partial_\nu U = (n-2)U^{n/(n-2)} & \text{on } \mathbb{R}^{n-1} \times \{t = 0\}. \end{cases} \tag{2-5}$$

We set $U_\delta(x,t) := U_{\delta,0}(x,t)$.

We also need to introduce the linear problem

$$\begin{cases} -\Delta V = 0 & \text{on } \mathbb{R}^{n-1} \times \mathbb{R}_+, \\ \partial_\nu V = n U_1^{2/(n-2)} V & \text{on } \mathbb{R}^{n-1} \times \{t = 0\}. \end{cases} \tag{2-6}$$

In [Almaraz 2011b] it has been proved that the n-dimensional space of solutions of (2-6) is generated by the functions

$$V_i = \frac{\partial U_1}{\partial x_i} = (2-n) \frac{x_i}{((1+t)^2 + |x|^2)^{n/2}} \quad \text{for } i = i, \dots, n-1,$$

$$V_0 = \frac{\partial U_\delta}{\partial \delta}\Big|_{\delta=1} = \frac{n-2}{2} \left(\frac{1}{(1+t)^2 + |x|^2} \right)^{n/2} \big(t^2 + |x|^2 - 1 \big).$$

Next, for a point $q \in \partial M$ and the $(n-1)$-dimensional unitary ball $B^{n-1}(0, R)$ in $\mathbb{R}^{n-1}$, we introduce the Fermi coordinates $\psi_q^\partial : B^{n-1}(0, R) \times [0, R) \to M$. We read the bubble on the manifold as the function

$$W_{\delta,q}(\xi) = U_\delta\big((\psi_q^\partial)^{-1} \xi\big) \chi\big((\psi_q^\partial)^{-1} \xi\big),$$

and the functions V_i on the manifold as the functions

$$Z_{\delta,q}^i(\xi) = \frac{1}{\delta^{(n-2)/2}} V_i\Big(\frac{1}{\delta} (\psi_q^\partial)^{-1} \xi\Big) \chi\big((\psi_q^\partial)^{-1} \xi\big) \quad \text{for } i = 0, \dots, n-1,$$

where $\chi(x, t) = \tilde{\chi}(|x|)\tilde{\chi}(t)$, for $\tilde{\chi}$ a smooth cut off function, $\tilde{\chi}(s) \equiv 1$ for $0 \le s < R/2$ and $\tilde{\chi}(s) \equiv 0$ for $s \ge R$. Then, it is necessary to split the Hilbert space H into the sum of the orthogonal spaces

$$K_{\delta,q} = \operatorname{Span}\big\langle Z_{\delta,q}^0, \dots, Z_{\delta,q}^{n-1} \big\rangle$$

and

$$K_{\delta,q}^\perp = \big\{ \varphi \in H^1(M) \mid \langle\!\langle \varphi, Z_{\delta,q}^i \rangle\!\rangle_H = 0 \text{ for all } i = 0, \dots, n-1 \big\}.$$

Finally, we can look for a solution to problem (2-4) in the form

$$u_\varepsilon(x) = W_{\delta,q}(x) + \phi(x)$$

where the blow-up point q is in ∂M, the blowing-up rate δ satisfies

$$\delta := d\varepsilon \quad \text{for some } d > 0 \tag{2-7}$$

and the remainder term ϕ belongs to the infinite dimensional space $K_{\delta,q}^\perp \cap \mathcal{H}$ of codimension n. We are led to solve the system

$$\Pi_{\delta,q}^\perp \big\{ W_{\delta,q}(x) + \phi(x) - i^*\big(f_\varepsilon(W_{\delta,q}(x) + \phi(x)) \big) \big\} = 0, \tag{2-8}$$

$$\Pi_{\delta,q} \big\{ W_{\delta,q}(x) + \phi(x) - i^*\big(f_\varepsilon(W_{\delta,q}(x) + \phi(x)) \big) \big\} = 0, \tag{2-9}$$

$\Pi_{\delta,q}^\perp$ and $\Pi_{\delta,q}$ being the projections on $K_{\delta,q}^\perp$ and $K_{\delta,q}$, respectively.

3. The finite dimensional reduction

In this section we perform the finite dimensional reduction. We rewrite the auxiliary equation (2-8) in the equivalent form

$$L(\phi) = N(\phi) + R, \tag{3-1}$$

where $L = L_{\delta,q} : K^{\perp}_{\delta,q} \cap \mathcal{H} \to K^{\perp}_{\delta,q} \cap \mathcal{H}$ is the linear operator

$$L(\phi) = \Pi^{\perp}_{\delta,q}\big\{\phi(x) - i^*(f'_\varepsilon(W_{\delta,q})[\phi])\big\},$$

$N(\phi)$ is the nonlinear term

$$N(\phi) = \Pi^{\perp}_{\delta,q}\big\{i^*\big(f_\varepsilon(W_{\delta,q}(x)+\phi(x))\big) - i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*(f'_\varepsilon(W_{\delta,q})[\phi])\big\} \tag{3-2}$$

and the error term R is defined by

$$R = \Pi^{\perp}_{\delta,q}\big\{i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\big\}. \tag{3-3}$$

3.1. *The invertibility of the linear operator L.*

Lemma 2. *For $a, b \in \mathbb{R}$ with $0 < a < b$, there exists a positive constant $C_0 = C_0(a, b)$ such that, for ε small, for any $q \in \partial M$, for any $d \in [a, b]$ and for any $\phi \in K^{\perp}_{\delta,q} \cap \mathcal{H}$, we have*

$$\|L_{\delta,q}(\phi)\|_{\mathcal{H}} \geq C_0 \|\phi\|_{\mathcal{H}}.$$

Proof. We argue by contradiction. Suppose that there exist two sequences of real numbers $\varepsilon_m \to 0$ and $d_m \in [a, b]$, a sequence of points $q_m \in \partial M$ and a sequence of functions $\phi_{\varepsilon_m d_m, q_m} \in K^{\perp}_{\varepsilon_m d_m, q_m} \cap \mathcal{H}$ such that

$$\|\phi_{\varepsilon_m d_m, q_m}\|_{\mathcal{H}} = 1 \quad \text{and} \quad \|L_{\varepsilon_m d_m, q_m}(\phi_{\varepsilon_m d_m, q_m})\|_{\mathcal{H}} \to 0 \text{ as } m \to +\infty.$$

For the sake of simplicity, we set $\delta_m = \varepsilon_m d_m$ and define

$$\tilde{\phi}_m := \delta_m^{(n-2)/2} \phi_{\delta_m, q_m}(\psi^{\partial}_{q_m}(\delta_m \eta))\chi(\delta_m \eta) \quad \text{for } \eta = (z, t) \in \mathbb{R}^n_+,\ z \in \mathbb{R}^{n-1},\ t \geq 0.$$

Since $\|\phi_{\varepsilon_m d_m, q_m}\|_H \leq 1$, by a change of variables we easily get that $\{\tilde{\phi}_m\}_m$ is bounded in $D^{1,2}(\mathbb{R}^n_+)$ (but not in $H^1(\mathbb{R}^n_+)$). Therefore, there exists $\tilde{\phi} \in D^{1,2}(\mathbb{R}^n_+)$ such that $\tilde{\phi}_m \rightharpoonup \tilde{\phi}$ almost everywhere, weakly in $D^{1,2}(\mathbb{R}^n_+)$, in $L^{2n/(n-2)}(\mathbb{R}^n_+)$ and strongly in $L^{2(n-1)/(n-2)}_{\text{loc}}(\partial \mathbb{R}^n_+)$.

Since $\phi_{\delta_m, q_m} \in K^{\perp}_{\delta_m, q_m}$, and taking (2-6) into account, for $i = 0, \dots, n-1$ we get

$$o(1) = \int_{\mathbb{R}^n_+} \nabla\tilde{\phi}\nabla V_i \, dz\, dt = n \int_{\mathbb{R}^{n-1}} U_1^{2/(n-2)}(z, 0) V_i(z, 0) \tilde{\phi}(z, 0)\, dz. \tag{3-4}$$

Indeed, by a change of variables we have

$$
\begin{aligned}
0 &= \langle\!\langle \phi_{\delta_m,q_m}, Z^i_{\delta_m,q_m} \rangle\!\rangle_H \\
&= \int_M \big(\nabla_g \phi_{\delta_m,q_m} \nabla_g Z^i_{\delta_m,q_m} + a(x)\phi_{\delta_m,q_m} Z^i_{\delta_m,q_m}\big)\, d\mu_g \\
&\quad + \frac{n-2}{2}\int_{\partial M} b(x)\phi_{\delta_m,q_m} Z^i_{\delta_m,q_m}\, d\sigma \\
&= \int_{\mathbb{R}^n_+} |g_{q_m}(\delta\eta)|^{1/2}\delta^{(n-2)/2} g^{\alpha\beta}_{q_m}(\delta\eta)\frac{\partial}{\partial\eta_\alpha} V_i(\eta)\chi(\delta\eta)\frac{\partial}{\partial\eta_\alpha}\phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m\eta))\,\delta\eta \\
&\quad + \int_{\mathbb{R}^n_+} |g_{q_m}(\delta\eta)|^{1/2}\delta^{(n+2)/2} a(\psi^\partial_{q_m}(\delta\eta))V_i(\eta)\phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m\eta))\,\delta\eta \\
&\quad + \int_{\partial\mathbb{R}^n_+} |g_{q_m}(\delta z,0)|^{1/2}\delta^{n/2} b(\psi^\partial_{q_m}(\delta\eta))\phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m z,0))V_i(\delta_m z,0)\,dz \\
&= \int_{\mathbb{R}^n_+} \nabla V_i(\eta)\nabla\tilde\phi_m(\eta) + \delta^2 a(q_m)V_i(\eta)\tilde\phi_m(\eta)\,\delta\eta \\
&\quad + \delta\int_{\partial\mathbb{R}^n_+} b(q_m)V_i(z,0)\tilde\phi_m(z,0)\,\delta\eta + O(\delta) \\
&= \int_{\mathbb{R}^n_+} \nabla V_i(\eta)\nabla\tilde\phi_m(\eta) + O(\delta) = \int_{\mathbb{R}^n_+} \nabla V_i(\eta)\nabla\tilde\phi(\eta) + o(1),
\end{aligned}
$$

By definition of L_{δ_m,q_m} we have

$$
\phi_{\delta_m,q_m} - i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]) - L_{\delta_m,q_m}(\phi_{\delta_m,q_m}) = \sum_{i=0}^{n-1} c^i_m Z^i_{\delta_m,q_m}. \tag{3-5}
$$

We want to prove that, for all $i = 0, \dots, n-1$, $c^i_m \to 0$ as $m \to \infty$. Multiplying (3-5) by $Z^j_{\delta_m,q_m}$ we obtain, by definition of i^*,

$$
\begin{aligned}
\sum_{i=0}^{n-1} c^i_m \langle\!\langle Z^i_{\delta_m,q_m}, Z^j_{\delta_m,q_m}\rangle\!\rangle_H &= \langle\!\langle i^*(f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]), Z^j_{\delta_m,q_m}\rangle\!\rangle_H \\
&= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]Z^j_{\delta_m,q_m}\,d\sigma.
\end{aligned}
$$

Moreover, by multiplying (3-5) by ϕ_{δ_m,q_m} we obtain that

$$
\|\phi_{\delta_m,q_m}\|_H - \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})\phi^2_{\delta_m,q_m}\,d\sigma \to 0.
$$

Thus $(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2}\phi_{\delta_m,q_m}$ is bounded and weakly convergent in $L^2(\partial M)$. With this consideration we easily get

$$\begin{aligned}\int_{\partial M} & f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]Z^j_{\delta_m,q_m}\,d\sigma \\ &= \int_{\partial M}(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2}\phi_{\delta_m,q_m}(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2}Z^j_{\delta_m,q_m}\,d\sigma \\ &= n\int_{\mathbb{R}^{n-1}} U_1^{2/(n-2)}(z,0)\tilde{\phi}(z,0)V_i(z,0)\,dz + o(1) = o(1),\end{aligned}$$

once we take (3-4) into account.

Now, it is easy to prove that

$$\langle\!\langle Z^i_{\delta_m,q_m}, Z^j_{\delta_m,q_m}\rangle\!\rangle_H = C\delta_{ij} + o(1),$$

hence we can conclude that $c^i_m \to 0$ as $m\to\infty$ for each $i = 0,\dots,n-1$. This, combined with (3-5) and using $\|L_{\varepsilon_m d_m,q_m}(\phi_{\varepsilon_m d_m,q_m})\|_{\mathcal{H}} \to 0$, gives us that

$$\big\|\phi_{\delta_m,q_m} - i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}])\big\|_{\mathcal{H}} = \sum_{i=0}^{n-1} c^i_m\|Z^i\|_{\mathcal{H}} + o(1) = o(1). \tag{3-6}$$

Choose a smooth function $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ and define

$$\varphi_m(x) = \frac{1}{\delta_m^{(n-2)/2}}\varphi\Big(\frac{1}{\delta_m}\big(\psi^\partial_{q_m}\big)^{-1}(x)\Big)\chi\big(\big(\psi^\partial_{q_m}\big)^{-1}(x)\big) \quad \text{for } x\in M.$$

We have that $\|\varphi_m\|_H$ is bounded and, by (3-6), that

$$\begin{aligned}\langle\!\langle & \phi_{\delta_m,q_m},\varphi_m\rangle\!\rangle_H \\ &= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m\,d\sigma + \langle\!\langle \phi_{\delta_m,q_m} - i^*(f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]),\varphi_m\rangle\!\rangle_H \\ &= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m\,d\sigma + o(1) \\ &= (n\pm\varepsilon_m(n-2))\int_{\mathbb{R}^{n-1}} \frac{1}{\delta_m^{\pm\varepsilon n/(n-2)}} U_1^{2/(n-2)\pm\varepsilon_m}(z,0)\tilde{\phi}_m(z,0)\varphi\,dz + o(1) \\ &= n\int_{\mathbb{R}^{n-1}} U_1^{2/(n-2)}(z,0)\tilde{\phi}(z,0)\varphi(z,0)\,dz + o(1),\end{aligned}$$

by the strong $L^{2(n-1)/(n-2)}_{\mathrm{loc}}(\partial\mathbb{R}^n_+)$ convergence of $\tilde{\phi}_m$. On the other hand,

$$\langle\!\langle \phi_{\delta_m,q_m},\varphi_m\rangle\!\rangle_H = \int_{\mathbb{R}^n_+} \nabla\tilde{\phi}\nabla\varphi\,\delta\eta + o(1),$$

so $\tilde{\phi}$ is a weak solution of (2-5) and we conclude that

$$\tilde{\phi} \in \operatorname{Span}\{V_0, V_1, \dots, V_n\}.$$

This, combined with (3-4), gives that $\tilde{\phi} = 0$. Proceeding as before we have

$$\begin{aligned}
&\langle\langle \phi_{\delta_m,q_m}, \phi_{\delta_m,q_m}\rangle\rangle_H \\
&= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\phi_{\delta_m,q_m}\, d\sigma + o(1) \\
&= (n \pm \varepsilon_m(n-2)) \int_{\mathbb{R}^{n-1}} \frac{1}{\delta_m^{\pm\varepsilon n/(n-2)}} U_1^{2/(n-2)\pm\varepsilon_m}(z,0)\tilde{\phi}_m^2(z,0)\varphi\, dz + o(1) = o(1).
\end{aligned}$$

In a similar way, by (3-6) we have

$$|\phi_{\delta_m,q_m}|_{L^{s_\varepsilon}} = \left|i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}])\right|_{L^{s_\varepsilon}} + o(1) = o(1),$$

which gives $\|\phi_{\delta_m,q_m}\|_{\mathcal{H}} \to 0$, which is a contradiction. □

3.2. ***The estimate of the error term R.***

Lemma 3. *For $a, b \in \mathbb{R}$ with $0 < a < b$, there exists a positive constant $C_1 = C_1(a,b)$ such that, for ε small, for any $q \in \partial M$ and for any $d \in [a,b]$ we have*

$$\|R_{\varepsilon,\delta,q}\|_{\mathcal{H}} \le C_1\varepsilon|\ln\varepsilon|$$

Proof. We estimate

$$\begin{aligned}
&\left\|i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H \\
&\qquad \le \left\|i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\right\|_H + \left\|i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H.
\end{aligned}$$

By definition of i^* there exists Γ which solves the equation

$$\begin{cases} -\Delta_g\Gamma + a(x)\Gamma = 0 & \text{on } M, \\ \dfrac{\partial}{\partial\nu}\Gamma + \dfrac{n-2}{2}b(x)\Gamma = f_0(W_{\delta,q}) & \text{on } \partial M, \end{cases} \tag{3-7}$$

so, by (3-7), we have

$$\begin{aligned}
&\left\|i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H \\
&\qquad = \|\Gamma(x) - W_{\delta,q}(x)\|_H^2 \\
&\qquad = \int_M [-\Delta_g(\Gamma - W_{\delta,q}) + a(\Gamma - W_{\delta,q})](\Gamma - W_{\delta,q})\, d\mu_g \\
&\qquad\qquad + \int_{\partial M}\Big[\frac{\partial}{\partial\nu}(\Gamma - W_{\delta,q}) + \frac{(n-2)}{2}b(x)(\Gamma - W_{\delta,q})\Big](\Gamma - W_{\delta,q})\, d\mu_g \\
&\qquad = \int_M [\Delta_g W_{\delta,q} - aW_{\delta,q}](\Gamma - W_{\delta,q})\, d\mu_g \\
&\qquad\qquad + \int_{\partial M}\Big[f_0(W_{\delta,q}) - \frac{\partial}{\partial\nu}W_{\delta,q}\Big](\Gamma - W_{\delta,q})\, d\mu_g \\
&\qquad\qquad - \frac{n-2}{2}\int_{\partial M} b(x)W_{\delta,q}(\Gamma - W_{\delta,q})\, d\mu_g := I_1 + I_2 + I_3.
\end{aligned}$$

We obtain

$$I_1 = \|\Gamma - W_{\delta,q}\|_H O(\delta). \tag{3-8}$$

In fact,

$$\begin{aligned} I_1 &\le |\Delta_g W_{\delta,q} - aW_{\delta,q}|_{L^{2n/(n+2)}(M)} |\Gamma - W_{\delta,q}|_{L^{2n/(n-2)}(M)} \\ &\le |\Delta_g W_{\delta,q} - aW_{\delta,q}|_{L^{2n/(n+2)}(M)} \|\Gamma - W_{\delta,q}\|_H. \end{aligned}$$

We easily have that $|W_{\delta,q}|_{L^{2n/(n+2)}} = O(\delta^2)$. For the other term we have, in coordinates,

$$\Delta_g W_{\delta,q} = \Delta[U_\delta \chi] + (g^{ab} - \delta_{ab})\partial_{ab}[U_\delta \chi] - g^{ab}\Gamma_{ab}^k \partial_k [U_\delta \chi], \tag{3-9}$$

Γ_{ab}^k being the Christoffel symbols. Using the expansion of the metric g^{ab} given by (4-2) and (4-3) we have that

$$\begin{aligned} \big|(g^{ab} - \delta_{ab})\partial_{ab}[U_\delta \chi]\big|_{L^{2n/(n+2)}(M)} &= O(\delta), \\ \big|g^{ab}\Gamma_{ab}^k \partial_k [U_\delta \chi]\big|_{L^{2n/(n+2)}(M)} &= O(\delta^2). \end{aligned} \tag{3-10}$$

Since U_δ is a harmonic function we deduce

$$|\Delta[U_\delta \chi]|_{L^{2n/(n+2)}(M)} = |U_\delta \Delta \chi + 2\nabla U_\delta \nabla \chi|_{L^{2n/(n+2)}(M)} = O(\delta^2). \tag{3-11}$$

For the second integral I_2 we have

$$I_2 = \|\Gamma - W_{\delta,q}\|_H O(\delta^2), \tag{3-12}$$

since

$$\begin{aligned} I_2 &\le \Big| f_0(W_{\delta,q}) - \frac{\partial}{\partial \nu} W_{\delta,q} \Big|_{L^{2(n-1)/n}(\partial M)} |\Gamma - W_{\delta,q}|_{L^{2(n-1)/n-2}(\partial M)} \\ &\le C \Big| f_0(W_{\delta,q}) - \frac{\partial}{\partial \nu} W_{\delta,q} \Big|_{L^{2(n-1)/n}(\partial M)} \|\Gamma - W_{\delta,q}\|_H, \end{aligned}$$

and, using the boundary condition for (2-5), we have

$$\begin{aligned} &\Big| f_0(W_{\delta,q}) - \frac{\partial}{\partial \nu} W_{\delta,q} \Big|_{L^{2(n-1)/n}(\partial M)} \\ &= \frac{1}{\delta^{n/2}} \bigg(\int_{\mathbb{R}^{n-1}} |g(\delta z, 0)|^{1/2} \Big[(n-2) U^{n/(n-2)}(z,0) \chi^{n/(n-2)}(\delta z, 0) \\ &\qquad\qquad - \chi(\delta z, 0) \frac{\partial U}{\partial t}(z,0) \Big]^{\frac{2(n-1)}{n}} \delta^{n-1}\, dz \bigg)^{\frac{n}{2(n-1)}} \\ &\le C \bigg(\int_{\mathbb{R}^{n-1}} \big[(n-2) U^{n/(n-2)}(z,0) [\chi^{n/(n-2)}(\delta z, 0) \\ &\qquad\qquad - \chi(\delta z, 0)] \big]^{\frac{2(n-1)}{n}} dz \bigg)^{\frac{n}{2(n-1)}} = O(\delta^2). \end{aligned} \tag{3-13}$$

Lastly,

$$I_3 \le |W_{\delta,q}|_{L^{2(n-1)/n}(\partial M)} |\Gamma - W_{\delta,q}|_{L^{2(n-1)/(n-2)}(\partial M)} = \|\Gamma - W_{\delta,q}\|_H O(\delta). \tag{3-14}$$

By (3-8), (3-12) and (3-14) we conclude that

$$\big\| i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\big\|_H = \|\Gamma(x) - W_{\delta,q}(x)\|_H = O(\delta).$$

To conclude the proof we estimate the term $\big\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\big\|_H$. We have, by the properties of i^*, that

$$\begin{aligned}
&\big\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\big\|_H \\
&\le \big| W_{\delta,q}(x)^{n/(n-2)\pm\varepsilon} - W_{\delta,q}^{n/(n-2)}(x)\big|_{L^{2(n-1)/n}(\partial M)} \\
&= \left(\int_{\mathbb{R}^{n-1}} \left[\left(\frac{1}{\delta^{\pm\varepsilon(n-2)/2}} U^{\pm\varepsilon}(z,0) - 1 \right) U^{n/(n-2)}(z,0) \right]^{\frac{2(n-1)}{n}} dz \right)^{\frac{n}{2(n-1)}} + O(\delta^2).
\end{aligned}$$

To estimate the last integral, we first recall two Taylor expansions with respect to ε:

$$U^{\pm\varepsilon} = 1 \pm \varepsilon \ln U + \frac{1}{2}\varepsilon^2 \ln^2 U + o(\varepsilon^2), \tag{3-15}$$

$$\delta^{\mp\varepsilon(n-2)/2} = 1 \mp \varepsilon \frac{n-2}{2} \ln\delta + \varepsilon^2 \frac{(n-2)^2}{8} \ln^2\delta + o(\varepsilon^2 \ln^2\delta). \tag{3-16}$$

In light of (3-15) and (3-16) we have

$$\begin{aligned}
&\big\| i^*(f_\varepsilon(W_{\delta,q})) - i^*(f_0(W_{\delta,q}))\big\|_H \\
&\quad \le \left(\int_{\mathbb{R}^{n-1}} \left| \left(\mp \frac{n-2}{2} \varepsilon \ln\delta \pm \varepsilon \ln U(z,0) + O(\varepsilon^2) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + O(\varepsilon^2 \ln\delta) \right) U^{n/(n-2)}(z,0) \right|^{\frac{2(n-1)}{n}} dz \right)^{\frac{n}{2(n-1)}} + O(\delta^2) \\
&\quad = \frac{n-2}{2} \varepsilon \ln\delta \big| U(z,0)\big|^{n/(n-2)}_{L^{2(n-1)/(n-2)}(\mathbb{R}^{n-1})} \\
&\qquad\qquad + \varepsilon \left(\int_{\mathbb{R}^{n-1}} U^{2(n-1)/(n-2)}(z,0) \ln U(z,0)\, dz \right)^{\frac{n}{2(n-1)}} \\
&\qquad\qquad\qquad + O(\varepsilon^2) + O(\varepsilon^2 |\ln\delta|) + O(\delta^2) \\
&\quad = O(\varepsilon) + O(\varepsilon|\ln\delta|) + O(\delta^2).
\end{aligned} \tag{3-17}$$

Choosing $\delta = d\varepsilon$ concludes the proof of Lemma 3 for the subcritical case.

For the supercritical case, we have to control $|R_{\varepsilon,\delta,q}|_{L^{s_\varepsilon}(\partial M)}$. As in the previous case we consider

$$|R_{\varepsilon,\delta,q}|_{L^{s_\varepsilon}(\partial M)} \le \left|i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\right|_{L^{s_\varepsilon}(\partial M)} + \left|i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right|_{L^{s_\varepsilon}(\partial M)}.$$

As before, set $\Gamma = i^*(f_0(W_{\delta,q}(x))$. Since Γ solves (3-7), $\Gamma - W_{\delta,q}$ solves

$$\begin{cases} -\Delta_g(\Gamma - W_{\delta,q}) + a(x)(\Gamma - W_{\delta,q}) = -\Delta_g W_{\delta,q} + a(x)W_{\delta,q} & \text{on } M, \\ \dfrac{\partial}{\partial\nu}(\Gamma - W_{\delta,q}) + \dfrac{n-2}{2}b(x)(\Gamma - W_{\delta,q}) \\ \qquad = f_0(\Gamma) + \dfrac{\partial}{\partial\nu}W_{\delta,q} + \dfrac{n-2}{2}b(x)W_{\delta,q} & \text{on } \partial M. \end{cases}$$

We choose q as in (2-3), and $r = \varepsilon$. Thus, by Theorem 3.14 in [Nittka 2011], we have

$$|\Gamma - W_{\delta,q}|_{L^{s_\varepsilon}(\partial M)} \le |-\Delta_g W_{\delta,q} + a(x)W_{\delta,q}|_{L^{q+\varepsilon}(M)} + \left|f_0(\Gamma) + \frac{\partial}{\partial\nu}W_{\delta,q} + \frac{n-2}{2}b(x)W_{\delta,q}\right|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)}.$$

We remark that

$$q = \frac{2n + n^2\big(\frac{n-2}{n-1}\big)\varepsilon}{n+2+2n\big(\frac{n-2}{n-1}\big)\varepsilon} = \frac{2n}{n+2} + O^+(\varepsilon) \quad \text{with } 0 < O^+(\varepsilon) < C\varepsilon$$

for some positive constant C. By direct computation we have

$$|a(x)W_{\delta,q}|_{L^{q+\varepsilon}(M)} \le C\delta^{2-O^+(\varepsilon)},$$
$$|b(x)W_{\delta,q}|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)} \le C\delta^{1-O^+(\varepsilon)}.$$

Moreover, proceeding as in (3-9), (3-10), (3-11) and (3-13) we get

$$|\Delta_g W_{\delta,q}|_{L^{q+\varepsilon}(M)} \le C\delta^{2-O^+(\varepsilon)},$$
$$\left|f_0(\Gamma) + \frac{\partial}{\partial\nu}W_{\delta,q}\right|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)} \le C\delta^{1-O^+(\varepsilon)}.$$

Since $i^*(f_\varepsilon(W_{\delta,q}))$ solves (1-5), and $i^*(f_\varepsilon|u|^{n/(n-2)+\varepsilon}(W_{\delta,q}))$ solves (1-5), we again use Theorem 3.14 in [Nittka 2011]. Taking (3-15) and (3-16) into account, we finally get

$$
\begin{aligned}
(3\text{-}18)\quad & \left|i^*(f_\varepsilon(W_{\delta,q})) - i^*(f_0(W_{\delta,q}))\right|_{L^{s_\varepsilon}(\partial M)} \\
&\le |f_\varepsilon(W_{\delta,q}) - f_0(W_{\delta,q})|_{L^{2(n-1)/n+O^+(\varepsilon)}(\partial M)} \\
&\le \delta^{-O^+(\varepsilon)}\biggl(\int_{\mathbb{R}^{n-1}}\biggl[\Bigl(\frac{1}{\delta^{\varepsilon(n-2)/2}}U^\varepsilon(z,0)-1\Bigr) \\
&\qquad\qquad \cdot U^{n/(n-2)}(z,0)\biggr]^{\frac{2(n-1)}{n}+O^+(\varepsilon)} dz\biggr)^{\frac{1}{2(n-1)/n+O^+(\varepsilon)}} + O(\delta^2) \\
&= \delta^{-O^+(\varepsilon)}(O(\varepsilon|\ln\delta|)+O(\varepsilon)) + O(\delta^2).
\end{aligned}
$$

Now, choosing $\delta = d\varepsilon$, we can conclude the proof, since

$$\delta^{-O^+(\varepsilon)} = 1 + O^+(\varepsilon)|\ln(\varepsilon d)| = 1 + O^+(\varepsilon|\ln\varepsilon|) = O(1). \qquad \square$$

3.3. *Solving* (2-8)*: the remainder term* ϕ.

Proposition 4. *For $a, b \in \mathbb{R}$ with $0 < a < b$, there exists a positive constant $C = C(a,b)$ such that, for ε small, for any $q \in \partial M$ and for any $d \in [a,b]$ there exists a unique $\phi_{\delta,q}$ which solves* (2-8). *This solution satisfies*

$$\|\phi_{\delta,q}\|_{\mathcal{H}} \le C\varepsilon|\ln\varepsilon|.$$

Moreover the map $q \mapsto \phi_{\delta,q}$ is a $C^1(\partial M, \mathcal{H})$ map.

Proof. First of all, we point out that N is a contraction mapping. We remark that the conjugate exponent of s_ε is

$$s'_\varepsilon = \begin{cases} \dfrac{2(n-1)}{n} & \text{in the subcritical case,} \\[2mm] \dfrac{2(n-1)+\varepsilon n(n-2)}{n+\varepsilon n(n-2)} & \text{in the supercritical case.} \end{cases}$$

By the properties of i^* and using the expansion of $f_\varepsilon(W_{\delta,q}+\phi_1)$ centered in $W_{\delta,q}+\phi_2$ we have

$$
\begin{aligned}
\|N(\phi_1)-N(\phi_2)\|_{\mathcal{H}} &\le \bigl\| f_\varepsilon(W_{\delta,q}+\phi_1) - f_\varepsilon(W_{\delta,q}+\phi_2) - f'_\varepsilon(W_{\delta,q})[\phi_1-\phi_2]\bigr\|_{L^{s'_\varepsilon}(\partial M)} \\
&\le \bigl\|\bigl(f'_\varepsilon(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2) - f'_\varepsilon(W_{\delta,q})\bigr)[\phi_1-\phi_2]\bigr\|_{L^{s'_\varepsilon}(\partial M)}
\end{aligned}
$$

and, since $|\phi_1-\phi_2|^{s'_\varepsilon} \in L^{s_\varepsilon/s'_\varepsilon}(\partial M)$ and $|f'_\varepsilon(\cdot)|^{s'_\varepsilon} \in L^{(s_\varepsilon/s'_\varepsilon)'}(\partial M)$ as $f'_\varepsilon(\cdot) \in L^{s_\varepsilon}(\partial M)$, we have

$$
\begin{aligned}
&\|N(\phi_1)-N(\phi_2)\|_{\mathcal{H}} \\
&\qquad \le \bigl\|\bigl(f'_\varepsilon(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2) - f'_\varepsilon(W_{\delta,q})\bigr)\bigr\|_{L^{s_\varepsilon}(\partial M)}\|\phi_1-\phi_2\|_{L^{s_\varepsilon}(\partial M)} \\
&\qquad = \gamma\|\phi_1-\phi_2\|_{\mathcal{H}},
\end{aligned}
$$

where

$$\gamma = \big\| \big(f_\varepsilon'(W_{\delta,q} + \theta\phi_1 + (1-\theta)\phi_2) - f_\varepsilon'(W_{\delta,q}) \big) \big\|_{L^{s_\varepsilon}(\partial M)} < 1$$

provided $\|\phi_1\|_{\mathcal{H}}$ and $\|\phi_2\|_{\mathcal{H}}$ are sufficiently small.

In the same way we can prove that $\|N(\phi)\|_{\mathcal{H}} \le \gamma \|\phi\|_{\mathcal{H}}$ with $\gamma < 1$ if $\|\phi\|_{\mathcal{H}}$ is sufficiently small.

Next, by Lemmas 2 and 3 we have

$$\|L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})\|_{\mathcal{H}} \le C(\gamma\|\phi\|_{\mathcal{H}} + \varepsilon|\ln \varepsilon|),$$

where $C = \max\{C_0, C_0 C_1\} > 0$, for the constants C_0, C_1 which appear in Lemmas 2 and 3. Notice that, given $C > 0$, it is possible (up to a choice of $\|\phi\|_{\mathcal{H}}$ sufficiently small) to choose $0 < C\gamma < \frac{1}{2}$.

Now, if $\|\phi\|_{\mathcal{H}} \le 2C\varepsilon|\ln\varepsilon|$, then the map

$$T(\phi) := L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})$$

is a contraction from the ball $\|\phi\|_{\mathcal{H}} \le 2C\varepsilon|\ln\varepsilon|$ in itself, so, by the fixed point theorem, there exists a unique $\phi_{\delta,q}$ with $\|\phi_{\delta,q}\|_{\mathcal{H}} \le 2C\varepsilon|\ln\varepsilon|$ solving (3-1), and hence (2-8). The regularity of the map $q \mapsto \phi_{\delta,q}$ can be proven via the implicit function theorem. □

4. The reduced problem

Problem (1-5) has a variational structure. Weak solutions to (1-5) are critical points of the energy functional $J_\varepsilon : \mathcal{H} \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{2}\int_M (|\nabla u|^2 + a(x)u^2)\, d\mu_g + \frac{n-2}{4}\int_{\partial M} b(x)u^2\, d\sigma - \frac{(n-2)^2}{2n-2 \pm \varepsilon(n-2)} \int_{\partial M} u^{(2n-2)/(n-2)\pm\varepsilon}\, d\sigma.$$

Let us introduce the reduced energy $I_\varepsilon : (0, +\infty) \times \partial M \to \mathbb{R}$ by

$$I_\varepsilon(d, q) := J_\varepsilon(W_{\varepsilon d,q} + \phi_{\varepsilon d,q}), \tag{4-1}$$

where the remainder term $\phi_{\varepsilon d,q}$ has been found in Proposition 4.

4.1. ***The reduced energy.*** Here we use the following expansion for the metric tensor on M:

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y_n + O(|y|^2) \quad \text{for } i, j = 1, \dots, n-1, \tag{4-2}$$

$$g^{in}(y) = \delta_{in} \quad \text{for } i = 1, \dots, n-1, \tag{4-3}$$

$$\sqrt{g}(y) = 1 - (n-1)H(0)y_n + O(|y|^2), \tag{4-4}$$

where $(y_1, \dots, y_n)$ are the Fermi coordinates and, by definition of h_{ij},

$$H = \frac{1}{n-1} \sum_i^{n-1} h_{ii}. \tag{4-5}$$

We also recall that on ∂M the Fermi coordinates coincide with the exponential ones, so we have that

$$\sqrt{g}(y_1, \dots, y_{n-1}, 0) = 1 + O(|y|^2). \tag{4-6}$$

To improve the readability of this paper, hereafter we write $z = (z_1, \dots, z_{n-1})$ to indicate the first $n-1$ Fermi coordinates and t to indicate the last one, so $(y_1, \dots, y_{n-1}, y_n) = (z, t)$. Moreover, indices i, j conventionally refer to sums from 1 to $n-1$, while l, m usually refer to sums from 1 to n.

Proposition 5. (i) *If* $(d_0, q_0) \in (0, +\infty) \times \partial M$ *is a critical point for the reduced energy* I_ε *defined in* (4-1), *then* $W_{\varepsilon d_0, q_0} + \phi_{\varepsilon d_0, q_0} \in \mathcal{H}$ *solves problem* (1-5).

(ii) *It holds true that*

$$\begin{cases} I_\varepsilon(d, q) = c_n(\varepsilon) + \varepsilon[\alpha_n d\varphi(q) - \beta_n \ln d] + o(\varepsilon) & \text{in the subcritical case,} \\ I_\varepsilon(d, q) = c_n(\varepsilon) + \varepsilon[\alpha_n d\varphi(q) + \beta_n \ln d] + o(\varepsilon) & \text{in the supercritical case,} \end{cases}$$

C^0*-uniformly with respect to* d *in compact subsets of* $(0, +\infty)$ *and* $q \in \partial M$. *Here* $c_n(\varepsilon)$ *is a constant which only depends on* ε *and* n, α_n *and* β_n *are positive constants which only depend on* n, *and* $\varphi(q) = h(q) - H_g(q)$ *is the function defined in* (1-4) .

Proof. (i) Set $q := q(y) = \psi^\partial_{q_0}(y)$. Since (d_0, q_0) is a critical point, we have, for any $h \in 1, \dots, n-1$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial y_h} I_\varepsilon\big(d, \psi^\partial_{q_0}(y)\big)\Big|_{y=0} \\ &= \Big\langle\!\Big\langle W_{\varepsilon d, q(y)} + \phi_{\varepsilon d, q(y)} - i^*(f_\varepsilon(W_{\varepsilon d, q(y)} + \phi_{\varepsilon d, q(y)})), \\ &\qquad\qquad\qquad \frac{\partial}{\partial y_h} W_{\varepsilon d, q(y)} + \frac{\partial}{\partial y_h}\phi_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H \Big|_{y=0} \\ &= \sum_{i=0}^{n-1} c^i_\varepsilon \Big\langle\!\Big\langle Z^i_{\varepsilon d, q(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon d, q(y)} + \frac{\partial}{\partial y_h}\phi_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H \Big|_{y=0} \\ &= \sum_{i=0}^{n-1} c^i_\varepsilon \Big\langle\!\Big\langle Z^i_{\varepsilon d, q(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H \Big|_{y=0} - \sum_{i=0}^{n-1} c^i_l \Big\langle\!\Big\langle \frac{\partial}{\partial y_h} Z^i_{\varepsilon d, q(y)}, \phi_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H \Big|_{y=0}, \end{aligned}$$

using that $\phi_{\varepsilon d, q(y)}$ is a solution of (2-8) and that

$$\Big\langle\!\Big\langle Z^i_{\varepsilon d, q(y)}, \frac{\partial}{\partial y_h}\phi_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H = -\Big\langle\!\Big\langle \frac{\partial}{\partial y_h} Z^i_{\varepsilon d, q(y)}, \phi_{\varepsilon d, q(y)} \Big\rangle\!\Big\rangle_H$$

since $\phi_{\varepsilon d,q(y)} \in K^{\perp}_{\varepsilon d,q(y)}$ for all y. Now it is enough to observe that

$$\Big\langle\!\Big\langle \frac{\partial}{\partial y_h} Z^i_{\varepsilon d,q(y)}, \phi_{\varepsilon d,q(y)} \Big\rangle\!\Big\rangle_H \le \Big\| \frac{\partial}{\partial y_h} Z^i_{\varepsilon d,q(y)} \Big\|_H \|\phi_{\varepsilon d,q(y)}\|_H = o(1),$$

$$\Big\langle\!\Big\langle Z^i_{\varepsilon d,q(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon d,q(y)} \Big\rangle\!\Big\rangle_H = \frac{1}{\varepsilon d} \langle\!\langle Z^i_{\varepsilon d,q(y)}, Z^h_{\varepsilon d,q(y)} \rangle\!\rangle_H = \frac{1}{\varepsilon d} \delta^{ih} + o(1),$$

to conclude that

$$0 = \frac{1}{\varepsilon d} \sum_{i=0}^{n-1} c^i_\varepsilon (\delta^{ih} + o(1)),$$

and so $c^i_\varepsilon = 0$ for all $i = 0, \dots, n-1$. This concludes the proof of (i).

(ii) We prove (ii) in two steps.

Step 1. *We prove that for ε small enough and for any $q \in \partial M$,*

$$|J_\varepsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\varepsilon(W_{\delta,q})| \le \|\phi_{\delta,q}\|^2_{\mathcal{H}} + C\varepsilon |\ln \varepsilon| \, \|\phi_{\delta,q}\|_{\mathcal{H}} = o(\varepsilon).$$

We have

$$\begin{aligned}
&|J_\varepsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\varepsilon(W_{\delta,q})| \\
&\quad = \left| \int_M [-\Delta_g W_{\delta,q} + a(x) W_{\delta,q}] \phi_{\delta,q} \, d\mu_g \right| + \frac{1}{2} \|\phi_{\delta,q}\|^2_H \\
&\qquad + \left| \int_{\partial M} \Big[\frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q} - f_0(W_{\delta,q}) \Big] \phi_{\delta,q} \, d\sigma \right| \\
&\qquad + \left| \int_{\partial M} \big[f_0(W_{\delta,q}) - f_\varepsilon(W_{\delta,q}) \big] \phi_{\delta,q} \, d\sigma \right| \\
&\qquad + \left| \int_{\partial M} \frac{(n-2)^2}{2n-2 \pm \varepsilon(n-2)} \big[(W_{\delta,q} + \phi_{\delta,q})^{(2n-2)/(n-2) \pm \varepsilon} - W_{\delta,q}^{(2n-2)/(n-2) \pm \varepsilon} \big] \right. \\
&\qquad\qquad \left. - f_\varepsilon(W_{\delta,q}) \phi_{\delta,q} \, d\sigma \right|.
\end{aligned}$$

With the same estimate of I_1 in Lemma 3 we obtain that

$$\left| \int_M [-\Delta_g W_{\delta,q} + a(x) W_{\delta,q}] \phi_{\delta,q} \, d\mu_g \right| = O(\delta) \|\phi_{\delta,q}\|_H,$$

and in light of the estimate of I_2 and I_3 in Lemma 3 we get

$$\left| \int_{\partial M} \Big[\frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q} - f_0(W_{\delta,q}) \Big] \phi_{\delta,q} \, d\sigma \right| = O(\delta) \|\phi_{\delta,q}\|_H.$$

In the subcritical case, following the computation in (3-17) we obtain

$$\begin{aligned}\left|\int_{\partial M}[f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})]\phi_{\delta,q}\,d\sigma\right|&\\ &\le C|f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})|_{L^{2(n-1)/n}(\partial M)}|\phi_{\delta,q}|_{L^{2(n-1)/(n-2)}(\partial M)}\\ &=[O(\varepsilon)+O(\varepsilon\ln\delta)]\|\phi_{\delta,q}\|_H=O(\varepsilon|\ln\varepsilon|)\|\phi_{\delta,q}\|_H,\end{aligned}$$

and in a similar way, for the supercritical case, in light of (3-18) we get

$$\begin{aligned}\left|\int_{\partial M}[f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})]\phi_{\delta,q}\,d\sigma\right|&\\ &\le C|f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})|_{L^{2(n-1)/n+O^+(\varepsilon)}(\partial M)}|\phi_{\delta,q}|_{L^{2(n-1)/(n-2)-O^+(\varepsilon)}(\partial M)}\\ &\le\big(\delta^{-O^+(\varepsilon)}(O(\varepsilon\ln\delta)+O(\varepsilon))+O(\delta^2)\big)\|\phi_{\delta,q}\|_H=O(\varepsilon|\ln\varepsilon|)\|\phi_{\delta,q}\|_H.\end{aligned}$$

Finally, by the Taylor expansion formula, for some $\theta\in(0,1)$ we immediately have

$$\begin{aligned}\left|\int_{\partial M}\frac{(n-2)^2}{2n-2\pm\varepsilon(n-2)}\left[\big(W_{\delta,q}+\phi_{\delta,q}\big)^{\frac{2n-2}{n-2}\pm\varepsilon}-W_{\delta,q}^{\frac{2n-2}{n-2}\pm\varepsilon}\right]-f_\varepsilon(W_{\delta,q})\phi_{\delta,q}\,d\sigma\right|&\\ =\left|\frac{n\pm\varepsilon(n-2)}{2}\int_{\partial M}\big(W_{\delta,q}+\theta\phi_{\delta,q}\big)^{\frac{2}{n-2}\pm\varepsilon}\phi_{\delta,q}^2\,d\sigma\right|&\\ \le C\left[\int_{\partial M}\big|W_{\delta,q}+\theta\phi_{\delta,q}\big|^{\left(\frac{2}{n-2}\pm\varepsilon\right)\frac{s_\varepsilon}{s_\varepsilon-2}}\,d\sigma\right]^{\frac{s_\varepsilon-2}{s_\varepsilon}}\left[\int_{\partial M}|\phi_{\delta,q}|^{s_\varepsilon}\,d\sigma\right]^{\frac{2}{s_\varepsilon}}&\\ \le C|W_{\delta,q}+\theta\phi_{\delta,q}|_{L^{s_\varepsilon}}^{s_\varepsilon-2}\|\phi_{\delta,q}\|_{\mathcal{H}}^2\le C\|\phi_{\delta,q}\|_{\mathcal{H}}^2.&\end{aligned}$$

Choosing $\delta=d\varepsilon$, and recalling that, by Proposition 4, $\|\phi_{\delta,q}\|_{\mathcal{H}}=O(\varepsilon|\ln\varepsilon|)$ concludes the proof.

Step 2. *We prove that*

$$\begin{aligned}&J_\varepsilon(W_{\delta,q})\\ &\quad=C(\varepsilon)+\varepsilon\Big(d\frac{n-2}{4}[b(q)-H(q)]\pm\ln d\,\frac{(n-2)^3(n-3)}{4(n-2)(2n-2)}\Big)\omega_{n-1}I_{n-2}^{n-2}+o(\varepsilon)\end{aligned}$$

C^0-uniformly with respect to d in compact subsets of $(0,+\infty)$ *and* $q\in\partial M$, *where*

$$\begin{aligned}C(\varepsilon)&=\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U(y)|^2\,dy\\ &\quad-\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz\pm\varepsilon\frac{(n-2)^3}{2n-2}\int_{\mathbb{R}^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz\\ &\quad\mp\varepsilon\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\ln U(z,0)\,dz\\ &\quad\mp\varepsilon|\ln\varepsilon|\frac{(n-2)^3}{2(2n-2)}\int_{\mathbb{R}^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz,\end{aligned}$$

and

$$I_{n-2}^{n-2}=\int_0^{\infty}\frac{s^{n-2}}{(1+s^2)^{n-2}}\,dz,$$

and ω_{n-1} *is the volume of the* $(n-1)$*-dimensional unit ball.*

We compute each term separately. First, we have, by a change of variables and by (4-2), (4-3) and (4-4),

$$\begin{aligned}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g &= \sum_{l,m=1}^{n}\int_{\mathbb{R}^n_+} g^{lm}(\delta y)\frac{\partial}{\partial y_l}U(y)\frac{\partial}{\partial y_m}U(y)\sqrt{g}(\delta y)\,dy+o(\delta)\\ &=\int_{\mathbb{R}^n_+}|\nabla U(y)|^2\,dy-\delta(n-1)H(q)\int_{\mathbb{R}^n_+}y_n|\nabla U(y)|^2\,dy\\ &\quad +2\delta\sum_{i,j=1}^{n-1}\int_{\mathbb{R}^n_+}y_n h_{ij}(q)\frac{\partial}{\partial y_i}U(y)\frac{\partial}{\partial y_j}U(y)\,dy+o(\delta).\end{aligned}$$

By a symmetry argument we can simplify the last integral to obtain, in a more compact form,

$$\begin{aligned}\frac{1}{2}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g &= \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2-\delta\frac{(n-1)H(q)}{2}\int_{\mathbb{R}^n_+}y_n|\nabla U|^2\\ &\quad +\delta\sum_{i=1}^{n-1}h_{ii}(q)\int_{\mathbb{R}^n_+}y_n\Big(\frac{\partial U}{\partial y_i}(y)\Big)^2+o(\delta).\end{aligned}$$

Since $\dfrac{\partial U}{\partial y_i}=\dfrac{\partial U}{\partial y_l}$ for all $i,l=1,\dots,n-1$, by (4-9) we get

$$\begin{aligned}\sum_{i=1}^{n-1}h_{ii}(q)\int_{\mathbb{R}^n_+}y_n\Big(\frac{\partial U}{\partial y_i}(y)\Big)^2\,dy &= \frac{1}{n-1}\sum_{i=1}^{n-1}h_{ii}(q)\int_{\mathbb{R}^n_+}y_n\sum_{l=1}^{n-1}\Big(\frac{\partial U}{\partial y_l}(y)\Big)^2\,dy\\ &=\frac{H(q)}{4}\int_{\mathbb{R}^{n-1}}U^2(z,0)\,dz,\end{aligned}$$

and in light of (4-7) we conclude that

$$\frac{1}{2}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g=\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2-\delta\frac{(n-2)H(q)}{4}\int_{\mathbb{R}^{n-1}}U^2(z,0)\,dz+o(\delta).$$

By a change of variables, we immediately obtain

$$\frac{1}{2}\int_M a(x)|W_{\delta,q}|^2\,d\mu_g=\frac{\delta^2}{2}\int_{\mathbb{R}^n_+}a(x)U^2(y)\sqrt{g}(\delta y)\,dy+o(\delta^2)=O(\delta^2).$$

Coming to the boundary integral, we get, by a change of variables, by (4-6), and by expanding b,

$$\frac{n-2}{4}\int_{\partial M} b(z)|W_{\delta,q}|^2\,d\sigma = \delta\,\frac{n-2}{4}\int_{\mathbb{R}^{n-1}} b(\delta z)U^2(z,0)\sqrt{g}(\delta z)\,dz + O(\delta^2)$$
$$= \delta b(q)\,\frac{n-2}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz + O(\delta^2).$$

Introducing the abbreviation $U_n(z) = U^{(2n-2)/(n-2)}(z,0)$, by (3-15), (3-16) and (4-6), we have

$$\int_{\partial M} |W_{\delta,q}|^{(2n-2)/(n-2)\pm\varepsilon}\,d\sigma$$
$$= \int_{\mathbb{R}^{n-1}} \delta^{\mp\varepsilon(n-2)/2}U_n(z)U^{\pm\varepsilon}(z,0)\sqrt{g}(\delta z)\,dz + o(\delta)$$
$$= \int_{\mathbb{R}^{n-1}} U_n(z)\,dz \pm \varepsilon\int_{\mathbb{R}^{n-1}} U_n(z)\ln U(z,0)\,dz \mp \frac{n-2}{2}\varepsilon\ln\delta\int_{\mathbb{R}^{n-1}} U_n(z)\,dz$$
$$+ o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta),$$

and, since $\dfrac{(n-2)^2}{2n-2\pm\varepsilon(n-2)} = \dfrac{(n-2)^2}{2n-2} \mp \varepsilon\,\dfrac{(n-2)^3}{2n-2}$, we get

$$-\frac{(n-2)^2}{2n-2\pm\varepsilon(n-2)}\int_{\partial M} |W_{\delta,q}|^{(2n-2)/(n-2)-\varepsilon}\,d\sigma$$
$$= -\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\,dz \pm \varepsilon\,\frac{(n-2)^3}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\,dz$$
$$\mp \varepsilon\,\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\ln U(z,0)\,dz \pm \frac{(n-2)^3}{2(2n-2)}\varepsilon\ln\delta\int_{\mathbb{R}^{n-1}} U_n(z)\,dz$$
$$+ o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta).$$

Notice that, with the choice $\delta = d\varepsilon$ it holds that $o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta) = o(\varepsilon)$ and $\varepsilon\ln\delta = \varepsilon\ln d - \varepsilon|\ln\varepsilon|$. At this point we have

$$J_\varepsilon(W_{\delta,q}) = C(\varepsilon) + \varepsilon d\,\frac{n-2}{4}\,[b(q) - H(q)]\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz$$
$$\pm \varepsilon\,\frac{(n-2)^3}{2(2n-2)}\ln d\int_{\mathbb{R}^{n-1}} U_n(z)\,dz + o(\varepsilon|\ln\varepsilon|).$$

To conclude, observe that

$$\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz = \omega_{n-1}I_{n-2}^{n-2} \quad\text{and}\quad \int_{\mathbb{R}^{n-1}} U_n(z)\,dz = \omega_{n-1}I_{n-1}^{n-2},$$

where

$$I_\beta^\alpha = \int_0^\infty \frac{s^\alpha}{(1+s^2)^\beta}\,ds.$$

The conclusion follows after we observe that $I_{n-1}^{n-2} = \dfrac{n-3}{2(n-2)} I_{n-2}^{n-2}$ (for a proof, see [Almaraz 2011b, Lemma 9.4(b)]). □

4.2. *Proof of Theorem 1.* Let us introduce

$$\hat{I}(d,q) = \alpha_n d\varphi(q) - \beta_n \ln d.$$

If q_0 is a local minimizer of $\varphi(q)$ with $\varphi(q_0) > 0$, set $d_0 = \beta_n/(\alpha_n\varphi(q_0)) > 0$. Thus the pair (d_0, q_0) is a critical point for $\hat{I}$. Moreover, since there exists a neighborhood B such that $\varphi(q) > \varphi(q_0)$ on ∂B, it is possible to find a neighborhood $\widetilde{B} \subset [a,b] \times \partial M$, $(d_0, q_0) \in \widetilde{B}$ such that $\hat{I}(d,q) > \hat{I}(d_0,q_0)$ for $(d,q) \in \partial\widetilde{B}$. Since, in the subcritical case, by (i) of Proposition 5 we have

$$I_\varepsilon(d,q) = c_n(\varepsilon) + \varepsilon\hat{I}(d,q) + o(\varepsilon),$$

we get that for ε sufficiently small there is a $(d^*, q^*) \in \widetilde{B}$ such that $W_{\varepsilon d^*,q^*} + \phi_{\varepsilon d^*,q^*}$ is a critical point for I_ε. Then, by (i) of Proposition 5, $W_{\varepsilon d^*,q^*} + \phi_{\varepsilon d^*,q^*} \in \mathcal{H}$ is a solution for problem (1-5) in the subcritical case.

The proof for the supercritical case follows in a similar way. □

4.3. *Some technicalities.* *If U is a solution of* (2-5), *then the following hold:*

$$\int_{\mathbb{R}^n_+} t|\nabla U|^2\,dz\,dt = \frac{1}{2}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz, \tag{4-7}$$

$$\int_{\mathbb{R}^n_+} t|\nabla U|^2\,dz\,dt = 2\int_{\mathbb{R}^n_+} t|\partial_t U|^2\,dz\,dt, \tag{4-8}$$

$$\int_{\mathbb{R}^n_+} t\sum_{i=1}^{n-1} |\partial_{z_i} U|^2\,dz\,dt = \frac{1}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz. \tag{4-9}$$

Proof. To simplify the notation, we set

$$\eta = (z,t) \in \mathbb{R}^n_+ \quad \text{where } z \in \mathbb{R}^{n-1} \text{ and } t \geq 0.$$

The first estimate can be obtained by integration by parts, taking into account that $\Delta U = 0$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2\,\delta\eta &= -\sum_{l=1}^{n}\int_{\mathbb{R}^n_+} U\partial_l[\eta_n\partial_l U]\,\delta\eta = -\int_{\mathbb{R}^n_+} U\partial_n U\,\delta\eta - \int_{\mathbb{R}^n_+} \eta_n U\Delta U\,\delta\eta \\ &= -\frac{1}{2}\int_{\mathbb{R}^n_+} \partial_n[U^2]\,\delta\eta = \frac{1}{2}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz.\end{aligned}$$

To obtain (4-8), we proceed in a similar way: since $\Delta U = 0$ we have

$$\begin{aligned}
0 &= -\int_{\mathbb{R}^n_+} \Delta U \eta_n^2 \partial_n U \, \delta\eta = \sum_{l=1}^{n} \int_{\mathbb{R}^n_+} \partial_l U \partial_l [\eta_n^2 \partial_n U] \, \delta\eta \\
&= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 \, \delta\eta + \sum_{l=1}^{n} \int_{\mathbb{R}^n_+} \eta_n^2 \partial_l U \partial_{ln}^2 U \, \delta\eta \\
&= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 \, \delta\eta + \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n^2 \partial_n |\nabla U|^2 \, \delta\eta \\
&= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_t U|^2 \, \delta\eta - \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \, \delta\eta,
\end{aligned}$$

so (4-8) is proved. Now (4-9) is a direct consequence of the first two equalities. In fact, by (4-8) we have

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \, \delta\eta &= \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \, \delta\eta + \int_{\mathbb{R}^n_+} \eta_n |\partial_n U|^2 \, \delta\eta \\
&= \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \, \delta\eta + \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \, \delta\eta.
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \, \delta\eta = \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \, \delta\eta,$$

and in light of (4-7) we get the proof. □

References

[Almaraz 2010] S. d. M. Almaraz, "An existence theorem of conformal scalar-flat metrics on manifolds with boundary", *Pacific J. Math.* **248**:1 (2010), 1–22. MR Zbl

[Almaraz 2011a] S. d. M. Almaraz, "Blow-up phenomena for scalar-flat metrics on manifolds with boundary", *J. Differential Equations* **251**:7 (2011), 1813–1840. MR Zbl

[Almaraz 2011b] S. d. M. Almaraz, "A compactness theorem for scalar-flat metrics on manifolds with boundary", *Calc. Var. Partial Differential Equations* **41**:3-4 (2011), 341–386. MR Zbl

[Aubin 1976] T. Aubin, "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", *J. Math. Pures Appl.* (9) **55**:3 (1976), 269–296. MR Zbl

[Brendle 2008] S. Brendle, "Blow-up phenomena for the Yamabe equation", *J. Amer. Math. Soc.* **21**:4 (2008), 951–979. MR Zbl

[Brendle and Chen 2014] S. Brendle and S.-Y. S. Chen, "An existence theorem for the Yamabe problem on manifolds with boundary", *J. Eur. Math. Soc.* (*JEMS*) **16**:5 (2014), 991–1016. MR Zbl

[Brendle and Marques 2009] S. Brendle and F. C. Marques, "Blow-up phenomena for the Yamabe equation, II", *J. Differential Geom.* **81**:2 (2009), 225–250. MR Zbl

[Cherrier 1984] P. Cherrier, "Problèmes de Neumann non linéaires sur les variétés riemanniennes", *J. Funct. Anal.* **57**:2 (1984), 154–206. MR Zbl

[Druet 2003] O. Druet, "From one bubble to several bubbles: The low-dimensional case", *J. Differential Geom.* **63**:3 (2003), 399–473. MR Zbl

[Druet 2004] O. Druet, "Compactness for Yamabe metrics in low dimensions", *Int. Math. Res. Not.* **2004**:23 (2004), 1143–1191. MR Zbl

[Druet and Hebey 2005a] O. Druet and E. Hebey, "Blow-up examples for second order elliptic PDEs of critical Sobolev growth", *Trans. Amer. Math. Soc.* **357**:5 (2005), 1915–1929. MR Zbl

[Druet and Hebey 2005b] O. Druet and E. Hebey, "Elliptic equations of Yamabe type", *Int. Math. Res. Surv.* 1 (2005), 1–113. MR Zbl

[Escobar 1992a] J. F. Escobar, "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary", *Ann. of Math.* (2) **136**:1 (1992), 1–50. MR Zbl

[Escobar 1992b] J. F. Escobar, "The Yamabe problem on manifolds with boundary", *J. Differential Geom.* **35**:1 (1992), 21–84. MR Zbl

[Esposito and Pistoia 2014] P. Esposito and A. Pistoia, "Blowing-up solutions for the Yamabe equation", *Port. Math.* **71**:3-4 (2014), 249–276. MR Zbl

[Esposito et al. 2014] P. Esposito, A. Pistoia, and J. Vétois, "The effect of linear perturbations on the Yamabe problem", *Math. Ann.* **358**:1-2 (2014), 511–560. MR Zbl

[Felli and Ould Ahmedou 2003] V. Felli and M. Ould Ahmedou, "Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries", *Math. Z.* **244**:1 (2003), 175–210. MR Zbl

[Felli and Ould Ahmedou 2005] V. Felli and M. Ould Ahmedou, "A geometric equation with critical nonlinearity on the boundary", *Pacific J. Math.* **218**:1 (2005), 75–99. MR Zbl

[Han and Li 1999] Z.-C. Han and Y. Li, "The Yamabe problem on manifolds with boundary: Existence and compactness results", *Duke Math. J.* **99**:3 (1999), 489–542. MR Zbl

[Khuri et al. 2009] M. A. Khuri, F. C. Marques, and R. M. Schoen, "A compactness theorem for the Yamabe problem", *J. Differential Geom.* **81**:1 (2009), 143–196. MR Zbl

[Marques 2005] F. C. Marques, "Existence results for the Yamabe problem on manifolds with boundary", *Indiana Univ. Math. J.* **54**:6 (2005), 1599–1620. MR Zbl

[Marques 2007] F. C. Marques, "Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary", *Comm. Anal. Geom.* **15**:2 (2007), 381–405. MR Zbl

[Micheletti et al. 2009] A. M. Micheletti, A. Pistoia, and J. Vétois, "Blow-up solutions for asymptotically critical elliptic equations on Riemannian manifolds", *Indiana Univ. Math. J.* **58**:4 (2009), 1719–1746. MR Zbl

[Nittka 2011] R. Nittka, "Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains", *J. Differential Equations* **251**:4-5 (2011), 860–880. MR Zbl

[Schoen 1984] R. Schoen, "Conformal deformation of a Riemannian metric to constant scalar curvature", *J. Differential Geom.* **20**:2 (1984), 479–495. MR Zbl

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa* (3) **22** (1968), 265–274. MR Zbl

[Yamabe 1960] H. Yamabe, "On a deformation of Riemannian structures on compact manifolds", *Osaka Math. J.* **12** (1960), 21–37. MR Zbl

Received May 29, 2015. Revised January 17, 2016.

MARCO GHIMENTI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PISA
VIA F. BUONARROTI 1/C
56127 PISA
ITALY
marco.ghimenti@dma.unipi.it

ANNA MARIA MICHELETTI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PISA
VIA F. BUONARROTI 1/C
56127 PISA
ITALY
a.micheletti@dma.unipi.it

ANGELA PISTOIA
DIPARTIMENTO SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA
UNIVERSTÀ DI ROMA "LA SAPIENZA"
VIA ANTONIO SCARPA 16
00161 ROMA
ITALY
angela.pistoia@uniroma1.it

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.103

QUANTIFYING SEPARABILITY IN VIRTUALLY SPECIAL GROUPS

MARK F. HAGEN AND PRIYAM PATEL

We give a new, effective proof of the separability of cubically convex-cocompact subgroups of special groups. As a consequence, we show that if G is a virtually compact special hyperbolic group, and $Q \leq G$ is a K-quasiconvex subgroup, then any $g \in G - Q$ of word length at most n is separated from Q by a subgroup whose index is polynomial in n and exponential in K. This generalizes a result of Bou-Rabee and the authors on residual finiteness growth (*Math. Z.* 279 (2015), 297–310) and a result of Patel on surface groups (*Proc. Amer. Math. Soc.* 142 (2014), 2891–2906).

Introduction

Early motivation for studying residual finiteness and subgroup separability was a result of the relevance of these properties to decision problems in group theory. An observation of Dyson [1964] and Mostowski [1966], related to earlier ideas of McKinsey [1943], states that finitely presented residually finite groups have a solvable word problem. The word problem is a special case of the membership problem, i.e., the problem of determining whether a given $g \in G$ belongs to a particular subgroup H of G. *Separability* can produce a solution to the membership problem in essentially the same way that a solution to the word problem is provided by residual finiteness; see, e.g., the discussion in [Aschenbrenner et al. 2015]. A subgroup $H \leq G$ is *separable* in G if, for all $g \in G - H$, there exists $G' \leq_{\text{f.i.}} G$ with $H \leq G'$ and $g \notin G'$. Producing an upper bound, in terms of g and H, on the minimal index of such a subgroup G' is what we mean by *quantifying separability* of H in G. Quantifying separability is related to the membership problem; see Remark D below.

Recently, separability has played a crucial role in low-dimensional topology, namely in the resolutions of the virtually Haken and virtually fibered conjectures [Agol 2013; Wise 2011]. Its influence in topology is a consequence of the seminal paper of Scott [1978], which establishes a topological reformulation of subgroup separability. Roughly, Scott's criterion allows one to use separability to promote

MSC2010: primary 20E26; secondary 20F36.
Keywords: subgroup separable, right-angled Artin groups, quantifying, virtually special groups.

(appropriately construed) immersions to embeddings in finite covers. Agol [2013] proved the virtually special conjecture of Wise, an outstanding component of the proofs of the above conjectures. Agol's theorem shows that every word hyperbolic cubical group virtually embeds in a right-angled Artin group (hereafter, *RAAG*). Cubically convex-cocompact subgroups of RAAGs are separable [Hsu and Wise 2002; Haglund 2008] and Agol's theorem demonstrates that word hyperbolic cubical groups inherit this property via the virtual embeddings (separability properties are preserved under passing to subgroups and finite index supergroups). In fact, since quasiconvex subgroups of hyperbolic cubical groups are cubically convex-cocompact [Haglund 2008; Sageev and Wise 2015], all quasiconvex subgroups of such groups are separable. In this paper, we give a new, effective proof of the separability of cubically convex-cocompact subgroups of special groups. Our main technical result is:

Theorem A. *Let Γ be a simplicial graph and let Z be a compact connected cube complex, based at a 0-cube x, with a based local isometry $Z \to S_\Gamma$. For all $g \in A_\Gamma - \pi_1 Z$, there is a cube complex (Y, x) such that*

(1) $Z \subset Y$,

(2) *there is a based local isometry $Y \to S_\Gamma$ such that $Z \to S_\Gamma$ factors as*

$$Z \hookrightarrow Y \to S_\Gamma,$$

(3) *any closed based path representing g lifts to a nonclosed path at x in Y,*

(4) $|Y^{(0)}| \le |Z^{(0)}|(|g|+1)$,

where $|g|$ is the word length of g with respect to the standard generators.

Via Haglund–Wise's canonical completion [2008], Theorem A provides the following bounds on the separability growth function (defined in Section 1) of the class of cubically convex-cocompact subgroups of a (virtually) special group. Roughly, separability growth quantifies separability of all subgroups in a given class.

Corollary B. *Let $G \cong \pi_1 X$, with X a compact special cube complex, and let $\mathcal{Q}_R$ be the class of subgroups represented by compact local isometries to X whose domains have at most R vertices. Then*

$$\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}_R}(Q, n) \le PRn$$

for all $Q \in \mathcal{Q}_R$ and $n \in \mathbb{N}$, where the constant P depends only on the generating set $\mathcal{S}$. Hence, letting $\mathcal{Q}'_K$ be the class of subgroups $Q \le G$ such that the convex hull of $Q\tilde{x}$ lies in $\mathcal{N}_K(Q\tilde{x})$ and $\tilde{x} \in \tilde{X}^{(0)}$,

$$\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}'_K}(Q, n) \le P' \operatorname{gr}_{\tilde{X}}(K) n,$$

where P' depends only on G, $\tilde{X}$, $\mathcal{S}$, and where $\operatorname{gr}_{\tilde{X}}$ is the growth function of $\tilde{X}^{(0)}$.

In the hyperbolic case, where cubically convex-cocompactness is equivalent to quasiconvexity, we obtain a bound that is polynomial in the length of the word and exponential in the quasiconvexity constant:

Corollary C. *Let G be a group with an index-J special subgroup. Fixing a word length $\|-\|_{\mathcal{S}}$ on G, suppose that $(G, \|-\|_{\mathcal{S}})$ is δ-hyperbolic. For each $K \geq 1$, let $\mathcal{Q}_K$ be the set of subgroups $Q \leq G$ such that Q is K-quasiconvex with respect to $\|-\|_{\mathcal{S}}$. Then there exists a constant $P = P(G, \mathcal{S})$ such that for all $K \geq 0$, $Q \in \mathcal{Q}_K$, and $n \geq 0$,*

$$\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}_K}(Q, n) \leq P \operatorname{gr}_G(PK)^{J!} n^{J!},$$

where gr_G is the growth function of G.

Corollary C says that if G is a hyperbolic cubical group, the subgroup $Q \leq G$ is K-quasiconvex, and $g \in G - Q$, then g is separated from Q by a subgroup of index bounded by a function polynomial in $\|g\|_{\mathcal{S}}$ and exponential in K.

The above results fit into a larger body of work dedicated to quantifying residual finiteness and subgroup separability of various classes of groups; see, e.g., [Bou-Rabee and Kaletha 2012; Bou-Rabee and McReynolds 2015; Kassabov and Matucci 2011; Buskin 2009; Patel 2014; 2013; Rivin 2012; Bou-Rabee 2011; Bou-Rabee and McReynolds 2014; Kozma and Thom 2016]. When G is the fundamental group of a hyperbolic surface, compare Corollary C to [Patel 2014, Theorem 7.1]. Combining various cubulation results with [Agol 2013], the groups covered by Corollary C include fundamental groups of hyperbolic 3-manifolds [Bergeron and Wise 2012; Kahn and Markovic 2012], hyperbolic Coxeter groups [Haglund and Wise 2010], simple-type arithmetic hyperbolic lattices [Bergeron et al. 2011], hyperbolic free-by-cyclic groups [Hagen and Wise 2015], hyperbolic ascending HNN extensions of free groups with irreducible monodromy [Hagen and Wise 2013], hyperbolic groups with a quasiconvex hierarchy [Wise 2011], $C'\left(\frac{1}{6}\right)$ small cancellation groups [Wise 2004], and hence random groups at low enough density [Ollivier and Wise 2011], among many others.

Bou-Rabee, Hagen and Patel [2015] quantified residual finiteness for virtually special groups, by working in RAAGs and appealing to the fact that upper bounds on residual finiteness growth are inherited by finitely generated subgroups and finite index supergroups. Theorem A generalizes a main theorem of [loc. cit.], and accordingly the proof is reminiscent of the one in that reference. However, residual finiteness is equivalent to separability of the trivial subgroup, and thus it is not surprising that quantifying separability for an arbitrary convex-cocompact subgroup of a RAAG entails engagement with a more complex geometric situation. Our techniques thus significantly generalize those of [loc. cit.].

Remark D (membership problem). If H is a finitely generated separable subgroup of the finitely presented group G, and one has an upper bound on $\mathbf{Sep}_{G,\mathcal{S}}^{\{H\}}(|g|)$ for some finite generating set $\mathcal{S}$ of G, then the following procedure decides if $g \in H$: first, enumerate all subgroups of G of index at most $\mathbf{Sep}_{G,\mathcal{S}}^{\{H\}}(|g|)$ using a finite presentation of G. Second, for each such subgroup, test whether it contains g; if so, ignore it, and if not, proceed to the third step. Third, for each finite index subgroup not containing g, test whether it contains each of the finitely many generators of H; if so, we have produced a finite index subgroup containing H but not g, whence $g \notin H$. If we exhaust the subgroups of index at most $\mathbf{Sep}_{G,\mathcal{S}}^{\{H\}}(|g|)$ without finding such a subgroup, then $g \in H$. In particular, Corollary C gives an effective solution to the membership problem for quasiconvex subgroups of hyperbolic cubical groups, though it does not appear to be any more efficient than the more general solution to the membership problem for quasiconvex subgroups of (arbitrary) hyperbolic groups recently given by Kharlampovich, Myasnikov and Weil [Kharlampovich et al. 2014].

The paper is organized as follows. In Section 1, we define the separability growth of a group with respect to a class $\mathcal{Q}$ of subgroups, which generalizes the residual finiteness growth introduced in [Bou-Rabee 2010]. We also provide some necessary background on RAAGs and cubical geometry. In Section 2, we discuss corollaries to the main technical result, including Corollary C, before concluding with a proof of Theorem A in Section 3.

1. Background

Separability growth. Let G be a group generated by a finite set $\mathcal{S}$ and let $H \leq G$ be a subgroup. Let $\Omega_H = \{\Delta \leq G : H \leq \Delta\}$, and define a map $D_G^{\Omega_H} : G - H \to \mathbb{N} \cup \{\infty\}$ by

$$D_G^{\Omega_H}(g) = \min\{[G:\Delta] : \Delta \in \Omega_H,\, g \notin \Delta\}.$$

This is a special case of the notion of a *divisibility function* defined in [Bou-Rabee 2010] and discussed in [Bou-Rabee and McReynolds 2015]. Note that H is a separable subgroup of G if and only if $D_G^{\Omega_H}$ takes only finite values.

The *separability growth* of G with respect to a class $\mathcal{Q}$ of subgroups is a function $\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}} : \mathcal{Q} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ given by

$$\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}}(Q, n) = \max\bigl\{D_G^{\Omega_Q}(g) : g \in G - Q,\, \|g\|_{\mathcal{S}} \leq n\bigr\}.$$

If $\mathcal{Q}$ is a class of separable subgroups of G, then the separability growth measures the index of the subgroup to which one must pass in order to separate Q from an element of $G - Q$ of length at most n. For example, when G is residually finite and $\mathcal{Q} = \{\{1\}\}$, then $\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}}$ is the residual finiteness growth function. The following fact is explained in greater generality in [Bou-Rabee et al. 2015, Section 2]. (In the notation of that reference, $\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}}(Q, n) = \mathrm{RF}_{G,\mathcal{S}}^{\Omega_Q}(n)$ for all $Q \in \mathcal{Q}$ and $n \in \mathbb{N}$.)

Proposition 1.1. *Let G be a finitely generated group and let $\mathcal{Q}$ be a class of subgroups of G. If $\mathcal{S}, \mathcal{S}'$ are finite generating sets of G, then there exists a constant $C > 0$ with*

$$\mathbf{Sep}^{\mathcal{Q}}_{G,\mathcal{S}'}(Q, n) \leq C \cdot \mathbf{Sep}^{\mathcal{Q}}_{G,\mathcal{S}}(Q, Cn)$$

for $Q \in \mathcal{Q}$, $n \in \mathbb{N}$. Hence the asymptotic growth rate of $\mathbf{Sep}^{\mathcal{Q}}_{G,\mathcal{S}}$ is independent of $\mathcal{S}$.

(Similar statements assert that upper bounds on separability growth are inherited by finite index supergroups and arbitrary finitely generated subgroups but we do not use, and thus omit, these.)

Nonpositively curved cube complexes. We assume familiarity with nonpositively curved and CAT(0) cube complexes and refer the reader to, e.g., [Hagen 2014; Haglund 2008; Wise 2012; 2011] for background. We now make explicit some additional notions and terminology, related to convex subcomplexes, which are discussed in greater depth in [Behrstock et al. 2014]. We also discuss some basic facts about RAAGs and Salvetti complexes. Finally, we will use the method of *canonical completion*, introduced in [Haglund and Wise 2008], and refer the reader to [Bou-Rabee et al. 2015, Lemma 2.8] for the exact statement needed here.

Local isometries, convexity, and gates. A *local isometry* $\phi : Y \to X$ of cube complexes is a locally injective combinatorial map with the property that, if $e_1, \dots, e_n$ are 1-cubes of Y all incident to a 0-cube y, and the (necessarily distinct) 1-cubes $\phi(e_1), \dots, \phi(e_n)$ all lie in a common n-cube c (containing $\phi(y)$), then $e_1, \dots, e_n$ span an n-cube c' in Y with $\phi(c') = c$. If $\phi : Y \to X$ is a local isometry and X is nonpositively curved, then Y is as well. Moreover, ϕ lifts to an embedding $\tilde{\phi} : \tilde{Y} \to \tilde{X}$ of universal covers, and $\tilde{\phi}(\tilde{Y})$ is *convex* in $\tilde{X}$ in the following sense.

Let $\tilde{X}$ be a CAT(0) cube complex. The subcomplex $K \subseteq \tilde{X}$ is *full* if K contains each n-cube of $\tilde{X}$ whose 1-skeleton appears in K. If K is full, then K is *isometrically embedded* if $K \cap \bigcap_i H_i$ is connected whenever $\{H_i\}$ is a set of pairwise-intersecting hyperplanes of $\tilde{X}$. Equivalently, the inclusion $K^{(1)} \hookrightarrow \tilde{X}^{(1)}$ is an isometric embedding with respect to the graph-metric. If the inclusion $K \hookrightarrow \tilde{X}$ of the full subcomplex K is a local isometry, then K is *convex*. Note that a convex subcomplex is necessarily isometrically embedded, and in fact K is convex if and only if $K^{(1)}$ is metrically convex in $\tilde{X}^{(1)}$. A convex subcomplex K is a CAT(0) cube complex in its own right, and its hyperplanes have the form $H \cap K$, where K is a hyperplane of $\tilde{X}$. Moreover, if K is convex, then hyperplanes $H_1 \cap K$, $H_2 \cap K$ of K intersect if and only if $H_1 \cap H_2 \neq \varnothing$. We often say that the hyperplane H *crosses* the convex subcomplex K to mean that $H \cap K \neq \varnothing$ and we say the hyperplanes H, H' *cross* if they intersect.

Hyperplanes are an important source of convex subcomplexes, in two related ways. First, recall that for all hyperplanes H of $\tilde{X}$, the carrier $\mathcal{N}(H)$ is a convex subcomplex. Second, $\mathcal{N}(H) \cong H \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, and the subcomplexes $H \times \left\{\pm\frac{1}{2}\right\}$ of $\tilde{X}$ "bounding" $\mathcal{N}(H)$ are convex subcomplexes isomorphic to H (when H is given the cubical structure in which its n-cubes are midcubes of $(n+1)$-cubes of $\tilde{X}$). A subcomplex of the form $H \times \left\{\pm\frac{1}{2}\right\}$ is a *combinatorial hyperplane*. The *convex hull* of a subcomplex $\mathcal{S} \subset \tilde{X}$ is the intersection of all convex subcomplexes that contain $\mathcal{S}$; see [Haglund 2008].

Let $K \subseteq \tilde{X}$ be a convex subcomplex. Then there is a map $\mathfrak{g}_K : \tilde{X}^{(0)} \to K$ such that for all $x \in \tilde{X}^{(0)}$, the point $\mathfrak{g}_K(x)$ is the unique closest point of K to x. (This point is often called the *gate* of x in K; gates are discussed further in [Chepoi 2000; Bandelt and Chepoi 2008].) This map extends to a cubical map $\mathfrak{g}_K : \tilde{X} \to K$, the *gate map*. See, e.g., [Behrstock et al. 2014] for a detailed discussion of the gate map in the language used here; we use only that it extends the map on 0-cubes and has the property that for all x, y, if $\mathfrak{g}_K(x), \mathfrak{g}_K(y)$ are separated by a hyperplane H, then the same H separates x from y. Finally, the hyperplane H separates x from $\mathfrak{g}_K(x)$ if and only if H separates x from K. The gate map allows us to define the *projection* of the convex subcomplex K' of $\tilde{X}$ onto K to be $\mathfrak{g}_{K'}(K)$, which is the convex hull of the set $\{\mathfrak{g}_K(x) \in K : x \in K'^{(0)}\}$. Convex subcomplexes K, K' are *parallel* if $\mathfrak{g}_{K'}(K) = K'$ and $\mathfrak{g}_K(K') = K$. Equivalently, K, K' are *parallel* if and only if, for each hyperplane H, we have $H \cap K \neq \varnothing$ if and only if $H \cap K' \neq \varnothing$. Note that parallel subcomplexes are isomorphic.

Remark 1.2. We often use the following facts. Let K, K' be convex subcomplexes of $\tilde{X}$. Then the convex hull C of $K \cup K'$ contains the union of K, K' and a convex subcomplex of the form $G_K(K') \times \hat{\gamma}$, where $G_K(K')$ is the image of the gate map discussed above and $\hat{\gamma}$ is the convex hull of a geodesic segment γ joining a closest pair of 0-cubes in K, K', by [Behrstock et al. 2014, Lemma 2.4]. A hyperplane H crosses K and K' if and only if H crosses $G_K(K')$; the hyperplane H separates K, K' if and only if H crosses $\hat{\gamma}$. All remaining hyperplanes either cross exactly one of K, K' or fail to cross C. Observe that the set of hyperplanes separating K, K' contains no triple H, H', H'' of disjoint hyperplanes, none of which separates the other two. (Such a configuration is called a *facing triple*.)

Salvetti complexes and special cube complexes. Let Γ be a simplicial graph and let A_Γ be the corresponding right-angled Artin group (RAAG), i.e., the group presented by

$$\langle V(\Gamma) \mid [v, w],\ \{v, w\} \in E(\Gamma)\rangle,$$

where $V(\Gamma)$ and $E(\Gamma)$ respectively denote the vertex- and edge-sets of Γ. The phrase *generator of* Γ refers to this presentation; we denote each generator of A_Γ by the corresponding vertex of Γ.

The RAAG A_Γ is isomorphic to the fundamental group of the *Salvetti complex* S_Γ, introduced in [Charney and Davis 1995], which is a nonpositively curved cube complex with one 0-cube x, an oriented 1-cube for each $v \in V(\Gamma)$, labeled by v, and an n-torus (an n-cube with opposite faces identified) for every n-clique in Γ.

A cube complex X is *special* if there exists a simplicial graph Γ and a local isometry $X \to S_\Gamma$ inducing a monomorphism $\pi_1 X \to A_\Gamma$ and a $\pi_1 X$-equivariant embedding $\tilde{X} \to \tilde{S}_\Gamma$ of universal covers whose image is a convex subcomplex. Specialness allows one to study geometric features of $\pi_1 X$ by working inside of $\tilde{S}_\Gamma$, which has useful structure not necessarily present in general CAT(0) cube complexes; see the next section. Following Haglund and Wise [2008], a group G is *(virtually) [compact] special* if G is (virtually) isomorphic to the fundamental group of a [compact] special cube complex.

Cubical features particular to Salvetti complexes. Let Γ be a finite simplicial graph and let Λ be an induced subgraph of Γ. The inclusion $\Lambda \hookrightarrow \Gamma$ induces a monomorphism $A_\Lambda \to A_\Gamma$. In fact, there is an injective local isometry $S_\Lambda \to S_\Gamma$ inducing $A_\Lambda \to A_\Gamma$. Hence each conjugate A_Λ^g of A_Λ in A_Γ is the stabilizer of a convex subcomplex $g\tilde{S}_\Lambda \subseteq \tilde{S}_\Gamma$. A few special cases warrant extra consideration.

When $\Lambda \subset \Gamma$ is an n-clique, for some $n \geq 1$, then $S_\Lambda \subseteq S_\Gamma$ is an n-torus, which is the Salvetti complex of the sub-RAAG isomorphic to $\mathbb{Z}^n$ generated by n pairwise-commuting generators. In this case, S_Λ is a *standard n-torus* in S_Γ. (When $n = 1$, S_Λ is a *standard circle*.) Each lift of $\tilde{S}_\Lambda$ to $\tilde{S}_\Gamma$ is a *standard flat*; when $n = 1$, we use the term *standard line*; a compact connected subcomplex of a standard line is a *standard segment*. The labels and orientations of 1-cubes in S_Γ pull back to $\tilde{S}_\Gamma$; a standard line is a convex subcomplex isometric to $\mathbb{R}$, all of whose 1-cubes have the same label, such that each 0-cube has one incoming and one outgoing 1-cube.

When $\mathrm{Lk}(v)$ is the link of a vertex v of Γ, the subcomplex $S_{\mathrm{Lk}(v)}$ is an immersed combinatorial hyperplane in the sense that $\tilde{S}_{\mathrm{Lk}(v)}$ is a combinatorial hyperplane of $\tilde{S}_\Gamma$. There is a corresponding hyperplane, whose carrier is bounded by $\tilde{S}_{\mathrm{Lk}(v)}$ and $v\tilde{S}_{\mathrm{Lk}(v)}$, that intersects only 1-cubes labeled by v. Moreover, $\tilde{S}_{\mathrm{Lk}\,v}$ is contained in $\tilde{S}_{\mathrm{St}(v)}$, where $\mathrm{St}(v)$ is the star of v, i.e., the join of v and $\mathrm{Lk}(v)$. It follows that

$$\tilde{S}_{\mathrm{St}(v)} \cong \tilde{S}_{\mathrm{Lk}(v)} \times \tilde{S}_v,$$

where $\tilde{S}_v$ is a standard line. Note that the combinatorial hyperplane $\tilde{S}_{\mathrm{Lk}(v)}$ is parallel to $v^k\tilde{S}_{\mathrm{Lk}(v)}$ for all $k \in \mathbb{Z}$. Likewise, $\tilde{S}_v$ is parallel to $g\tilde{S}_v$ exactly when $g \in A_\Lambda$, and parallel standard lines have the same labels. We say $\tilde{S}_v$ is a *standard line dual to* $\tilde{S}_{\mathrm{Lk}(v)}$, and is a standard line dual to any hyperplane H such that $N(H)$ has $\tilde{S}_{\mathrm{Lk}(v)}$ as one of its bounding combinatorial hyperplanes.

Remark 1.3. We warn the reader that a given combinatorial hyperplane may correspond to distinct hyperplanes whose dual standard lines have different labels; this occurs exactly when there exist multiple vertices in Γ whose links are the same subgraph. However, the standard line dual to a genuine (noncombinatorial) hyperplane is uniquely determined up to parallelism.

Definition 1.4 (frame). Let $K \subseteq \tilde{S}_\Gamma$ be a convex subcomplex and let H be a hyperplane. Let L be a standard line dual to H. The *frame* of H is the convex subcomplex $H' \times L \subseteq \tilde{S}_\Gamma$ described above, where H' is a combinatorial hyperplane bounding $N(H)$. If $K \subseteq \tilde{S}_\Gamma$ is a convex subcomplex, and H intersects K, then the *frame of H in K* is the complex $K \cap (H \times L)$. It is shown in [Bou-Rabee et al. 2015] that the frame of H in K has the form $(H \cap K) \times (L \cap K)$, provided that L is chosen in its parallelism class to intersect K. Note that the frame of H is in fact well-defined, since all possible choices of L are parallel.

2. Consequences of Theorem A

Assuming Theorem A, we quantify separability of cubically convex-cocompact subgroups of special groups with the proofs of Corollaries B and C, before proving Theorem A in the next section.

Proof of Corollary B. Let Γ be a finite simplicial graph so that there is a local isometry $X \to S_\Gamma$. Let $Q \in \mathcal{Q}_R$ be represented by a local isometry $Z \to X$. Then for all $g \in \pi_1 X - \pi_1 Z$, by Theorem A, there is a local isometry $Y \to S_\Gamma$ such that Y contains Z as a locally convex subcomplex, $g \notin \pi_1 Y$, and $|Y^{(0)}| \leq |Z^{(0)}|(|g|+1)$. Applying canonical completion [Haglund and Wise 2008] to $Y \to S_\Gamma$ yields a cover $\hat{S}_\Gamma \to S_\Gamma$ in which Y embeds; this cover has degree $|Y^{(0)}|$ by [Bou-Rabee et al. 2015, Lemma 2.8]. Let $H' = \pi_1 \hat{S}_\Gamma \cap \pi_1 X$, so that $\pi_1 Z \leq H'$, $g \notin H'$, and $[\pi_1 X : H'] \leq |Z^{(0)}|(|g|+1)$. The first claim follows.

Let $G \cong \pi_1 X$ with X compact special, $Q \leq G$, and let the convex hull of $Q\tilde{x}$ in $\tilde{X}$ lie in $\mathcal{N}_K(Q\tilde{X})$. Then the second claim follows since we can choose Z to be the quotient of the hull of $Q\tilde{x}$ by the action of Q, and $|Z^{(0)}| \leq \mathrm{gr}_{\tilde{X}}(K)$. □

In general, the number of 0-cubes in Z is computable from the quasiconvexity constant of a Q-orbit in $\tilde{X}^{(1)}$ by [Haglund 2008, Theorem 2.28]. In the hyperbolic case, we obtain Corollary C in terms of the quasiconvexity constant, without reference to any particular cube complex:

Proof of Corollary C. We use Corollary B when $J = 1$, and promote the result to a polynomial bound when $J \geq 1$. Let $Q \in \mathcal{Q}_K$ and let $g \in G - Q$.

The special case: Suppose $J = 1$ and let X be a compact special cube complex with $G \cong \pi_1 X$. Let $Z \to X$ be a compact local isometry representing the inclusion $Q \to G$. Such a complex exists by quasiconvexity of Q and [Haglund 2008, Theorem 2.28], although we shall use the slightly more computationally explicit proof in [Sageev and Wise 2015]. Let $A' \geq 1$, $B' \geq 0$ be constants such that an orbit map $(G, \|-\|_{\mathcal{S}}) \to (\tilde{X}^{(1)}, \mathsf{d})$ is an (A', B')-quasi-isometric embedding, where d is the graph-metric. Then there exist constants A, B, depending only on A', B' and hence on $\| - \|_{\mathcal{S}}$, such that Qx is $(AK + B)$-quasiconvex, where x is a 0-cube in $\tilde{Z} \subset \tilde{X}$. By the proof of of [op. cit., Proposition 3.3], the convex hull $\tilde{Z}$ of Qx lies in the ρ-neighborhood of Qx, where

$$\rho = AK + B + \sqrt{\dim X} + \delta'\left(\csc\left(\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{\dim X}}\right) + 1\right)$$

and $\delta' = \delta'(\delta, A', B')$. Corollary B provides $G' \le G$ with $g \notin G'$, and the bound $[G : G'] \le |Z^{(0)}|(|g| + 1)$. But $|g| + 1 \le A'\|g\|_{\mathcal{S}} + B' + 1$, while $|Z^{(0)}| \le \mathrm{gr}_{\tilde{X}}(\rho)$. Thus $[G : G'] \le \mathrm{gr}_{\tilde{X}}(\rho)A'\|g\|_{\mathcal{S}} + \mathrm{gr}_{\tilde{X}}(\rho)B' + \mathrm{gr}_{\tilde{X}}(\rho)$, so there exists P_1 such that

$$\mathbf{Sep}_{G,\mathcal{S}}^{\mathcal{Q}_K}(Q, n) \le P_1 \,\mathrm{gr}_{\tilde{X}}(P_1 K)n$$

for all K, $Q \in \mathcal{Q}_K$, $n \in \mathbb{N}$, where P_1 depends only on X.

The virtually special case: Now suppose that $J \ge 1$. We have a compact special cube complex X, and $[G : G'] \le J!$, where $G' \cong \pi_1 X$ and $G' \lhd G$. Let $Q \le G$ be a K-quasiconvex subgroup. By Lemma 2.1, there exists $C = C(G, \mathcal{S})$ such that $Q \cap G'$ is $CJ!(K + 1)$-quasiconvex in G, and thus is $P_2 CJ!(K + 1)$-quasiconvex in G', where P_2 depends only on G and $\mathcal{S}$.

Let $g \in G - Q$. Since $G' \lhd G$, the product QG' is a subgroup of G of index at most $J!$ that contains Q. Hence, if $g \notin QG'$, then we are done. We thus assume $g \in QG'$. Hence we can choose a left transversal $\{q_1, \dots, q_s\}$ for $Q \cap G'$ in Q, with $s \le J!$ and $q_1 = 1$. Write $g = q_i g'$ for some $i \le s$, with $g' \in G'$. Suppose that we have chosen each q_i to minimize $\|q_i\|_{\mathcal{S}}$ among all elements of $q_i(Q \cap G')$, so that, by Lemma 2.3, $\|q_i\| \le J!$ for all i. Hence $\|g'\|_{\mathcal{S}} \le (\|g\|_{\mathcal{S}} + J!)$.

By the first part of the proof, there exists a constant P_1, depending only on G, G', $\mathcal{S}$, and a subgroup $G'' \le G'$ such that $Q \cap G' \le G''$ and $g' \notin G''$, and

$$[G' : G''] \le P_1 \,\mathrm{gr}_{G'}(P_1 P_2 CJ!(K + 1))\|g'\|_{\mathcal{S}} \le P_1 \,\mathrm{gr}_G(P_1 P_2 CJ!(K + 1))\|g'\|_{\mathcal{S}}.$$

Let $G''' = \bigcap_{i=1}^{s} q_i G'' q_i^{-1}$, so that $g' \notin G'''$ and $Q \cap G' \le G'''$ (since G' is normal), and

$$[G' : G'''] \le \big(P_1 \,\mathrm{gr}_G(P_1 P_2 CJ!(K + 1))\|g'\|_{\mathcal{S}}\big)^s.$$

Finally, let $H = QG'''$. This subgroup clearly contains Q. Suppose that $g = q_i g' \in H$. Then $g' \in QG'''$, i.e., $g' = ag'''$ for some $a \in Q$ and $g''' \in G'''$. Since $g' \in G'$ and $G''' \le G'$, we have $a \in Q \cap G'$, whence $a \in G'''$, by construction. This implies that $g' \in G''' \le G''$, a contradiction. Hence H is a subgroup of G separating g from Q. Finally,

$$[G : H] \le [G : G'''] \le J!\big[P_1 \,\mathrm{gr}_G(P_1 P_2 CJ!(K + 1))(\|g\|_{\mathcal{S}} + J!)\big]^{J!},$$

and the proof is complete. ☐

Lemma 2.1. *Let the group G be generated by a finite set $\mathcal{S}$ and let $(G, \|-\|_{\mathcal{S}})$ be δ-hyperbolic. Let $Q \leq G$ be K-quasiconvex, and let $G' \leq G$ be an index-I subgroup. Then $Q \cap G'$ is $CI(K+1)$-quasiconvex in G for some C depending only on δ and $\mathcal{S}$.*

Proof. Since Q is K-quasiconvex in G, it is generated by a set $\mathcal{T}$ of $q \in Q$ with $\|q\|_{\mathcal{S}} \leq 2K+1$ by [Bridson and Haefliger 1999, Lemma III.Γ.3.5]. A standard argument shows $(Q, \|-\|_{\mathcal{T}}) \hookrightarrow (G, \|-\|_{\mathcal{S}})$ is a $(2K+1, 0)$-quasi-isometric embedding. Lemma 2.3 shows that $Q \cap G'$ is I-quasiconvex in $(Q, \|-\|_{\mathcal{T}})$, since $[Q : Q \cap G'] \leq I$. Hence $Q \cap G'$ has a generating set making it $((2I+1)(2K+1), 0)$-quasi-isometrically embedded in $(G, \|-\|_{\mathcal{S}})$. Apply Lemma 2.2 to conclude. □

The next lemma is standard, but we include it to highlight the constants involved:

Lemma 2.2. *Let G be a group generated by a finite set $\mathcal{S}$ and suppose that $(G, \|-\|_{\mathcal{S}})$ is δ-hyperbolic. Then there exists a (sub)linear function $f : \mathbb{N} \to \mathbb{N}$, depending on $\mathcal{S}$ and δ, such that $\sigma \subseteq \mathcal{N}_{f(\lambda)}(\gamma)$ whenever $\gamma : [0, L] \to G$ is a $(\lambda, 0)$-quasigeodesic and σ is a geodesic joining $\gamma(0)$ to $\gamma(L)$.*

Proof. See, e.g., the proof of [Bridson and Haefliger 1999, Theorem III.H.1.7]. □

Lemma 2.3. *Let Q be a group generated by a finite set $\mathcal{S}$ and let $Q' \leq Q$ be a subgroup with $[Q : Q'] = s < \infty$. Then there exists a left transversal $\{q_1, \ldots, q_s\}$ for Q' such that $\|q_i\|_{\mathcal{S}} \leq s$ for $1 \leq i \leq s$. Hence Q' is s-quasiconvex in Q.*

Proof. Suppose that $q_k = s_{i_k} \cdots s_{i_1}$ is a geodesic word in $\mathcal{S} \cup \mathcal{S}^{-1}$ and that q_k is a shortest representative of $q_k Q'$. Let $q_j = s_{i_j} \cdots s_{i_1}$ be the word in Q consisting of the last j letters of q_k for all $1 < j < k$, and let $q_1 = 1$. We claim that each q_j is a shortest representative for $q_j Q'$. Otherwise, there would exist p with $\|p\|_{\mathcal{S}} < j$ such that $q_j Q' = pQ'$. But then $s_k \cdots s_{j+1} p Q' = q_k Q'$, and thus q_k was not a shortest representative. It also follows immediately that $q_j Q' \neq q_{j'} Q'$ for $j \neq j'$. Thus, $q_1, q_2, \ldots, q_k$ represent distinct left cosets of Q' provided $k \leq s$, and the claim follows. □

Remark 2.4 (embeddings in finite covers). Given a compact special cube complex X and a compact local isometry $Z \to X$, Theorem A gives an upper bound on the minimal degree of a finite cover in which Z embeds; indeed, producing such an embedding entails separating $\pi_1 Z$ from finitely many elements in $\pi_1 X$. However, it is observed in [Bou-Rabee et al. 2015, Lemma 2.8] that the Haglund–Wise canonical completion construction [2008] produces a cover $\hat{X} \to X$ of degree $|Z^{(0)}|$ in which Z embeds.

3. Proof of Theorem A

In this section, we give a proof of the main technical result.

Definition 3.1. Let S_Γ be a Salvetti complex and let $\tilde{S}_\Gamma$ be its universal cover. The hyperplanes H, H' of $\tilde{S}_\Gamma$ are *collateral* if they have a common dual standard line (equivalently, the same frame). Clearly collateralism is an equivalence relation, and collateral hyperplanes are isomorphic and have the same stabilizer.

Being collateral implies that the combinatorial hyperplanes bounding the carrier of H are parallel to those bounding the carrier of H'. However, the converse is not true when Γ contains multiple vertices whose links coincide. In the proof of Theorem A, we will always work with hyperplanes, rather than combinatorial hyperplanes, unless we explicitly state otherwise.

Proof of Theorem A. Let $\tilde{x} \in \tilde{S}_\Gamma$ be a lift of the base 0-cube x in S_Γ, and let $\tilde{Z} \subseteq \tilde{S}_\Gamma$ be the lift of the universal cover of Z containing $\tilde{x}$. Since $Z \to S_\Gamma$ is a local isometry, $\tilde{Z}$ is convex. Let $\hat{Z} \subset \tilde{Z}$ be the convex hull of a compact connected fundamental domain for the action of $\pi_1 Z \leq A_\Gamma$ on $\tilde{Z}$. Denote by K the convex hull of $\hat{Z} \cup \{g\tilde{x}\}$ and let $\mathfrak{S}$ be the set of hyperplanes of $\tilde{S}_\Gamma$ intersecting K. We will form a quotient of K, restricting to $\hat{Z} \to Z$ on $\hat{Z}$, whose image admits a local isometry to S_Γ.

The subcomplex $\lfloor\hat{Z}\rfloor$. Let $\mathfrak{L}$ be the collection of standard segments ℓ in K that map to standard circles in S_Γ with the property that $\ell \cap \hat{Z}$ has noncontractible image in Z. Let $\lfloor\hat{Z}\rfloor$ be the convex hull of $\hat{Z} \cup \bigcup_{\ell \in \mathfrak{L}} \ell$, so that $\hat{Z} \subseteq \lfloor\hat{Z}\rfloor \subseteq K$.

Partitioning $\mathfrak{S}$. We now partition $\mathfrak{S}$ according to the various types of frames in K. First, let $\mathfrak{Z}$ be the set of hyperplanes intersecting $\hat{Z}$. Second, let $\mathfrak{N}$ be the set of $N \in \mathfrak{S} - \mathfrak{Z}$ such that the frame $(N \cap K) \times (L \cap K)$ of N in K has the property that for some choice of $x_0 \in N^{(0)}$, the segment $(\{x_0\} \times L) \cap \hat{Z}$ maps to a nontrivial cycle of 1-cubes in Z. Let $n_N \geq 1$ be the length of that cycle. By convexity of $\hat{Z}$, the number n_N is independent of the choice of the segment L within its parallelism class. Note that $\mathfrak{N}$ is the set of hyperplanes that cross $\lfloor\hat{Z}\rfloor$, but do not cross $\hat{Z}$. Hence each $N \in \mathfrak{N}$ is collateral to some $W \in \mathfrak{Z}$. Third, fix a collection $\{H_1, \dots H_k\} \subset \mathfrak{S} - \mathfrak{Z}$ such that:

(1) For $1 \leq i \leq k-1$, the hyperplane H_i separates H_{i+1} from $\lfloor\hat{Z}\rfloor$.

(2) For $1 \leq i < j \leq k$, if a hyperplane H separates H_i from H_j, then H is collateral to H_ℓ for some $\ell \in [i, j]$. Similarly, if H separates H_1 from $\lfloor\hat{Z}\rfloor$, then H is collateral to H_1, and if H separates H_k from $g\tilde{x}$, then H is collateral to H_k.

(3) For each i, the frame $(H_i \cap K) \times L_i$ of H_i in K has the property that for every $h \in H_i^{(0)}$, the image in Z of the segment $(\{h\} \times L_i) \cap \hat{Z}$ is empty or contractible. (Here, L_i is a standard segment of a standard line dual to H_i.)

Let $\mathfrak{H}$ be the set of all hyperplanes of $\mathfrak{S} - \mathfrak{Z}$ that are collateral to H_i for some i. Condition (3) above ensures that $\mathfrak{H} \cap \mathfrak{N} = \varnothing$, while $\mathfrak{H} = \varnothing$ only if $K = \lfloor\hat{Z}\rfloor$. Finally, let $\mathfrak{B} = \mathfrak{S} - (\mathfrak{Z} \cup \mathfrak{N} \cup \mathfrak{H})$. Note that each $B \in \mathfrak{B}$ crosses some H_i. Figure 1 shows a possible K and various families of hyperplanes crossing it.

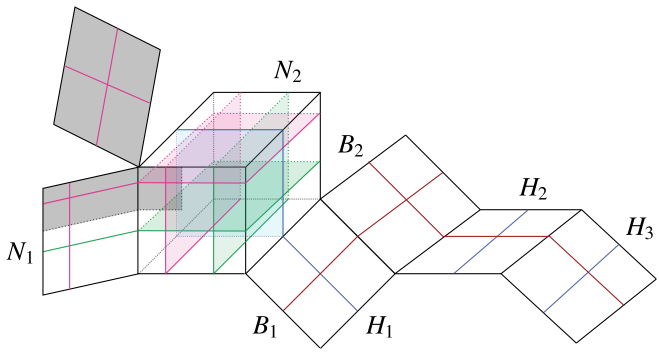

Figure 1. Hyperplanes crossing K (the dark shaded area on the left is $\hat{Z}$).

Mapping $\lfloor \hat{Z} \rfloor$ *to* Z. We now define a quotient map $\mathfrak{q} : \lfloor \hat{Z} \rfloor \to Z$ extending the restriction $\hat{Z} \to Z$ of $\tilde{Z} \to Z$. Note that if $\mathfrak{N} = \varnothing$, then $\lfloor \hat{Z} \rfloor = \hat{Z}$, and $\mathfrak{q}$ is just the map $\hat{Z} \to Z$. Hence suppose $\mathfrak{N} \neq \varnothing$ and let $\mathfrak{N}_1, \ldots, \mathfrak{N}_s$ be the collateralism classes of hyperplanes in $\mathfrak{N}$, and for $1 \leq i \leq s$, let $\mathfrak{N}_i'$ be the collateralism class of $\mathfrak{N}_i$ in $\mathfrak{S}$, i.e., $\mathfrak{N}_i$ together with a nonempty set of collateral hyperplanes in $\mathfrak{Z}$. For each i, let L_i be a maximal standard line segment of $\lfloor \hat{Z} \rfloor$, each of whose 1-cubes is dual to a hyperplane in $\mathfrak{N}_i'$ and which crosses each element of $\mathfrak{N}_i'$. For each i, let $N_i \in \mathfrak{N}_i$ be a hyperplane separating $\hat{Z}$ from $g\tilde{x}$. Then $N_i \cap N_j \neq \varnothing$ for $i \neq j$, since neither separates the other from $\hat{Z}$. We can choose the L_i so that there is an isometric embedding $\prod_{i=1}^{k} L_i \to \lfloor \hat{Z} \rfloor$, since whether or not two hyperplanes of $\tilde{S}_\Gamma$ cross depends only on their collateralism classes.

For each nonempty $I \subseteq \{1, \ldots, k\}$, a hyperplane $W \in \mathfrak{Z}$ crosses some $U \in \bigcup_{i \in I} \mathfrak{N}_i'$ if and only if W crosses each hyperplane collateral to U. Hence, by Lemma 7.11 of [Hagen 2014], there is a maximal convex subcomplex $Y(I) \subset \hat{Z}$, defined up to parallelism, such that a hyperplane W crosses each $U \in \bigcup_{i \in I} \mathfrak{N}_i'$ if and only if $W \cap Y(I) \neq \varnothing$. Let $\mathfrak{A}(I)$ be the set of hyperplanes crossing $Y(I)$. By the definition of $Y(I)$ and the lemma just cited, there is a combinatorial isometric embedding $Y(I) \times \prod_{i \in I} L_i \to \lfloor \hat{Z} \rfloor$, whose image we denote by $F(I)$ and refer to as a *generalized frame*. Moreover, for any 0-cube $z \in \lfloor \hat{Z} \rfloor$ that is not separated from a hyperplane in $\bigcup_{i \in I} \mathfrak{N}_i' \cup \mathfrak{A}(I)$ by a hyperplane not in that set, we can choose $F(I)$ to contain z; this follows from the proof of the same lemma of Hagen. Figures 2 and 3 show possible collateralism classes $\mathfrak{N}_i'$ and generalized hyperplane frames.

To build $\mathfrak{q}$, we will express $\lfloor \hat{Z} \rfloor$ as the union of $\hat{Z}$ and a collection of generalized frames, define $\mathfrak{q}$ on each generalized frame, and check that the definition is compatible where multiple generalized frames intersect. Let $z \in \lfloor \hat{Z} \rfloor$ be a 0-cube. Either $z \in \hat{Z}$, or there is a nonempty set $I \subset \{1, \ldots, k\}$ such that the set of hyperplanes separating z from $\hat{Z}$ is contained in $\bigcup_{i \in I} \mathfrak{N}_i'$, and each $\mathfrak{N}_i'$ contains a hyperplane

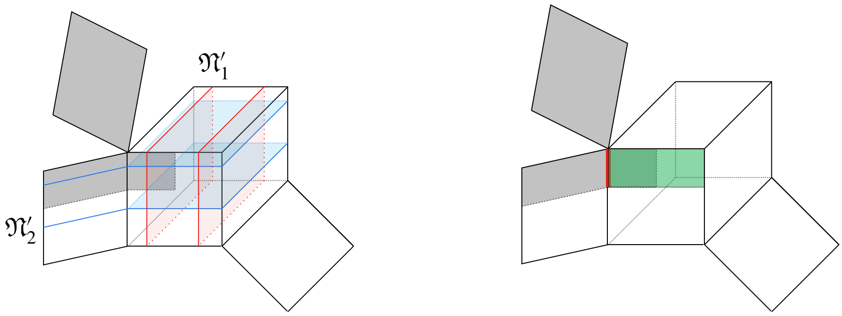

Figure 2. Collateral families $\mathfrak{N}_1'$ and $\mathfrak{N}_2'$ (left) and $Y(\{1\}) \times L_1$ (right).

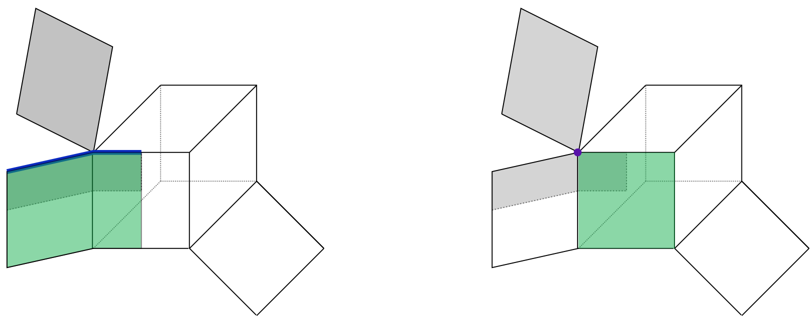

Figure 3. $Y(\{2\}) \times L_2$ (left) and $Y(\{1, 2\}) \times (L_1 \times L_2)$ (right).

separating z from $\hat{Z}$. If $H \in \mathfrak{A}(I) \cup \bigcup_{i \in I} \mathfrak{N}_i'$ is separated from z by a hyperplane U, then $U \in \mathfrak{A}(I) \cup \bigcup_{i \in I} \mathfrak{N}_i'$, whence we can choose $F(I)$ to contain z. Hence $\lfloor \hat{Z} \rfloor$ is the union of $\hat{Z}$ and a finite collection of generalized frames $F(I_1), \ldots, F(I_t)$.

For any $p \in \{1, \ldots, t\}$, we have $F(I_p) = Y(I_p) \times \prod_{j \in I_p} L_j$ and we define $\bar{Y}(I_p) = \operatorname{im}(Y(I_p) \to Z)$. Also, let $\bar{L}_j = \operatorname{im}(L_j \cap \hat{Z} \to Z)$ be the cycle of length n_{N_j} to which L_j maps, for each $j \in I_p$. Note that Z contains $\bar{F}(I_p) = \bar{Y}(I_p) \times \prod_{j \in I_P} \bar{L}_j$ and so we define the quotient map $\mathfrak{q}_p : F(I_p) \to Z$ as the product of the above combinatorial quotient maps, namely, $\mathfrak{q}_p(y, (r_j)_{j \in I_P}) = (\bar{y}, (r_j \mod n_{N_j})_{j \in I_p})$ for $y \in Y(I_p)$ and $r_j \in L_j$.

To ensure that $\mathfrak{q}_p(F(I_p) \cap F(I_j)) = \mathfrak{q}_j(F(I_p) \cap F(I_j))$ for all $i, j \leq t$, it suffices to show that

$$F(I_p) \cap F(I_j) := \left(Y(I_p) \times \prod_{k \in I_p} L_k \right) \cap \left(Y(I_j) \times \prod_{\ell \in I_j} L_\ell \right) = [Y(I_p) \cap Y(I_j)] \times \prod_{k \in I_p \cap I_j} L_k.$$

This in turn follows from [Caprace and Sageev 2011, Proposition 2.5]. Hence, the quotient maps $\mathfrak{q}_p$ are compatible and thus define a combinatorial quotient map $\mathfrak{q} : \lfloor \hat{Z} \rfloor \to Z$ extending the maps $\mathfrak{q}_p$.

Observe that if $\mathfrak{H} = \varnothing$, i.e., $K = \lfloor \hat{Z} \rfloor$, then we take $Y = Z$. By hypothesis, Z admits a local isometry to S_Γ and has the desired cardinality. Moreover, our hypothesis on g ensures that $g \notin \pi_1 Y$, but the map $\mathfrak{q}$ shows that any closed combinatorial path in S_Γ representing g lifts to a (nonclosed) path in Z, so the proof of the theorem is complete. Thus we can and shall assume that $\mathfrak{H} \neq \varnothing$.

Quotients of $\mathfrak{H}$-frames. To extend $\mathfrak{q}$ to the rest of K, we now describe quotient maps, compatible with the map $\hat{Z} \to Z$, on frames associated to hyperplanes in $\mathfrak{H}$. An *isolated $\mathfrak{H}$-frame* is a frame $(H \cap K) \times L$, where $H \in \mathfrak{H}$ and H crosses no hyperplane of $\hat{Z}$ (and hence crosses no hyperplane of $\lfloor \hat{Z} \rfloor$). An *interfered $\mathfrak{H}$-frame* is a frame $(H \cap K) \times L$, where $H \in \mathfrak{H}$ and H crosses an element of $\mathfrak{Z}$. Equivalently, $(H \cap K) \times L$ is interfered if $\mathfrak{g}_{N(H)}(\hat{Z})$ contains a 1-cube and is isolated otherwise.

Define quotient maps on isolated $\mathfrak{H}$-frames by the same means as was used for arbitrary frames in [Bou-Rabee et al. 2015]. Let $(H \cap K) \times L$ be an isolated $\mathfrak{H}$-frame. Let $\overline{H}$ be the immersed hyperplane in S_Γ to which H is sent by $\tilde{S}_\Gamma \to S_\Gamma$, and let $\overline{H \cap K}$ be the image of $H \cap K$. We form a quotient $Y_H = \overline{H \cap K} \times L$ of every isolated $\mathfrak{H}$-frame $(H \cap K) \times L$.

Now we define the quotients of interfered $\mathfrak{H}$-frames. Let $\hat{A} = \mathfrak{g}_{N(H)}(\hat{Z})$ and let A be the image of $\hat{A}$ under $\hat{Z} \to Z$. There is a local isometry $A \to S_\Gamma$, to which we apply canonical completion to produce a finite cover $\dddot{S}_\Gamma \to S_\Gamma$ where A embeds. By [Bou-Rabee et al. 2015, Lemma 2.8], $\deg(\dddot{S}_\Gamma \to S_\Gamma) = |\dddot{S}_\Gamma^{(0)}| = |A^{(0)}| \leq |Z^{(0)}|$. Let $\overline{H \cap K} = \operatorname{im}(H \cap K \to \dddot{S}_\Gamma)$, and map the interfered $\mathfrak{H}$-frame $(H \cap K) \times L$ to $Y_H = \overline{H \cap K} \times L$.

Constructing Y. We now construct a compact cube complex Y' from Z and the various quotients Y_H. A hyperplane W in K separates H_1 from $\hat{Z}$ only if $W \in \mathfrak{N}$. Each $\mathfrak{H}$-hyperplane frame has the form $(H_i \cap K) \times L_i = (H_i \cap K) \times [0, m_i]$, parametrized so that $(H_i \cap K) \times \{0\}$ is the closest combinatorial hyperplane in the frame to $\hat{Z}$. We form $Y'(1)$ by gluing Y_{H_1} to Z along the image of $\mathfrak{g}_{\hat{Z}}((H_1 \cap K) \times \{0\})$, enabled by the fact that the quotients of interfered $\mathfrak{H}$-frames are compatible with $\hat{Z} \to Z$. In a similar manner, form $Y'(i)$ from $Y'(i-1)$ and Y_{H_i} by identifying the image of $(H_{i-1} \cap K) \times \{m_{i-1}\} \cap (H_i \cap K) \times \{0\}$ in $Y_{H_{i-1}} \subset Y'(i-1)$ with its image in Y_{H_i}. Let $Y' = Y'(k)$.

Let

$$K' = \lfloor \hat{Z} \rfloor \cup \bigcup_{H_i \in \mathfrak{H}} (H_i \cap K) \times L_i.$$

Since $H_i \cap H_j = \varnothing$ for $i \neq j$, there exists a map $(K', \tilde{x}) \to (Y', x)$ and a map $(Y', x) \to (S_\Gamma, x)$ such that the composition is precisely the restriction to K' of the covering map $(\tilde{S}_\Gamma, \tilde{x}) \to (S_\Gamma, x)$.

If $Y' \to S_\Gamma$ fails to be a local isometry, then there exists i and nontrivial open cubes $e \subset \overline{H_{i-1} \cap K} \times \{m_{i-1}\}$ (or Z if $i = 1$) and $c \subset \overline{H_i \cap K} \times \{0\}$ such that S_Γ

contains an open cube $\bar{e} \times \bar{c}$, where $\bar{e}, \bar{c}$ are the images of e, c under $\tilde{S}_\Gamma \to S_\Gamma$, respectively. Moreover, since $\mathfrak{g}_{\hat{Z}}(H_i \cap K) \subseteq \mathfrak{g}_{\hat{Z}}(H_{i-1} \cap K)$, we can assume that $\bar{c}$ is disjoint from each immersed hyperplane of S_Γ crossing Z. Hence the closure $Cl(\bar{c})$ is a standard torus. Glue $Cl(\bar{e}) \times Cl(\bar{c})$ to Y', if necessary, in the obvious way. Note that this gluing adds no new 0-cubes to Y'. Indeed, every 0-cube of $Cl(\bar{e}) \times Cl(\bar{c})$ is identified with an existing 0-cube of Y' lying in $\overline{H_{i-1} \cap K} \times \{m_{i-1}\}$. Adding $Cl(\bar{e}) \times Cl(\bar{c})$ also preserves the existence of a local injection from our cube complex to S_Γ. Either this new complex admits a local isometry to S_Γ, or there is a missing cube of the form $\bar{e} \times \bar{c}$ where $Cl(\bar{c})$ is a standard torus and $\bar{e}$ lies in Y'. We add cubes of this type until we have no missing corners. That the process terminates in a local isometry with compact domain Y is a consequence of the following facts: at each stage, every missing cube has the form $\bar{e} \times \bar{c}$ where $\bar{e}$ lies in Y' and $Cl(\bar{c})$ is a standard torus, so the number of 0-cubes remains unchanged; each gluing preserves the existence of a local injection to S_Γ; each gluing increases the number of positive dimensional cubes containing some 0-cube; cubes that we add are images of cubes in K, which is compact.

There exists a combinatorial path γ in K' joining $\tilde{x}$ to $g\tilde{x}$. It follows from the existence of γ that the convex hull of K' is precisely equal to K. Hence, there exists a based cubical map $(K, \tilde{x}) \to (Y, x) \to (S_\Gamma, x)$, so that the composition is the restriction of the covering map $(\tilde{S}_\Gamma, \tilde{x}) \to (S_\Gamma, x)$. Therefore, any closed path in S_Γ representing g lifts to a nonclosed path at x in Y. It is easily verified that the number of 0-cubes in Y is bounded by $|Z^{(0)}|(m_1 + \cdots + m_k)$, where each m_i is the length of L_i, and hence $|Y^{(0)}| \leq |Z^{(0)}|(|g| + 1)$. Thus, Y is the desired cube complex. □

Remark 3.2. When $\dim S_\Gamma = 1$, arguing as above shows that Y can be chosen so that $|Y^{(0)}| \leq |Z^{(0)}| + |g|$. Hence, if F is freely generated by $\mathcal{S}$, with $|\mathcal{S}| = r$, then $\mathbf{Sep}_{F,\mathcal{S}}^{Q_K}(Q, n) \leq (2r)^K + n$.

Acknowledgments

We thank K. Bou-Rabee, S. Dowdall, F. Haglund, D.B. McReynolds, N. Miller, H. Wilton, and D.T. Wise for helpful discussions about issues related to this paper. We also thank an anonymous referee for astute corrections. The authors acknowledge travel support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation Varieties" (the GEAR Network) and from grant NSF 1045119. Hagen was supported by the National Science Foundation under Grant Number NSF 1045119.

References

[Agol 2013] I. Agol, "The virtual Haken conjecture", *Doc. Math.* **18** (2013), 1045–1087. MR 3104553 Zbl 1286.57019

[Aschenbrenner et al. 2015] M. Aschenbrenner, S. Friedl, and H. Wilton, "Decision problems for 3-manifolds and their fundamental groups", *Geom. Topol. Monogr.* **19** (2015), 201–236. Zbl 06537400

[Bandelt and Chepoi 2008] H.-J. Bandelt and V. Chepoi, "Metric graph theory and geometry: a survey", pp. 49–86 in *Surveys on discrete and computational geometry*, edited by J. E. Goodman et al., Contemporary Mathematics **453**, American Mathematical Society, Providence, RI, 2008. MR 2405677 Zbl 1169.05015

[Behrstock et al. 2014] J. Behrstock, M. F. Hagen, and A. Sisto, "Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups", preprint, 2014. arXiv 1412.2171

[Bergeron and Wise 2012] N. Bergeron and D. T. Wise, "A boundary criterion for cubulation", *Amer. J. Math.* **134**:3 (2012), 843–859. MR 2931226 Zbl 1279.20051

[Bergeron et al. 2011] N. Bergeron, F. Haglund, and D. T. Wise, "Hyperplane sections in arithmetic hyperbolic manifolds", *J. Lond. Math. Soc.* (2) **83**:2 (2011), 431–448. MR 2776645 Zbl 1236.57021

[Bou-Rabee 2010] K. Bou-Rabee, "Quantifying residual finiteness", *J. Algebra* **323**:3 (2010), 729–737. MR 2574859 Zbl 1222.20020

[Bou-Rabee 2011] K. Bou-Rabee, "Approximating a group by its solvable quotients", *New York J. Math.* **17** (2011), 699–712. MR 2851069 Zbl 1243.20038

[Bou-Rabee and Kaletha 2012] K. Bou-Rabee and T. Kaletha, "Quantifying residual finiteness of arithmetic groups", *Compos. Math.* **148**:3 (2012), 907–920. MR 2925403 Zbl 1256.20030

[Bou-Rabee and McReynolds 2014] K. Bou-Rabee and D. B. McReynolds, "Characterizing linear groups in terms of growth properties", preprint, 2014. To appear in *Mich. Math. J.* arXiv 1403.0983

[Bou-Rabee and McReynolds 2015] K. Bou-Rabee and D. B. McReynolds, "Extremal behavior of divisibility functions", *Geom. Dedicata* **175** (2015), 407–415. MR 3323650 Zbl 1314.20023

[Bou-Rabee et al. 2015] K. Bou-Rabee, M. F. Hagen, and P. Patel, "Residual finiteness growths of virtually special groups", *Math. Z.* **279**:1-2 (2015), 297–310. MR 3299854 Zbl 1317.20036

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer, Berlin, 1999. MR 1744486 Zbl 0988.53001

[Buskin 2009] N. V. Buskin, "Экономная отделимость в свободных группах", *Sibirsk. Mat. Zh.* **50**:4 (2009), 765–771. Translated as "Economical separability in free groups" in *Siberian Math. J.* **50**:4 (2009), 603–608. MR 2583614 Zbl 1212.20055

[Caprace and Sageev 2011] P.-E. Caprace and M. Sageev, "Rank rigidity for CAT(0) cube complexes", *Geom. Funct. Anal.* **21**:4 (2011), 851–891. MR 2827012 Zbl 1266.20054

[Charney and Davis 1995] R. Charney and M. W. Davis, "Finite $K(\pi, 1)$'s for Artin groups", pp. 110–124 in *Prospects in topology* (Princeton, NJ, 1994), edited by F. Quinn, Annals of Mathematics Studies **138**, Princeton University Press, 1995. MR 1368655 Zbl 0930.55006

[Chepoi 2000] V. Chepoi, "Graphs of some CAT(0) complexes", *Adv. in Appl. Math.* **24**:2 (2000), 125–179. MR 1748966 Zbl 1019.57001

[Dyson 1964] V. Dyson, "The word problem and residually finite groups", *Not. Amer. Math. Soc.* **11** (1964), 743.

[Hagen 2014] M. F. Hagen, "Weak hyperbolicity of cube complexes and quasi-arboreal groups", *J. Topol.* **7**:2 (2014), 385–418. MR 3217625 Zbl 06366501

[Hagen and Wise 2013] M. F. Hagen and D. T. Wise, "Cubulating hyperbolic free-by-cyclic groups: the irreducible case", preprint, 2013. arXiv 1311.2084

[Hagen and Wise 2015] M. F. Hagen and D. T. Wise, "Cubulating hyperbolic free-by-cyclic groups: the general case", *Geom. Funct. Anal.* **25**:1 (2015), 134–179. MR 3320891 Zbl 06422799

[Haglund 2008] F. Haglund, "Finite index subgroups of graph products", *Geom. Dedicata* **135** (2008), 167–209. MR 2413337 Zbl 1195.20047

[Haglund and Wise 2008] F. Haglund and D. T. Wise, "Special cube complexes", *Geom. Funct. Anal.* **17**:5 (2008), 1551–1620. MR 2377497 Zbl 1155.53025

[Haglund and Wise 2010] F. Haglund and D. T. Wise, "Coxeter groups are virtually special", *Adv. Math.* **224**:5 (2010), 1890–1903. MR 2646113 Zbl 1195.53055

[Hsu and Wise 2002] T. Hsu and D. T. Wise, "Separating quasiconvex subgroups of right-angled Artin groups", *Math. Z.* **240**:3 (2002), 521–548. MR 1924020 Zbl 1006.20028

[Kahn and Markovic 2012] J. Kahn and V. Markovic, "Immersing almost geodesic surfaces in a closed hyperbolic three manifold", *Ann. of Math.* (2) **175**:3 (2012), 1127–1190. MR 2912704 Zbl 1254.57014

[Kassabov and Matucci 2011] M. Kassabov and F. Matucci, "Bounding the residual finiteness of free groups", *Proc. Amer. Math. Soc.* **139**:7 (2011), 2281–2286. MR 2784792 Zbl 1230.20045

[Kharlampovich et al. 2014] O. Kharlampovich, A. Myasnikov, and P. Weil, "Stallings graphs for quasi-convex subgroups", preprint, 2014. arXiv 1408.1917

[Kozma and Thom 2016] G. Kozma and A. Thom, "Divisibility and laws in finite simple groups", *Math. Ann.* **364**:1-2 (2016), 79–95. MR 3451381 Zbl 06540649

[McKinsey 1943] J. C. C. McKinsey, "The decision problem for some classes of sentences without quantifiers", *J. Symbolic Logic* **8** (1943), 61–76. MR 0008991 Zbl 0063.03864

[Mostowski 1966] A. W. Mostowski, "On the decidability of some problems in special classes of groups", *Fund. Math.* **59** (1966), 123–135. MR 0224693 Zbl 0143.03701

[Ollivier and Wise 2011] Y. Ollivier and D. T. Wise, "Cubulating random groups at density less than $1/6$", *Trans. Amer. Math. Soc.* **363**:9 (2011), 4701–4733. MR 2806688 Zbl 1277.20048

[Patel 2013] P. Patel, *Quantifying algebraic properties of surface groups and* 3*-manifold groups*, thesis, Rutgers University, New Brunswick, NJ, 2013. MR 3192999

[Patel 2014] P. Patel, "On a theorem of Peter Scott", *Proc. Amer. Math. Soc.* **142**:8 (2014), 2891–2906. MR 3209342 Zbl 1311.57003

[Rivin 2012] I. Rivin, "Geodesics with one self-intersection, and other stories", *Adv. Math.* **231**:5 (2012), 2391–2412. MR 2970452 Zbl 1257.57024

[Sageev and Wise 2015] M. Sageev and D. T. Wise, "Cores for quasiconvex actions", *Proc. Amer. Math. Soc.* **143**:7 (2015), 2731–2741. MR 3336599 Zbl 06428953

[Scott 1978] P. Scott, "Subgroups of surface groups are almost geometric", *J. Lond. Math. Soc.* (2) **17**:3 (1978), 555–565. MR 0494062 Zbl 0412.57006

[Wise 2004] D. T. Wise, "Cubulating small cancellation groups", *Geom. Funct. Anal.* **14**:1 (2004), 150–214. MR 2053602 Zbl 1071.20038

[Wise 2011] D. T. Wise, "The structure of groups with a quasiconvex hierarchy", preprint, 2011, available at https://drive.google.com/file/d/0B45cNx80t5-2T0twUDFxVXRnQnc/view.

[Wise 2012] D. T. Wise, *From riches to raags:* 3*-manifolds, right-angled Artin groups, and cubical geometry* (New York, NY, 2011), CBMS Regional Conference Series in Mathematics **117**, American Mathematical Society, Providence, RI, 2012. MR 2986461 Zbl 1278.20055

Received August 7, 2015. Revised September 29, 2015.

MARK F. HAGEN
DEPARTMENT OF PURE MATHEMATICSS AND MATHEMATICAL STATISTICS
UNIVERSITY OF CAMBRIDGE
WILBERFORCE RD.
CAMBRIDGE
CB3 0WB
UNITED KINGDOM
markfhagen@gmail.com

PRIYAM PATEL
DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
150 N. UNIVERSITY ST.
WEST LAFAYETTE, IN 47907
UNITED STATES
patel376@purdue.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.121

CONFORMAL DESIGNS AND MINIMAL CONFORMAL WEIGHT SPACES OF VERTEX OPERATOR SUPERALGEBRAS

TOMONORI HASHIKAWA

We give equivalent conditions for conformal designs of the minimal conformal weight spaces of SVOAs, and show that if the minimal conformal weight space of an SVOA forms a conformal $2m$-design, then it also forms a conformal $(2m + 1)$-design. Also, we derive trace formulae for the zero-modes of elements of the conformal weight 2 space on the minimal conformal weight space when the minimal conformal weight space forms a conformal 4-design. As an application of the trace formulae, we classify code SVOAs whose minimal conformal weight spaces form conformal 4-designs. Moreover, we show that the classified code SVOAs are of class $\mathcal{S}^5$.

1. Introduction

Vertex operator algebras (VOA) and vertex operator superalgebras (SVOA) have deep connections to binary codes, integral lattices, and other combinatorial objects. The notion of conformal designs was introduced in [Höhn 2008], and is an analogue of the notions of combinatorial and spherical designs based on binary codes and integral lattices, respectively. Also, an analogue of the theorems of Assmus and Mattson [1969] and Venkov [2001] was presented in the same work. Due to this result, we expect that analogues of other properties of combinatorial and spherical designs hold in the theory of conformal designs. An integral lattice whose set of minimum norm vectors is a spherical design has been studied in [loc. cit.]. As one of the results, it was proved that an integral lattice whose set of minimum norm vectors forms a spherical 4-design, which is called a *strongly perfect lattice*, is isomorphic to the root lattices A_1, A_2, D_4, E_6, E_7, or E_8 if its minimum norm is 2. Also, strongly perfect lattices with minimum norm 3 have been classified in the same paper. Due to these circumstances, we speculate that a structural symmetry of an algebraic object is dominated by a subset which has a design structure. From this point of view, our purpose of this study is to clarify how the symmetry of the minimal conformal weight space of an SVOA influences a structural symmetry of

MSC2010: 17B69.
Keywords: vertex operator superalgebra, conformal design, binary code.

the SVOA. VOAs of class $\mathcal{S}^n$ with minimal conformal weight 2 have been discussed in [Matsuo 2001]. The notion of VOAs of class $\mathcal{S}^n$ was introduced in the same paper, and gives a sufficient condition that the minimal conformal spaces of VOAs form conformal n-designs. The conformal designs have been studied in [Yamauchi 2014] under the assumption for introducing the notion of extended Griess algebras. Considering these known results, we maintain the theory of the minimal conformal weight spaces of SVOAs and conformal designs. The following are the main results obtained in Section 3 of this paper.

Main Result 1 (Theorems 3.5 and 3.6). *Let V be an SVOA and μ the minimal conformal weight of V. Assume that $\mu < \infty$. Then the following hold*:

(1) *The space V_μ forms a conformal t-design based on the even part of V if and only if the t-th Casimir vector*, *introduced in* [Matsuo 2001], *belongs to the sub-VOA V_ω generated by the Virasoro element ω.*

(2) *If V_μ is a conformal $2m$-design based on the even part of V*, *then it is also a conformal $(2m+1)$-design.*

By using the computation of traces and invariant bilinear forms in [Yamauchi 2014], we have (1) of Main Result 1. The crucial point of the proof of (2) of Main Result 1 is that the $(2m+1)$-th Casimir vector can be determined from the n-th Casimir vectors for $n \leq 2m$ and the action of $L(-1)$. Note that (2) is an analogue of a well-known result in the theory of integral lattices and spherical designs. Moreover, trace formulae of the zero-modes of elements of the conformal weight-2 space on the minimal conformal weight space of an SVOA are obtained when the minimal conformal weight space forms a conformal 4-design.

As another related topic of conformal designs and SVOAs, there are classification problems of SVOAs whose minimal conformal weight spaces form conformal t-designs. This problem has been solved in [Höhn 2008] for the case that the minimal conformal weight is 1 and $t = 6$. More precisely, SVOAs with minimal conformal weight 1 are isomorphic to lattice VOAs associated to the root lattices of type A_1 and E_8 if the conformal weight-1 space forms a conformal 6-design. Also, it was proved in [Tuite 2009] that a VOA whose 4th Casimir vector belongs to V_ω is isomorphic to one of the simple affine VOA associated to the Deligne exceptional series of the simple Lie algebras $A_1, A_2, G_2, D_4, F_4, E_6, E_7$, and E_8 at level 1 if the minimal conformal weight is 1. Using (1) of Main Result 1, this classification result can be obtained under the condition that the conformal weight-1 space forms a conformal 4-design. This result is actually an analogue of the result in [Venkov 2001], as already mentioned. Due to the classification in [Tuite 2009], one can consider the classification problem in the case of SVOAs with minimal conformal weight $\frac{3}{2}$ and $t = 4$. The commutant superalgebra (see [Yamauchi 2005]) of an Ising vector in the lattice type VOA $V_{\sqrt{2}E_8}^{+}$ is included in the list of candidates of

SVOAs with minimal conformal weight $\frac{3}{2}$ whose 4th Casimir vector belongs to V_ω (see [Tuite and Van 2014]), and is isomorphic to the code SVOA $V_{\mathcal{H}_4}$ associated to the Hamming code $\mathcal{H}_4$ (see [Miyamoto 1996a] for the definition of code SVOAs). The notion of SVOAs of class $\mathcal{S}^n$ is an analogue of the ordinary notion introduced in [Matsuo 2001] and gives a sufficient condition that the minimal conformal weight spaces of SVOAs form conformal n-designs in the same way as the cases of VOAs of class $\mathcal{S}^n$.

In this paper, we show that $V_{\mathcal{H}_4}$ is of class $\mathcal{S}^5$. Moreover, we classify code SVOAs whose minimal conformal weight spaces form conformal 4-designs as an application of the results in Section 3 and show that the classified code SVOAs, which contain $V_{\mathcal{H}_4}$, are of class $\mathcal{S}^5$. We obtain the following.

Main Result 2 (Theorems 4.8 and 5.9). *Let C be a binary code. Assume that the minimal conformal weight μ of the code SVOA V_C is not ∞. Then*:

(1) *If $(V_C)_\mu$ forms a conformal 4-design based on the even part, then C is equivalent to one of*

$$\{(0^1),(1^1)\},\ \widehat{\mathcal{H}}_3,\ \mathcal{E}_8,\ E(\mathcal{H}_4),\ \mathcal{H}_4,\ \textit{and}\ \widehat{\mathcal{H}}_4,$$

where $\mathcal{H}_m$, $E(\mathcal{H}_m)$, $\widehat{\mathcal{H}}_m$, and $\mathcal{E}_8$ are the Hamming code of length 2^m-1, the even subcode of $\mathcal{H}_m$, the extended Hamming code of $\mathcal{H}_m$, and the set of all even weight vectors in $\mathbb{F}_2^8$, respectively.

(2) *The code SVOAs associated to the codes in* (1) *are of class $\mathcal{S}^5$.*

We see that for a code SVOA the minimal conformal weight space forms a conformal 4-design if and only if the SVOA is of class $\mathcal{S}^5$.

In the following, we sketch the proof of Main Result 2. Let C be a binary code of length n and μ the minimal conformal weight of V_C. Obviously, we can exclude the case $\mu > 2$. Considering the trace formulae on the minimal conformal weight space, $n = 1, 8$, and 15 if $\mu = \frac{1}{2}$, 1, and $\frac{3}{2}$, respectively. In case $\mu = \frac{1}{2}$, C must be $\{(0^1),(1^1)\}$ because it has a weight-1 vector. In cases $\mu = 1, \frac{3}{2}$, and 2, we show that $(\Omega_n, C(2\mu))$ is a combinatorial 2-design if $(V_C)_\mu$ is a conformal 4-design, where $\Omega_n := \{1,\dots,n\}$ and $C(2\mu)$ is the set of all weight 2μ vectors in C. By this result, $C \cong \mathcal{E}_8$ if $\mu = 1$, and $C \cong \mathcal{H}_4$ if $\mu = \frac{3}{2}$. Also, we have $C \cong \widehat{\mathcal{H}}_3, E(\mathcal{H}_4)$, or $\widehat{\mathcal{H}}_4$ by using fundamental techniques of algebraic coding theory and a list of possible central charges of VOAs with $\mu = 2$ which is obtained in [Matsuo 2001]. Thus (1) of Main Result 2 holds. Now we turn to (2) of that result. Obviously,

$$V_{\{(0^1),(1^1)\}} = L\left(\tfrac{1}{2},0\right) \oplus L\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

is of class $\mathcal{S}^\infty$. Note that the code SVOAs $V_{\mathcal{E}_8}$, $V_{\widehat{\mathcal{H}}_3}$, and $V_{\widehat{\mathcal{H}}_4}$ have already been proved; see [Maruoka et al. 2016; Hashikawa and Shimakura 2016]). Also, if $V_{\mathcal{H}_4}$ is of class $\mathcal{S}^5$, then so is $V_{E(\mathcal{H}_4)}$ because $V_{E(\mathcal{H}_4)}$ is the even part of $V_{\mathcal{H}_4}$. Hence it is sufficient to show that $V_{\mathcal{H}_4}$ is of class $\mathcal{S}^5$. Using the same method as in [Lam et al. 2007, Propositions 3.13], one can show that the automorphism group of a code SVOA is generated by σ-involutions and the lift of the automorphism group of the binary code if the minimum weight of the code is greater than or equal to 3. Considering the action of a σ-involution associated to an Ising vector of σ-type which is not included in the standard Ising frame of $V_{\mathcal{H}_4}$, we prove that $V_{\mathcal{H}_4}$ is of class $\mathcal{S}^5$. Therefore, (2) holds.

This paper is organized as follows. In Section 2, we recall the notions of SVOAs and Ising vectors of SVOAs. In Section 3, we recall the notions of conformal designs, give necessary and sufficient conditions for conformal designs of minimal conformal weight spaces, and show that conformal $2m$-designs imply conformal $(2m+1)$-designs. Also, we give trace formulae on the minimal conformal weight space of an SVOA by using the same argument as in [Matsuo 2001]. In Section 4, using the trace formulae obtained in Section 3, we classify code SVOAs whose minimal conformal weight spaces form conformal 4-designs. In Section 5, we show that the code SVOAs associated to the codes in the classification of Section 4 are of class $\mathcal{S}^5$.

2. Preliminaries

In this section, we recall the notion of vertex operator superalgebras and Ising vectors. Additionally, we show an analogue of [Höhn et al. 2012, Lemma 2.6], which will be used in Section 3.

Vertex operator superalgebras. A *vertex operator superalgebra* (SVOA)

$$V = V^0 \oplus V^1$$

is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space equipped with a linear map

$$Y(\,\cdot\,,z) : V \to \operatorname{End}(V)[\![z,z^{-1}]\!], \qquad v \mapsto \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$$

and two nonzero vectors $\mathbf{1}$ and ω in V^0, which are called the *vacuum vector* and the *Virasoro element*, respectively, satisfying certain conditions; see [Frenkel et al. 1993; Kac 1998] for details. As one of the conditions, the *Virasoro relation* holds on V:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\,\delta_{m+n,0}\,c$$

for $m,n \in \mathbb{Z}$, where $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, δ_{ij} is the Kronecker symbol, and $c \in \mathbb{C}$ is the *central charge* of V. The subspaces V^0 and V^1 are called the *even part* and the *odd part* of V, respectively. Throughout the paper, we assume that an SVOA V has the following grading:

$$V^0 = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n \quad\text{and}\quad V^1 = \bigoplus_{n\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} V_n,$$

where V_n is the eigenspace of the $L(0)$-operator with eigenvalue n. We also assume that V is of CFT-type, i.e., $V_0 = \mathbb{C}\mathbf{1}$. An SVOA $V = V^0 \oplus V^1$ is said to be a *vertex operator algebra* (VOA) if $V^1 = 0$. If $u \in V_m$, then we write $\mathrm{wt}(u) := m$. Define the *zero-mode* of a homogeneous element u by $o(u) := u_{(\mathrm{wt}(u)-1)}$, and extend linearly.

Let V_ω denote the sub-VOA of an SVOA V generated by the Virasoro element ω. Then the *minimal conformal weight* of V is defined by $\min\{n \in \frac{1}{2}\mathbb{Z}_{\geq 0} \mid V_n \neq (V_\omega)_n\}$ if $V \neq V_\omega$ and ∞ if $V = V_\omega$. Since we assume that V is of CFT-type, the minimal conformal weight of V is always greater than zero throughout this paper.

An element σ of $\mathrm{GL}(V)$ is called an *automorphism* of an SVOA V if it satisfies

$$\sigma(u_{(m)}v) = \sigma(u)_{(m)}\sigma(v) \quad \text{for all } u, v \in V,\ m \in \mathbb{Z}, \text{ and } \sigma(\omega) = \omega.$$

Let $\mathrm{Aut}(V)$ denote the group of all automorphisms of V.

Ising vectors of SVOAs. Let V be an SVOA. An element $e \in V_2$ is called an *Ising vector* of V if it satisfies $e_{(1)}e = 2e$, $e_{(3)}e = \frac{1}{4}\mathbf{1}$, and the subalgebra $\mathrm{Vir}(e)$ generated by e is isomorphic to the simple Virasoro VOA $L\left(\frac{1}{2}, 0\right)$ with central charge $\frac{1}{2}$. It is known that $L\left(\frac{1}{2}, 0\right)$ is rational and has three irreducible modules

$$L\left(\tfrac{1}{2}, 0\right), \quad L\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad \text{and} \quad L\left(\tfrac{1}{2}, \tfrac{1}{16}\right);$$

see [Dong et al. 1994, Theorem 3.4] for details. Let e be an Ising vector of V. Note that $\{L^e(n) := e_{(n+1)} \mid n \in \mathbb{Z}\}$ satisfies the Virasoro relation with central charge $\frac{1}{2}$. Since $\mathrm{Vir}(e) \cong L\left(\frac{1}{2}, 0\right)$, we have a decomposition

$$V = V_e(0) \oplus V_e\left(\tfrac{1}{2}\right) \oplus V_e\left(\tfrac{1}{16}\right),$$

where $V_e(k)$ for $k \in \left\{0, \frac{1}{2}, \frac{1}{16}\right\}$, is the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L\left(\frac{1}{2}, k\right)$. Let μ be the minimal conformal weight of V. Set

$$W^e_\mu(k) := \{u \in V_\mu \mid o(e)u = ku\}. \tag{2-1}$$

The following lemma is an analogue of [Höhn et al. 2012, Lemma 2.6].

Lemma 2.1. *Let V be an SVOA and e an Ising vector. If $\mu \in \{1\} \cup \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)$, then*

$$V_\mu = W^e_\mu(0) \oplus W^e_\mu\left(\tfrac{1}{2}\right) \oplus W^e_\mu\left(\tfrac{1}{16}\right).$$

Proof. Since $o(e)$ preserves V_μ and acts semisimply on V, the space can be decomposed into the direct sum of the eigenspaces of $o(e)$. Let $v \in V_\mu$ be an eigenvector of $o(e)$ with eigenvalue λ. It is sufficient to show that $\lambda \in \left\{0, \frac{1}{2}, \frac{1}{16}\right\}$. In case $\mu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ we have $L^e(m)v \in V_{\mu-m} = 0$ for $m \geq 1$, and hence $\mathrm{Vir}(e)v$ is a $\mathrm{Vir}(e)$-module whose top weight is λ. Since $\mathrm{Vir}(e) \cong L\left(\frac{1}{2}, 0\right)$, this case holds. For $\mu = 1$, we have $L^e(m)v \in V_{1-m} = 0$ for $m \geq 2$. If we suppose that $L^e(1)v = 0$, then this case also holds by using the same method as in the case of $\mu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. We show $L^e(1)v = 0$. Suppose the claim is not true. Since $o(e)L^e(1)v = (\lambda-1)L^e(1)v$

and $o(e)\mathbf{1} = e_{(1)}\mathbf{1} = 0$, we have $\lambda = 1$. However, this contradicts the nonexistence of an $o(e)$-weight-1 vector. $\square$

Here, we give the definitions of Ising vectors of σ-type and Ising frames, which will be used later. An Ising vector e of an SVOA V is said to be of *σ-type* if $V_e\big(\frac{1}{16}\big) = 0$. For an Ising vector e of σ-type, the linear map

$$\sigma_e := \begin{cases} 1 & \text{on } V_e(0), \\ -1 & \text{on } V_e\big(\frac{1}{2}\big), \end{cases} \tag{2-2}$$

is an automorphism of V; see [Miyamoto 1996b, Theorem 4.8]. A subset $\{e^1, \dots, e^n\}$ of V_2 such that $\omega = e^1 + \cdots + e^n$ is called an *Ising frame* if e^i is an Ising vector of V for each $1 \le i \le n$ and $[Y(e^i, z),\ Y(e^j, z)] = 0$ for $i \ne j$.

3. Conformal designs

In this section, we first review the notion of conformal designs, and obtain necessary and sufficient conditions in the case where the minimal conformal weight spaces of SVOAs form conformal designs. Also, we show that if the minimal conformal weight space of an SVOA forms a conformal $2m$-design, then it also forms a conformal $(2m + 1)$-design. Afterward, we give trace formulae of the composition of the zero-modes of elements of V_2 on the minimal conformal weight space when the space forms a conformal 4-design.

Conditions of SVOAs. Set $\zeta^r = e^{\pi\sqrt{-1}r}$ for $r \in \mathbb{Q}$, and let V be an SVOA. A bilinear form $(\cdot \mid \cdot)$ on V is said to be *invariant* if it satisfies

$$(Y(a, z)u \mid v) = \big(u \mid Y(e^{zL(1)} z^{-2L(0)} \zeta^{L(0)+2L(0)^2} a, -z)v\big)$$

for $a, u, v \in V$. It was proved in [Frenkel et al. 1993; Li 1994; Yamauchi 2014] that any invariant bilinear form on an SVOA is symmetric and there is a one-to-one correspondence between invariant bilinear forms and elements of the dual space of $V_0/L(1)V_1$. In this paper, we assume that V has a nondegenerate invariant bilinear form $(\cdot \mid \cdot)$. Due to the results above, the bilinear form is unique up to scalar since V is of CFT-type. Moreover, we assume that V as a V_ω-module is a direct sum of highest weight modules. Hence

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\ge 0}} V[n],$$

where $V[n]$ is the sum of highest weight V_ω-submodules of V with highest weight $n \in \frac{1}{2}\mathbb{Z}$. Note that $V[0] = V_\omega$ holds. We have the following lemma.

Lemma 3.1. *The spaces $V[0]$ and $V[m]$ for $m \ne 0$ are orthogonal with respect to $(\cdot \mid \cdot)$.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$, $u \in V[0] \cap V_n$, and $v \in V[m] \cap V_n$. By the invariance of $(\cdot \mid \cdot)$,

$$(u \mid v) = \sum_{\ell \geq 0} \frac{\zeta^{n+2n^2}}{\ell!} \big(\mathbf{1} \mid (L(1)^{\ell} u)_{(2n-1-\ell)} v\big). \tag{3-1}$$

Since $V[m]$ is a V_ω-module, $(L(1)^\ell u)_{(2n-1-\ell)} v$ belongs to $V[m] \cap V_0$ for each $\ell \geq 0$. Because $V_0 \subset V[0]$ and $V[0] \cap V[m] = 0$, the right hand side of (3-1) is 0. Therefore, we have this lemma because $(V_k \mid V_{k'}) = 0$ for $k \neq k'$. □

Define the projection map

$$\pi : V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V[n] \to V[0] = V_\omega,$$

which is a V_ω-module homomorphism.

Minimal conformal weight spaces and conformal designs. The notion of conformal designs was introduced by Höhn in [2008].

Definition 3.2 [Höhn 2008, Section 2]. Let U be a VOA and M a U-module. An $L(0)$-homogeneous subspace X of M is called a *conformal t-design* based on U if $\operatorname{tr}|_X o(a) = \operatorname{tr}|_X o(\pi(a))$ holds for any $a \in \bigoplus_{0 \leq n \leq t} U_n$.

Let $V = V^0 \oplus V^1$ be an SVOA. Clearly, V^0 and V^1 are V^0-modules. From now on, we assume that the minimal conformal weight μ of V is not ∞.

Remark 3.3. Assume that V has an involution g. Set $V^{\pm} := \{u \in V \mid g(u) = \pm u\}$. Yamauchi [2014] considered that the top weight space of V^- forms a conformal design based on $V^0 \cap V^+$ under some assumptions, and obtained various results. However, these results do not contain the general cases $\mu \in \{\frac{1}{2}, 1\}$. We are going to include these general cases in our discussion.

By Lemma 3.1, $(\cdot \mid \cdot)$ is also nondegenerate on $(V_\omega)_\mu$ and P_μ, where

$$P_\mu := \{u \in V_\mu \mid L(k)u = 0 \text{ for all } k \in \mathbb{Z}_{\geq 0}\}.$$

Moreover, $V_\mu = (V_\omega)_\mu \oplus P_\mu$ holds because

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V[n].$$

Let $\{v_i\}_{i=1}^{p_\mu}$ be a basis of P_μ, and $\{v^i\}_{i=1}^{p_\mu}$ the dual basis of $\{v_i\}_{i=1}^{p_\mu}$ with respect to $(\cdot \mid \cdot)$, where $p_\mu := \dim P_\mu$. We consider the vector

$$\lambda_\mu^m := \zeta^{\mu+2\mu^2} \sum_{i=1}^{p_\mu} v^i_{(2\mu-1-m)} v_i \in V_m.$$

Note that $\zeta^{\mu+2\mu^2}\lambda_\mu^m$ is called the quadratic Casimir vector in [Tuite 2009; Tuite and Van 2014]. We also note that if $\lambda_\mu^m \in V_\omega$, then $\lambda_\mu^\ell \in V_\omega$ for $\ell \le m$; see [loc. cit.]. Set $q_\mu := \dim(V_\omega)_\mu$. Let $\{w_i\}_{i=1}^{q_\mu}$ be a basis of $(V_\omega)_\mu$ and $\{w^i\}_{i=1}^{q_\mu}$ the dual basis of $\{w_i\}_{i=1}^{q_\mu}$ with respect to $(\cdot \mid \cdot)$. The following lemma holds.

Lemma 3.4. *Let V be an SVOA with minimal conformal weight μ. Then*

$$\mathrm{tr}|_{V_\mu} o(u) = (-1)^{\mathrm{wt}(u)}(u \mid \lambda_\mu^{\mathrm{wt}(u)})$$

for a homogeneous element $u \in \bigoplus_{n\in\mathbb{Z}_{>0}} V[n]$.

Proof. Because $\{v^i\}_{i=1}^{p_\mu} \cup \{w^i\}_{i=1}^{q_\mu}$ is the dual basis of $\{v_i\}_{i=1}^{p_\mu} \cup \{w_i\}_{i=1}^{q_\mu}$ with respect to $(\cdot \mid \cdot)$,

$$\mathrm{tr}|_{V_\mu} o(u) = \sum_{i=1}^{p_\mu}(o(u)v^i \mid v_i) + \sum_{i=1}^{q_\mu}(o(u)w^i \mid w_i). \tag{3-2}$$

Since $o(u)w^i \in \bigoplus_{n>0} V[n]$ and $w_i \in V[0]$, the second summation of (3-2) is 0 by Lemma 3.1, i.e., $\mathrm{tr}|_{V_\mu} o(u) = \sum_{i=1}^{p_\mu}(o(u)v^i \mid v_i)$ holds. By the same computation as in [Yamauchi 2014, Section 4.1, Lemma 5], we obtain the statement. □

Set $d_\mu := \dim V_\mu$, let $\{u_i\}_{i=1}^{d_\mu}$ be a basis of V_μ, and let $\{u^i\}_{i=1}^{d_\mu}$ be its dual basis with respect to $(\cdot \mid \cdot)$. We also consider the following Casimir vector (see [Matsuo 2001; Yamauchi 2014]):

$$\kappa_\mu^m := \zeta^{\mu+2\mu^2}\sum_{i=1}^{d_\mu} u^i_{(2\mu-1-m)}u_i = \lambda_\mu^m + \zeta^{\mu+2\mu^2}\sum_{i=1}^{q_\mu} w^i_{(2\mu-1-m)}w_i \in V_m. \tag{3-3}$$

We obtain the following equivalent conditions to define conformal designs. It has already been discussed in [Yamauchi 2014] for $\mu \in \frac{1}{2} + \mathbb{Z}_{\ge 1}$.

Theorem 3.5. *Let V be an SVOA with minimal conformal weight μ. Then the following are equivalent*: (1) *V_μ is a conformal t-design based on V^0*, (2) *$\kappa_\mu^t \in V_\omega$*, *and* (3) *$\lambda_\mu^t \in V_\omega$.*

Proof. By (3-3), (2) $\iff$ (3) holds. We show (1) $\iff$ (3). Let $a \in V_t^0$. Set $\bar{a} := a - \pi(a)$. Then $\mathrm{tr}|_{V_\mu} o(a) = \mathrm{tr}|_{V_\mu} o(\pi(a)) + (-1)^t(\bar{a} \mid \lambda_\mu^t)$ by Lemma 3.4. Therefore, $\mathrm{tr}|_{V_\mu} o(a) = \mathrm{tr}|_{V_\mu} o(\pi(a))$ if and only if $(\bar{a} \mid \lambda_\mu^t) = 0$. We see from Lemma 3.1 that $(\bar{a} \mid \lambda_\mu^t) = 0$ for any $a \in V_t^0$ if and only if $\lambda_\mu^t \in V[0] = V_\omega$. □

It is known that if the set of minimum norm vectors of an integral lattice forms a spherical $2m$-design, then it also forms $(2m+1)$-design; see [Venkov 2001, Section 5, p. 23]. The assertion of the following theorem is an analogy of this particular result in the case of a conformal design. A method to prove this when $\mu = 2$ was mentioned briefly in [Matsuo 2001, Section 2, p. 573].

Theorem 3.6. *Let V be an SVOA with minimal conformal weight μ. If V_μ forms a conformal $2m$-design based on the even part, then it also forms a conformal $(2m+1)$-design.*

Proof. By the skew symmetry,

$$\kappa_\mu^{2m+1} = \sum_{\ell\geq 0} \frac{(-1)^{2\mu-(2m+1)+\ell}}{\ell!} L(-1)^\ell \Bigl(\zeta^{\mu+2\mu^2} \sum_{i=1}^{d_\mu} (-1)^{|u^i||u_i|} (u_i)_{(2\mu-1-(2m+1)+\ell)} u^i\Bigr),$$

where $|a|$ equals 0 if $a \in V^0$, and 1 if $a \in V^1$. Since $|u^i||u_i| = 2\mu \bmod 2$, we have

$$\kappa_\mu^{2m+1} = \sum_{\ell\geq 0} \frac{(-1)^{1+\ell}}{\ell!} L(-1)^\ell \kappa_\mu^{2m+1-\ell}.$$

Hence,

$$\kappa_\mu^{2m+1} = \frac{1}{2} \sum_{\ell\geq 1} \frac{(-1)^{1+\ell}}{\ell!} L(-1)^\ell \kappa_\mu^{2m+1-\ell}. \tag{3-4}$$

If V_μ forms a conformal $2m$-design based on the even part, then by Theorem 3.5 $\kappa_\mu^s \in V_\omega$ for $1 \leq s \leq 2m$. Therefore, using (3-4) and Theorem 3.5, we are done. □

From now on, we assume that the central charge of an SVOA is neither 0 nor $-\frac{22}{5}$. This assumption implies that the degree m subspace of V_ω with $m \leq 5$ has a basis

$$\bigl\{L(-n_1)\cdots L(-n_r)\mathbf{1} \bigm| n_1 \geq \cdots \geq n_r \geq 2,\ \textstyle\sum_{i=1}^r n_i = m\bigr\};$$

see [Kac and Raina 1987, Lecture 8].

We also assume that the bilinear form is normalized by $(\mathbf{1} \,|\, \mathbf{1}) = 1$. The following lemma holds.

Lemma 3.7. *Let V be an SVOA, $a, b \in V_2$, $m \in \mathbb{Z}_{\geq 2}$, and $n \in \mathbb{Z}$. Then:*

(1) $(L(-m)\mathbf{1} \,|\, a_{(n)}b) = (2m-2)\delta_{m+n,3}(a \,|\, b) - \frac{m^2-3m+4}{2}\delta_{m+n,3}(L(1)a \,|\, L(1)b)$,

(2) $(L(-2)^2\mathbf{1} \,|\, a_{(-1)}b) = 2(a \,|\, \omega)(b \,|\, \omega) + 8(a \,|\, b) - 4(L(1)a \,|\, L(1)b)$.

Proof. By the commutator formula, for $k, \ell \in \mathbb{Z}$,

$$[L(k), a_{(\ell)}] = (k-\ell+1)a_{(k+\ell)} + \tbinom{k+1}{2}(L(1)a)_{(k+\ell-1)} + \tbinom{k+1}{3}\delta_{k+\ell,1}(a \,|\, \omega). \tag{3-5}$$

Since V is of CFT-type and has a nondegenerate invariant bilinear form, $L(1)V_1 = 0$. We compute $X := (L(-m)\mathbf{1} \mid a_{(n)}b)$:

$$X = \underbrace{(\mathbf{1} \mid a_{(n)}L(m)b)}_{=0} + (\mathbf{1} \mid [L(m), a_{(n)}]b) \qquad \text{(by invariance)}$$

$$= (m-n+1)(\mathbf{1} \mid a_{(m+n)}b) + \tbinom{m+1}{2}(\mathbf{1} \mid (L(1)a)_{(m+n-1)}b) \qquad \text{(by (3-5))}$$

$$= \delta_{m+n,3}\big((2m-2)(\mathbf{1} \mid a_{(3)}b) + \tbinom{m+1}{2}(\mathbf{1} \mid (L(1)a)_{(2)}b)\big)$$
$$\text{(since } (V_k \mid V_\ell) = 0 \text{ if } k \neq \ell)$$

$$= \delta_{m+n,3}\big((2m-2)(a \mid b) - \tfrac{m^2-3m+4}{2}((L(1)a)_{(-2)}\mathbf{1} \mid b)\big)$$
$$\text{(by invariance, } L(1)V_1 = 0)$$

$$= \delta_{m+n,3}\big((2m-2)(a \mid b) - \tfrac{m^2-3m+4}{2}(L(1)a \mid L(1)b)\big).$$
$$\text{(since } (L(1)a)_{(-2)}\mathbf{1} = L(-1)L(1)a)$$

Hence, we obtain (1). Next, we show (2). By (1) for $(m,n) = (2,1)$, the invariance, and (3-5),

$$\text{(3-6)} \quad (L(-2)^2\mathbf{1} \mid a_{(-1)}b)$$
$$= 2(a \mid \omega)(b \mid \omega) + 8(a \mid b) - 4(L(1)a \mid L(1)b) + 3(\omega \mid (L(1)a)_{(0)}b).$$

We show $(\omega \mid (L(1)a)_{(0)}b) = 0$. By (3-5),

$$\text{(3-7)} \quad (\omega \mid [L(1), a_{(0)}]b) = 4(a \mid b) - 2(L(1)a \mid L(1)b) + (\omega \mid (L(1)a)_{(0)}b).$$

On the other hand, by (1) for $(m,n) = (3,0)$ and the Virasoro relation,

$$\text{(3-8)} \quad (\omega \mid [L(1), a_{(0)}]b) = (L(-1)\omega \mid a_{(0)}b) - (\omega \mid a_{(0)}L(1)b)$$
$$= 4(a \mid b) - 2(L(1)a \mid L(1)b) - (\omega \mid a_{(0)}L(1)b).$$

We see from (3-5), (3-7), and (3-8) that $(\omega \mid (L(1)a)_{(0)}b)$ is computed as follows:

$$\begin{aligned}(\omega \mid (L(1)a)_{(0)}b) &= -(\omega \mid a_{(0)}L(1)b)\\ &= -(\mathbf{1} \mid [L(2), a_{(0)}]L(1)b)\\ &= -3((\mathbf{1} \mid a_{(2)}L(1)b) + (\mathbf{1} \mid (L(1)a)_{(1)}L(1)b))\\ &= -3(L(1)a \mid L(1)b) + 3(L(1)a \mid L(1)b) = 0.\end{aligned}$$

Therefore (2) holds by (3-6). □

Let V be an SVOA with minimal conformal weight $\mu \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$. We give trace formulae of the zero-modes of elements of V_2 on V_μ by using the same method as in [Matsuo 2001, Section 2.3]. Let $a, b \in V_2$, and $w \in V_\mu$. In general, by the Borcherds–Jacobi identity (see [Kac 1998]), for $u, v \in V^0$ and $p, q, r \in \mathbb{Z}$,

$$\text{(3-9)} \quad \sum_{\ell \geq 0} \binom{p}{\ell} (u_{(r+\ell)}v)_{(p+q-\ell)}$$
$$= \sum_{\ell \geq 0} (-1)^\ell \binom{r}{\ell} (u_{(p+r-\ell)}v_{(q+\ell)} - (-1)^r v_{(q+r-\ell)}u_{(p+\ell)}).$$

By (3-9) for $p = 2, q = 1$, and $r = -1$,

$$a_{(1)}b_{(1)}w = \sum_{\ell=0}^{2} \binom{2}{\ell} (a_{(-1+\ell)}b)_{(3-\ell)}w - a_{(-1)}b_{(3)}w - b_{(-1)}a_{(3)}w.$$

Therefore,

$$\operatorname{tr}|_{V_\mu} o(a)o(b) = \sum_{\ell=0}^{2} \binom{2}{\ell} \operatorname{tr}|_{V_\mu} o(\pi(a_{(-1+\ell)}b)) - 2\delta_{\mu,2}(a \mid b)$$

if V_μ forms a conformal 4-design based on the even part. Then, by Lemma 3.7 one can compute the trace because $\pi(a_{(-\ell+1)}b) \in (V_\omega)_{4-\ell}$, yielding

$$\operatorname{tr}|_{V_\mu} o(L(-4)\mathbf{1}) = 3\mu d_\mu, \qquad \operatorname{tr}|_{V_\mu} o(L(-2)^2\mathbf{1}) = \mu d_\mu(\mu+2) + c\delta_{\mu,2},$$
$$\operatorname{tr}|_{V_\mu} o(L(-3)\mathbf{1}) = -2\mu d_\mu, \qquad \operatorname{tr}|_{V_\mu} o(L(-2)\mathbf{1}) = \mu d_\mu.$$

Note that the cases $\mu \in \frac{1}{2} + \mathbb{Z}_{\geq 1} \cup \{2\}$ have already been obtained in [Matsuo 2001; Yamauchi 2014].

Proposition 3.8 [Matsuo 2001, Theorem 2.1; Yamauchi 2014, Theorem 1]. *Let V be an SVOA of central charge c with minimal conformal weight $\mu \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$.*

(1) *If V_μ forms a conformal* 2*-design, then for $a \in V_2$,*

$$\operatorname{tr}|_{V_\mu} o(a) = \frac{2\mu d_\mu}{c}(a \mid \omega).$$

(2) *If V_μ forms a conformal* 4*-design, then for $a, b \in V_2$,*

$$\begin{aligned}\operatorname{tr}|_{V_\mu} o(a)o(b) = {}& \frac{2(\mu d_\mu(22\mu - c) - 5c^2\delta_{\mu,2})}{c(5c+22)}(a \mid b) \\ & - \frac{2(\mu d_\mu(c+6+8\mu) + 8c\delta_{\mu,2})}{c(5c+22)}(L(1)a \mid L(1)b) \\ & + \frac{4(\mu d_\mu(5\mu+1) + 5c\delta_{\mu,2})}{c(5c+22)}(a \mid \omega)(b \mid \omega).\end{aligned}$$

Remark 3.9. The reason why we consider the cases where $\mu \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$ is that the trace of $o(a)$ on V_μ for $a \in V_2$ is a multiple of μd_μ because $V_2 = (V_\omega)_2$ if $\mu > 2$. Hence, we consider trace formulae on V_μ in the cases $\mu \leq 2$ only.

Set $d^e_\mu(k) := \dim W^e_\mu(k)$, where $W^e_\mu(k)$ is defined in (2-1). The following corollary holds. It has already been mentioned in the introduction of [loc. cit.] for $\mu = 2$.

Corollary 3.10. *Let V be an SVOA with minimal conformal weight $\mu \in \{\frac{1}{2}, 1, \frac{3}{2}\}$, and e an Ising vector. If V_μ forms a conformal* 4*-design based on V^0, then*

$$d^e_\mu(0) = \frac{d_\mu(c(5c+22-61\mu) + 2\mu(196\mu - 95))}{c(5c+22)}.$$

Also,

$$d^e_\mu\left(\frac{1}{2}\right)=\frac{\mu d_\mu(56\mu-3c-2)}{c(5c+22)}\quad \textit{and}\quad d^e_\mu\left(\frac{1}{16}\right)=\frac{64\mu d_\mu(c+3-7\mu)}{c(5c+22)}. \tag{3-10}$$

Proof. Obviously, $\mathrm{tr}|_{V_\mu} o(\mathbf{1})=d_\mu$. Note that $L(1)e=0$ since V is of CFT-type; see [Yamauchi 2004, Lemma 8.1.2]. Since $(\omega \mid e)=(e\mid e)=\frac{1}{4}$, by Proposition 3.8,

$$\mathrm{tr}|_{V_\mu} o(e)=\frac{\mu d_\mu}{2c}\quad\text{and}\quad \mathrm{tr}|_{V_\mu} o(e)^2=\frac{\mu d_\mu(49\mu-2c+1)}{4c(5c+22)}.$$

By Lemma 2.1,

$$\begin{bmatrix}1 & 1 & 1\\ 0 & \frac{1}{2} & \frac{1}{16}\\ 0 & (\frac{1}{2})^2 & (\frac{1}{16})^2\end{bmatrix}\begin{bmatrix}d^e_\mu(0)\\ d^e_\mu(\frac{1}{2})\\ d^e_\mu(\frac{1}{16})\end{bmatrix}=\begin{bmatrix}d_\mu\\ \frac{\mu d_\mu}{2c}\\ \frac{\mu d_\mu(49\mu-2c+1)}{4c(5c+22)}\end{bmatrix}.$$

Therefore, we obtain this corollary by direct computation. □

The following corollary is obtained from (3-10) immediately.

Corollary 3.11. *Let V be an SVOA of central charge c with minimal conformal weight $\mu\in\{\frac{1}{2},1,\frac{3}{2}\}$, and e an Ising vector of V. Assume that V_μ forms a conformal 4-design based on the even part. Then $c=7\mu-3$ if and only if $d^e_\mu\big(\frac{1}{16}\big)=0$.*

An $\mathbb{R}$-*form* W of an SVOA V is an $\mathbb{R}$-subalgebra of V with the same Virasoro element such that $V\cong\mathbb{C}\otimes W$. As an application of the trace formulae, we have the following theorem.

Theorem 3.12. *Let V be an SVOA with minimal conformal weight $\frac{1}{2}$ and let W be an $\mathbb{R}$-form which has a positive definite invariant bilinear form. If $V_{1/2}$ forms a conformal 4-design based on the even part, then V is isomorphic to*

$$L\big(\tfrac{1}{2},0\big)\oplus L\big(\tfrac{1}{2},\tfrac{1}{2}\big).$$

Remark 3.13. [Höhn 2008, Theorem 4.1(a)] shows that V, as in Theorem 3.12, is isomorphic to $L(\frac{1}{2},0)\oplus L\big(\frac{1}{2},\frac{1}{2}\big)$ if the minimal conformal weight space forms a conformal 6-design. Hence, Theorem 3.12 is more general than the theorem in that reference.

Proof of Theorem 3.12. Since the bilinear form is positive definite on W, we can take an orthogonal basis $\{x^i\}_{i=1}^{d_{1/2}}$ of $W_{1/2}$ such that $(x^i\mid x^j)=\frac{1}{2}\delta_{ij}$. Since $V\cong\mathbb{C}\otimes W$, $\{x^i\}_{i=1}^{d_{1/2}}$ is also a basis of $V_{1/2}$. Set

$$e^i:=-x^i_{(-2)}x^i\quad\text{and}\quad L^{e^i}(n)=e^i_{(n+1)}$$

for each $1\le i\le d_{1/2}$ and $n\in\mathbb{Z}$. Then for each $1\le i\le d_{1/2}$ we can check by direct computation that $\{L^{e^i}(n)\}_{n\in\mathbb{Z}}$ satisfies the Virasoro relation with central charge $\frac{1}{2}$.

Since $e^1, \dots, e^{d_{1/2}}$ belong to W, the vectors are Ising vectors; see [Miyamoto 1996b, Section 6, p. 540]. Also by direct computation,

$$L^{e^i}(0)x^j = \tfrac{1}{2}\delta_{ij}x^j$$

for $1 \le i, j \le d_{1/2}$. Because $\{x^i\}_{i=1}^{d_{1/2}}$ is a basis of $V_{1/2}$, the central charge of V is $\frac{1}{2}$ by Corollary 3.11. Then by Proposition 3.8 we also have $d_{1/2} = 1$ since $\operatorname{tr}|_{V_{1/2}} o(e^i) = \frac{1}{2}$. Since the central charge is $\frac{1}{2}$ and ω is the Virasoro element of W, we may conclude that ω is an Ising vector of V. Because the $L(0)$-weights of V are half-integers, $V = V[0] \oplus V\big[\frac{1}{2}\big]$, where $V[k]$ is the V_ω-submodule of V defined at the beginning of Section 3. Therefore,

$$V = V_\omega \oplus V\big[\tfrac{1}{2}\big] \cong L\big(\tfrac{1}{2}, 0\big) \oplus L\big(\tfrac{1}{2}, \tfrac{1}{2}\big)$$

because $V[0] = V_\omega$ and $d_{1/2} = 1$. □

4. Conformal designs and code SVOAs

In this section, we first review the notion of binary codes and combinatorial designs. Next, we recall the definition of code SVOAs, and classify the code SVOAs whose minimal conformal weight spaces form conformal 4-designs.

Binary codes. A *binary code* C of length n is a subspace of $\mathbb{F}_2^n$. The *support* $\operatorname{supp}(x)$ and the *weight* $\operatorname{wt}(x)$ of $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$ are defined by

$$\operatorname{supp}(x) := \{1 \le i \le n \mid x_i \ne 0\} \quad \text{and} \quad \operatorname{wt}(x) := \#\operatorname{supp}(x),$$

respectively. A binary code C is said to be *even* if $\operatorname{wt}(c) \in 2\mathbb{Z}$ for all $c \in C$. Let (0^n) and (1^n) denote the vectors $(0, \dots, 0) \in \mathbb{F}_2^n$ and $(1, \dots, 1) \in \mathbb{F}_2^n$, respectively. The *minimum weight* of C is $\min\{\operatorname{wt}(c) \mid c \in C \setminus \{(0^n)\}\}$ if $C \ne \{(0^n)\}$ and ∞ if $C = \{(0^n)\}$. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{F}_2^n$, let $x * y$ denote the vector $(x_1 y_1, \dots, x_n y_n) \in \mathbb{F}_2^n$. For C a binary code with minimum weight $d \ne \infty$,

$$\#C \le 2^n \Big/ \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i}, \tag{4-1}$$

where $\big\lfloor \frac{1}{2}(d-1) \big\rfloor$ is the largest integer not greater than $\frac{1}{2}(d-1)$. The upper bound of $\#C$ is called the *sphere-packing bound* of C; see [Assmus and Key 1992, Theorem 2.1.3]. It is easy to see that d is an odd integer if equality holds in (4-1).

Set $\widehat{\mathcal{H}}_2 := \{(0^4), (1^4)\}$. Define the binary code $\widehat{\mathcal{H}}_m$ for $m \in \mathbb{Z}_{\ge 3}$ by

$$\{(u, u+v) \mid u \in \mathcal{E}_{2^{m-1}}, v \in \widehat{\mathcal{H}}_{m-1}\},$$

where $\mathcal{E}_\ell$ is the set of all even weight vectors in $\mathbb{F}_2^\ell$. Set

$$\mathcal{H}_m := \{(c_1, \dots, c_{2^m-1}) \mid (c_1 \dots, c_{2^m}) \in \widehat{\mathcal{H}}_m\} \subset \mathbb{F}_2^{2^m-1}$$

for $m \in \mathbb{Z}_{\geq 2}$. The binary codes $\mathcal{H}_m$ and $\widehat{\mathcal{H}}_m$ are called the *Hamming code* of length $2^m - 1$ and the *extended Hamming code* of length 2^m, respectively; see [MacWilliams and Sloane 1977, Chapters 1 and 13]. Denote by $E(\mathcal{H}_m)$ the even subcode of $\mathcal{H}_m$. Note that the dimensions of $\mathcal{H}_m$ and $\widehat{\mathcal{H}}_m$ are $2^m - m - 1$. The following lemmas are obtained by a basic method of algebraic coding theory. For the reader's convenience we include the proof.

Lemma 4.1. *Let $m \in \mathbb{Z}_{\geq 2}$ and D a binary code of length 2^m whose minimum weight is greater than or equal to 3. If D has a subcode equivalent to $\widehat{\mathcal{H}}_m$, then D is equivalent to $\widehat{\mathcal{H}}_m$.*

Proof. By (4-1),

$$\#D \leq 2^{2^m} \Big/ \sum_{i=0}^{1} \binom{2^m}{i} < 2^{2^m - m}.$$

Hence $\dim D$ is less than or equal to $2^m - m - 1$. Since $\dim \widehat{\mathcal{H}}_m = 2^m - m - 1$, the assertion holds. □

Lemma 4.2. *Let $m \in \mathbb{Z}_{\geq 2}$ and D a binary code of length $2^m - 1$ whose minimum weight is greater than or equal to 3. If D has a subcode equivalent to $E(\mathcal{H}_m)$, then D is equivalent to $E(\mathcal{H}_m)$ or $\mathcal{H}_m$.*

Proof. We see from (4-1) that $\#D \leq 2^{2^m-1} / \sum_{i=0}^{1} \binom{2^m-1}{i} = 2^{2^m-1-m}$. Hence $\dim D$ is $2^m - 1 - m$ or $2^m - 2 - m$ because D has a subcode equivalent to $E(\mathcal{H}_m)$. Clearly, D is equivalent to $E(\mathcal{H}_m)$ if $\dim D$ is $2^m - 2 - m$. If $\dim D$ equals $2^m - 1 - m$, then equality holds in (4-1). As already mentioned before, the minimum weight of D must be an odd integer, and hence it must be 3. Thus D is a binary code of length $2^m - 1$ whose dimension and minimum weight are $2^m - 1 - m$ and 3, respectively. It is known that such a code is equivalent to $\mathcal{H}_m$; see [MacWilliams and Sloane 1977, Chapter 1, Section 7, Problem (28)]. Therefore, this lemma holds. □

Combinatorial designs and binary codes. Set $\Omega_n := \{1, \dots, n\}$. Let k be a nonnegative integer such that $k \leq n$. Denote the set of all k-subsets of Ω_n by $\binom{\Omega_n}{k}$. Let $\mathcal{B}$ be a subset of $\binom{\Omega_n}{k}$. A pair $(\Omega_n, \mathcal{B})$ is a t-(n, k, λ)-*design* if there exists a constant λ such that $\#\{B \in \mathcal{B} \mid X \subset B\} = \lambda$ for all $X \in \binom{\Omega_n}{t}$. For $D \subset \mathbb{F}_2^n$, set $D(k) := \{u \in D \mid \mathrm{wt}(u) = k\}$. We often say that $(\Omega_n, D(k))$ is a t-(n, k, λ)-design if the pair of Ω_n and $\{\mathrm{supp}(u) \mid u \in D(k)\}$ forms a t-(n, k, λ)-design. By using a basic method for algebraic coding theory, the following proposition holds. We include the proof for the reader's convenience.

Proposition 4.3. *Let $m \in \mathbb{Z}_{\geq 2}$ and C a binary code of length 2^m with minimum weight* 4. *If $(\Omega_{2^m}, C(4))$ forms a* 3-$(2^m, 4, 1)$-*design, then C is equivalent to $\widehat{\mathcal{H}}_m$. Analogously, let C be a binary code of length $2^m - 1$ with minimum weight* 3. *If $(\Omega_{2^m-1}, C(3))$ forms a* 2-$(2^m - 1, 3, 1)$-*design, then C is equivalent to $\mathcal{H}_m$.*

Proof. We show the $\widehat{\mathcal{H}}_m$ case only because the $\mathcal{H}_m$ case is obtained by the same method. If we show that $\#(\mathbb{F}_2^{2^m}/C) = 2^{m+1}$, then we obtain the statement because a binary code of length 2^m whose dimension and minimum weight are $2^m - 1 - m$ and 4, respectively, is equivalent to $\widehat{\mathcal{H}}_m$ [MacWilliams and Sloane 1977, Chapter 1, Section 9, Problem (41)]. Let $u \in \mathbb{F}_2^{2^m}$ such that $\mathrm{wt}(u) > 2$. Then there exists $v \in C(4)$ such that $\mathrm{wt}(u + v) \leq \mathrm{wt}(u) - 2$ because $(\Omega_{2^m}, C(4))$ is a 3-$(2^m, 4, 1)$-design. Hence, every element of $\mathbb{F}_2^{2^m}/C$ is represented by an element of weight at most 2. Also, since the weight of the sum of vectors $x, x' \in \mathbb{F}_2^{2^m}$ such that $\mathrm{wt}(x) \leq 1, \mathrm{wt}(x') \leq 2$, and $x \neq x'$ is less than 4 and the minimum weight of C is 4, $x + C$ and $x' + C$ are distinct. Set $X_i := \{y + C \in \mathbb{F}_2^{2^m}/C \mid \mathrm{wt}(y) = i\}$. By the argument above, we have

$$\mathbb{F}_2^{2^m}/C = X_0 \amalg X_1 \amalg X_2, \quad \#X_0 = 1, \quad \text{and} \quad \# X_1 = 2^m.$$

Hence, it is sufficient to show that $\#X_2 = 2^m - 1$. Let $y + C \in X_2$. It is easy to check that

$$(y + C)(2) = \{y\} \amalg \{y + c \mid c \in C(4) \text{ such that } \mathrm{supp}(y) \subset \mathrm{supp}(c)\}.$$

Hence, $\#(y + C)(2) = 2^{m-1}$ because $(\Omega_{2^m}, C(4))$ is also a 2-$(2^m, 4, 2^{m-1} - 1)$-design [op. cit., Chapter 2, Section 5, Theorem 9]. Since

$$\mathbb{F}_2^{2^m}(2) = \coprod_{z+C \in X_2} (z + C)(2),$$

we have $\#X_2 = \frac{1}{2^{m-1}}\binom{2^m}{2} = 2^m - 1$, completing the proof of this proposition. $\square$

In order to prove our main result, we need the following two lemmas.

Lemma 4.4. *Let $t \in \mathbb{Z}_{\geq 2}$ and let C be a binary code of length $n \geq 3$ with minimum weight $t + 1$. Then the cardinality of $C(t + 1)$ is at most $\frac{1}{t+1}\binom{n}{t}$. Moreover, equality holds in the inequality if and only if $(\Omega_n, C(t + 1))$ forms a t-$(n, t + 1, 1)$-design.*

Proof. Consider the cardinality of

$$S := \left\{X \in \binom{\Omega_n}{t} \;\middle|\; \text{there exists } u \in C(t + 1) \text{ such that } X \subset \mathrm{supp}(u)\right\}.$$

Since $t \geq 2$ and the minimum weight of C is $t + 1$, for $X \in \binom{\Omega_n}{t}$ the cardinality of $\{u \in C(t + 1) \mid X \subset \mathrm{supp}(u)\}$ is at most 1. Hence

$$S = \coprod_{u \in C(t+1)} \binom{\mathrm{supp}(u)}{t}. \tag{4-2}$$

By (4-2),

$$\binom{n}{t} \geq \#S = \sum_{u \in C(t+1)} \#\binom{\mathrm{supp}(u)}{t} = (t+1)\,\#\,C(t+1).$$

Therefore, the first claim of this lemma holds. Also, from (4-2), that equality holds in the inequality if and only if $\binom{\Omega_n}{t} = S$. Hence we have the second claim because $\binom{\Omega_n}{t} = S$ implies that $(\Omega_n, C(t+1))$ forms a t-$(n, t+1, 1)$-design. □

Lemma 4.5. *Let C be an even code of length $2^m - 1$ ($m \geq 2$) with minimum weight 4. If $(\Omega_{2^m-1}, C(4))$ forms a 2-$(2^m-1, 4, 2^{m-1}-2)$-design, then $C \cong E(\mathcal{H}_m)$.*

Proof. Set $D := \langle C, (1^{2^m-1})\rangle_{\mathbb{F}_2} = C \amalg ((1^{2^m-1}) + C)$. Note that if we show that the minimum weight of D is 3 and $(\Omega_{2^m-1}, D(3))$ forms a 2-$(2^m-1, 3, 1)$-design, then this lemma follows from Lemma 4.2 since the even subcode of D is C.

First we show that the minimum weight of D is 3. Fix $X \in \binom{\Omega_{2^m-1}}{2}$. Set $C_X := \{u \in C(4) \mid X \subset \mathrm{supp}(u)\}$ and $w_X := \sum_{u \in C_X} u$. Let $u, v \in C_X$ such that $u \neq v$. Then $\mathrm{supp}(u) \cap \mathrm{supp}(v) = X$ because the minimum weight of C is 4. Since $(\Omega_{2^m-1}, C(4))$ forms a 2-$(2^m-1, 4, 2^{m-1}-2)$-design, the cardinality of C_X is $2^{m-1}-2$, and hence we have $\mathrm{wt}(w_X) = 2^m - 4$ and $(1^{2^m-1}) + w_X \in D(3)$. If we suppose that D has a weight-1 vector v, then $(1^{2^m-1}) + w_Y + v \in D(2)$ for $Y \in \binom{\Omega_{2^m-1}}{2}$ such that $\mathrm{supp}(v) \subset Y$, which contradicts $D(2) = C(2) = \varnothing$. Thus, the minimum weight of D is 3.

Next we show that $(\Omega_{2^m-1}, D(3))$ forms a 2-$(2^m-1, 3, 1)$-design. Set $z_X := (1^{2^m-1}) + w_X$ for $X \in \binom{\Omega_{2^m-1}}{2}$. Then $\mathrm{wt}(w_X + w_Y) = 6 - 2\,\mathrm{wt}(z_X * z_Y)$ for $X, Y \in \binom{\Omega_{2^m-1}}{2}$. Since $C(2) = \varnothing$, we have $\mathrm{wt}(z_X * z_Y) \leq 1$ if $w_X \neq w_Y$, and $\mathrm{wt}(z_X * z_Y) = 3$ if $w_X = w_Y$. Because the support of $z_X * z_Y$ is

$$(\Omega_{2^m-1} \setminus \mathrm{supp}(w_X)) \cap (\Omega_{2^m-1} \setminus \mathrm{supp}(w_Y)),$$

we have $w_X = w_Y$ if and only if $Y \subset \Omega_{2^m-1} \setminus \mathrm{supp}(w_X)$. By this argument, $\#\{w_X \mid X \in \binom{\Omega_{2^m-1}}{2}\}$ is exactly $\frac{1}{3}\binom{2^m-1}{2}$. Hence $\#D(3) \geq \frac{1}{3}\binom{2^m-1}{2}$. We see from Lemma 4.4 that the assertion holds. □

Code SVOAs. Let X be the free fermionic SVOA of central charge $\frac{1}{2}$, i.e., $X = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$. Set $X^k := L(\frac{1}{2}, \frac{k}{2})$ for $k = 0, 1$. Then $X^{\otimes n}$ is an SVOA as a tensor product of SVOAs. Set $V^\alpha := X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_2^n$. Note that V^α is a $V^{(0^n)}$-module. For a binary code C of length n, set

$$V_C := \bigoplus_{\alpha \in C} V^\alpha,$$

which is a sub-SVOA of $X^{\otimes n}$. The SVOA V_C is called the *code SVOA* associated to C; see [Miyamoto 1996a; Lam et al. 2007] for details. We remark that the central charge of V_C is half of the length of C. Let $u^0 = \mathbf{1}$ be the vacuum vector of X^0 and u^1 a highest weight vector of X^1 such that $u^1_{(-2)}u^1 = 2\omega$, where ω is the Virasoro element of X^0. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}_2^n$, set

$$u^\alpha := u^{\alpha_1} \otimes \cdots \otimes u^{\alpha_n} \in V^\alpha. \tag{4-3}$$

Note that u^α is a highest weight vector of V^α. For $1 \le i \le n$, set

$$e^i := \mathbf{1} \otimes \cdots \otimes \omega \otimes \cdots \otimes \mathbf{1} \in V_C,$$

where the Virasoro element ω of X^0 is the i-th tensor factor. It is known that e^i is an Ising vector of σ-type; see [Miyamoto 1996a]. Let $(\cdot \mid \cdot)$ be the invariant bilinear form on V_C such that $(\mathbf{1} \mid \mathbf{1}) = 1$. Then $(e^i \mid e^j) = \frac{1}{4}\delta_{ij}$ obviously holds.

Set $N := \bigoplus_{1 \le i,j \le 2n} \mathbb{Z}(x_i + x_j)$, where $\{x_i\}_{i=1}^{2n}$ is an orthonormal basis of $\mathbb{R}^{2n}$. Let $(V_N)_\mathbb{R}$ be the lattice VOA over $\mathbb{R}$ associated to N. In [Miyamoto 2004], it was proved that if a binary code C of length n is even, then the code VOA over $\mathbb{R}$ is embedded into the VOA $(V_N^+)_\mathbb{R} \oplus \sqrt{-1}(V_N^-)_\mathbb{R}$, where $(V_N^\pm)_\mathbb{R} \subset (V_N)_\mathbb{R}$ is the eigenspace of a lift of the -1 isometry of N with eigenvalue ± 1, respectively. The VOA $(V_N^+)_\mathbb{R} \oplus \sqrt{-1}(V_N^-)_\mathbb{R}$ has a positive definite invariant bilinear form; see [op. cit., Proposition 2.7]. Replacing N by $L := \bigoplus_{i=1}^{2n} \mathbb{Z}x_i$, one can show the case that C has an odd weight vector, and hence the following holds.

Proposition 4.6 [Miyamoto 2004, Corollary 3.6]. *Let C be a binary code. Then V_C has an $\mathbb{R}$-form which has a positive definite invariant bilinear form. In particular, V_C satisfies the assumptions in Section 3 on page 126.*

Conformal* 4*-designs and code SVOAs. Let C be a binary code and μ the minimal conformal weight of the code SVOA V_C. Assume that $\mu < \infty$. We show that C is equivalent to $\{(0^1), (1^1)\}, \widehat{\mathcal{H}}_3, \mathcal{E}_8, E(\mathcal{H}_4), \mathcal{H}_4$, or $\widehat{\mathcal{H}}_4$ if $(V_C)_\mu$ forms a conformal 4-design based on V_C^0. The next lemma plays an important role in our main result.

Lemma 4.7. *Let C be a binary code of length n and μ the minimal conformal weight of V_C. Assume that $\mu \in \{1, \frac{3}{2}, 2\}$. If $(V_C)_\mu$ forms a conformal 4-design based on V_C^0, then $(\Omega_n, C(2\mu))$ forms a 2-$(n, 2\mu, \nu)$-design, where*

$$\nu = \frac{4\mu(5\mu+1)\,\#C(2\mu) + 98n\delta_{\mu,2}}{n(5n+44)}.$$

Proof. A basis of $(V_C)_\mu$ is given by

$$\begin{cases} \{u^\alpha \mid \alpha \in C(2\mu)\} & \text{if } \mu = 1, \frac{3}{2}, \\ \{e^i \mid 1 \le i \le n\} \cup \{u^\alpha \mid \alpha \in C(4)\} & \text{if } \mu = 2, \end{cases}$$

where e^i and u^α are defined in Section 4 on page 137. Let $i, j \in \Omega_n$ with $i \ne j$. Due to $(e^i \mid e^j) = 0$, $L(1)e^i = L(1)e^j = 0$, and Proposition 3.8(2),

$$\operatorname{tr}|_{(V_C)_\mu} o(e^i)o(e^j) = \frac{\mu d_\mu(5\mu+1) + 5c\delta_{\mu,2}}{4c(5c+22)}, \tag{4-4}$$

where c is the central charge of V_C, i.e., $c = \frac{n}{2}$ and

$$d_\mu = \dim(V_C)_\mu = \#C(2\mu) + n\delta_{\mu,2}.$$

On the other hand,

$$\text{(4-5)} \qquad \operatorname{tr}|_{(V_C)_\mu} o(e^i)o(e^j) = \tfrac{1}{4}\,\#\{\alpha \in C(2\mu) \mid i, j \in \operatorname{supp}(\alpha)\}$$

since

$$o(e^i)o(e^j)u^\alpha = \begin{cases} \frac{1}{4}u^\alpha & \text{if } i, j \in \operatorname{supp}(\alpha), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad o(e^i)o(e^j)e^k = 0$$

for $\alpha \in C$ and $k \in \Omega_n$. It follows from (4-4) and (4-5) that

$$\text{(4-6)} \qquad \#\{\alpha \in C(2\mu) \mid i, j \in \operatorname{supp}(\alpha)\} = \frac{\mu d_\mu(5\mu+1) + 5c\delta_{\mu,2}}{c(5c+22)} = \frac{4\mu(5\mu+1)\,\#C(2\mu) + 98n\delta_{\mu,2}}{n(5n+44)},$$

concluding the proof. □

One of our main results is the following.

Theorem 4.8. *Let C be a binary code and μ the minimal conformal weight of V_C. Assume that $\mu < \infty$. If $(V_C)_\mu$ forms a conformal 4-design based on V_C^0, then C is equivalent to $\{(0^1), (1^1)\}$, $\widehat{\mathcal{H}}_3$, $\mathcal{E}_8$, $E(\mathcal{H}_4)$, $\mathcal{H}_4$, or $\widehat{\mathcal{H}}_4$.*

Proof. Let n be the length of C. Note that μ must be ∞ if $\mu > 2$, because $(V_C)_2 = (V_\omega)_2$ implies that $V_C = L\big(\frac{1}{2}, 0\big)$ by the construction of code SVOAs. Hence our assumption implies $\mu \le 2$.

Recall that the Ising vectors e^i are of σ-type. We see from Corollary 3.11 that the central charge is uniquely determined by μ if $\mu \in \big\{\frac{1}{2}, 1, \frac{3}{2}\big\}$. In case $\mu = \frac{1}{2}$, the central charge is $\frac{1}{2}$, i.e., $n = 1$. Moreover the minimum weight of C is 1 because $(V_C)_{\frac{1}{2}} = \operatorname{Span}_{\mathbb{C}}\{u^\alpha \mid \alpha \in C(1)\}$. Hence C must be $\{(0^1), (1^1)\}$. It follows from Lemma 4.7 that $(\Omega_n, C(2\mu))$ forms a 2-$(n, 2\mu, \nu)$-design, where

$$\nu = \frac{4\mu(5\mu+1)\,\#C(2\mu) + 98n\delta_{\mu,2}}{n(5n+44)},$$

if $\mu \in \big\{1, \frac{3}{2}, 2\big\}$. Also, the length of C is 8 if $\mu = 1$, and 15 if $\mu = \frac{3}{2}$ by Corollary 3.11. For $\mu = 1$, $C(2)$ is equal to $\mathbb{F}_2^8(2)$ because $(\Omega_8, C(2))$ forms a 2-$(8, 2, \#C(2)/28)$-design. Hence $\langle C(2)\rangle_{\mathbb{F}_2}$ is equivalent to $\mathcal{E}_8$, and so is C because the minimum weight of C is 2. If $\mu = \frac{3}{2}$, then $(\Omega_{15}, C(3))$ forms a 2-$(15, 3, \#C(3)/35)$-design. More precisely, $(\Omega_{15}, C(3))$ forms a 2-$(15, 3, 1)$-design because the minimum weight of C is 3. Then it follows from Proposition 4.3 that $\langle C(3)\rangle_{\mathbb{F}_2}$ is equivalent to $\mathcal{H}_4$, and hence we have $C \cong \mathcal{H}_4$ by Lemma 4.2. In case $\mu = 2$, a list of possible pairs of the central charge and $\dim(V_C)_2$ has been obtained in [Matsuo 2001, Section 3.2, Table 3.2] since the central charge is a half-integer. Using this list, we obtain another list of possible pairs n and $\#C(4)$ since $\dim(V_C)_2 = n + \#C(4)$. The two lists are given as follows:

c	$\dim(V_C)_2$	c	$\dim(V_C)_2$
4	22	$\frac{19}{2}$	418
$\frac{15}{2}$	120	10	685
8	156	$\frac{21}{2}$	1491

$\Longrightarrow$

n	$\#C(4)$	n	$\#C(4)$
8	14	19	399
15	105	20	665
16	140	21	1470

However, $(n, \#C(4))$ cannot be $(19, 399), (20, 665), (21, 1470)$ because these pairs do not satisfy the inequality in Lemma 4.4. By using Lemma 4.4 again, $(\Omega_8, C(4))$ (resp., $(\Omega_{16}, C(4))$) forms a 3-$(8, 4, 1)$- design (resp., 3-$(16, 4, 1)$-design). Hence it follows from Proposition 4.3 that $\langle C(4)\rangle_{\mathbb{F}_2}$ is equivalent to $\widehat{\mathcal{H}}_3$ (resp., $\widehat{\mathcal{H}}_4$). Since the minimum weight of C is 4, C must be $\widehat{\mathcal{H}}_3$ (resp., $\widehat{\mathcal{H}}_4$) by Lemma 4.1. Also, by (4-6) the pair $(\Omega_{15}, C(4))$ forms a 2-$(15, 4, 6)$-design if $(n, \#C(4)) = (15, 105)$. We see from Lemma 4.5 that $\langle C(4)\rangle_{\mathbb{F}_2}$ is equivalent to $E(\mathcal{H}_4)$. Hence $C \cong E(\mathcal{H}_4)$ by Lemma 4.2. This finishes the proof of the theorem. $\square$

Remark 4.9. The $\mu = \frac{1}{2}$ case in Theorem 4.8 has been obtained in Theorem 3.12. Nevertheless, we provided a second proof, because this method is easier than the method of Theorem 3.12 when we consider only code SVOAs.

Remark 4.10. It is known that $V_{\mathcal{E}_8}$ is isomorphic to the lattice VOA V_{D_4} associated to the root lattice of D_4 type; see [Dong et al. 1998]. It was proved in [Tuite 2009, Theorem 2.8] that a VOA with minimal conformal weight 1 whose 4th Casimir element belongs to V_ω is isomorphic to one of the level 1 affine VOAs associated to the Deligne exceptional series of simple Lie algebras. Thanks to Theorem 3.5, we see that this classification, which contains V_{D_4}, can be obtained under the condition that V_1 forms a conformal 4-design. In fact, V_{D_4} is the only VOA in the classification by [op. cit.] which is a code SVOA.

5. Code SVOAs of class $\mathcal{S}^5$

In this section, we show that the code SVOAs associated to the codes in Theorem 4.8 are of class $\mathcal{S}^5$. In particular, their minimal conformal weight spaces form conformal 5-designs.

SVOAs of class $\mathcal{S}^n$. The notion of SVOAs of class $\mathcal{S}^n$ is an analogue of the notion of VOAs of class $\mathcal{S}^n$ introduced by Matsuo.

Definition 5.1 [Matsuo 2001, Definition 1.1]. An SVOA $V = V^0 \oplus V^1$ is said to be *of class* $\mathcal{S}^n$ if $(V^0)^{\mathrm{Aut}(V)}$ coincides with V_ω up to degree n subspace, i.e.,

$$(V^0)_m^{\mathrm{Aut}(V)} = (V_\omega)_m \quad \text{for} \quad 0 \le m \le n.$$

Clearly, the definition above is the ordinary definition in [loc. cit.] when V is a VOA. Note that the fixed point subspace of V^1 is always 0 since an SVOA has an involution which is the identity on the even part and acts as -1 on the odd part.

Proposition 5.2 [Hashikawa and Shimakura 2016, Proposition 2.12]. *Let U be a VOA and W a sub-VOA of U with the same Virasoro element ω. Assume that U is completely reducible as a V_ω-module. If $W_n = (V_\omega)_n$, then $W_{n-1} = (V_\omega)_{n-1}$. In particular, an SVOA V is of class $\mathcal{S}^n$ if V^0 is completely reducible as a V_ω-module and $(V^0)_n^{\mathrm{Aut}(V)} = (V_\omega)_n$.*

The following lemma holds.

Lemma 5.3. *Let $V = V^0 \oplus V^1$ be an SVOA of class $\mathcal{S}^n$. Then*:

(1) *The even part V^0 is also of class $\mathcal{S}^n$.*

(2) *The minimal conformal weight space of V forms a conformal n-design based on V^0.*

Proof. Since $\mathrm{Aut}(V)$ preserves V^0, there exists a group homomorphism

$$\varphi : \mathrm{Aut}(V) \to \mathrm{Aut}(V^0), \quad g \mapsto g|_{V^0}.$$

Then we have $\mathrm{Aut}(V)/\ker\varphi \cong \mathrm{Im}\,\varphi \subset \mathrm{Aut}(V^0)$. Hence (1) is proved because $(V^0)_n^{\mathrm{Aut}(V)}$ contains $(V^0)_n^{\mathrm{Aut}(V^0)}$. Also, since $\kappa_\mu^n \in (V^0)_n^{\mathrm{Aut}(V)} = (V_\omega)_n$, we obtain (2) by Theorem 3.5. □

Automorphism groups of code SVOAs. The symmetric group S_n of degree n acts on $\mathbb{F}_2^n$ by $\sigma(x_1, \dots, x_n) := (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$ for $\sigma \in S_n$ and $(x_1, \dots, x_n) \in \mathbb{F}_2^n$. Let C be a binary code of length n. An element $\sigma \in S_n$ is called an *automorphism* of C if $\sigma(C) = C$. Let $\mathrm{Aut}(C)$ denote the group of automorphisms of C. Every $\sigma \in \mathrm{Aut}(C)$ induces an automorphism $\tilde{\sigma}$ of V_C [Miyamoto 1996a, Section 5]. We call $\tilde{\sigma}$ a *lift* of σ. In particular, $\tilde{\sigma}$ acts as a permutation on $V^{(0^n)}$, that is,

$$\tilde{\sigma}(v^1 \otimes \cdots \otimes v^n) = v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(n)} \quad \text{for } v^1 \otimes \cdots \otimes v^n \in V^{(0^n)}.$$

Set

$$t^\alpha := \frac{1}{8}\sum_{i=1}^{8} e^i + \frac{1}{8}\sum_{\beta \in \widehat{\mathcal{H}}_3(4)} (-1)^{\mathrm{wt}(\beta * \alpha)} u^\beta \in V_{\widehat{\mathcal{H}}_3} \tag{5-1}$$

for $\alpha \in \mathbb{F}_2^8$. It is known that t^α is an Ising vector of σ-type of $V_{\widehat{\mathcal{H}}_3}$; see [op. cit.]. Set $\nu_i := (0^{i-1}1\,0^{n-i}) \in \mathbb{F}_2^n$.

Proposition 5.4 [Matsuo and Matsuo 2000, Proposition 2.4.1; Miyamoto 1999, Lemma 2.3]. *The Hamming code VOA $V_{\widehat{\mathcal{H}}_3}$ has exactly three Ising frames*:

$$I_0 := \{e^i \mid 1 \le i \le 8\}, \quad I_1 := \{t^{\nu_i} \mid 1 \le i \le 8\}, \quad \textit{and} \quad I_2 := \{t^{\nu_1+\nu_i} \mid 1 \le i \le 8\}.$$

Moreover, *if $f \in I_a$, then $\sigma_f(I_b) = I_c$ if $\{a,b,c\} = \{0,1,2\}$, where σ_f is the involution defined in* (2-2).

Let C be a binary code. Set

$$\mathcal{D}(C) := \{D \subset C \mid D \cong \widehat{\mathcal{H}}_3 \text{ and } \#(\mathrm{supp}(\alpha) \cap \mathrm{supp}(D)) \in 2\mathbb{Z} \text{ for all } \alpha \in C\},$$

where

$$\mathrm{supp}(D) := \bigcup_{d \in D} \mathrm{supp}(d).$$

Let $I(V_C)$ denote the set of all Ising vectors of σ-type of V_C.

Proposition 5.5 [Lam et al. 2007, Proposition 3.8, Lemma 3.10]. *Let C be a binary code of length n whose minimum weight is at least* 3, *and $f \in I(V_C)$. If $f \notin \{e^i\}_{i=1}^n$, then there exists $D \in \mathcal{D}(C)$ such that $f \in V_D \subset V_C$ and f is of the form* (5-1) *in V_D. Also*, *if $f \in V_D \subset V_C$ is an Ising vector of σ-type for $D \in \mathcal{D}(C)$, then $f \in I(V_C)$.*

The following proposition for the VOA case has been obtained in [Lam et al. 2007, Proposition 3.13]. Using the same argument, one can also show the SVOA case.

Proposition 5.6. *Let C be a binary code whose minimum weight is at least* 3. *Then* $\mathrm{Aut}(V_C)$ *is generated by $\{\sigma_f \mid f \in I(V_C)\}$ and the lift of* $\mathrm{Aut}(C)$.

Examples of code SVOAs of class $\mathcal{S}^5$. The SVOA $L\left(\frac{1}{2},0\right) \oplus L\left(\frac{1}{2},\frac{1}{2}\right)$ is clearly of class $\mathcal{S}^\infty$. Note that $V_{\mathcal{E}_8}$ is isomorphic to the lattice VOA V_{D_4}, and $V_{\widehat{\mathcal{H}}_3}$ and $V_{\widehat{\mathcal{H}}_4}$ are isomorphic to the lattice-type VOAs $V^+_{\sqrt{2}D_4}$ and $V^+_{\sqrt{2}E_8}$, respectively; see [Dong et al. 1998; Lam et al. 2007]. It was shown in [Maruoka et al. 2016; Hashikawa and Shimakura 2016] that

$$V_{D_4}, \quad V^+_{\sqrt{2}D_4}, \quad \text{and} \quad V^+_{\sqrt{2}E_8}$$

are of class $\mathcal{S}^5$. Therefore the code VOAs are also of class $\mathcal{S}^5$. By Lemma 5.3(2), their minimal conformal weight spaces are conformal 5-designs. Hence we show that the remaining code SVOAs $V_{E(\mathcal{H}_4)}$ and $V_{\mathcal{H}_4}$ are also of class $\mathcal{S}^5$.

Note that $\widehat{\mathcal{H}}_3$ and $\mathcal{H}_4$ are generated by the rows of the following matrices.

$$\text{(5-2)} \qquad \widehat{\mathcal{H}}_3 : \begin{bmatrix} 0000 & 1111 \\ 1111 & 0000 \\ 0011 & 0011 \\ 0101 & 0101 \end{bmatrix}, \qquad \mathcal{H}_4 : \begin{bmatrix} 0001 & 0001 & 0001 & 000 \\ 0010 & 0010 & 0010 & 001 \\ 0100 & 0100 & 0100 & 010 \\ 1000 & 1000 & 1000 & 100 \\ 0101 & 0000 & 0101 & 000 \\ 1010 & 0000 & 1010 & 000 \\ 1100 & 0000 & 1100 & 000 \\ 0000 & 1111 & 0000 & 000 \\ 1111 & 0000 & 0000 & 000 \\ 0011 & 0011 & 0000 & 000 \\ 0101 & 0101 & 0000 & 000 \end{bmatrix}.$$

It is easily seen from (5-2) that $\mathcal{D}(\mathcal{H}_4) \neq \varnothing$. Also, it is known that $\mathrm{Aut}(\mathcal{H}_4)$ acts doubly transitively on Ω_{15} [MacWilliams and Sloane 1977, Chapter 13, Theorem 9.24, and Problem (9)].

Theorem 5.7. *The code SVOAs $V_{\mathcal{H}_4}$ and $V_{E(\mathcal{H}_4)}$ are of class $\mathcal{S}^5$.*

Proof. Obviously, $V_{E(\mathcal{H}_4)}$ is the even part of $V_{\mathcal{H}_4}$. Now by Proposition 5.2 and Lemma 5.3(1), it is sufficient to show that

$$(V^0_{\mathcal{H}_4})_5^{\mathrm{Aut}(V_{\mathcal{H}_4})} = (V_\omega)_5.$$

A basis of $(V_\omega)_5$ is given by $\{L(-5)\mathbf{1}, L(-3)L(-2)\mathbf{1}\}$ because the central charge of $V_{\mathcal{H}_4}$ is neither 0 nor $-\frac{22}{5}$ (see Section 3.2). Note that for $n \in \mathbb{Z}$,

$$L(n) = \sum_{i=1}^{15} L^{e^i}(n), \quad \text{where } L^{e^i}(n) = e^i_{(n+1)} \quad \text{for } 1 \le i \le 15.$$

Let P be the subgroup of $\mathrm{Aut}(V_C)$ generated by $\{\sigma_{e^i} \mid 1 \le i \le 15\}$. Since σ_{e^i} acts as $(-1)^{\mathrm{wt}(\alpha * \nu_i)}$ on V^α for $\alpha \in C$, the fixed point subspace of P in $(V_C)_5$ is

$$V_5^{(0^{15})} = \big\langle L^{e^i}(-5)\mathbf{1},\ L^{e^i}(-3)\,L^{e^j}(-2)\mathbf{1} \bigm| 1 \le i, j \le 15 \big\rangle_{\mathbb{C}}.$$

Set $X := \sum_{i=1}^{15} L^{e^i}(-3)\,L^{e^i}(-2)\mathbf{1}$. Then

$$L(-3)L(-2)\mathbf{1} = X + \sum_{1 \le i \neq j \le 15} L^{e^i}(-3)\,L^{e^j}(-2)\mathbf{1}.$$

The double transitivity of $\mathrm{Aut}(\mathcal{H}_4)$ gives

$$\begin{aligned}(V^{(0^{15})})_5^{\mathrm{Aut}(\mathcal{H}_4)} &= \Big\langle L(-5)\mathbf{1}, X, \sum_{1 \le i \neq j \le 15} L^{e^i}(-3)\,L^{e^j}(-2)\mathbf{1} \Big\rangle_{\mathbb{C}} \\ &= \langle L(-5)\mathbf{1}, L(-3)\,L(-2)\mathbf{1}, X \rangle_{\mathbb{C}}.\end{aligned}$$

Hence we also have $(V_{\mathcal{H}_4}^0)_5^{\mathrm{Aut}(V_{\mathcal{H}_4})} \subset (V_\omega)_5 \oplus \langle X \rangle_{\mathbb{C}}$ because P and the lift of $\mathrm{Aut}(\mathcal{H}_4)$ are subgroups of $\mathrm{Aut}(V_{\mathcal{H}_4})$. We show $X \notin (V_{\mathcal{H}_4}^0)_5^{\mathrm{Aut}(V_{\mathcal{H}_4})}$. Let $D \in \mathcal{D}(\mathcal{H}_4)$ such that $\mathrm{supp}(D) = \{1, \dots, 8\}$. By (5-2), we can take such a subcode. Let $\{f^i\}_{i=1}^8$ and $\{g^i\}_{i=1}^8$ be distinct Ising frames of V_D except for $\{e^i\}_{i=1}^8$ (see Proposition 5.4). We see from Proposition 5.5 that f^i and g^i are also Ising vectors of σ-type of $V_{\mathcal{H}_4}$. By Proposition 5.4,

$$\sigma_{f^1}(X) = \sum_{i=1}^{8} L^{g^i}(-3)\, L^{g^i}(-2)\mathbf{1} + \sum_{i=9}^{15} L^{e^i}(-3)\, L^{e^i}(-2)\mathbf{1},$$

where $L^{g^i}(n) = g^i_{(n+1)}$ for $1 \le i \le 8$ and $n \in \mathbb{Z}$. By direct computation, $\sigma_{f^1}(X) \neq X$. Therefore the assertion holds. $\square$

Remark 5.8. As already mentioned before, the cases of $V_{\widehat{\mathcal{H}}_3}$ and $V_{\widehat{\mathcal{H}}_4}$ have already been obtained in [Hashikawa and Shimakura 2016]. By using the same method as in Theorem 5.7, one can also show these cases.

In conclusion, we obtain the following.

Theorem 5.9. *The code SVOAs associated to the codes in Theorem 4.8 are of class $\mathcal{S}^5$.*

As a corollary of Theorems 4.8 and 5.9, the following holds.

Corollary 5.10. *Let C be a binary code. Then the minimal conformal weight space of V_C is a conformal 4-design based on V_C^0 if and only if V_C is of class $\mathcal{S}^5$.*

Remark 5.11. It is known that the code VOA $V_{E(\mathcal{H}_4)}$ is isomorphic to the commutant subalgebra of an Ising vector in the VOA $V_{\sqrt{2}E_8}^+$ [Lam et al. 2007, Section 4 and Corollary 5.6].

Acknowledgments

The author would like to thank Hiroki Shimakura for discussions and many valuable suggestions. He also thanks Hiroshi Yamauchi for helpful comments and Yuta Watanabe for discussions of coding theory and combinatorial design theory.

References

[Assmus and Key 1992] E. F. Assmus, Jr. and J. D. Key, *Designs and their codes*, Cambridge Tracts in Mathematics **103**, Cambridge University Press, 1992. MR 1192126 Zbl 0762.05001

[Assmus and Mattson 1969] E. F. Assmus, Jr. and H. F. Mattson, Jr., “New 5-designs”, *J. Combinatorial Theory* **6** (1969), 122–151. MR 0272647 Zbl 0179.02901

[Dong et al. 1994] C. Dong, G. Mason, and Y. Zhu, “Discrete series of the Virasoro algebra and the moonshine module”, pp. 295–316 in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), edited by W. J. Haboush and B. J.

Parshall, Proc. Sympos. Pure Math. **56**, Amer. Math. Soc., Providence, RI, 1994. MR 1278737 Zbl 0813.17019

[Dong et al. 1998] C. Dong, R. L. Griess, Jr., and G. Höhn, "Framed vertex operator algebras, codes and the Moonshine module", *Comm. Math. Phys.* **193**:2 (1998), 407–448. MR 1618135 Zbl 0908.17018

[Frenkel et al. 1993] I. B. Frenkel, Y.-Z. Huang, and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, vol. 104, Mem. Amer. Math. Soc. **494**, 1993. MR 1142494 Zbl 0789.17022

[Hashikawa and Shimakura 2016] T. Hashikawa and H. Shimakura, "Classification of the vertex operator algebras V_L^+ of class $\mathcal{S}^4$", *J. Algebra* **456** (2016), 151–181. MR 3484139 Zbl 1335.17015

[Höhn 2008] G. Höhn, "Conformal designs based on vertex operator algebras", *Adv. Math.* **217**:5 (2008), 2301–2335. MR 2388095 Zbl 1157.17008

[Höhn et al. 2012] G. Höhn, C. H. Lam, and H. Yamauchi, "McKay's E_7 observation on the Baby Monster", *Int. Math. Res. Not.* **2012**:1 (2012), 166–212. MR 2874931 Zbl 1267.17033

[Kac 1998] V. Kac, *Vertex algebras for beginners*, 2nd ed., University Lecture Series **10**, American Mathematical Society, Providence, RI, 1998. MR 1651389 Zbl 0924.17023

[Kac and Raina 1987] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Adv. Ser. Math. Phys. **2**, World Scientific, Teaneck, NJ, 1987. MR 1021978 Zbl 0668.17012

[Lam et al. 2007] C. H. Lam, S. Sakuma, and H. Yamauchi, "Ising vectors and automorphism groups of commutant subalgebras related to root systems", *Math. Z.* **255**:3 (2007), 597–626. MR 2270290 Zbl 1139.17010

[Li 1994] H. S. Li, "Symmetric invariant bilinear forms on vertex operator algebras", *J. Pure Appl. Algebra* **96**:3 (1994), 279–297. MR 1303287 Zbl 0813.17020

[MacWilliams and Sloane 1977] F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes, I*, North-Holland, Amsterdam–New York–Oxford, 1977. MR 0465509 Zbl 0369.94008

[Maruoka et al. 2016] H. Maruoka, A. Matsuo, and H. Shimakura, "Classification of vertex operator algebras of class $\mathcal{S}^4$ with minimal conformal weight one", preprint, 2016. arXiv math.QA/1509.05529

[Matsuo 2001] A. Matsuo, "Norton's trace formulae for the Griess algebra of a vertex operator algebra with larger symmetry", *Comm. Math. Phys.* **224**:3 (2001), 565–591. MR 1871901

[Matsuo and Matsuo 2000] A. Matsuo and M. Matsuo, "The automorphism group of the Hamming code vertex operator algebra", *J. Algebra* **228**:1 (2000), 204–226. MR 1760962 Zbl 0978.17023

[Miyamoto 1996a] M. Miyamoto, "Binary codes and vertex operator (super)algebras", *J. Algebra* **181**:1 (1996), 207–222. MR 1382033 Zbl 0857.17026

[Miyamoto 1996b] M. Miyamoto, "Griess algebras and conformal vectors in vertex operator algebras", *J. Algebra* **179**:2 (1996), 523–548. MR 1367861 Zbl 0964.17021

[Miyamoto 1999] M. Miyamoto, "A Hamming code vertex operator algebra and construction of vertex operator algebras", *J. Algebra* **215**:2 (1999), 509–530. MR 1686204 Zbl 0929.17035

[Miyamoto 2004] M. Miyamoto, "A new construction of the Moonshine vertex operator algebra over the real number field", *Ann. of Math.* (2) **159**:2 (2004), 535–596. MR 2081435 Zbl 1133.17017

[Tuite 2009] M. P. Tuite, "Exceptional vertex operator algebras and the Virasoro algebra", pp. 213–225 in *Vertex operator algebras and related areas* (Normal, IL, 2008), edited by M. Bergvelt et al., Contemp. Math. **497**, Amer. Math. Soc., Providence, RI, 2009. MR 2568410 Zbl 1225.17034

[Tuite and Van 2014] M. P. Tuite and H. D. Van, "On exceptional vertex operator (super)algebras", pp. 351–384 in *Developments and retrospectives in Lie theory*, edited by G. Mason et al., Dev. Math. **38**, Springer, 2014. MR 3308791 Zbl 06463592

[Venkov 2001] B. Venkov, "Réseaux et designs sphériques", pp. 10–86 in *Réseaux euclidiens, designs sphériques et formes modulaires*, edited by J. Martinet, Monogr. Enseign. Math. **37**, Enseignement Math., Geneva, 2001. MR 1878745 Zbl 1139.11320

[Yamauchi 2004] H. Yamauchi, *A theory of simple current extensions of vertex operator algebras and applications to the Moonshine vertex operator algebra*, Ph.D. thesis, University of Tsukuba, 2004, available at http://www.math.twcu.ac.jp/~yamauchi/math/phd/myphd.pdf.

[Yamauchi 2005] H. Yamauchi, "2A-orbifold construction and the baby-monster vertex operator superalgebra", *J. Algebra* **284**:2 (2005), 645–668. MR 2114573 Zbl 1147.17314

[Yamauchi 2014] H. Yamauchi, "Extended Griess algebras and Matsuo–Norton trace formulae", pp. 75–107 in *Conformal field theory, automorphic forms and related topics* (Heidelberg, Germany, 2011), edited by W. Kohnen and R. Weissauer, Springer, Berlin, 2014. Zbl 06489870

Received November 15, 2015.

TOMONORI HASHIKAWA
TOHOKU UNIVERSITY
#113 49-154
AOBA-KU SENDAI-SHI MIYAGI-KEN
SENDAI 980-0866
JAPAN

t.hashikawa@ims.is.tohoku.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.147

COACTION FUNCTORS

S. KALISZEWSKI, MAGNUS B. LANDSTAD AND JOHN QUIGG

A certain type of functor on a category of coactions of a locally compact group on C^*-algebras is introduced and studied. These functors are intended to help in the study of the crossed-product functors that have been recently introduced in relation to the Baum–Connes conjecture. The most important coaction functors are the ones induced by large ideals of the Fourier–Stieltjes algebra. It is left as an open problem whether the "minimal exact and Morita compatible crossed-product functor" is induced by a large ideal.

1. Introduction

In [Baum et al. 2016], with an eye toward expanding the class of locally compact groups G for which the Baum–Connes conjecture holds, the authors study "crossed-product functors" that take an action of G on a C^*-algebra and produce an "exotic crossed product" between the full and reduced ones, in a functorial manner.

In [KLQ 2013], inspired by [Brown and Guentner 2013], we studied certain quotients of $C^*(G)$ that lie "above" $C^*_r(G)$ — namely those that carry a quotient coaction. We characterized these intermediate (which we now call "large") quotients as those for which the annihilator E, in the Fourier–Stieltjes algebra $B(G)$, of the kernel of the quotient map is a G-invariant weak*-closed ideal containing the reduced Fourier–Stieltjes algebra $B_r(G)$ (which we now call "large ideals" of $B(G)$). We went on to show how, if α is an action of G on a C^*-algebra B, large ideals E induce exotic crossed products $B \rtimes_{\alpha,E} G$ intermediate between the full and reduced crossed products $B \rtimes_\alpha G$ and $B \rtimes_{\alpha,r} G$. One of the reasons this interested us is the possibility of "E-crossed-product duality" for a coaction δ of G on a C^*-algebra A: namely, that the canonical surjection

$$\Phi : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$$

descends to an isomorphism

$$A \rtimes_\delta G \rtimes_{\widehat{\delta},E} G \cong A \otimes \mathcal{K}.$$

MSC2010: primary 46L55; secondary 46M15.
Keywords: crossed product, action, coaction, Fourier–Stieltjes algebra, exact sequence, Morita compatible.

Crossed-product duality

$$A \rtimes_\delta G \rtimes_{\widehat{\delta},r} G \cong A \otimes \mathcal{K}$$

for normal coactions and

$$A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \cong A \otimes \mathcal{K}$$

for maximal coactions are the extreme cases with $E = B_r(G)$ and $B(G)$, respectively. We (rashly) conjectured that every coaction satisfies E-crossed-product duality for some E, and moreover that the dual coaction on every E-crossed product $B \rtimes_{\alpha,E} G$ satisfies E-crossed-product duality.

Buss and Echterhoff [2014] disproved the first of the above conjectures and proved the second, and in [KLQ 2016] we independently proved the second conjecture. (Note: in that paper we wrote "We originally wondered whether every coaction satisfies E-crossed product duality for some E. In [KLQ 2013, Conjecture 6.12] we even conjectured that this would be true for dual coactions." This is slightly inaccurate — [KLQ 2013, Conjecture 6.14] concerns dual coactions, while Conjecture 6.12 says "Every coaction satisfies E-crossed-product duality for some E.")

In [KLQ 2016, Section 3] we showed that every large ideal E of $B(G)$ induces a transformation $(A,\delta) \mapsto (A^E, \delta^E)$ of G-coactions, where $A^E = A/A_E$ and $A_E = \ker(\mathrm{id} \otimes q_E) \circ \delta$, and where in turn

$$q_E : C^*(G) \to C^*_E(G) := C^*(G)/{}^\perp E$$

is the quotient map.

In this paper we further study this assignment $(A,\delta) \mapsto (A^E, \delta^E)$. When $(A,\delta) = (B \rtimes_\alpha G, \widehat{\alpha})$, the composition

$$(B,\alpha) \mapsto (B \rtimes_\alpha G, \widehat{\alpha}) \mapsto (B \rtimes_{\alpha,E} G, \widehat{\alpha}^E)$$

was shown to be functorial in [Buss and Echterhoff 2014, Corollary 6.5]; here we show that $(A,\delta) \mapsto (A^E, \delta^E)$ is functorial, giving an alternate proof of the Buss–Echterhoff result.

In fact, we study more general functors on the category of coactions of G, of which the functors induced by large ideals of $B(G)$ are special cases. We are most interested in the connection with the crossed-product functors of [Baum et al. 2016]. In particular, we introduce a "minimal exact and Morita compatible" coaction functor. When this functor is composed with the full-crossed-product functor for actions, the result is a crossed-product functor in the sense of [loc. cit.]. We briefly discuss various possibilities for how these functors are related: for example, is the composition mentioned in the preceding sentence equal to the minimal exact and Morita compatible crossed-product functor of [loc. cit.]? Also, is the greatest lower bound of the coaction functors defined by large ideals itself defined by a large ideal?

These are just two among others that arise naturally from these considerations. Unfortunately, at this early stage we have more questions than answers.

After a short section on preliminaries, in Section 3 we define the categories we will use for our functors. In numerous previous papers, we have used "nondegenerate categories" of C^*-algebras and their equivariant counterparts. But these categories are inappropriate for the current paper, primarily due to our need for short exact sequences. Rather, here we must use "classical" categories, where the homomorphisms go between the C^*-algebras themselves, not into multiplier algebras. In order to avail ourselves of tools that have been developed for the equivariant nondegenerate categories, we include a brief summary of how the basic theory works for the classical categories. Interestingly, the crossed products are the same in both versions of the categories (see Corollaries 3.9 and 3.13).

In Section 4 we define *coaction functors*, which are a special type of functor on the classical category of coactions. Composing such a coaction functor with the full-crossed-product functor on actions, we get crossed-product functors in the sense of Baum, Guentner and Willett [loc. cit.]; it remains an open problem whether every such crossed-product functor is of this form. Maximalization and normalization are examples of coaction functors, but there are lots more — for example, the functors induced by large ideals of the Fourier–Stieltjes algebra (see Section 6). In Section 4 we also define a partial ordering on coaction functors, and prove in Theorem 4.9 that the class of coaction functors is complete in the sense that every nonempty collection of them has a greatest lower bound. We also introduce the general notions of *exact* or *Morita compatible* coaction functors, and prove in Theorem 4.22 that they are preserved by greatest lower bounds. We show in Proposition 4.24 that our partial order, exactness and Morita compatibility are consistent with those of [loc. cit.].

To help prepare for the study of coaction functors associated to large ideals, in Section 5 we introduce *decreasing coaction functors*, and show how Morita compatibility takes a particularly simple form for these functors in Proposition 5.5.

In Section 6 we study the coaction functors τ_E induced by large ideals E of $B(G)$. Perhaps interestingly, maximalization is not among these functors. We show that these functors τ_E are decreasing in Proposition 6.2, and how the test for exactness simplifies significantly for them in Proposition 6.7. Moreover, τ_E is automatically Morita compatible (see Proposition 6.10). Composing maximalization followed by τ_E, we get a related functor that we call *E-ization*. We show that these functors are also Morita compatible in Theorem 6.14. Although E-ization and τ_E have similar properties, they are not naturally isomorphic functors (see Remark 6.15). The outputs of E-ization are precisely the coactions we call *E-coactions*, namely those for which *E-crossed-product duality* holds [KLQ 2016, Theorem 4.6] (see also [Buss and Echterhoff 2014, Theorem 5.1]). Theorem 6.17 shows that τ_E gives an equivalence of maximal coactions with E-coactions. We close Section 6 with some open problems that mainly concern the application of the coaction functors τ_E to the theory of [Baum et al. 2016].

Finally, the Appendix supplies a few tools that show how some properties of coactions can be more easily handled using the associated $B(G)$-module structure.

2. Preliminaries

We refer to [Echterhoff et al. 2004; 2006, Appendix A] for background material on coactions of locally compact groups on C^*-algebras, and [Echterhoff et al. 2006, Chapters 1–2] for imprimitivity bimodules and their linking algebras. Throughout, G will denote a locally compact group, and $A, B, C, \dots$ will denote C^*-algebras.

Recall from [loc. cit., Definition 1.14] that the *multiplier bimodule* of an $A-B$ imprimitivity bimodule X is defined as $M(X) = \mathcal{L}_B(B, X)$, where B is regarded as a Hilbert module over itself in the canonical way. Also recall [loc. cit., Corollary 1.13] that $M(X)$ becomes an $M(A)-M(B)$ correspondence in a natural way. The *linking algebra* of an $A-B$ imprimitivity bimodule X is

$$L(X) = \begin{pmatrix} A & X \\ \widetilde{X} & B \end{pmatrix},$$

where $\widetilde{X}$ is the *dual* $B-A$ imprimitivity bimodule. A, B and X are recovered from $L(X)$ via the *corner projections*

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M(L(X)).$$

The multiplier algebra of $L(X)$ decomposes as

$$M(L(X)) = \begin{pmatrix} M(A) & M(X) \\ M(\widetilde{X}) & M(B) \end{pmatrix}.$$

We usually omit the lower left corner of the linking algebra, writing $L(X) = \left(\begin{smallmatrix} A & X \\ * & B \end{smallmatrix}\right)$, since it takes care of itself. Also recall from [loc. cit., Lemma 1.52] (see also [Echterhoff and Raeburn 1995, Remark (2), p. 307]) that nondegenerate homomorphisms of imprimitivity bimodules correspond bijectively to nondegenerate homomorphisms of their linking algebras.

For an action (A, α) of G, we use the following notation for the (full) crossed product $A \rtimes_\alpha G$:

- $i_A = i_A^\alpha : A \to M(A \rtimes_\alpha G)$ and $i_G = i_G^\alpha : G \to M(A \rtimes_\alpha G)$ make up the universal covariant homomorphism (i_A, i_G).
- $\widehat{\alpha}$ is the dual coaction on $A \rtimes_\alpha G$.

On the other hand, for the reduced crossed product $A \rtimes_{\alpha,r} G$ we use the following notation:

- $\Lambda : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is the regular representation.
- $i_A^r = i_A^{\alpha,r} = \Lambda \circ i_A$ and $i_G^r = i_G^{\alpha,r} = \Lambda \circ i_G$ are the canonical maps into $M(A \rtimes_{\alpha,r} G)$.
- $\widehat{\alpha}^n$ is the dual coaction on $A \rtimes_{\alpha,r} G$.

We will need to work extensively with morphisms between coactions, in particular (but certainly not only) with maximalization and normalization. In the literature, the notation for these maps has not yet stabilized. Recall that a coaction (A, δ) is called *normal* if the canonical surjection

$$\Phi : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$$

factors through an isomorphism of the reduced crossed product

$$\Phi_r : A \rtimes_\delta G \rtimes_{\widehat{\delta},r} G \to A \otimes \mathcal{K}(L^2(G)),$$

and *maximal* if Φ itself is an isomorphism. One convention is, for a coaction (A, δ) of G, to write

$$q_A^m : (A^m, \delta^m) \to (A, \delta)$$

for a maximalization, and

$$q_A^n : (A, \delta) \to (A^n, \delta^n)$$

for a normalization. We will use this convention for maximalization, but we will need the letter "q" for other similar purposes, and it would be confusing to keep using it for normalization. Instead, we will use

$$\Lambda = \Lambda_A : (A, \delta) \to (A^n, \delta^n)$$

for normalization — this is supposed to remind us that for crossed products by actions the regular representation

$$\Lambda : (A \rtimes_\alpha G, \widehat{\alpha}) \to (A \rtimes_{\alpha,r} G, \widehat{\alpha}^n)$$

is a normalization.

B(G)-modules. Every coaction (A, δ) of G induces $B(G)$-module structures on both A and A^*: for $f \in B(G)$, define

$$\begin{aligned} f \cdot a &= (\mathrm{id} \otimes f) \circ \delta(a) && \text{for } a \in A, \\ (\omega \cdot f)(a) &= \omega(f \cdot a) && \text{for } \omega \in A^*,\ a \in A. \end{aligned}$$

Many properties of coactions can be handled using these module structures rather than the coactions themselves. For example (see the Appendix), letting (A,δ) and (B,ε) be coactions of G:

(1) A homomorphism $\phi : A \to B$ is $\delta - \varepsilon$ equivariant, meaning $\varepsilon \circ \phi = \overline{\phi \otimes \mathrm{id}} \circ \delta$, if and only if

$$\phi(f \cdot a) = f \cdot \phi(a) \quad \text{for all } f \in B(G),\ a \in A.$$

(2) An ideal I of A is *weakly* δ-invariant, meaning $I \subset \ker \overline{q \otimes \mathrm{id}} \circ \delta$, where $q : A \to A/I$ is the quotient map, if and only if

$$B(G) \cdot I \subset I,$$

because the proof of [KLQ 2013, Lemma 3.11] shows that

$$\ker(q \otimes \mathrm{id}) \circ \delta = \{a \in A : B(G) \cdot a \subset I\}.$$

If I is a weakly δ-invariant ideal of A, then in fact $I = \ker(q \otimes \mathrm{id}) \circ \delta$, and the quotient map q is $\delta - \delta^I$ equivariant for a unique coaction δ^I on A/I, which we call the *quotient coaction*. Since the slice map $\mathrm{id} \otimes f : M(A \otimes C^*(G)) \to M(A)$ is strictly continuous [Landstad et al. 1987, Lemma 1.5], the $B(G)$-module structure extends to $M(A)$, and moreover $m \mapsto f \cdot m$ is strictly continuous on $M(A)$ for every $f \in B(G)$.

Short exact sequences. Several times we will need the following elementary lemma.

Lemma 2.1. *Let*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\phi_1"] \arrow[d, "\iota_A"'] & B_1 \arrow[r, "\psi_1"] \arrow[d, "\iota_B"'] & C_1 \arrow[r] \arrow[d, "\iota_C"'] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\pi_A"'] & B_2 \arrow[r, "\psi_2"] \arrow[d, "\pi_B"'] & C_2 \arrow[r] \arrow[d, "\pi_C"'] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "\phi_3"] \arrow[d] & B_3 \arrow[r, "\psi_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram of C^*-algebras, where the columns and the middle row are exact. Suppose that the $\iota_\bullet$ are inclusions of ideals and the $\pi_\bullet$ are quotient maps.

Then the bottom (interesting) row is exact if and only if both

$$\phi_2(A_1) = \phi_2(A_2) \cap B_1 \tag{2-1}$$

and

$$\phi_2(A_2) + B_1 \supset \psi_2^{-1}(C_1). \tag{2-2}$$

Proof. Since $\psi_3 \circ \pi_B = \pi_C \circ \psi_2$ and ψ_B and ϕ_2 are both surjective, ψ_3 is surjective, so the bottom row is automatically exact at C_3.

Thus, the only items to consider are exactness of the bottom row at A_3 and B_3, i.e., whether ϕ_3 is injective and $\phi_3(A_3) = \ker \psi_3$.

The map ϕ_3 is injective if and only if $\ker \pi_A = \ker \pi_B \circ \phi_2$, which, since ϕ_2 is injective, is equivalent to (2-1).

Since $\psi_2 \circ \phi_2 = 0$ and π_A is surjective, $\psi_3 \circ \phi_3 = 0$, so $\phi_3(A_3) \subset \ker \psi_3$ automatically. Since π_B is surjective, $\phi_3(A_3) \supset \ker \psi_3$ if and only if

$$\pi_B^{-1}(\phi_3(A_3) \supset \pi_B^{-1}(\ker \psi_3).$$

Since $\pi_B^{-1}(\phi_3(A_3))$ consists of all $b \in B_2$ for which

$$\pi_B(a) \in \phi_3(A_3) = \phi_3(\pi_A(A_2)) = \pi_B(\phi_2(A_2)),$$

equivalently for which

$$b \in \phi_2(A_2) + B_1,$$

we see that

$$\pi_B^{-1}(\phi_3(A_3)) = \phi_2(A_2) + B_1.$$

On the other hand,

$$\pi_B^{-1}(\ker \psi_3) = \ker \psi_3 \circ \pi_B = \ker \pi_C \circ \psi_2 = (\psi_2)^{-1}(C_1).$$

Thus, the bottom row is exact at B_3 if and only if (2-2) holds. □

Remark 2.2. In this lemma, we were interested in characterizing exactness of the bottom (interesting) row of the diagram. Lemma 3.5 of [Baum et al. 2016] does this in terms of subsets of the spectrum $\widehat{B}_2$, which could just as well be done with subsets of Prim B_2, but we instead did it directly in terms of ideals of B_2. Note that, although the $\iota_\bullet$ were inclusion maps of ideals and the $\pi_\bullet$ were the associated quotient maps, for technical reasons we did *not* make the analogous assumptions regarding the middle row.

There is a standard characterization from homological algebra, namely that the bottom row is exact if and only if the top row is — this is sometimes called the nine lemma, and is an easy consequence of the snake lemma. However, this doesn't seem to lead to a simplification of the proof.

3. The categories and functors

We want to study coaction functors. Among other things, we want to apply the theory we've developed in [KLQ 2013; 2016] concerning large ideals E of $B(G)$. On the other hand, it is important to us in this paper for our theory to be consistent with the crossed-product functors of [Baum et al. 2016]. In particular, we want to be able to apply our coaction functors to short exact sequences.

But now a subtlety arises: some of us working in noncommutative duality for C^*-dynamical systems have grown accustomed to doing everything in the "nondegenerate" categories, where the morphisms are nondegenerate homomorphisms into multiplier algebras (possibly preserving some extra structure). But the maps in a short exact sequence

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

are not of this type, most importantly ϕ. So, we must replace the nondegenerate category by something else. We can't just allow arbitrary homomorphisms into multiplier algebras, because they wouldn't be composable. We can't require "extendible homomorphisms" into multiplier algebras, because the inclusion of an ideal won't typically have that property. Thus, it seems we need to use the "classical category" of homomorphisms between the C^*-algebras, not into multiplier algebras. This is what [Baum et al. 2016] uses, so presumably our best chance of seamlessly connecting with their work is to do likewise.

Since most of the existing categorical theory of coactions uses nondegenerate categories, it behooves us to establish the basic theory we need in the context of the classical categories, which we do below.

One drawback to this is that the covariant homomorphisms and crossed products can't be constructed using morphisms from the classical C^*-category — so, it seems we have to abandon some of the appealing features of the nondegenerate category.

Definition 3.1. A morphism $\phi : A \to B$ in the *classical category* $\mathbf{C}^*$ *of C^*-algebras* is a *-homomorphism from A to B in the usual sense (no multipliers).

Definition 3.2. A morphism $\phi : (A, \delta) \to (B, \varepsilon)$ in the *classical category* **Coact** *of coactions* is a morphism $\phi : A \to B$ in $\mathbf{C}^*$ such that the diagram

$$\begin{CD}
A @>{\delta}>> \widetilde{M}(A \otimes C^*(G)) \\
@V{\phi}VV @VV{\overline{\phi\otimes\mathrm{id}}}V \\
B @>>{\varepsilon}> \widetilde{M}(B \otimes C^*(G))
\end{CD}$$

commutes, and we call ϕ a $\delta - \varepsilon$ *equivariant* homomorphism.

To make sense of the above commuting diagram, recall that for any C^*-algebra C,

$$\widetilde{M}(A \otimes C) = \{m \in M(A \otimes C) : m(1 \otimes C) \cup (1 \otimes C)m \subset A \otimes C\},$$

and that for any homomorphism $\phi : A \to B$ there is a canonical extension to a homomorphism

$$\overline{\phi \otimes \mathrm{id}} : \widetilde{M}(A \otimes C) \to \widetilde{M}(B \otimes C),$$

by [Echterhoff et al. 2006, Proposition A.6]. It is completely routine to verify that $\mathbf{C}^*$ and **Coact** are categories, i.e., there are identity morphisms and there is an associative composition.

Remark 3.3. Thus, a coaction is not itself a morphism in the classical category; this will cause no trouble.

To work in the classical category of coactions, we need to be just a little bit careful with covariant homomorphisms and crossed products. We write w_G for the unitary element of $M(C_0(G) \otimes C^*(G)) = C_b(G, M^\beta(C^*(G)))$ defined by $w_G(s) = s$, where we have identified G with its canonical image in $M(C^*(G))$, and where the superscript β means that we use the strict topology on $M(C^*(G))$.

Definition 3.4. A *degenerate covariant homomorphism* of a coaction (A, δ) to a C^*-algebra B is a pair (π, μ), where $\pi : A \to M(B)$ and $\mu : C_0(G) \to M(B)$ are homomorphisms such that μ is nondegenerate and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\delta\quad} & \widetilde{M}(A \otimes C^*(G)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi\otimes\mathrm{id}}} \\ M(B) & \xrightarrow[\mathrm{Ad}(\mu\otimes\mathrm{id})(w_G)\circ(\cdot\otimes 1)]{} & M(B \otimes C^*(G)) \end{array}$$

commutes, where the bottom arrow is the map $b \mapsto \mathrm{Ad}(\mu \otimes \mathrm{id})(w_G)(b \otimes 1)$. If $\pi : A \to M(B)$ happens to be nondegenerate, we sometimes refer to (π, μ) as a *nondegenerate covariant homomorphism* for clarity.

Remark 3.5. The homomorphisms π and μ are not morphisms in the classical category $\mathbf{C}^*$; this will cause no trouble, but does present a danger of confusion.

Remark 3.6. Thus, in our new definition of degenerate covariant homomorphism, we include all the usual nondegenerate covariant homomorphisms, and we add more, allowing the homomorphism π of A (but not the homomorphism μ of $C_0(G)$) to be degenerate.

Remark 3.7. We wrote $M(B \otimes C^*(G))$, rather than the relative multiplier algebra $\widetilde{M}(B \otimes C^*(G))$, in the above diagram, because $\overline{\pi \otimes \mathrm{id}}$ will in general not map $\widetilde{M}(A \otimes C^*(G))$ into $\widetilde{M}(B \otimes C^*(G))$ since π does not map A into B.

Although we have apparently enlarged the supply of covariant homomorphisms, in some sense we have not. In Lemma 3.8 below we use the following terminology: given C^*-algebras $A \subset B$, the *idealizer* of A in B is $\{b \in B : bA \cup Ab \subset A\}$.

Lemma 3.8. *Let (π, μ) be a degenerate covariant homomorphism of (A, δ) to B, as in Definition 3.4. Put*

$$B_0 = \overline{\operatorname{span}}\{\pi(A)\mu(C_0(G))\}.$$

Then:

(1) $B_0 = \overline{\operatorname{span}}\{\mu(C_0(G))\pi(A)\}$.

(2) *B_0 is a C^*-subalgebra of $M(B)$.*

(3) *π and μ map into the idealizer D of B_0 in $M(B)$. Let $\rho : D \to M(B_0)$ be the homomorphism given by*

$$\rho(m)b_0 = mb_0 \quad \text{for } m \in D \subset M(B),\ b_0 \in B_0 \subset B,$$

and let $\pi_0 = \rho \circ \pi : A \to M(B_0)$ and $\mu_0 = \rho \circ \mu : C_0(G) \to M(B_0)$. Then (π_0, μ_0) is a nondegenerate covariant homomorphism of (A, δ) to B_0.

(4) *For all $a \in A$ and $f \in C_0(G)$ we have*

$$\pi_0(a)\mu_0(f) = \pi(a)\mu(f) \in B_0.$$

Proof. For (1), by symmetry it suffices to show that for $a \in A$ and $f \in C_0(G)$ we have

$$\mu(f)\pi(a) \in B_0,$$

and we use an old trick from [Landstad et al. 1987, proof of Lemma 2.5]: since $A(G)$ is dense in $C_0(G)$, it suffices to take $f \in A(G)$, and then since $A(G)$ is a nondegenerate $C^*(G)$-module via $\langle y, g\cdot x\rangle = \langle xy, g\rangle$ for $x, y \in C^*(G)$, $g \in A(G)$, by Cohen's factorization theorem we can write $f = g \cdot x$. Then the following approximation suffices:

$$\begin{aligned}
\mu(f)\pi(a) &= \langle(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes f\rangle\pi(a)\\
&= \langle(\mu \otimes \mathrm{id})(w_G)(\pi(a) \otimes 1), \mathrm{id} \otimes f\rangle\\
&= \langle\overline{\pi \otimes \mathrm{id}}(\delta(a))(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes g \cdot x\rangle\\
&= \langle(\pi \otimes \mathrm{id})((1 \otimes x)\delta(a))(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes g\rangle\\
&\approx \sum_i \langle(\pi \otimes \mathrm{id})(a_i \otimes x_i)(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes g\rangle\\
&\qquad\qquad \text{for finitely many } a_i \in A,\ x_i \in C^*(G)\\
&= \sum_i \langle(\pi(a_i) \otimes x_i)(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes g\rangle
\end{aligned}$$

$$= \sum_i \pi(a_i)\langle(\mu \otimes \mathrm{id})(w_G), \mathrm{id} \otimes g \cdot x_i\rangle$$
$$= \sum_i \pi(a_i)\mu(g \cdot x_i).$$

From (1) it follows that B_0 is a $*$-subalgebra of B, giving (2).

(3) It is now clear that

$$\pi(A)B_0 \cup B_0\pi(A) \subset B_0,$$

and similarly for μ, so both π and μ map into D. It is also clear that π_0 and μ_0 map nondegenerately into $M(B_0)$. The covariance property for (π_0, μ_0) follows quickly from that of (π, μ): if $a \in A$ then

$$\begin{aligned}\mathrm{Ad}(\mu_0 \otimes \mathrm{id})(w_G)(\pi_0(a) \otimes 1) &= (\rho \otimes \mathrm{id}) \circ \mathrm{Ad}(\mu \otimes \mathrm{id})(w_G)(\pi(a) \otimes 1)\\ &= (\rho \otimes \mathrm{id}) \circ \overline{\pi \otimes \mathrm{id}} \circ \delta(a)\\ &= \overline{\pi_0 \otimes \mathrm{id}} \circ \delta(a).\end{aligned}$$

(4) This follows from the construction. □

Let $(A \rtimes_\delta G, j_A, j_G)$ be the usual crossed product of the coaction (A, δ), i.e., (j_A, j_G) is a nondegenerate covariant homomorphism of (A, δ) to $A \rtimes_\delta G$ that is universal in the sense that if (π, μ) is any nondegenerate covariant homomorphism of (A, δ) to a C^*-algebra B, then there is a unique homomorphism $\pi \times \mu : A \rtimes_\delta G \to M(B)$ such that

$$\pi \times \mu \circ j_A = \pi,$$
$$\pi \times \mu \circ j_G = \mu,$$

equivalently such that

$$\pi \times \mu\big(j_A(a) j_G(f)\big) = \pi(a)\mu(f) \quad \text{for all } a \in A,\ f \in C_0(G). \tag{3-1}$$

Corollary 3.9. *With the above notation, (j_A, j_G) is also universal among degenerate covariant homomorphisms (in the sense of Definition 3.4). More precisely: for any degenerate covariant homomorphism (π, μ) of (A, δ) to B as in Definition 3.4, there is a unique homomorphism $\pi \times \mu : A \rtimes_\delta G \to M(B)$ satisfying* (3-1).

Proof. Let π_0, μ_0, B_0 be as in the preceding lemma. Then we have a unique homomorphism $\pi_0 \times \mu_0 : A \rtimes_\delta G \to M(B_0)$ such that

$$\pi_0 \times \mu_0\big(j_A(a) j_G(f)\big) = \pi_0(a)\mu_0(f) \quad \text{for all } a \in A,\ f \in C_0(G).$$

By construction we have $\pi \times \mu(A \rtimes_\delta G) \subset B_0$. Since $B_0 \subset M(B)$, we can regard π_0 as a homomorphism $\pi : A \to M(B)$, and similarly for $\mu : C_0(G) \to M(B)$. Then we regard $\pi_0 \times \mu_0$ as a homomorphism $\pi \times \mu : A \rtimes_\delta G \to M(B)$, and trivially (3-1) holds. Since $\pi_0(a)\mu_0(f) = \pi(a)\mu(f) \in B_0$ for all $a \in A$, $f \in C_0(G)$, the homomorphism $\pi \times \mu$ is unique. □

Similarly, and more easily, for actions:

Definition 3.10. A morphism $\phi : (A, \alpha) \to (B, \beta)$ in the *classical category* **Act** *of actions* is a morphism $\phi : A \to B$ in $\mathbf{C}^*$ such that

$$\beta_s \circ \phi = \phi \circ \alpha_s \quad \text{for all } s \in G.$$

Definition 3.11. A *degenerate covariant homomorphism* of an action (A, α) to a C^*-algebra is a pair (π, u), where $\pi : A \to M(B)$ is a homomorphism and $u : G \to M(B)$ is a strictly continuous unitary homomorphism such that

$$\pi \circ \alpha_s = \operatorname{Ad} u_s \circ \pi \quad \text{for all } s \in G.$$

We call (π, u) *nondegenerate* if $\pi : A \to M(B)$ is.

Lemma 3.12. *Let (π, u) be a degenerate covariant homomorphism of an action (A, α) to B, and put*

$$B_0 = \overline{\operatorname{span}}\{\pi(A)u(C^*(G))\},$$

where we use the same notation u for the associated nondegenerate homomorphism $u : C^*(G) \to M(B)$. Then:

(1) $B_0 = \overline{\operatorname{span}}\{u(C^*(G))\pi(A)\}$.

(2) *B_0 is a C^*-subalgebra of $M(B)$.*

(3) *π and u map into the idealizer D of B_0 in $M(B)$. Let $\rho : D \to M(B_0)$ be the homomorphism given by*

$$\rho(m)b_0 = mb_0 \quad \textit{for } m \in D \subset M(B),\ b_0 \in B_0 \subset B,$$

and let $\pi_0 = \rho \circ \pi : A \to M(B_0)$ and $u_0 = \rho \circ u : G \to M(B_0)$. Then (π_0, u_0) is a nondegenerate covariant homomorphism of (A, α) to B_0.

(4) *For all $a \in A$ and $c \in C^*(G)$ we have*

$$\pi_0(a)u_0(c) = \pi(a)u(c) \in B_0.$$

Let $(A \rtimes_\alpha G, i_A, i_G)$ be the usual crossed product of the action (A, α), i.e., (i_A, i_G) is a nondegenerate covariant homomorphism of (A, α) to $A \rtimes_\alpha G$ that is universal in the sense that if (π, u) is any nondegenerate covariant homomorphism of (A, α) to a C^*-algebra B, then there is a unique homomorphism $\pi \times u : A \rtimes_\alpha G \to M(B)$ such that

$$\pi \times u\big(i_A(a)i_G(c)\big) = \pi(a)u(c) \quad \text{for all } a \in A,\ c \in C^*(G). \tag{3-2}$$

Corollary 3.13. *With the above notation,* (i_A, i_G) *is also universal among degenerate covariant homomorphisms* (*in the sense of Definition 3.4*)*: for any degenerate covariant homomorphism* (π, u) *of* (A, α) *to* B *as in Definition 3.11, there is a unique homomorphism* $\pi \times u : A \rtimes_\alpha G \to M(B)$ *satisfying* (3-2).

If $\phi : (A, \delta) \to (B, \varepsilon)$ is a morphism in **Coact**, then a routine adaptation of the usual arguments shows that we get a morphism

$$\phi \rtimes G = (j_B \circ \phi) \times j_G^B : (A \rtimes_\delta G, \widehat{\delta}) \to (B \rtimes_\varepsilon G, \widehat{\varepsilon})$$

in **Act**, and similarly if $\phi : (A, \alpha) \to (B, \beta)$ is a morphism in **Act** we get a morphism

$$\phi \rtimes G = (i_B \circ \phi) \times i_G^B : (A \rtimes_\alpha G, \widehat{\alpha}) \to (B \rtimes_\beta G, \widehat{\beta})$$

in **Coact**. Thus we have crossed-product functors between the classical categories of coactions and actions.

It is also routine to verify that if (A, δ) is a coaction then the canonical surjection

$$\Phi : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \otimes \mathcal{K}$$

is a natural transformation between the double crossed-product functor and stabilization.[1]

We need to check that normalization and maximalization behave appropriately in the new coaction category.

Maximalization. A *maximalization* of a coaction (A, δ) consists of a maximal coaction (A^m, δ^m) and a surjective morphism $q^m : (A^m, \delta^m) \to (A, \delta)$ in **Coact** such that

$$q^m \rtimes G : A^m \rtimes_{\delta^m} G \to A \rtimes_\delta G$$

is an isomorphism. Existence of maximalizations is established in [Fischer 2004, Theorem 6.4; Echterhoff et al. 2004, Theorem 3.3].

To make maximalization into a functor on the classical category of coactions, we note that the argument of [Fischer 2004, proof of Lemma 6.2] carries over to give an appropriate version of the universal property: given coactions (A, δ) and (B, ε), with ε maximal, and a morphism $\phi : (B, \varepsilon) \to (A, \delta)$ in **Coact**, there is a unique morphism $\widetilde{\phi}$ in **Coact** making the diagram

$$\begin{tikzcd} (B,\varepsilon) \arrow[r, dashed, "\widetilde{\phi}"] \arrow[dr, "\phi"'] & (A^m,\delta^m) \arrow[d, "q^m"] \\ & (A,\delta) \end{tikzcd}$$

[1] It is completely routine to verify that stabilization $A \mapsto A \otimes \mathcal{K}$ is a functor on the classical category $\mathbf{C}^*$.

commute. Thus, given a morphism $\phi : (A, \delta) \to (B, \varepsilon)$ in **Coact**, there is a unique morphism ϕ^m making the diagram

$$\begin{tikzcd}
(A^m, \delta^m) \arrow[r, dashed, "\phi^m"] \arrow[d, "q_A^m"'] & (B^m, \varepsilon^m) \arrow[d, "q_B^m"] \\
(A, \delta) \arrow[r, "\phi"'] & (B, \varepsilon)
\end{tikzcd}$$

commute in **Coact**. Uniqueness makes the assignments $\phi \mapsto \phi^m$ functorial, and the *maximalizing maps* q^m give a natural transformation from the maximalization functor to the identity functor. Also, the universal property implies that the maximalization functor is faithful, i.e., if $\phi, \psi : (A, \delta) \to (B, \varepsilon)$ are distinct morphisms in **Coact**, then the maximalizations $\phi^m, \psi^m : (A^m, \delta^m) \to (B^m, \varepsilon^m)$ are also distinct.

Remark 3.14. It is important for us that maximalization is a *functor*; however, when we refer to (A^m, δ^m) as "the" maximalization of a coaction (A, δ), we do not have in mind a specific C^*-algebra A^m, rather we regard the maximalization as being characterized up to isomorphism by its universal properties, but for the purpose of having a functor we imagine that a choice of maximalization has been made for every coaction — any other choices would give a naturally isomorphic functor. On the other hand, whenever we have a maximal coaction (B, ε), we may call a morphism $\phi : (B, \varepsilon) \to (A, \delta)$ with the defining property *a maximalization* of (A, δ).

Normalization. A *normalization* of a coaction (A, δ) consists of a normal coaction (A^n, δ^n) and a surjective morphism $\Lambda : (A, \delta) \to (A^n, \delta^n)$ in **Coact** such that

$$\Lambda \rtimes G : A \rtimes_\delta G \to A^n \rtimes_{\delta^n} G$$

is an isomorphism. Existence of normalizations is established in [Quigg 1994, Proposition 2.6].

To make normalization into a functor on the classical category of coactions, we note that [Echterhoff et al. 2004, Lemma 2.1] says that, given a morphism $\phi : (A, \delta) \to (B, \varepsilon)$ in **Coact**, there is a unique morphism ϕ^n making the diagram

$$\begin{tikzcd}
(A, \delta) \arrow[r, "\phi"] \arrow[d, "\Lambda_A"'] & (B, \varepsilon) \arrow[d, "\Lambda_B"] \\
(A^n, \delta^n) \arrow[r, dashed, "\phi^n"'] & (B^n, \varepsilon^n)
\end{tikzcd}$$

commute in **Coact**. Uniqueness makes the assignments $\phi \mapsto \phi^n$ functorial, and the *normalizing maps* Λ give a natural transformation from the identity functor to the normalization functor.

Remark 3.15. The comments of Remark 3.14 can be adapted in an obvious way to normalization, and also to crossed products, etc. There are numerous "natural" relationships among such functors; for example, maximalization is naturally isomorphic to the composition

$$(A,\delta)\mapsto(A^n,\delta)\mapsto(A^{nm},\delta^{nm})$$

of normalization followed by maximalization, and the dual coaction $\widehat{\alpha}^n$ on the reduced crossed product $A\rtimes_{\alpha,r}G$ of an action (A,α) is naturally isomorphic to the normalization of the dual coaction $\widehat{\alpha}$ on the full crossed product $A\rtimes_\alpha G$ [Echterhoff et al. 2006, Proposition A.61].

The normalization $\Lambda:(A,\delta)\to(A^n,\delta^n)$ of a maximal coaction is also a maximalization of the normal coaction δ^n. It follows that the normalization functor is faithful, i.e., if $\phi,\psi:(A,\delta)\to(B,\varepsilon)$ are distinct morphisms in **Coact**, then the normalizations $\phi^n,\psi^n:(A^n,\delta^n)\to(B^n,\varepsilon^n)$ are also distinct. It follows from this and surjectivity of the normalizing maps $\Lambda_A:(A,\delta)\to(A^n,\delta^n)$ that the normalizing maps are monomorphisms in the category **Coact**, i.e., if $\phi,\psi:(A,\delta)\to(B,\varepsilon)$ are distinct morphisms in **Coact**, then the compositions $\Lambda_B\circ\phi,\Lambda_B\circ\psi:(A,\delta)\to(B^n,\varepsilon^n)$ are also distinct.[2]

Exact sequences. It is crucial for us to note that in each of the classical categories $\mathbf{C}^*$, **Coact**, and **Act** there is an obvious concept of short exact sequence. Nilsen [1999] develops the basic theory of short exact sequences for coactions and crossed products. We briefly outline the essential facts here.

Definition 3.16. Let (A,δ) be a coaction. An ideal I of A is *strongly δ-invariant* if

$$\overline{\operatorname{span}}\{\delta(I)(1\otimes C^*(G))\}=I\otimes C^*(G).$$

We will normally just write *invariant* to mean strongly invariant.

Nilsen proves [1999, Propositions 2.1 and 2.2, Theorem 2.3] (see also [Landstad et al. 1987, Proposition 4.8]) that, using her conventions, if I is strongly invariant then:

(1) δ restricts to a coaction δ_I on I.

(2) $I\rtimes_{\delta_I}G$ is (canonically isomorphic to) an ideal of $A\rtimes_\delta G$.

(3) I is *weakly* δ-invariant, i.e., δ descends to a coaction δ^I on A/I.

(4) $0\to I\rtimes_{\delta_I}G\to A\rtimes_\delta G\to(A/I)\rtimes_{\delta^I}G\to 0$ is a short exact sequence in the classical category $\mathbf{C}^*$.

[2] The analogous fact for the nondegenerate category of coactions is [Bédos et al. 2011, Corollary 6.1.20].

We point out that Nilsen had to do a bit of work to map $I \rtimes_{\delta_I} G$ into $A \rtimes_\delta G$; in our framework with the classical categories, we just note that the inclusion $\phi : I \hookrightarrow A$ is $\delta_I - \delta$ equivariant, hence gives a morphism in **Coact**, so we can apply the functor CP to get a morphism

$$\phi \rtimes G : I \rtimes_{\delta_I} G \to A \rtimes_\delta G \quad \text{in } \mathbf{C}^*.$$

Definition 3.17. A functor between any two of the categories $\mathbf{C}^*$, **Coact**, **Act** is *exact* if it preserves short exact sequences.

Example 3.18. The full crossed-product functor

$$(A, \alpha) \mapsto (A \rtimes_\alpha G, \widehat{\alpha}),$$
$$\phi \mapsto \phi \rtimes G$$

from **Act** to **Coact** is exact [Green 1978, Proposition 12]. However, the reduced crossed-product functor is not exact, due to Gromov's examples of nonexact groups.

Example 3.19. The crossed-product functor

$$(A, \delta) \mapsto (A \rtimes_\delta G, \widehat{\delta}),$$
$$\phi \mapsto \phi \rtimes G$$

from **Coact** to **Act** is exact [Nilsen 1999, Theorem 2.3].

Example 3.20. The stabilization functor

$$A \mapsto A \otimes \mathcal{K},$$
$$\phi \mapsto \phi \otimes \mathrm{id}$$

on $\mathbf{C}^*$ is exact.

4. Coaction functors

Baum, Guentner and Willett [Baum et al. 2016] defined a *crossed-product* as a functor $(B, \alpha) \mapsto B \rtimes_{\alpha,\tau} G$, from the category of actions to the category of C^*-algebras, equipped with natural transformations

$$\begin{tikzcd} B \rtimes_\alpha G \arrow[r] \arrow[d] & B \rtimes_{\alpha,\tau} G \arrow[dl] \\ B \rtimes_{\alpha,r} G & \end{tikzcd}$$

where the vertical arrow is the regular representation, such that the horizontal arrow is surjective.

Our predilection is to decompose such a crossed-product functor as a composition

$$(B,\alpha) \mapsto (B \rtimes_\alpha G, \widehat{\alpha}) \mapsto B \rtimes_{\alpha,\tau} G,$$

where the first arrow is the full crossed product and the second arrow depends only upon the dual coaction $\widehat{\alpha}$. Our approach will require the target C^*-algebra $B \rtimes_{\alpha,\tau} G$ to carry a quotient of the dual coaction. Thus, it is certainly not obvious that our techniques can handle all crossed-product functors of [Baum et al. 2016], because that paper does not require the crossed products $B \rtimes_{\alpha,\tau} G$ to have coactions, and even if they all do, there is no reason to believe that the crossed-product functor factors in this way. Nevertheless, we think that it is useful to study crossed-product functors that do factor, and thus we can focus upon the second functor, where all the action stays within the realm of coactions. The following definition is adapted more or less directly from [loc. cit., Definition 2.1]:

Definition 4.1. A *coaction functor* is a functor $\tau : (A,\delta) \mapsto (A^\tau, \delta^\tau)$ on the category of coactions, together with a natural transformation q^τ from maximalization to τ such that for every coaction (A,δ),

(1) $q^\tau_A : A^m \to A^\tau$ is surjective, and

(2) $\ker q^\tau_A \subset \ker \Lambda_{A^m}$.

Example 4.2. (1) Maximalization $(A,\delta) \mapsto (A^m, \delta^m)$ is a coaction functor, with natural surjections given by the identity maps id_{A^m}.

(2) Normalization $(A,\delta) \mapsto (A^n, \delta^n)$ is a coaction functor, with natural surjections $\Lambda_{A^m} : A^m \to A^n$.

(3) The identity functor is a coaction functor, with natural surjections $q^m_A : A^m \to A$.

Lemma 4.3. *If τ is a coaction functor, then for every coaction (A,δ) there is a unique $\delta^\tau - \delta^n$ equivariant surjection Λ^τ_A making the diagram*

(4-1)
$$\begin{tikzcd} A^m \arrow[r, "q^\tau_A"] \arrow[d, "\Lambda_{A^m}"'] & A^\tau \arrow[dl, dashed, "\Lambda^\tau_A"] \\ A^n & \end{tikzcd}$$

commute. Moreover, Λ^τ is a natural transformation from τ to normalization.

Proof. The first statement follows immediately from the definitions. To verify that Λ^τ is a natural transformation, we must show that the homomorphisms Λ^τ

(1) are morphisms of coactions, and

(2) are natural.

(1) In the commuting triangle (4-1), we must show that Λ^τ_A is a $B(G)$-module map, but this follows since Λ_{A^m} and q^τ_A are module maps and q^τ_A is surjective.

(2) For the naturality, let $\phi : (A,\delta) \to (B,\varepsilon)$ be a morphism in the category of coactions. Consider the diagram

$$\begin{tikzcd}
A^m \arrow[rr, "\phi^m"] \arrow[dr, "q^\tau_A"] \arrow[dd, "\Lambda_{A^m}"'] & & B^m \arrow[dr, "q^\tau_B"] \arrow[dd, "\Lambda_{B^m}"' near end] & \\
& A^\tau \arrow[rr, "\phi^\tau" near start] \arrow[dl, "\Lambda^\tau_A"] & & B^\tau \arrow[dl, "\Lambda^\tau_B"] \\
A^n \arrow[rr, "\phi^n"'] & & B^n &
\end{tikzcd}$$

We need to know that the lower quadrilateral, with horizontal and southwest arrows, commutes, and this follows from surjectivity of q^τ_A and commutativity of the other two quadrilaterals and the two triangles. $\square$

Corollary 4.4. *If τ is a coaction functor, then in* (4-1) *we have*

(1) *$q^\tau : A^m \to A^\tau$ is a maximalization of δ^τ, and*

(2) *$\Lambda^\tau : A^\tau \to A^n$ is a normalization of δ^τ.*

Proof. Taking crossed products in (4-1), we get a commutative diagram

$$\begin{tikzcd}
A^m \rtimes_{\delta^m} G \arrow[r, "q^\tau \rtimes G", "\simeq"'] \arrow[d, "\Lambda \rtimes G"', "\simeq"] & A^\tau \rtimes_{\delta^\tau} G \arrow[dl, "\simeq"', "\Lambda^\tau \rtimes G"] \\
A^n \rtimes_{\delta^n} G &
\end{tikzcd}$$

where the horizontal arrow is surjective because q^τ is, and is injective because of the vertical isomorphism, and then the diagonal arrow is an isomorphism because the other two arrows are. Thus q^τ and Λ^τ satisfy the defining properties of maximalization and normalization, respectively. $\square$

Remark 4.5. Caution: it might seem that τ should factor through the maximalization functor, at least up to natural isomorphism. This would entail, in particular, that

$$(A^{m\tau}, \delta^{m\tau}) \cong (A^\tau, \delta^\tau) \quad \text{for every coaction } (A,\delta).$$

But this is violated with $\tau = \mathrm{id}$.

Notation 4.6. With the above notation, we define an ideal of A^m by

$$A^m_\tau := \ker q^\tau_A.$$

Note that for the maximalization functor m we have $A^m_m = \{0\}$, while for the normalization functor n the associated ideal A^m_n is the kernel of the normalization map $\Lambda_{A^m} : A^m \to A^{mn} \cong A^n$.

Partial ordering of coaction functors. Baum, Guentner and Willett [Baum et al. 2016, p. 8] define one crossed-product functor σ to be *smaller* than another one τ if the natural surjection $A \rtimes_{\alpha,\tau} G \to A \rtimes_{\alpha,r} G$ factors through the σ-crossed product.

We adapt this definition of partial order to coaction functors, but "from the top rather than toward the bottom".

Definition 4.7. If σ and τ are coaction functors, then σ is *smaller* than τ, written $\sigma \leq \tau$, if for every coaction (A, δ) we have

$$A^m_\tau \subset A^m_\sigma.$$

Lemma 4.8. *For coaction functors σ, τ, the following are equivalent*:

(1) $\sigma \leq \tau$.

(2) *For every coaction (A, δ) there is a homomorphism $\Gamma^{\tau,\sigma}$ making the diagram*

$$\begin{tikzcd} A^m \arrow[r, "q^\tau"] \arrow[dr, "q^\sigma"'] & A^\tau \arrow[d, dashed, "\Gamma^{\tau,\sigma}"] \\ & A^\sigma \end{tikzcd}$$

commute.

(3) *For every coaction (A, δ) there is a homomorphism $\Gamma^{\tau,\sigma}$ making the diagram*

$$\begin{tikzcd} & A^\tau \arrow[dl, "\Lambda^\tau"'] \arrow[d, dashed, "\Gamma^{\tau,\sigma}"] \\ A^n & A^\sigma \arrow[l, "\Lambda^\sigma"] \end{tikzcd}$$

commute.

Moreover, if these equivalent conditions hold then $\Gamma^{\tau,\sigma}$ is unique, is surjective, and is a natural transformation from τ to σ.

Proof. (1) is equivalent to (2) since $A^m_\tau = \ker q^\tau$ and $A^m_\sigma = \ker q^\sigma$. Moreover, (1) implies that $\Gamma^{\tau,\sigma}$ is unique and is surjective, since the maps q^τ are surjective.

Assume (3). Consider the combined diagram

$$
\begin{tikzcd}
A^m \arrow[r, "q^t"] \arrow[d, "\Lambda_{A^m}"'] \arrow[dr, "q^\sigma" near start] & A^\tau \arrow[d, "\Gamma^{\tau,\sigma}"] \arrow[dl, "\Lambda^\tau"' near start] \\
A^n & A^\sigma \arrow[l, "\Lambda^\sigma"]
\end{tikzcd}
\tag{4-2}
$$

The upper left and lower left triangles commute by definition of coaction functor, and the lower right triangle commutes by assumption. Thus the upper right triangle commutes after postcomposing with Λ^σ. Since the latter map is a normalizer, by [Bédos et al. 2011, Corollary 6.1.20] it is a monomorphism in the category of coactions. Thus the upper right triangle commutes.

Similarly (but more easily), assuming (2), the lower right triangle in the diagram (4-2) commutes because it commutes after precomposing with the surjection q^τ.

Naturality of $\Gamma^{\tau,\sigma}$ is proved by virtually the same argument as in Lemma 4.3. □

The following is a coaction-functor analogue of [Baum et al. 2016, Lemma 3.7], and we adapt their argument:

Theorem 4.9. *Every nonempty collection $\mathcal{T}$ of coaction functors has a greatest lower bound σ with respect to the above partial ordering, characterized by*

$$A^m_\sigma = \overline{\operatorname{span}}_{\tau\in\mathcal{T}} A^m_\tau$$

for every coaction (A,δ).

Proof. Let (A,δ) be a coaction, Then the ideal

$$A^m_\sigma := \overline{\operatorname{span}}_{\tau\in\mathcal{T}} A^m_\tau$$

of A^m is contained in the kernel of the normalization map Λ_{A^m}. Put

$$A^\sigma = A^m/A^m_\sigma,$$

and let

$$q^\sigma_A : A^m \to A^\sigma$$

be the quotient map.

A^m_τ is a weakly δ^m-invariant ideal of A^m for all $\tau \in \mathcal{T}$, so for all $f \in B(G)$ we have

$$f \cdot A^m_\tau \subset A^m_\tau \subset A^m_\sigma,$$

and it follows that $f \cdot A^m_\sigma \subset A^m_\sigma$, i.e., A^m_σ is a weakly δ^m-invariant ideal. Thus q^σ is equivariant for δ^m and a unique coaction δ^σ on A^σ.

We now have assignments

$$(A,\delta)\mapsto(A^\sigma,\delta^\sigma)$$

on objects, and we need to handle morphisms. Thus, let $\phi:(A,\delta)\to(B,\varepsilon)$ be a morphism of coactions; i.e., $\phi:A\to B$ is a $\delta-\varepsilon$ equivariant homomorphism. Since

$$A^m_\tau\subset(\phi^m)^{-1}(B^m_\tau)\subset(\phi^m)^{-1}(B^m_\sigma)\quad\text{for all }\tau\in\mathcal{T},$$

we have

$$\ker q^\sigma_A=A^m_\sigma=\overline{\operatorname{span}}_{\tau\in\mathcal{T}}A^m_\tau\subset(\phi^m)^{-1}(B^m_\sigma)=\ker q^\sigma_B\circ\phi^m.$$

Thus there is a unique homomorphism ϕ^σ making the diagram

(4-3)
$$\begin{tikzcd}
A^m \arrow[r, "\phi^m"] \arrow[d, "q^\sigma_A"'] & B^m \arrow[d, "q^\sigma_B"] \\
A^\sigma \arrow[r, dashed, "\phi^\sigma"'] & B^\sigma
\end{tikzcd}$$

commute. Moreover, ϕ^σ is $\delta^\sigma-\varepsilon^\sigma$ equivariant because the other three maps are and q^σ_A is surjective.

We need to verify that the assignments $\phi\mapsto\phi^\sigma$ of morphisms are functorial. Obviously identity morphisms are preserved. For compositions, let

$$\begin{tikzcd}
(A,\delta) \arrow[r, "\phi"] \arrow[dr, "\nu"'] & (B,\varepsilon) \arrow[d, "\rho"] \\
& (C,\gamma)
\end{tikzcd}$$

be a commuting diagram of coactions. Consider the diagram

$$\begin{tikzcd}
A^m \arrow[rr, "\phi^m"] \arrow[dr, "\nu^m"] \arrow[dd, "q^\sigma_A"'] & & B^m \arrow[dl, "\rho^m"'] \arrow[dd, "q^\sigma_B"] \\
& C^m & \\
A^\tau \arrow[rr, "\phi^\tau" near start] \arrow[dr, "\nu^\tau"'] & & B^\tau \arrow[dl, "\rho^\tau"] \\
& C^\tau \arrow[from=uu, crossing over, "q^\sigma_C" near start] &
\end{tikzcd}$$

The three vertical quadrilaterals and the top triangle commute, and q^σ_A is surjective. It follows that the bottom triangle commutes, and we have shown that composition is preserved.

Thus we have a functor σ on the category of coactions. Moreover, σ is a coaction functor, since the surjections q^σ have small kernels and the commuting diagram (4-3) shows that q^σ gives a natural transformation from maximalization to σ. By construction, σ is a greatest lower bound for $\mathcal{T}$. □

Exact coaction functors. As a special case of our general Definition 3.17, we explicitly record:

Definition 4.10. A coaction functor τ is *exact* if for every short exact sequence

$$0 \longrightarrow (I,\gamma) \xrightarrow{\phi} (A,\delta) \xrightarrow{\psi} (B,\varepsilon) \longrightarrow 0$$

of coactions the associated sequence

$$0 \longrightarrow (I^\tau,\gamma^\tau) \xrightarrow{\phi^\tau} (A^\tau,\delta^\tau) \xrightarrow{\psi^\tau} (B^\tau,\varepsilon^\tau) \longrightarrow 0$$

is exact.

Theorem 4.11. *The maximalization functor is exact.*

Proof. Let

$$0 \longrightarrow (I,\gamma) \xrightarrow{\phi} (A,\delta) \xrightarrow{\psi} (B,\varepsilon) \longrightarrow 0$$

be an exact sequence of coactions. Taking crossed products twice, we get an exact sequence

$$0 \longrightarrow I \rtimes_\gamma G \rtimes_{\widehat{\gamma}} G \xrightarrow{\phi\rtimes G\rtimes G} A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \xrightarrow{\psi\rtimes G\rtimes G} B \rtimes_\varepsilon G \rtimes_{\widehat{\varepsilon}} G \longrightarrow 0.$$

Since the identity functor on coactions is a coaction functor, we get an isomorphic sequence

$$\begin{aligned} 0 \longrightarrow I^m \rtimes_{\gamma^m} G \rtimes_{\widehat{\gamma^m}} G \xrightarrow{\phi^m\rtimes G\rtimes G} A^m \rtimes_{\delta^m} G \rtimes_{\widehat{\delta^m}} G \\ \xrightarrow{\psi^m\rtimes G\rtimes G} B^m \rtimes_{\varepsilon^m} G \rtimes_{\widehat{\varepsilon^m}} G \longrightarrow 0, \end{aligned}$$

which is therefore also exact. Since the canonical surjection Φ is a natural transformation from the double crossed-product functor to the stabilization functor, and since the coactions are now maximal, we get an isomorphic sequence

$$0 \longrightarrow I^m \otimes \mathcal{K} \xrightarrow{\phi^m\otimes \mathrm{id}} A^m \otimes \mathcal{K} \xrightarrow{\psi^m\otimes \mathrm{id}} B^m \otimes \mathcal{K} \longrightarrow 0,$$

which is therefore also exact. Since $\mathcal{K}$ is an exact C^*-algebra,

$$(\ker \phi^m) \otimes \mathcal{K} = \ker(\phi^m \otimes \mathrm{id}) = \{0\},$$

so $\ker \phi^m = \{0\}$, and similarly

$$(\ker \psi^m) \otimes \mathcal{K} = \ker(\psi^m \otimes \mathrm{id}) = (\phi^m \otimes \mathrm{id})(I^m \otimes \mathcal{K}) = \phi^m(I^m) \otimes \mathcal{K},$$

so, because $\phi^m(I^m) \subset \ker \psi^m$ by functoriality, we must have $\phi^m(I^m) = \ker \psi^m$. Therefore the sequence

$$0 \longrightarrow I^m \xrightarrow{\phi^m} A^m \xrightarrow{\psi^m} B^m \longrightarrow 0$$

is exact. □

Theorem 4.12. *A coaction functor τ is exact if and only if for any short exact sequence*

$$0 \longrightarrow (I, \delta_I) \xrightarrow{\phi} (A, \delta) \xrightarrow{\psi} (B, \delta^I) \longrightarrow 0$$

of coactions, both

$$\phi^m(I^m_\tau) = \phi^m(I^m) \cap A^m_\tau$$

and

$$\phi^m(I^m) + A^m_\tau = (\psi^m)^{-1}(B^m_\tau)$$

hold.

Proof. We have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I^m_\tau \arrow[r, "\phi^m|"] \arrow[d, "\iota_I"'] & A^m_\tau \arrow[r, "\psi^m|"] \arrow[d, "\iota_A"'] & B^m_\tau \arrow[r] \arrow[d, "\iota_B"'] & 0 \\
0 \arrow[r] & I^m \arrow[r, "\phi^m"] \arrow[d, "q_I"'] & A^m \arrow[r, "\psi^m"] \arrow[d, "q_A"'] & B^m \arrow[r] \arrow[d, "q_B"'] & 0 \\
0 \arrow[r] & I^\tau \arrow[r, "\phi^\tau"] \arrow[d] & A^\tau \arrow[r, "\psi^\tau"] \arrow[d] & B^\tau \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4-4}$$

in which the columns are exact by definition, and the middle row is exact by Theorem 4.11. Thus the result follows immediately from Lemma 2.1. □

Morita compatible coaction functors. If we have coactions (A, δ) and (B, ε), and a $\delta - \varepsilon$ compatible coaction ζ on an $A - B$ imprimitivity bimodule X, we'll say that (X, ζ) is an $(A, \delta) - (B, \varepsilon)$ *imprimitivity bimodule*.

Example 4.13. The double dual bimodule coaction

$$(Y, \eta) := (X \rtimes_{\zeta} G \rtimes_{\widehat{\zeta}} G, \widehat{\widehat{\zeta}})$$

is an

$$(A \rtimes_{\delta} G \rtimes_{\widehat{\delta}} G, \widehat{\widehat{\delta}}) - (B \rtimes_{\varepsilon} G \rtimes_{\widehat{\varepsilon}} G, \widehat{\widehat{\varepsilon}})$$

imprimitivity bimodule. Since the identity functor on coactions is a coaction functor, (Y, η) becomes an

$$(A^m \rtimes_{\delta^m} G \rtimes_{\widehat{\delta^m}} G, \widehat{\widehat{\delta^m}}) - (B^m \rtimes_{\varepsilon^m} G \rtimes_{\widehat{\varepsilon^m}} G, \widehat{\widehat{\varepsilon^m}})$$

imprimitivity bimodule. Since maximalizations satisfy full-crossed-product duality, (Y, η) becomes, after replacing the double dual coactions by exterior equivalent coactions, an

$$(A^m \otimes \mathcal{K}, \delta^m \otimes_* \mathrm{id}) - (B^m \otimes \mathcal{K}, \varepsilon^m \otimes_* \mathrm{id})$$

imprimitivity bimodule (see [Echterhoff et al. 2004, Lemma 3.6]).

We need the following basic lemma, which is probably folklore, although we could not find it in the literature. Our formulation is partially inspired by Fischer's treatment of relative commutants of $\mathcal{K}$ [Fischer 2004, Section 3].

Lemma 4.14. *Let A and B be C^*-algebras, and let Y be an $(A \otimes \mathcal{K}) - (B \otimes \mathcal{K})$ imprimitivity bimodule. Define*

$$X = \{m \in M(Y) : (1_A \otimes k) \cdot m = m \cdot (1_B \otimes k) \in Y \text{ for all } k \in \mathcal{K}\}.$$

Then:

1. *X is an $(A \otimes 1_{\mathcal{K}}) - (B \otimes 1_{\mathcal{K}})$ submodule of $M(Y)$.*
2. *$\overline{\mathrm{span}}\langle X, X\rangle_{M(B \otimes \mathcal{K})} = B \otimes 1_{\mathcal{K}}$.*
3. *$\overline{\mathrm{span}}_{M(A \otimes \mathcal{K})}\langle X, X\rangle = A \otimes 1_{\mathcal{K}}$.*

Thus X becomes an $A - B$ imprimitivity bimodule in an obvious way, and moreover there is a unique $(A \otimes \mathcal{K}) - (B \otimes \mathcal{K})$ imprimitivity bimodule isomorphism

$$\theta : X \otimes \mathcal{K} \xrightarrow{\cong} Y$$

such that

$$\theta(m \otimes k) = m \cdot (1_B \otimes k) \quad \text{for } m \in X,\ k \in \mathcal{K}.$$

Lemma 4.15. *Given coactions (A, δ) and (B, ε), and a $\delta - \varepsilon$ compatible coaction ζ on an $A - B$ imprimitivity bimodule X, let (Y, η) be the*

$$(A^m \otimes \mathcal{K}, \delta^m \otimes_* \mathrm{id}) - (B^m \otimes \mathcal{K}, \varepsilon^m \otimes_* \mathrm{id})$$

imprimitivity bimodule from Example 4.13, and let X^m denote the associated $A^m - B^m$ imprimitivity bimodule as in Lemma 4.14, with an $(A^m \otimes \mathcal{K}) - (B^m \otimes \mathcal{K})$ imprimitivity bimodule isomorphism $\theta : X^m \otimes \mathcal{K} \to Y$. Then there is a unique $\delta^m - \varepsilon^m$ compatible coaction ζ^m on X^m such that θ transports $\zeta^m \otimes_* \mathrm{id}$ to η.

Proof. The diagram

$$\begin{tikzcd} X^m \otimes \mathcal{K} \arrow[r, dashed, "\kappa"] \arrow[d, "\theta"', "\simeq"] & M(X^m \otimes \mathcal{K} \otimes C^*(G)) \arrow[d, "\simeq"', "\theta\otimes\mathrm{id}"] \\ Y \arrow[r, "\eta"'] & M(Y \otimes C^*(G)) \end{tikzcd}$$

certainly has a unique commuting completion, and κ is a $(\delta^m \otimes_* \mathrm{id}) - (\varepsilon^m \otimes_* \mathrm{id})$ compatible coaction on $X^m \otimes \mathcal{K}$. In order to recognize that κ is of the form $\zeta^m \otimes_* \mathrm{id}$, we need to know that, letting $\Sigma : \mathcal{K} \otimes C^*(G) \to C^*(G) \otimes \mathcal{K}$ be the flip isomorphism, for every $\xi \in X^m$, the element

$$m := (\mathrm{id}_{X^m} \otimes \Sigma) \circ (\theta \otimes \mathrm{id})^{-1} \circ \eta \circ \theta(\xi \otimes 1_{\mathcal{K}})$$

of the multiplier bimodule $M(X^m \otimes C^*(G) \otimes \mathcal{K})$ is contained in the subset $M(X^m \otimes C^*(G)) \otimes 1_{\mathcal{K}}$, and for this we need only check that for all $k \in \mathcal{K}$ we have

$$(1_{A\otimes C^*(G)} \otimes k) \cdot m = m \cdot (1_{B\otimes C^*(G)} \otimes k) \in X^m \otimes C^*(G) \otimes \mathcal{K},$$

which follows from the properties of the maps involved. Then it is routine to check that the resulting map ζ^m is a $\delta^m - \varepsilon^m$ compatible coaction on X^m. □

Definition 4.16. A coaction functor τ is *Morita compatible* if whenever (X, ζ) is an $(A, \delta) - (B, \varepsilon)$ imprimitivity bimodule, with associated $A^m - B^m$ imprimitivity bimodule X^m as above, the Rieffel correspondence of ideals satisfies

$$X^m\text{-Ind } B^m_\tau = A^m_\tau. \tag{4-5}$$

We will use without comment the simple observation that if (A, δ) (and hence also (B, ε)) is maximal, then we can replace X^m by X and regard the natural surjection q^τ_A as going from A to A^τ (and similarly for B), since the maximalizing maps q^m_A and q^m_B can be combined to give an isomorphism of the $A^m - B^m$ imprimitivity bimodule X^m onto X.

Remark 4.17. Caution: Definition 4.16 is not a direct analogue of the definition of Morita compatibility in [Baum et al. 2016, Definition 3.2], but it suits our purposes in working with coaction functors, as we will see in Proposition 4.24.

Remark 4.18. Lemma 4.15 says in particular that maximalization preserves Morita equivalence of coactions. This is almost new: it also follows from first applying the cross-product functor, noting that the dual actions are “weakly proper $G \rtimes G$-algebras” in the sense of [Buss and Echterhoff 2014], then applying [Buss and Echterhoff 2015, Corollary 4.6] with the universal crossed-product norm (denoted by u in [Buss and Echterhoff 2014]).

Lemma 4.19. *A coaction functor τ is Morita compatible if and only if whenever (X, ζ) is an $(A, \delta)-(B, \varepsilon)$ imprimitivity bimodule, there are an $A^\tau - B^\tau$ imprimitivity bimodule X^τ and a $q_A^\tau - q_B^\tau$ compatible imprimitivity-bimodule homomorphism $q_X^\tau : X^m \to X^\tau$.*

Proof. Given X^τ and q_X^τ with the indicated properties, by [Echterhoff et al. 2006, Lemma 1.20] we have

$$X^m\text{-Ind}\, B_\tau^m = X^m\text{-Ind} \ker q_B^\tau = \ker q_A^\tau = A_\tau^m.$$

It follows that τ is Morita compatible.

Conversely, suppose τ is Morita compatible, and let (X^m, ζ^m) be as above. Then, by the Rieffel correspondence, $X^\tau := X^m / X^m \cdot B_\tau^m$ is an $A^m/A_\tau^m - B^m/B_\tau^m$ imprimitivity bimodule, and the quotient map $q_X^\tau : X^m \to X^\tau$ is compatible with the quotient maps $A^m \mapsto A^m/A_\tau^m$ and $B^m \mapsto B_\tau^m$. Via the unique isomorphisms making the diagrams

$$\begin{tikzcd} A^m \arrow[d, "\text{quotient map}"'] \arrow[dr, "q_A^\tau"] & \\ A^m/A_\tau^m \arrow[r, dashed, "\simeq"'] & A^\tau \end{tikzcd} \qquad \begin{tikzcd} B^m \arrow[d, "\text{quotient map}"'] \arrow[dr, "q_B^\tau"] & \\ B^m/B_\tau^m \arrow[r, dashed, "\simeq"'] & B^\tau \end{tikzcd}$$

commute, q_X^τ becomes $q_A^\tau - q_B^\tau$ compatible. □

Example 4.20. It follows trivially that the maximalization functor is Morita compatible.

Lemma 4.21. *The identity functor on coactions is Morita compatible.*

Proof. Let (X, ζ) be an $(A, \delta) - (B, \varepsilon)$ imprimitivity bimodule, and let (X^m, ζ^m) be the associated $(A^m, \delta^m) - (B^m, \varepsilon^m)$ imprimitivity bimodule from Lemma 4.15. By Lemma 4.19 it suffices to find a $q_A^m - q_B^m$ compatible imprimitivity-bimodule homomorphism $q_X^m : X^m \to X$. Now, X^m is the upper right corner of the 2×2 matrix representation of the linking algebra L^m, and the maximalization map q_L^m of the linking algebra L of X preserves the upper right corners. Thus q_L^m takes X^m onto X, and simple algebraic manipulations show that it has the right properties. □

Theorem 4.22. *The greatest lower bound of the collection of all exact and Morita compatible coaction functors is itself exact and Morita compatible.*

Proof. Let $\mathcal{T}$ be the collection of all exact and Morita compatible coaction functors, and let τ be the greatest lower bound of $\mathcal{T}$. As in the proof of Theorem 4.9, for every coaction (A,δ) we have

$$A^m_\tau = \overline{\operatorname{span}}_{\sigma\in\mathcal{T}} A^m_\sigma.$$

For exactness, we apply Definition 4.10. Let

$$0 \longrightarrow (I,\gamma) \xrightarrow{\ \phi\ } (A,\delta) \xrightarrow{\ \psi\ } (B,\varepsilon) \longrightarrow 0$$

be a short exact sequence of coactions. Then

$$\begin{aligned}
\phi^m(I^m_\tau) &= \phi^m\bigl(\overline{\operatorname{span}}_{\sigma\in\mathcal{T}} I^m_\sigma\bigr)\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}} \phi^m(I^m_\sigma)\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}}(\phi^m(I^m)\cap A^m_\sigma) \quad \text{(since σ is exact)}\\
&= \phi^m(I^m)\cap \overline{\operatorname{span}}_{\sigma\in\mathcal{T}} A^m_\sigma\\
&\qquad\qquad \text{(since all spaces involved are ideals in C^*-algebras)}\\
&= \phi^m(I^m)\cap A^m_\tau,
\end{aligned}$$

and

$$\begin{aligned}
\phi^m(I^m)+A^m_\tau &= \phi^m(I^m)+\overline{\operatorname{span}}_{\sigma\in\mathcal{T}} A^m_\sigma\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}}(\phi^m(I^m)+A^m_\sigma)\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}}(\psi^m)^{-1}(B^m_\sigma) \quad \text{(since σ is exact)}\\
&= (\psi^m)^{-1}\bigl(\overline{\operatorname{span}}_{\sigma\in\mathcal{T}} B^m_\sigma\bigr)\\
&= (\psi^m)^{-1}(B^m_\tau),
\end{aligned}$$

so τ is exact.

For Morita compatibility, let (X,ζ) be an $(A,\delta)-(B,\varepsilon)$ imprimitivity bimodule, with associated $A^m - B^m$ imprimitivity bimodule X^m. Then

$$\begin{aligned}
X^m\text{-Ind}\, B^m_\tau &= X^m\text{-Ind}\,\overline{\operatorname{span}}_{\sigma\in\mathcal{T}} B^m_\sigma\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}} X^m\text{-Ind}\, B^m_\sigma \quad \text{(by continuity of Rieffel induction)}\\
&= \overline{\operatorname{span}}_{\sigma\in\mathcal{T}} A^m_\sigma \quad \text{(since σ is Morita compatible)}\\
&= A^m_\tau,
\end{aligned}$$

so τ is Morita compatible. □

Definition 4.23. We call the above greatest lower bound of the collection of all exact and Morita compatible coaction functors the *minimal exact and Morita compatible coaction functor*.

***Comparison with* [Baum et al. 2016].** As we mentioned previously, [Baum et al. 2016, p. 8] defines one crossed-product functor σ_1 to be *smaller* than another one σ_2, written $\sigma_1 \le \sigma_2$, if the natural surjection $A \rtimes_{\alpha,\sigma_2} G \to A \rtimes_{\alpha,r} G$ factors through the σ_1-crossed product.

Let τ be a coaction functor, and let $\sigma = \tau \circ \mathrm{CP}$ be the associated crossed-product functor, i.e.,

$$(A,\alpha)^\sigma = A \rtimes_{\alpha,\sigma} G := (A \rtimes_\alpha G)^\tau.$$

For a morphism $\phi : (A,\alpha) \to (B,\beta)$ of actions, we write

$$\phi \rtimes_\sigma G = (\phi \rtimes G)^\tau : A \rtimes_{\alpha,\sigma} G \to B \rtimes_{\beta,\sigma} G$$

for the associated morphism of σ-crossed products.

Proposition 4.24. *With the above notation, if the coaction functor τ is exact or Morita compatible, then the associated crossed-product functor σ has the same property. Moreover, if $\tau_1 \le \tau_2$ then $\sigma_1 \le \sigma_2$.*

Proof. The last statement follows immediately from the definitions. For the other statement, first assume that τ is exact, and let

$$0 \longrightarrow (I,\gamma) \xrightarrow{\ \phi\ } (A,\alpha) \xrightarrow{\ \psi\ } (B,\beta) \longrightarrow 0$$

be a short exact sequence of actions. Then the sequence

$$0 \longrightarrow (I \rtimes_\gamma G, \widehat{\gamma}) \xrightarrow{\phi \rtimes G} (A \rtimes_\alpha G, \widehat{\alpha}) \xrightarrow{\psi \rtimes G} (B \rtimes_\beta G, \widehat{\beta}) \longrightarrow 0$$

of coactions is exact, since the full-crossed-product functor is exact. Then by exactness of τ we see that the sequence

$$0 \longrightarrow I \rtimes_{\gamma,\sigma} G \xrightarrow{\phi \rtimes_\sigma G} A \rtimes_{\alpha,\sigma} G \xrightarrow{\psi \rtimes_\sigma G} B \rtimes_{\beta,\sigma} G \longrightarrow 0$$

is also exact.

On the other hand, assume that the coaction functor τ is Morita compatible. As in [Baum et al. 2016, Section 3], the *unwinding isomorphism* Φ, which is the integrated form of the covariant pair

$$\pi(a \otimes k) = i_A(a) \otimes k,$$
$$u_s = i_G(s) \otimes \lambda_s,$$

fits into a diagram

$$
\text{(4-6)}\qquad
\begin{tikzcd}
(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda} G \arrow[r, "\Phi", "\simeq"'] \arrow[d, "q^\tau_{(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda}G}"'] & (A\rtimes_\alpha G)\otimes\mathcal{K} \arrow[d, "q^\tau_{A\rtimes_\alpha G}\otimes\mathrm{id}"] \\
(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda,\sigma} G \arrow[r, dashed, "\simeq", "\Upsilon"'] & (A\rtimes_{\alpha,\sigma} G)\otimes\mathcal{K}
\end{tikzcd}
$$

i.e.,

$$
\ker q^\tau_{(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda}G} = \ker(q^\tau_{A\rtimes_\alpha G}\otimes\mathrm{id})\circ\Phi.
$$

The diagram (4-6) fits into a more elaborate diagram

$$
\begin{tikzcd}
(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda} G \arrow[r, "\Phi", "\simeq"'] \arrow[d, "q^\tau_{(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda}G}"'] & (A\rtimes_\alpha G)\otimes\mathcal{K} \arrow[d, dashed, "q^\tau_{(A\rtimes_\alpha G)\otimes\mathcal{K}}"'] \arrow[ddr, bend left, "q^\tau_{A\rtimes_\alpha G}\otimes\mathrm{id}"] & \\
(A\otimes\mathcal{K})\rtimes_{\alpha\otimes\mathrm{Ad}\,\lambda,\sigma} G \arrow[r, dashed, "\simeq", "\Phi^\tau"'] \arrow[drr, dashed, "\simeq", "\Upsilon"'] & ((A\rtimes_\alpha G)\otimes\mathcal{K})^\tau \arrow[dr, dashed, "\theta", "\simeq"'] & \\
 & & (A\rtimes_{\alpha,\sigma} G)\otimes\mathcal{K}
\end{tikzcd}
$$

which we proceed to analyze. There is a unique

$$
(\widehat{\alpha\otimes\mathrm{Ad}\,\lambda})^\tau - (\widehat{\alpha}\otimes_*\mathrm{id})^\tau
$$

equivariant homomorphism Φ^τ making the upper left rectangle commute, since τ is functorial. Moreover, Φ^τ is an isomorphism since Φ is, again by functoriality. Applying Morita compatibility of τ to the equivariant $((A\rtimes_\alpha G)\otimes\mathcal{K}) - (A\rtimes_\alpha G)$ imprimitivity bimodule $(A\rtimes_\alpha G)\otimes L^2(G)$ shows that there is a unique

$$
(\widehat{\alpha}\otimes_*\mathrm{id})^\tau - (\widehat{\alpha}^\tau\otimes_*\mathrm{id})
$$

equivariant isomorphism θ that makes the upper right triangle commute. Thus there is a unique isomorphism Υ making the lower left triangle commute, and then the outer quadrilateral commutes, as desired. □

Question 4.25. (1) Is the minimal exact and Morita compatible crossed product of [Baum et al. 2016, Section 4] naturally isomorphic to the composition of the minimal exact and Morita compatible coaction functor and the full crossed product?

(2) More generally, given a crossed-product functor on actions, when does it decompose as a full crossed product followed by a coaction functor? Does it make any difference if the crossed-product functor is exact or Morita compatible?

5. Decreasing coaction functors

In this section we introduce a particular type of coaction functor with the convenient property that we do not need to check things by going through the maximalization functor, as we'll see in Propositions 5.4 and 5.5. Suppose that for each coaction (A,δ) we have a coaction (A^τ,δ^τ) and a $\delta-\delta^\tau$ equivariant surjection $Q^\tau : A \to A^\tau$, and further suppose that for each morphism $\phi : (A,\delta) \to (B,\varepsilon)$ we have

$$\ker Q^\tau_A \subset \ker Q^\tau_B \circ \phi,$$

so that there is a unique morphism ϕ^τ making the diagram

$$\begin{tikzcd}
(A,\delta) \arrow[r, "\phi"] \arrow[d, "Q^\tau_A"'] & (B,\varepsilon) \arrow[d, "Q^\tau_B"] \\
(A^\tau,\delta^\tau) \arrow[r, dashed, "!", "\phi^\tau"'] & (B^\tau,\delta^\tau)
\end{tikzcd}$$

commute. The uniqueness and surjectivity assumptions imply that τ constitutes a functor on the category of coactions, and moreover $Q^\tau : \mathrm{id} \to \tau$ is a natural transformation.

Definition 5.1. We call a functor τ as above *decreasing* if for each coaction (A,δ) we have

$$\ker Q^\tau_A \subset \ker \Lambda_A.$$

Lemma 5.2. *Every decreasing functor τ on coactions is a coaction functor*, *and moreover* $\tau \le \mathrm{id}$.

Proof. For each coaction (A,δ), define a homomorphism q^τ_A by the commutative diagram

$$\begin{tikzcd}
A^m \arrow[d, "q^m_A"'] \arrow[dr, "q^\tau_A"] & \\
A \arrow[r, "Q^\tau_A"'] & A^\tau
\end{tikzcd}$$

where q^m_A is the maximalization map. The map q^τ is natural and surjective since both q^m and Q^τ are. We have

$$\begin{aligned}
\ker q^\tau_A &= \{a \in A^m : q^m_A(a) \in \ker Q^\tau_A\} \\
&\subset \{a \in A^m : q^m_A(a) \in \ker \Lambda_A\} \\
&= \ker \Lambda_A \circ q^m_A \\
&= \ker \Lambda_{A^m}.
\end{aligned}$$

Thus τ is a coaction functor, and then $\tau \le \mathrm{id}$ by Lemma 4.8. $\square$

Notation 5.3. For a decreasing coaction functor τ and any coaction (A,δ) put

$$A_\tau = \ker Q_A^\tau.$$

Proposition 5.4. *A decreasing coaction functor τ is exact if and only if for any short exact sequence*

$$0 \longrightarrow (I,\delta_I) \xrightarrow{\phi} (A,\delta) \xrightarrow{\psi} (B,\delta^I) \longrightarrow 0 \tag{5-1}$$

of coactions, both

$$\phi(I_\tau) = \phi(I) \cap A_\tau$$

and

$$\phi(I) + A_\tau \supset \psi^{-1}(B_\tau)$$

hold.

Proof. The proof is very similar to, and slightly easier than, that of Theorem 4.12, using the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_\tau \arrow[r, "\phi|"] \arrow[d, "\iota_I"'] & A_\tau \arrow[r, "\psi|"] \arrow[d, "\iota_A"'] & B_\tau \arrow[r] \arrow[d, "\iota_B"'] & 0 \\
0 \arrow[r] & I \arrow[r, "\phi"] \arrow[d, "Q_I^\tau"'] & A \arrow[r, "\psi"] \arrow[d, "Q_A^\tau"'] & B \arrow[r] \arrow[d, "Q_B^\tau"'] & 0 \\
0 \arrow[r] & I^\tau \arrow[r, "\phi^\tau"] \arrow[d] & A^\tau \arrow[r, "\psi^\tau"] \arrow[d] & B^\tau \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

□

Proposition 5.5. *A decreasing coaction functor τ is Morita compatible if and only if whenever (X,ζ) is an $(A,\delta)-(B,\varepsilon)$ imprimitivity bimodule, there are an $A^\tau - B^\tau$ imprimitivity bimodule X^τ and a $Q_A^\tau - Q_B^\tau$ compatible imprimitivity-bimodule homomorphism $Q_X^\tau : X \to X^\tau$.*

Proof. First suppose τ is Morita compatible. Let (X,ζ) be an $(A,\delta)-(B,\varepsilon)$ imprimitivity bimodule,and let $q_X^\tau : X^m \to X^\tau$ be a $q_A^m - q_B^m$ compatible imprimitivity-bimodule homomorphism onto an $A^\tau - B^\tau$ imprimitivity bimodule X^τ, as in Lemma 4.19. By Lemmas 4.21 and 4.19 there is also a $q_A^m - q_B^m$ compatible imprimitivity bimodule homomorphism q_X^m of $X^m \to X$. By definition, we have

$$q_A^\tau = Q_A^\tau \circ q_A^m : A^m \to A^\tau.$$

Thus

$$\begin{aligned}\ker q_X^m &= (\ker q_A^m)\cdot X^m\\ &\subset (\ker Q_A^\tau \circ q_A^m)\cdot X^m\\ &= (\ker q_A^\tau)\cdot X^m\\ &= \ker q_X^\tau,\end{aligned}$$

and hence q_X^τ factors through a commutative diagram

$$\begin{tikzcd} X^m \arrow[dr, "q_X^m"] \arrow[dd, "q_X^\tau"'] & \\ & X \arrow[dl, dashed, "Q_X^\tau", "!"'] \\ X^\tau & \end{tikzcd}$$

for a unique imprimitivity bimodule homomorphism Q_X^τ. Moreover, Q_X^τ is compatible on the left with Q_A^τ by construction, and similar reasoning, using the Rieffel correspondence of ideals, shows that it is also Q_B^τ compatible on the right.

Conversely, suppose we have (X,ζ), X^τ and Q_X^τ as indicated, and let (X^m,ζ^m) be the associated $(A^m,\delta^m)-(B^m,\varepsilon^m)$ imprimitivity bimodule from Lemma 4.15. By Lemma 4.19 it suffices to find a $q_A^m - q_B^m$ compatible imprimitivity-bimodule homomorphism $q_X^\tau : X^m \to X^\tau$. Since $q^\tau = Q^\tau \circ q^m$ on both A^m and B^m, by Lemma 4.21 and our assumptions we can take $q_X^\tau = Q_X^\tau \circ q_X^m$. □

6. Coaction functors from large ideals

The most important source of examples of the decreasing coaction functors of the preceding section is large ideals. We recall some basic concepts from [KLQ 2013; 2016]. Let E be an ideal of $B(G)$ that is *large*, meaning it is nonzero, G-invariant, and weak*-closed. Then the preannihilator ${}^\perp E$ of E in $C^*(G)$ is an ideal contained in the kernel of the regular representation λ. Write $C_E^*(G) = C^*(G)/{}^\perp E$ for the quotient group C^*-algebra and $q_E : C^*(G) \to C_E^*(G)$ for the quotient map. The ideal ${}^\perp E = \ker q_E$ of $C^*(G)$ is *weakly* δ_G-invariant, i.e., δ_G descends to a coaction, which we denote by δ_G^E, on the quotient $C_E^*(G)$.

For any coaction (A,δ) and any large ideal E of $B(G)$,

$$A_E := \{a \in A : E\cdot a = \{0\}\} = \ker(\mathrm{id}\otimes q_E)\circ\delta$$

is a *small* ideal of A (that is, an ideal contained in $\ker j_A = \ker \Lambda_A$) and we write $A^E = A/A_E$ for the quotient C^*-algebra and $Q_A^E : A \to A^E$ for the quotient map. A_E is weakly δ-invariant [KLQ 2016, Lemma 3.5], and we write δ^E for the quotient coaction on A^E.

Remark 6.1. The properties of the $B(G)$-module structure (see the Appendix) allow for a shorter proof of invariance than in [KLQ 2016]: if $a \in A_E$, $f \in B(G)$, and $g \in E$ then

$$g \cdot (f \cdot a) = (gf) \cdot a = 0,$$

because E is an ideal, and it follows that $B(G) \cdot A_E \subset A_E$.

Proposition 6.2. *The functor $(A, \delta) \mapsto (A^E, \delta^E)$ is a decreasing coaction functor, which we denote by τ_E.*

Proof. By the above discussion and Lemma 5.2, it suffices to observe that for any morphism $\phi : (A, \delta) \to (B, \varepsilon)$ of coactions and for all $a \in \ker Q^E_A$ and $f \in E$,

$$f \cdot \phi(a) = \phi(f \cdot a) = 0,$$

which implies that $\ker Q^E_A \subset \ker Q^E_B \circ \phi$. □

Remark 6.3. Proposition 6.2 should be compared with [Buss and Echterhoff 2014, Corollary 6.5 and Lemma 7.1], [Buss and Echterhoff 2015, Lemma 2.3], and [Baum et al. 2016, Lemma A.3].

Example 6.4. The functor $\tau_{B(G)}$ is the identity functor.

Example 6.5. The functor $\tau_{B_r(G)}$ is naturally isomorphic to the normalization functor.

Example 6.6. The maximalization functor is not of the form $(A, \delta) \mapsto (A^E, \delta^E)$ for any large ideal E of $B(G)$, because the maximalization functor is not decreasing in the sense of Definition 5.1.

Proposition 6.7. *For a large ideal E of $B(G)$, the coaction functor τ_E is exact if and only if, for every coaction (A, δ) and every strongly invariant ideal I of A,*

$$I + A_E \supset \{a \in A : E \cdot a \subset I\}. \tag{6-1}$$

Proof. Let

$$0 \longrightarrow (I, \zeta) \xrightarrow{\phi} (A, \delta) \xrightarrow{\psi} (B, \varepsilon) \longrightarrow 0 \tag{6-2}$$

be a short exact sequence of coactions. Exactness of the associated sequence

$$0 \longrightarrow I^E \xrightarrow{\phi^E} A^E \xrightarrow{\psi^E} B^E \longrightarrow 0 \tag{6-3}$$

will not be affected if we replace the short exact sequence (6-2) by an isomorphic one, so without loss of generality ϕ is the inclusion of an ideal I of A and ψ is the quotient map onto $B = A/I$. By Proposition 5.4, the sequence (6-3) is exact if and only if

$$I_E = I \cap A_E \tag{6-4}$$

and

$$I + A_E \supset \psi^{-1}(B_E). \tag{6-5}$$

Since

$$I_E = \{a \in I : E \cdot a = \{0\}\},$$

(6-4) automatically holds in this context. On the other hand, (6-5) is equivalent to (6-1) because

$$\begin{aligned} B_E &= \{a + I \in B = A/I : E \cdot (a + I) = \{0\}\} \\ &= \{a + I : E \cdot a \subset I\}. \end{aligned} \qquad \square$$

Remark 6.8. Techniques similar to those used in the above proof, showing that (6-4) holds automatically, can also be used to show that the functor τ_E preserves injectivity of morphisms: if $\phi : A \to B$ is an injective equivariant homomorphism and $a \in \ker \phi^E$, then we can write $a = Q_A^E(a')$ for some $a' \in A$. We have

$$0 = \phi^E(a) = \phi^E \circ Q_A^E(a') = Q_B^E \circ \phi(a'),$$

so

$$\phi(a) \in \ker Q_B^E = B_E.$$

Thus for all $f \in E$ we have

$$0 = f \cdot \phi(a') = \phi(f \cdot a'),$$

so $f \cdot a' = 0$ since ϕ is injective. But then $a' \in A_E = \ker Q_A^E$, so $a = 0$. This remark should be compared with [Buss and Echterhoff 2014, Proposition 6.2].

Corollary 6.9. *Let E and F be large ideals of $B(G)$, and let $\langle EF\rangle$ denote the weak*-closed linear span of the set EF of products. If τ_E or τ_F is exact then $\langle EF\rangle = E \cap F$.*

Proof. Without loss of generality assume that τ_E is exact. Note that, since E is an ideal of $B(G)$,

$${}^{\perp}E = \{a \in C^*(G) : E \cdot a = \{0\}\},$$

and similarly for ${}^{\perp}F$. We claim that

$${}^{\perp}E + {}^{\perp}F = {}^{\perp}\langle EF\rangle.$$

To see this, note that, since E is exact, by Proposition 6.7 with $(A,\delta) = (C^*(G), \delta_G)$ and $I = {}^\perp F$ we have

$$ {}^\perp F + {}^\perp E \supset \{a \in C^*(G) : E \cdot a \subset {}^\perp F\}. $$

Now, for $a \in C^*(G)$ we have

$$ \begin{aligned} E \cdot a \subset {}^\perp F &\iff F \cdot (E \cdot a) = \{0\} \\ &\iff (EF) \cdot a = \{0\} \\ &\overset{*}{\iff} \langle EF \rangle \cdot a = \{0\} \\ &\iff a \in {}^\perp\langle EF \rangle, \end{aligned} $$

where the equivalence at * holds since for every $a \in C^*(G)$ the map from $B(G)$ to $C^*(G)$ defined by $f \mapsto f \cdot a$ is weak*-weak continuous. Thus ${}^\perp F + {}^\perp E \supset {}^\perp\langle EF \rangle$.

For the reverse containment, note that $EF \subset E$ because E is an ideal, so $\langle EF \rangle \subset E$ because E is weak*-closed, and hence ${}^\perp E \subset {}^\perp\langle EF \rangle$. Similarly, ${}^\perp F \subset {}^\perp\langle EF \rangle$, and so ${}^\perp E + {}^\perp F \subset {}^\perp\langle EF \rangle$, proving the claim.

Now, since ${}^\perp E$ and ${}^\perp F$ are closed ideals of $C^*(G)$, it follows from the elementary duality theory for Banach spaces that

$$ {}^\perp E + {}^\perp F = {}^\perp(E \cap F), $$

and the corollary follows upon taking annihilators. □

The following result should be compared with [Baum et al. 2016, Lemma A.5]:

Proposition 6.10. *The coaction functor τ_E is Morita compatible.*

Proof. Let (X,ζ) be an $(A,\delta)-(B,\varepsilon)$ imprimitivity bimodule. Since τ is decreasing, by Proposition 5.5, it suffices to show that $X\text{-Ind}\, B_E = A_E$. The external tensor product $X \otimes C^*_E(G)$ is an $(A \otimes C^*_E(G)) - (B \otimes C^*_E(G))$ imprimitivity bimodule, and we have an $(\mathrm{id}_A \otimes q_E) - (\mathrm{id}_B \otimes q_E)$ compatible imprimitivity bimodule homomorphism

$$ \mathrm{id}_X \otimes q_E : X \otimes C^*(G) \to X \otimes C^*_E(G). $$

The composition

$$ (\mathrm{id}_X \otimes q_E) \circ \zeta : X \to M(X \otimes C^*_E(G)) $$

is an $(\mathrm{id}_A \otimes q_E) \circ \delta - (\mathrm{id}_B \otimes q_E) \circ \varepsilon$ compatible imprimitivity bimodule homomorphism. We have

$$ \begin{aligned} \ker(\mathrm{id}_A \otimes q_E) \circ \delta &= A_E, \\ \ker(\mathrm{id}_B \otimes q_E) \circ \varepsilon &= B_E. \end{aligned} $$

Thus, by [Echterhoff et al. 2006, Lemma 1.20], A_E is the ideal of A associated to the ideal B_E of B via the Rieffel correspondence. $\square$

Remark 6.11. Proposition 6.10 subsumes [KLQ 2016, Lemma 4.8], which is the special case of exterior equivalent coactions. It is tempting to try to use this to simplify the proof of [loc. cit., Theorem 4.6], which says that (A, δ) satisfies E-crossed-product duality if and only if it is isomorphic to (A^{mE}, δ^{mE}), since we have Morita equivalences

$$(A^m, \delta^m) \sim_M (A^m \otimes \mathcal{K}, \delta \otimes_* \mathrm{id}) \sim_M (A \rtimes_\delta G \rtimes_{\widehat{\delta}} G, \widehat{\widehat{\delta}}).$$

However, it turns out that appealing to Proposition 6.10 would not shorten the proof much. Nevertheless, it is interesting to note that, by Proposition 6.10, we have

$$(A, \delta) = (A^{mE}, \delta^{mE}) \quad \iff \quad (A \otimes \mathcal{K}, \delta \otimes_* \mathrm{id}) = ((A^m \otimes \mathcal{K})^E, (\delta^m \otimes_* \mathrm{id})^E),$$

or, equivalently,

$$\ker \Phi = (A \rtimes_\delta G \rtimes_{\widehat{\delta}} G)_E,$$

which by definition is equivalent to E-crossed-product duality for (A, δ).

For some purposes, albeit not for the purposes of this paper, a more appropriate coaction functor associated to E is the following (see also [Buss and Echterhoff 2014, Theorem 5.1]):

Definition 6.12. The *E-ization* of a coaction (A, δ) is

$$(A^{E\text{-ize}}, \delta^{E\text{-ize}}) := ((A^m)^E, (\delta^m)^E).$$

E-ization is a functor on the category of coactions, being the composition of the functors maximalization and τ_E. The E-ization of a $\delta - \varepsilon$ equivariant homomorphism $\phi : A \to B$ is

$$\phi^{E\text{-ize}} = (\phi^m)^E : A^{mE} \to B^{mE}.$$

Proposition 6.13. *E-ization is a coaction functor.*

Proof. We must produce a suitable natural transformation $q^{E\text{-ize}} : (A^m, \delta^m) \to (A^{E\text{-ize}}, \delta^{E\text{-ize}})$, and we take

$$q_A^{E\text{-ize}} = Q_{A^m}^E : A^m \to A^{mE} = A^{E\text{-ize}}.$$

The map $q^{E\text{-ize}}$ is natural since τ_E is a decreasing coaction functor. $\square$

Theorem 6.14. *For any large ideal E of $B(G)$, the E-ization coaction functor is Morita compatible.*

Proof. Let (X, ζ) be an $(A, \delta) - (B, \varepsilon)$ imprimitivity bimodule, with associated $(A^m, \delta^m) - (B^m, \varepsilon^m)$ imprimitivity bimodule (X^m, ζ^m). We must show that

$$X^m\text{-Ind}\ker q_B^{E\text{-ize}} = \ker q_A^{E\text{-ize}}.$$

But this follows immediately by applying Proposition 6.10 to (X^m, ζ^m), since

$$q_A^{E\text{-ize}} = Q_E^{A^m} \quad \text{and} \quad q_B^{E\text{-ize}} = Q_E^{B^m}. \qquad \square$$

Remark 6.15. For any large ideal E, the two coaction functors τ_E and E-ization have similar properties; e.g., they are both Morita compatible (Proposition 6.10 and Theorem 6.14). However, in general they are not naturally isomorphic functors. For example, if $E = B(G)$ then τ_E is the identity functor and E-ization is maximalization. That being said, for $E = B_r(G)$ we do have $\tau_E \cong \tau_E \circ$ maximalization.

Note that, given a coaction (A, δ), we have two homomorphisms of the maximalization (A^m, δ^m):

$$\begin{tikzcd}
(A^m, \delta^m) \arrow[d, "q^m"] \arrow[dr, "q^{E\text{-ize}}"] & \\
(A, \delta) & (A^{E\text{-ize}}, \delta^{E\text{-ize}})
\end{tikzcd}$$

In [KLQ 2013, Definition 3.7] we said (A, δ) is *E-determined from its maximalization* if $\ker q^m = \ker q^{E\text{-ize}}$, in which case there is a natural isomorphism $(A, \delta) \cong (A^{E\text{-ize}}, \delta^{E\text{-ize}})$.

Given an action (B, α), in [KLQ 2013, Definition 6.1] we defined the *E-crossed product* as

$$B \rtimes_{\alpha,E} G = (B \rtimes_\alpha G)/(B \rtimes_\alpha G)_E = (B \rtimes_\alpha G)^E,$$

where in the last expression we have composed the full-crossed-product functor with τ_E.

As in [Buss and Echterhoff 2014, Definition 4.5], we say a coaction (A, δ) satisfies *E-duality* (called "E-crossed product duality" in [KLQ 2016, Definition 4.3]), or is an *E-coaction*, if there is an isomorphism θ making the diagram

$$\begin{tikzcd}
A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \arrow[r, "\Phi"] \arrow[d, "Q_E"'] & A \otimes \mathcal{K} \\
A \rtimes_\delta G \rtimes_{\widehat{\delta},E} G \arrow[ur, dashed, "\theta"', "\simeq"] &
\end{tikzcd}$$

commute, or, equivalently,

$$\ker \Phi = (A \rtimes_\delta G \rtimes_{\widehat{\delta}} G)_E,$$

where Φ is the canonical surjection.

In [KLQ 2016, Theorem 4.6] we proved that (A,δ) is an E-coaction if and only if it is E-determined from its maximalization. (Theorem 5.1 of [Buss and Echterhoff 2014] proves the converse direction.)

Lemma 6.16. *For a coaction (A,δ), the following are equivalent*:

(1) (A,δ) *is an E-coaction.*

(2) (A,δ) *is E-determined from its maximalization.*

(3) *There exists a maximal coaction (B,ε) such that $(A,\delta)\cong(B^E,\varepsilon^E)$.*

Proof. The equivalence of (1) and (2) is [KLQ 2016, Theorem 4.6], and (2) trivially implies (3). Assume (3), i.e., that (B,ε) is maximal and we have an isomorphism $\theta:(B^E,\varepsilon^E)\to(A,\delta)$. The surjection $Q_E^B:(B,\varepsilon)\to(B^E,\varepsilon^E)$ is a maximalization, since ε is maximal and $\ker Q_E^B\subset\ker q_B^n$. Thus $\theta\circ Q_E^B$ is a maximalization of (A,δ). Since any two maximalizations of (A,δ) are isomorphic, there is an isomorphism ψ making the diagram

$$\begin{tikzcd}
(A^m,\delta^m) \arrow[d, "q_A^m"'] & (B,\varepsilon) \arrow[l, dashed, "\psi"', "\simeq"] \arrow[d, "Q_E"] \\
(A,\delta) & (B^E,\varepsilon^E) \arrow[l, "\simeq"', "\theta"]
\end{tikzcd}$$

commute. Thus $q_A^m\circ\psi$ is also a maximalization of (A,δ). Therefore

$$\ker q_A^m=\psi(\ker Q_E)=\psi(B_E)=A_E^m,$$

giving (2). $\square$

Theorem 6.17. *The functor τ_E restricts to give an equivalence of the category of maximal coactions to the category of E-coactions.*

In this statement, we mean the *full* subcategories of the category of coactions.

Proof. By abstract nonsense, it suffices to show that the functor is essentially surjective and fully faithful, i.e.,

(1) every E-coaction (A,δ) is isomorphic to (B^E,ε^E) for some maximal coaction (B,ε), and

(2) for any two maximal coactions (A,δ) and (B,ε),

$$\phi\mapsto\phi^E$$

maps the set of equivariant homomorphisms $\phi:A\to B$ bijectively onto the set of equivariant homomorphisms $\psi:A^E\to B^E$.

Statement (1) is immediate from Lemma 6.16. For (2), given maximal coactions (A, δ) and (B, ε) and distinct nondegenerate equivariant homomorphisms $\phi, \psi : A \to B$, we have an equivariant commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "Q_A^E"] \arrow[dd, "\Lambda_A"'] & & B \arrow[dr, "Q_B^E"] \arrow[dd, "\Lambda_B"] & \\
& A^E \arrow[rr, "\phi^E"] \arrow[dl, "\Lambda_A^E"] & & B^E \arrow[dl, "\Lambda_B^E"] \\
A^n \arrow[rr, "\phi^n"'] & & B^n &
\end{tikzcd}$$

where Q_A^E is a maximalization of (A^E, δ^E), Λ_A is a normalization of (A, δ), and Λ_A^E is a normalization of (A^E, δ^E), and similarly for the right-hand triangle involving the Bs. There is a similar commutative diagram for ψ. Since the normalizations ϕ^n and ψ^n are distinct, by [Bédos et al. 2011, Corollary 6.1.19], we must have $\phi^E \neq \psi^E$ by commutativity of the diagram. This proves injectivity. For the surjectivity, let $\sigma : A^E \to B^E$ be an equivariant homomorphism. Then the maximalization $\sigma^m : A \to B$ of σ is the unique equivariant homomorphism making the diagram

$$\begin{tikzcd}
A \arrow[r, "\sigma^m"] \arrow[d, "Q_A^E"'] & B \arrow[d, "Q_B^E"] \\
A^E \arrow[r, "\sigma"'] & B^E
\end{tikzcd}$$

commute. Applying the functor τ_E, we see that the diagram

$$\begin{tikzcd}
A \arrow[r, "\sigma^m"] \arrow[d, "Q_A^E"'] & B \arrow[d, "Q_B^E"] \\
A^E \arrow[r, "(\sigma^m)^E"'] & B^E
\end{tikzcd}$$

also commutes, so we must have $\sigma^m = ((\sigma^m)^E)^m$ by the universal property of maximalization, and hence $\sigma = (\sigma^m)^E$ by [loc. cit.]. $\square$

Remark 6.18. Much of the development in this paper regarding “classical” categories carries over to the “nondegenerate” categories (involving multiplier algebras). The nondegenerate version of the above result resembles the “maximal-normal equivalence” of [Kaliszewski and Quigg 2009, Theorem 3.3], which says that normalization restricts to an equivalence between maximal and normal coactions.

However, there are some properties missing: for example, the functor τ_E is not a reflector in the categorical sense, because

$$Q_E : (A^E, \delta^E) \to (A^{EE}, \delta^{EE})$$

is not an isomorphism in general. Indeed, [KLQ 2016, Proposition 8.4] shows that if (A, δ) is a maximal coaction then the composition $(\mathrm{id} \otimes q_E) \circ \delta^E$ in the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "Q_E"] & A^E \arrow[r, "\delta^E"] \arrow[dr, "(\mathrm{id}\otimes q_E)\circ\delta^E"'] & M(A^E \otimes C^*(G)) \arrow[d, "\mathrm{id}\otimes q_E"] \\
& & M(A^E \otimes C^*_E(G))
\end{tikzcd}$$

need not be faithful. Thus we cannot characterize the E-coactions as the coactions that are "E-normal" in the sense that the map Q_E is faithful. Furthermore, unlike with normalization, Remark 6.15 shows that τ_E is not isomorphic to its composition with maximalization.

Question 6.19. Let $\mathcal{F}$ be a collection of large ideals of $B(G)$, and let

$$F = \bigcap_{E \in \mathcal{F}} E.$$

Then F is a large ideal of $B(G)$. Is τ_F a greatest lower bound for the coaction functors $\{\tau_E : E \in \mathcal{F}\}$? (It is easy to see that τ_F is a lower bound.) What if we take $\mathcal{F}$ to be the set of all large ideals E of $B(G)$ for which τ_E is exact?

Question 6.20. Given a coaction functor τ, is there a large ideal E of $B(G)$ such that, after restricting to maximal coactions, τ is naturally isomorphic to τ_E? Note that at the level of objects the statement is false: [Buss and Echterhoff 2014, Example 5.4] gives a source of examples of a maximal coaction (A, δ) and a weakly invariant ideal $I \subset \ker q_A^n$ such that the quotient coaction $(A/I, \delta^I)$ is not of the form (A^E, δ^E) for any large ideal E. (Theorem 6.10 of [KLQ 2016] gives related examples, albeit not involving maximal coactions.)

Here is a related question: do there exist coaction functors that include the Buss–Echterhoff examples? Such a functor could not be exact, since the Buss–Echterhoff examples are explicitly based upon short exact sequences whose image under the quotient maps are not exact. We could ask the same question for the functor τ_E, which, again, is exact for $E = B(G)$ but not for $E = B_r(G)$.

Question 6.21. For which large ideals E is the coaction functor E-ization exact? Exactness trivially holds for $E = B(G)$, since $B(G)$-ization coincides with maximalization. On the other hand, exactness does not always hold for $E = B_r(G)$, because Gromov has shown the existence of nonexact groups.

Question 6.22. Let τ be the minimal exact and Morita compatible coaction functor. Applying τ to the canonical coaction $(C^*(G), \delta_G)$, we get a coaction $(C^*(G)^\tau, \delta_G^\tau)$, with a canonical quotient map

$$q^\tau : C^*(G) \to C^*(G)^\tau.$$

Then

$$E_\tau := (\ker q^\tau)^\perp$$

is a large ideal of $B(G)$, by [KLQ 2013, Corollary 3.13].

Does the functor τ coincide with E_τ-ization? This is related to the following question: is

$$E_\tau = \bigcap \{E : E \text{ is a large ideal such that } E\text{-ization is exact}\}?$$

Again we could ask the analogous questions for τ_E. See also the discussion in [Baum et al. 2016, Section 8.1].

Remark 6.23. Related to Question 6.19 above, what if we consider only finitely many large ideals? Let E and F be two large ideals, and let $D = E \cap F$, which is also a large ideal. Suppose that the coaction functors τ_E and τ_F are both exact.

Is τ_D exact? We proved in Corollary 6.9 that exactness of E implies that D is the weak*-closed span of the set of products EF, and then we can deduce from this that if

$$0 \longrightarrow (I, \gamma) \xrightarrow{\phi} (A, \delta) \xrightarrow{\psi} (B, \varepsilon) \longrightarrow 0$$

is a short exact sequence of coactions, and if we assume that δ is *w-proper* in the sense that $(\omega \otimes \mathrm{id}) \circ \delta(A) \subset C^*(G)$ for all $\omega \in A^*$, then the sequence

$$0 \longrightarrow I^D \xrightarrow{\phi^D} A^D \xrightarrow{\psi^D} B^D \longrightarrow 0$$

is exact. We see a way to parlay this into a proof that τ_D is indeed exact, but this requires a somewhat more elaborate version of Morita compatibility, involving not only imprimitivity bimodules but more general C^*-correspondences. This will perhaps resemble the property that Buss, Echterhoff and Willett call *correspondence functoriality* (see [Buss et al. 2015, Theorem 4.9]). We plan to address this in a forthcoming publication.

Appendix: $B(G)$-module lemmas

Every coaction $\delta : A \to M(A \otimes C^*(G))$ gives rise to a $B(G)$-module structure on A via

$$f \cdot a = (\mathrm{id} \otimes f) \circ \delta(a) \quad \text{for } f \in B(G),\ a \in A.$$

We feel that this module structure is under-appreciated, and will point out here several situations in which it makes things easier, since it allows us to avoid computations with tensor products.

Proposition A.1. *Let (A,δ) and (B,ε) be coactions of G, and let $\phi : A \to B$ be a homomorphism. Then ϕ is $\delta-\varepsilon$ equivariant if and only if it is a module map, i.e.,*

$$\phi(f\cdot a) = f\cdot\phi(a) \quad \textit{for all } f\in B(G),\ a\in A.$$

Proof. First assume that ϕ is $\delta-\varepsilon$ equivariant, and let $f\in B(G)$ and $a\in A$. Then

$$\begin{aligned}\phi(f\cdot a) &= \phi((\mathrm{id}\otimes f)\circ\delta(a))\\ &= (\mathrm{id}\otimes f)((\phi\otimes\mathrm{id})\circ\delta(a))\\ &= (\mathrm{id}\otimes f)(\varepsilon\circ\phi(a))\\ &= f\cdot\phi(a).\end{aligned}$$

Conversely, assume that ϕ is a module map, and let $a\in A$. Then for every $f\in B(G)$ the above computation shows that

$$(\mathrm{id}\otimes f)((\phi\otimes\mathrm{id})\circ\delta(a)) = (\mathrm{id}\otimes f)(\varepsilon\circ\phi(a)),$$

and it follows that $(\phi\otimes\mathrm{id})\circ\delta(a) = \varepsilon\circ\phi(a)$ since slicing by $B(G) = C^*(G)^*$ separates points of $M(B\otimes C^*(G))$. □

Proposition A.2. *Let (A,δ) be a coaction, and let I be an ideal of A. Then I is weakly δ-invariant if and only if it is invariant for the module structure, i.e.,*

$$B(G)\cdot I\subset I.$$

Proof. First assume that I is δ-invariant, and let $f\in B(G)$ and $a\in I$. We must show that $f\cdot a\in I$. Let $q : A\to A/I$ be the quotient map. We have

$$\begin{aligned}q(f\cdot a) &= q((\mathrm{id}\otimes f)(\delta(a)))\\ &= (\mathrm{id}\otimes f)((q\otimes\mathrm{id})\circ\delta(a))\\ &= 0 \quad (\text{since } I\subset\ker(q\otimes\mathrm{id})\circ\delta).\end{aligned}$$

Conversely, assume that I is $B(G)$-invariant, and let $a\in I$. We need to show that $a\in\ker(q\otimes\mathrm{id})\circ\delta$. For every $f\in B(G)$ we have $f\cdot a\in I$, so

$$0 = q(f\cdot a) = (\mathrm{id}\otimes f)((q\otimes\mathrm{id})\circ\delta(a)).$$

It then follows that $(q\otimes\mathrm{id})\circ\delta(a) = 0$ since slicing by $B(G)$ separates points in $M(A\otimes C^*(G))$. □

Remark A.3. It has been noticed elsewhere in the literature that the $B(G)$-module structure can be useful in other ways. For example, δ is slice-proper [KLQ 2016, Definition 5.1] if and only if the maps

$$f \mapsto f \cdot a : B(G) \to A$$

are weak*-weak continuous (for $a \in A$) [KLQ 2016, Lemma 5.3]. Also, for any full coaction (A, δ),

$$A_0 := \overline{\operatorname{span}}\{A(G) \cdot A\}$$

is a C^*-subalgebra and a nondegenerate $A(G)$-submodule of A, where $A(G)$ is the Fourier algebra of A, and δ is nondegenerate if and only if $A_0 = A$ [Quigg 1994, Lemma 1.2, Corollary 1.5] (see also [Katayama 1984, Lemma 2]). In the same vein, [Quigg 1994, Corollary 1.7] says that if B is a nondegenerate $A(G)$-submodule of $M(A)$, then $\delta|_B$ is a nondegenerate coaction of G on B.

Acknowledgement

We thank the referee for comments that significantly improved our paper.

References

[Baum et al. 2016] P. Baum, E. Guentner, and R. Willett, "Expanders, exact crossed products, and the Baum–Connes conjecture", *Ann. K-Theory* **1**:2 (2016), 155–208. Zbl 1331.46064

[Bédos et al. 2011] E. Bédos, S. Kaliszewski, and J. Quigg, "Reflective-coreflective equivalence", *Theory Appl. Categ.* **25**:6 (2011), 142–179. MR 2805748 Zbl 1232.18002

[Brown and Guentner 2013] N. P. Brown and E. P. Guentner, "New C*-completions of discrete groups and related spaces", *Bull. Lond. Math. Soc.* **45**:6 (2013), 1181–1193. MR 3138486 Zbl 06237632

[Buss and Echterhoff 2014] A. Buss and S. Echterhoff, "Universal and exotic generalized fixed-point algebras for weakly proper actions and duality", *Indiana Univ. Math. J.* **63**:6 (2014), 1659–1701. MR 3298718 Zbl 1320.46052

[Buss and Echterhoff 2015] A. Buss and S. Echterhoff, "Imprimitivity theorems for weakly proper actions of locally compact groups", *Ergodic Theory Dynam. Systems* **35**:8 (2015), 2412–2457. MR 3456601 Zbl 06540094

[Buss et al. 2015] A. Buss, S. Echterhoff, and R. Willett, "Exotic crossed products and the Baum–Connes conjecture", *J. Reine Angew. Math.* (online publication October 2015).

[Echterhoff and Raeburn 1995] S. Echterhoff and I. Raeburn, "Multipliers of imprimitivity bimodules and Morita equivalence of crossed products", *Math. Scand.* **76**:2 (1995), 289–309. MR 1354585 Zbl 0843.46049

[Echterhoff et al. 2004] S. Echterhoff, S. Kaliszewski, and J. Quigg, "Maximal coactions", *Internat. J. Math.* **15**:1 (2004), 47–61. MR 2039211 Zbl 1052.46051

[Echterhoff et al. 2006] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*, Mem. Amer. Math. Soc. **850**, Providence, RI, 2006. MR 2203930 Zbl 1097.46042

[Fischer 2004] R. Fischer, "Maximal coactions of quantum groups", Preprint 350, University of Münster, SFB 478 Geometrische Strukturen in der Mathematik, 2004.

[Green 1978] P. Green, "The local structure of twisted covariance algebras", *Acta Math.* **140**:1 (1978), 191–250. MR 0493349 Zbl 0407.46053

[Kaliszewski and Quigg 2009] S. Kaliszewski and J. Quigg, "Categorical Landstad duality for actions", *Indiana Univ. Math. J.* **58**:1 (2009), 415–441. MR 2504419 Zbl 1175.46060

[Katayama 1984] Y. Katayama, "Takesaki's duality for a nondegenerate co-action", *Math. Scand.* **55**:1 (1984), 141–151. MR 769030 Zbl 0598.46042

[KLQ 2013] S. Kaliszewski, M. B. Landstad, and J. Quigg, "Exotic group C^*-algebras in noncommutative duality", *New York J. Math.* **19** (2013), 689–711. MR 3141810 Zbl 1294.46047

[KLQ 2016] S. Kaliszewski, M. B. Landstad, and J. Quigg, "Exotic coactions", *Proc. Edinburgh Math. Soc.* (2) **59**:2 (2016), 411–434. Zbl 06580770

[Landstad et al. 1987] M. B. Landstad, J. Phillips, I. Raeburn, and C. E. Sutherland, "Representations of crossed products by coactions and principal bundles", *Trans. Amer. Math. Soc.* **299**:2 (1987), 747–784. MR 869232 Zbl 0722.46031

[Nilsen 1999] M. Nilsen, "Full crossed products by coactions, $C_0(X)$-algebras and C^*-bundles", *Bull. London Math. Soc.* **31**:5 (1999), 556–568. MR 1703865 Zbl 0987.46046

[Quigg 1994] J. C. Quigg, "Full and reduced C^*-coactions", *Math. Proc. Cambridge Philos. Soc.* **116**:3 (1994), 435–450. MR 1291751 Zbl 0830.22005

Received June 6, 2015. Revised March 31, 2016.

S. KALISZEWSKI
SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES
ARIZONA STATE UNIVERSITY
TEMPE, AZ 85287
UNITED STATES

kaliszewski@asu.edu

MAGNUS B. LANDSTAD
DEPARTMENT OF MATHEMATICAL SCIENCES
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
NO-7491 TRONDHEIM
NORWAY

magnusla@math.ntnu.no

JOHN QUIGG
SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES
ARIZONA STATE UNIVERSITY
TEMPE, AZ 85287
UNITED STATES

quigg@asu.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.191

COHOMOLOGY AND EXTENSIONS OF BRACES

VICTORIA LEBED AND LEANDRO VENDRAMIN

Braces and linear cycle sets are algebraic structures playing a major role in the classification of involutive set-theoretic solutions to the Yang–Baxter equation. This paper introduces two versions of their (co)homology theories. These theories mix the Harrison (co)homology for the abelian group structure and the (co)homology theory for general cycle sets, developed earlier by the authors. Different classes of brace extensions are completely classified in terms of second cohomology groups.

1. Introduction

A (*left*) *brace* is an abelian group $(A, +)$ with an additional group operation $\circ$ such that for all $a, b, c \in A$, the following compatibility condition holds:

$$a \circ (b + c) + a = a \circ b + a \circ c. \tag{1-1}$$

The two group structures necessarily share the same neutral element, denoted by 0. Braces, in a slightly different but equivalent form, were introduced by Rump [2007]; the definition above goes back to Cedó, Jespers, and Okniński [Cedó et al. 2014]. To get a feeling of what braces look like, and to convince oneself that they are not as rare in practice as one might think, the reader is referred to Bachiller's classification of braces of order p^3 [2015a]. The growing interest into these structures is due to a number of reasons. First, braces generalize radical rings. Second, Catino and Rizzo [2009] and Catino, Colazzo, and Stefanelli [Catino et al. 2015; 2016] unveiled the role of an F-linear version of this notion into the classification problem for regular subgroups of affine groups over a field F. Third, braces are enriched cycle sets, and are therefore important in the study of set-theoretic solutions to the Yang–Baxter equation (YBE), as we now recall.

A *cycle set*, as defined by Rump [2005], is a set X with a binary operation $\cdot$ having bijective left translations $X \to X$, $a \mapsto b \cdot a$, and satisfying the relation

$$(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c). \tag{1-2}$$

MSC2010: 20E22, 20N02, 55N35, 16T25.
Keywords: brace, cycle set, Yang–Baxter equation, extension, cohomology.

Rump showed that *nondegenerate* cycle sets (i.e., with invertible squaring map $a \mapsto a \cdot a$) are in bijection with nondegenerate involutive set-theoretic solutions to the Yang–Baxter equation. Such solutions form a combinatorially rich class of structures, connected with many other domains of algebra: semigroups of I-type, Bieberbach groups, Hopf algebras, Garside groups, etc. The cycle set approach turned out to be extremely fruitful for elucidating the structure of such solutions and obtaining classification results (see, for instance, [Cedó et al. 2010a; 2014; Chouraqui 2010; Dehornoy 2015; Gateva-Ivanova 2015; Gateva-Ivanova and Majid 2008; Gateva-Ivanova and Van den Bergh 1998; Jespers and Okniński 2005; Rump 2007; 2008; 2014; Smoktunowicz 2015a; 2015b; Vendramin 2016] and references therein). In spite of the intensive ongoing research on cycle sets, their structure is still far from being completely understood. This can be illustrated by numerous conjectures and open questions in the area, many of which were formulated by Gateva-Ivanova and Cameron [Gateva-Ivanova 2004; Gateva-Ivanova and Cameron 2012] and by Cedó, Jespers, and del Río [Cedó et al. 2010b].

Etingof, Schedler, and Soloviev [Etingof et al. 1999] initiated the study of the structure group of a solution to the YBE — and in particular of a cycle set. These ideas were further explored in [Lu et al. 2000; Soloviev 2000] for noninvolutive solutions. Concretely, the *structure group* $G_{(X,\,\cdot\,)}$ of a cycle set $(X, \cdot\,)$ is the free group on the set X, modulo the relations

$$(a \cdot b)\,a = (b \cdot a)\,b$$

for all $a, b \in X$.[1] In [Etingof et al. 1999], the structure group of a nondegenerate cycle set $(X, \cdot\,)$ was shown to be isomorphic, as a set, to the free abelian group $\mathbb{Z}^{(X)}$ on X; see also [Lebed and Vendramin 2015] for an explicit graphical form of this isomorphism. The group $G_{(X,\,\cdot\,)}$ thus carries a second, abelian, group structure — the one pulled back from $\mathbb{Z}^{(X)}$ — and becomes a brace. Moreover, $G_{(X,\,\cdot\,)}$ inherits a cycle set structure from X, and yields a key example of the following notion. A *linear cycle set* is a cycle set $(A, \cdot\,)$ with an abelian group operation $+$ satisfying, for all $a, b, c \in A$, the compatibility conditions

$$a \cdot (b + c) = a \cdot b + a \cdot c, \tag{1-3}$$

$$(a + b) \cdot c = (a \cdot b) \cdot (a \cdot c). \tag{1-4}$$

This structure also goes back to Rump [2007], who showed it to be equivalent to the brace structure, via the relation

$$a \cdot b = a^{-1} \circ (a + b).$$

[1] Some authors prefer an alternative relation $a(a \cdot b) = b(b \cdot a)$, which defines an isomorphic group.

Understanding structure groups and certain classes of their quotients is often regarded as a reasonable first step towards understanding cycle sets. Even better: Bachiller, Cedó, and Jespers [Bachiller et al. 2015a] recently reduced the classification problem for cycle sets to that for braces. This explains the growing interest in braces and linear cycle sets. As pointed out by Bachiller, Cedó, Jespers, and Okniński [Bachiller et al. 2015b], an extension theory for braces would be crucial for classification purposes, as well as for elaborating new examples. This served as motivation for our paper.

Lebed and Vendramin [2015] developed a cohomology theory for general cycle sets, in which second cohomology groups were given particular attention: they were shown to encode central cycle set extensions. Here we propose homology and cohomology theories for linear cycle sets, and thus for braces. As usual, central linear cycle set extensions turn out to be classified by the second cohomology groups.

For pedagogical reasons, we first study extensions that are trivial on the level of abelian groups, together with a corresponding (co)homology theory (Sections 2–3). Such extensions are still of interest, since it is often the cycle set operation that is the most significant part of the linear cycle set structure (as in the example of structure groups). On the other hand, they are technically much easier to handle than the general extensions (Sections 4–5). We therefore found it instructive to present this "reduced" case before the general one.

When finishing this paper, we learned that an analogous extension theory was independently developed by Bachiller [2015b], using the language of braces. Some fragments of it in the F-linear setting also appeared in the work of Catino, Colazzo, and Stefanelli [2015]. An alternative approach to extensions was suggested earlier by Ben David and Ginosar [2016]. Concretely, they studied the lifting problem for bijective 1-cocycles — which is yet another avatar of braces. Their work was translated into the language of braces by Bachiller [2015a]. Our choice of the linear cycle set language leads to more transparent constructions. Moreover, it made possible the development of a full cohomology theory extending the degree 2 constructions motivated by the extension analysis. Such a theory was missing in all the previous approaches.

2. Reduced linear cycle set cohomology

From now on we work with linear cycle sets (LCS). As explained in the introduction, all constructions and results can be directly translated into the language of braces. We will perform this translation for major results only.

Take an LCS $(A, \cdot, +)$ and an abelian group Γ. For $n > 0$, let $RC_n(A; \Gamma)$ denote the abelian group $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}A^{\times n} \simeq \Gamma^{(A^{\times n})}$, modulo the linearity relation

$$\gamma(a_1, \ldots, a_{n-1}, a_n + a_n') = \gamma(a_1, \ldots, a_{n-1}, a_n) + \gamma(a_1, \ldots, a_{n-1}, a_n') \tag{2-1}$$

for the last copy of A. Denote by $RC_n^{\mathrm{D}}(A;\Gamma)$ the abelian subgroup of $RC_n(A;\Gamma)$ generated by the *degenerate n-tuples*, i.e., $\gamma(a_1,\dots,a_n)$ with $a_i=0$ for some $1\leqslant i\leqslant n$. Consider also the quotient $RC_n^{\mathrm{N}}(A;\Gamma)=RC_n(A;\Gamma)/RC_n^{\mathrm{D}}(A;\Gamma)$. Further, define the maps $\partial_n:\Gamma A^{\times n}\to\Gamma A^{\times(n-1)}$, $n>1$, as the linearizations of

$$\begin{aligned}\partial_n(a_1,\dots,a_n)=(a_1\cdot a_2,\dots,a_1\cdot a_n)&\\ +\sum_{i=1}^{n-2}(-1)^i(a_1,\dots,a_i+a_{i+1},\dots,a_n)&\\ +(-1)^{n-1}(a_1,\dots,a_{n-2},a_n).&\end{aligned}\tag{2-2}$$

Complete this family of maps by $\partial_1=0$. Dually, for $n>0$, let $RC^n(A;\Gamma)$ denote the set of maps $f:A^{\times n}\to\Gamma$ linear in the last coordinate:

$$f(a_1,\dots,a_{n-1},a_n+a_n')=f(a_1,\dots,a_{n-1},a_n)+f(a_1,\dots,a_{n-1},a_n'),\tag{2-3}$$

and let $RC^n_{\mathrm{N}}(A;\Gamma)\subset RC^n(A;\Gamma)$ comprise the maps vanishing on all degenerate n-tuples. Define the maps $\partial^n:\mathrm{Fun}(A^{\times n},\Gamma)\to\mathrm{Fun}(A^{\times(n+1)},\Gamma)$, $n\geqslant 1$, by

$$\begin{aligned}(\partial^n f)(a_1,\dots,a_{n+1})=f(a_1\cdot a_2,\dots,a_1\cdot a_{n+1})&\\ +\sum_{i=1}^{n-1}(-1)^i f(a_1,\dots,a_i+a_{i+1},\dots,a_{n+1})&\\ +(-1)^n f(a_1,\dots,a_{n-1},a_{n+1}).&\end{aligned}\tag{2-4}$$

These formulas resemble the group (co)homology construction for $(A,+)$. We will now show that they indeed define a (co)homology theory.

Proposition 2.1. *Let $(A,\cdot,+)$ be a linear cycle set and Γ be an abelian group.*

(1) *The maps $\partial_\bullet$ above*
   - *square to zero: $\partial_{n-1}\partial_n=0$ for all $n>1$;*
   - *induce maps $RC_n(A;\Gamma)\to RC_{n-1}(A;\Gamma)$;*
   - *and further restrict to maps $RC_n^{\mathrm{D}}(A;\Gamma)\to RC_{n-1}^{\mathrm{D}}(A;\Gamma)$.*

(2) *The maps $\partial^\bullet$ above*
   - *square to zero: $\partial^{n+1}\partial^n=0$ for all $n\geqslant 1$;*
   - *restrict to maps $RC^n(A;\Gamma)\to RC^{n+1}(A;\Gamma)$;*
   - *and further restrict to maps $RC^n_{\mathrm{N}}(A;\Gamma)\to RC^{n+1}_{\mathrm{N}}(A;\Gamma)$.*

The induced or restricted maps from the proposition will be abusively denoted by the same symbols $\partial_\bullet$, $\partial^\bullet$. In the proof we shall need the special properties of the zero element of an LCS.

Lemma 2.2. *In any LCS A, the relations $a\cdot 0=0$ and $0\cdot a=a$ hold for all $a\in A$.*

Proof. By the LCS axioms, one has $a\cdot 0 = a\cdot(0+0) = a\cdot 0 + a\cdot 0$ and hence $a\cdot 0 = 0$. Similarly, $0\cdot a = (0+0)\cdot a = (0\cdot 0)\cdot(0\cdot a) = 0\cdot(0\cdot a)$, and the relation $a = 0\cdot a$ follows by canceling out 0 (recall that the left translation $0\cdot -$ is bijective). $\square$

Proof of Proposition 2.1. We treat only the homological statements here; they imply the cohomological ones by duality.

The maps ∂_n can be presented as signed sums $\partial_n = \sum_{i=0}^{n-1}(-1)^i \partial_{n;i}$, where

$$\partial_{n;0}(a_1,\dots,a_n) = (a_1\cdot a_2,\dots,a_1\cdot a_n), \tag{2-5}$$

$$\partial_{n;i}(a_1,\dots,a_n) = (a_1,\dots,a_i+a_{i+1},\dots,a_n), \qquad 1\leqslant i\leqslant n-2, \tag{2-6}$$

$$\partial_{n;n-1}(a_1,\dots,a_n) = (a_1,\dots,a_{n-2},a_n). \tag{2-7}$$

The relation $\partial_{n-1}\partial_n = 0$ then classically reduces to the "almost commutativity" $\partial_{n-1;j}\partial_{n;i} = \partial_{n-1;i}\partial_{n;j+1}$ for all $i\leqslant j$. In the case $i>0$ this latter relation is either tautological, or follows from the associativity of $+$. For $i=0<j$, it follows from the left distributivity (1-3) for A. For $i=0=j$, it is a consequence of the second LCS relation (1-4) for A.

Further, using the linearity (1-3) of the left translations $a_n\mapsto a_1\cdot a_n$, one sees that when applied to expressions of type

$$(\dots,a_{n-1},a_n+a_n') - (\dots,a_{n-1},a_n) - (\dots,a_{n-1},a_n'),$$

all the maps $\partial_{n;i}$ yield expressions of the same type. Hence their signed sums $\partial_\bullet$ induce a differential on $RC_\bullet$. The possibility to further restrict to $RC_\bullet^{\mathrm{D}}$ is guaranteed by Lemma 2.2. $\square$

Proposition 2.1 legitimizes the following definition:

Definition 2.3. The *reduced* (*resp.*, *normalized*) *cycles*, *boundaries*, *and homology groups of a linear cycle set* $(A,\cdot,+)$ with coefficients in an abelian group Γ are those of the chain complex $(RC_\bullet(A;\Gamma),\partial_\bullet)$ (resp., $(RC_\bullet^{\mathrm{N}}(A;\Gamma),\partial_\bullet)$) above. Dually, the *reduced* (*resp.*, *normalized*) *cocycles*, *coboundaries*, *and cohomology groups of* $(A,\cdot,+)$ are those of the complex $(RC^\bullet(A;\Gamma),\partial^\bullet)$ (resp., $(RC_{\mathrm{N}}^\bullet(A;\Gamma),\partial^\bullet)$). We use the usual notation for these groups: $RQ_n(A;\Gamma)$, $RQ_n^{\mathrm{N}}(A;\Gamma)$, $RQ^n(A;\Gamma)$, $RQ_{\mathrm{N}}^n(A;\Gamma)$, where Q is one of the letters Z, B, or H.

Remark 2.4. We actually showed that our (co)homology constructions can be refined into (co)simplicial ones in the proof of Proposition 2.1.

Example 2.5. Recall from the introduction that for a nondegenerate cycle set $(X,\cdot)$, the free abelian group $(\mathbb{Z}^{(X)},+)$ can be seen as a linear cycle set, with the cycle set operation induced from $\cdot$. In this case $RC_1(\mathbb{Z}^{(X)};\Gamma)$ is simply the abelian group $\Gamma\otimes_{\mathbb{Z}}\mathbb{Z}^{(X)} = \Gamma^{(X)}$, and for $a_1,a_2\in\mathbb{Z}^{(X)}$ one calculates $\partial_1(a_1,a_2) = a_1\cdot a_2 - a_2$. Standard arguments from LCS theory then yield

$$RH_1(\mathbb{Z}^{(X)};\Gamma)\cong\Gamma^{(\mathrm{Orb}(X))},$$

where $\mathrm{Orb}(X)$ is the set of *orbits* of X, i.e., classes for the equivalence relation generated by $a_1 \cdot a_2 \sim a_2$ for all $a_1, a_2 \in X$. Similarly, one calculates the first reduced cohomology group:

$$RH^1(\mathbb{Z}^{(X)}; \Gamma) \cong \mathrm{Fun}(\mathrm{Orb}(X), \Gamma).$$

We finish with a comparison between the (co)homology of an LCS $(A, \cdot, +)$ and the (co)homology of its underlying cycle set $(A, \cdot)$, as defined in [Lebed and Vendramin 2015]. Recall that the homology $H_n^{\mathrm{CS}}(A; \Gamma)$ of $(A, \cdot)$ is computed by the complex $(\Gamma^{(A^{\times n})}, \partial_n^{\mathrm{CS}})$, where

$$\partial_n^{\mathrm{CS}}(a_1, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} \big((a_i \cdot a_1, \dots, a_i \cdot a_{i-1}, a_i \cdot a_{i+1}, \dots, a_i \cdot a_n) \\ - (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \big).$$

Dually, the cohomology $H^n_{\mathrm{CS}}(A; \Gamma)$ of $(A, \cdot)$ is computed from the complex

$$(\mathrm{Fun}(A^{\times n}, \Gamma), \partial^n_{\mathrm{CS}}), \quad \text{with } \partial^n_{\mathrm{CS}} f = f \circ \partial^{\mathrm{CS}}_{n+1}.$$

Denoting by $(-1)^\sigma$ the sign of the permutation σ, define $S_n : \Gamma^{(A^{\times n})} \to RC_n(A; \Gamma)$ as the composition of the antisymmetrization map

$$\gamma(a_1, \dots, a_n) \mapsto \sum_{\sigma \in \mathrm{Sym}_{n-1}} (-1)^\sigma \gamma(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, a_n)$$

and the obvious projection $\Gamma^{(A^{\times n})} \twoheadrightarrow RC_n(A; \Gamma)$.

Proposition 2.6. *Let $(A, \cdot, +)$ be a linear cycle set and Γ be an abelian group. The map S defined above yields a map of chain complexes*

$$S_n : (\Gamma^{(A^{\times n})}, \partial_n^{\mathrm{CS}}) \to (RC_n(A; \Gamma), \partial_n).$$

Proof. One has to compare the evaluations of the maps $\partial_n S_n$ and $S_{n-1} \partial_n^{\mathrm{CS}}$ on $\gamma(a_1, \dots, a_n)$. For this, it is convenient to use the decomposition

$$\partial_n = \sum_{i=0}^{n-1} (-1)^i \partial_{n;i}$$

from (2-5)–(2-7). For $0 < i < n-1$, the map $\partial_{n;i} S_n$ is zero: in its evaluation, the terms $\pm\gamma(\dots, a_j + a_k, \dots)$ and $\mp\gamma(\dots, a_k + a_j, \dots)$, with the sum at the i-th position, cancel. A careful sign inspection yields

$$\partial_{n;0} S_n(\gamma(a_1,\ldots,a_n)) \\ = \sum_{i=1}^{n-1}(-1)^{i-1} S_{n-1}(\gamma(a_i\cdot a_1,\ldots,a_i\cdot a_{i-1},a_i\cdot a_{i+1},\ldots,a_i\cdot a_n)),$$

$$\partial_{n;n-1} S_n(\gamma(a_1,\ldots,a_n)) \\ = \sum_{i=1}^{n-1}(-1)^{n-1-i} S_{n-1}(\gamma(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n)),$$

hence the maps $\partial_n S_n$ and $S_{n-1}\partial_n^{\mathrm{CS}}$ coincide. □

As a consequence, one obtains the dual map

$$S^n : (RC^n(A;\Gamma),\partial^n) \to (\mathrm{Fun}(A^{\times n},\Gamma),\partial^n_{\mathrm{CS}})$$

of cochain complexes, and the induced maps in (co)homology.

3. Cycle-type extensions vs. reduced 2-cocycles

We now turn to a study of the *reduced* 2*-cocycles* of a linear cycle set $(A,\cdot,+)$, i.e., maps $f : A\times A\to\Gamma$ (where Γ is an abelian group) satisfying

$$f(a,b+c) = f(a,b)+f(a,c), \tag{3-1}$$

$$f(a+b,c) = f(a\cdot b,a\cdot c)+f(a,c), \tag{3-2}$$

for all $a,b,c\in A$. The last relation, together with the commutativity of $+$, yields

$$f(a\cdot b,a\cdot c)+f(a,c) = f(b\cdot a,b\cdot c)+f(b,c), \tag{3-3}$$

implying $\partial^2_{\mathrm{CS}}(f)=0$, so our f is necessarily a cocycle of the cycle set $(A,\cdot)$.

Among the reduced 2-cocycles we distinguish the *reduced* 2*-coboundaries*

$$\partial^1(\theta)(a,b) = \theta(a\cdot b)-\theta(b),$$

where the map $\theta : A\to\Gamma$ is linear.

Example 3.1. Let A and Γ be abelian groups. Consider the *trivial linear cycle set* structure $a\cdot_{\mathrm{tr}} b = b$ over A. A map $f : A\times A\to\Gamma$ is a reduced 2-cocycle of this LCS if and only if f is a *bicharacter*, in the sense of the bilinearity relations

$$f(a+b,c) = f(a,c)+f(b,c) \quad\text{and}\quad f(a,b+c) = f(a,b)+f(a,c).$$

The reduced 2-coboundaries are all trivial in this case. Thus $RH^2(A;\Gamma)$ is the abelian group of bicharacters of A with values in Γ. Observe that for the cycle set $(A,\cdot_{\mathrm{tr}})$, all the differentials ∂^n_{CS} vanish. The second cohomology group $H^2_{\mathrm{CS}}(A;\Gamma)$

of this cycle set thus comprises all the maps $f : A \times A \to \Gamma$, and is strictly larger than $RH^2(A; \Gamma)$.

Example 3.2. Let $A = \{0, 1, 2, 3\} = \mathbb{Z}/4$ be the cyclic group of 4 elements written additively. Then A is a brace with

$$a \circ b = a + b + 2ab \quad \text{and} \quad a^{-1} = (2a - 1)a.$$

The corresponding linear cycle set structure on A is given by the operation

$$a \cdot b = a^{-1} \circ (a + b) = (1 + 2a)b,$$

which is b when one of a, b is even, and $b + 2$ otherwise. Take $\Gamma = \{0, 1\} = \mathbb{Z}/2$. For a map $f : \mathbb{Z}/4 \times \mathbb{Z}/4 \to \mathbb{Z}/2$, relation (3-1) means that f is of the form $f(a, b) = b\psi(a)$ (where the product is taken in $\mathbb{Z}/2$, and b is reduced modulo 2), for some $\psi : \mathbb{Z}/4 \to \mathbb{Z}/2$. Relation (3-2) then translates as

$$\psi(a + b) = \psi(b + 2ab) + \psi(a).$$

The substitution $b = 0$ yields $\psi(0) = 0$. Analyzing other values of a and b, one sees that $\psi(1)$ and $\psi(3)$ can be chosen arbitrarily, and $\psi(2)$ has to equal $\psi(1) + \psi(3)$. The reduced 2-coboundaries are again trivial: a linear map $\theta : \mathbb{Z}/4 \to \mathbb{Z}/2$ is necessarily of the form $\theta(a) = at$ for some constant $t \in \mathbb{Z}/2$, yielding

$$\theta(a \cdot b) = (a \cdot b)t = (1 + 2a)bt = bt = \theta(b)$$

(since $2a = 0$ in $\mathbb{Z}/2$). Summarizing, one gets

$$RH^2(\mathbb{Z}/4; \mathbb{Z}/2) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Let us now turn to the underlying cycle set $(\mathbb{Z}/4, \cdot\,)$. Playing with (3-3), one verifies that its 2-cocycles are maps $f : \mathbb{Z}/4 \times \mathbb{Z}/4 \to \mathbb{Z}/2$ verifying 3 linear relations:

$$\begin{aligned} f(0, 1) + f(0, 3) &= 0, \\ f(2, 1) + f(2, 3) &= 0, \\ f(1, 1) + f(1, 3) + f(3, 1) + f(3, 3) &= 0. \end{aligned}$$

Its only nontrivial 2-coboundary is $f(a, b) = ab \mod 2$. This implies

$$\begin{aligned} &Z^2_{\mathrm{CS}}(\mathbb{Z}/4; \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{4\times 4-3} = (\mathbb{Z}/2)^{13}, \\ &H^2_{\mathrm{CS}}(\mathbb{Z}/4; \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{12}. \end{aligned}$$

We will now construct extensions of our LCS A by Γ out of 2-cocycles, show that any central cycle-type extension is isomorphic to one of this type, and that reduced 2-cocycles, modulo reduced 2-coboundaries, classify such extensions.

Lemma 3.3. *Let $(A, \cdot, +)$ be a linear cycle set, let Γ be an abelian group, and let $f : A \times A \to \Gamma$ be a map. Then the abelian group $\Gamma \oplus A$ with the operation*

$$(\gamma, a) \cdot (\gamma', a') = (\gamma' + f(a, a'), a \cdot a'), \qquad \gamma, \gamma' \in \Gamma,\ a, a' \in A$$

is a linear cycle set if and only if f is a reduced 2-cocycle, i.e., $f \in RZ^2(A; \Gamma)$.

Notation 3.4. The LCS from the lemma is denoted by $\Gamma \oplus_f A$.

Proof. The left translation invertibility for $\Gamma \oplus_f A$ follows from to the same property for A. Properties (1-3) and (1-4) are equivalent for $\Gamma \oplus_f A$ to, respectively, properties (3-1) and (3-2) from the definition of a 2-cocycle for f. The cycle set property (1-2) follows from (1-4) and the commutativity of $+$. □

Lemma 3.3 and the correspondence between linear cycle sets and braces yield the following result.

Lemma 3.5. *Let $(A, \circ, +)$ be a brace, let Γ be an abelian group, and let $f : A \times A \to \Gamma$ be a map. Then the abelian group $\Gamma \oplus A$ with the product*

$$(\gamma, a) \circ (\gamma', a') = (\gamma + \gamma' + f(a, a'), a \circ a'), \qquad \gamma, \gamma' \in \Gamma,\ a, a' \in A,$$

is a brace if and only if for the corresponding linear cycle set $(A, \cdot, +)$, the map $\bar{f}(a, b) = f(a, a \cdot b)$ is a reduced 2-cocycle.

Before introducing the notion of LCS extensions, we need some preliminary definitions.

Definition 3.6. A *morphism* between linear cycle sets A and B is a map $\varphi : A \to B$ preserving the structure, i.e., for all $a, a' \in A$ one has $\varphi(a + a') = \varphi(a) + \varphi(a')$ and $\varphi(a \cdot a') = \varphi(a) \cdot \varphi(a')$. The *kernel* of φ is defined by $\operatorname{Ker} \varphi = \varphi^{-1}(0)$. The notions of the *image* $\operatorname{Im} \varphi = \varphi(A)$, of a *short exact sequence* of linear cycle sets, and of *linear cycle subsets*, are defined in the obvious way. A linear cycle subset A' of A is called *central* if for all $a \in A$, $a' \in A'$, one has $a \cdot a' = a'$ and $a' \cdot a = a$.

For a LSC morphism $\varphi : A \to B$, $\operatorname{Ker} \varphi$ and $\operatorname{Im} \varphi$ are clearly linear cycle subsets of A and B respectively. Lemma 2.2 can be rephrased by stating that $\{0\}$ is a central linear cycle subset of A.

Definition 3.7. A *central cycle-type extension* of a linear cycle set $(A, \cdot, +)$ by an abelian group Γ is the datum of a short exact sequence of linear cycle sets

$$0 \to \Gamma \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0, \tag{3-4}$$

where Γ is endowed with the trivial cycle set structure $\gamma \cdot \gamma' = \gamma'$, its image $\iota(\Gamma)$ is central in E (in the sense of Definition 3.6), and the short exact sequence of abelian groups underlying (3-4) splits.

The adjective *cycle-type* refers here to the fact that our extensions are interesting on the level of the cycle set operation $\cdot$ only, and trivial on the level of the additive operation $+$, since we require the short exact sequences to linearly split. More general extensions — those taking into account the additive operation as well — are postponed until the next section. Cycle-type extensions are important, for example, for comparing the LCS structures on the structure group of a cycle set before and after a cycle set extension; see the introduction for more detail on structure groups, and [Lebed and Vendramin 2015] for the cycle set extension theory.

The LCS $\Gamma \oplus_f A$ from Lemma 3.3 is an extension of A by Γ in the obvious way. We now show that this example is essentially exhaustive.

Definition 3.8. Two central cycle-type LCS extensions

$$\Gamma \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A \quad \text{and} \quad \Gamma \overset{\iota'}{\rightarrowtail} E' \overset{\pi'}{\twoheadrightarrow} A$$

are called *equivalent* if there exists an LCS isomorphism $\varphi : E \to E'$ making the following diagram commute:

(3-5)
$$\begin{tikzcd} & E \arrow[dd, "\varphi", "\sim"'] \arrow[dr, two heads, "\pi"] & \\ \Gamma \arrow[ur, tail, "\iota"] \arrow[dr, tail, "\iota'"'] & & A \\ & E' \arrow[ur, two heads, "\pi'"'] & \end{tikzcd}$$

The set of equivalence classes of central cycle-type extensions of A by Γ is denoted by $\mathrm{CTExt}(A, \Gamma)$.

Lemma 3.9. *Let $\Gamma \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A$ be a central cycle-type LCS extension, and $s : A \to E$ be a linear section of π. Then the map*

$$\tilde{f} : A \times A \to E, \qquad (a, a') \mapsto s(a) \cdot s(a') - s(a \cdot a')$$

takes values in $\iota(\Gamma)$ and defines a reduced cocycle $f \in RZ^2(A; \Gamma)$. Extensions E and $\Gamma \oplus_f A$ are equivalent. Furthermore, a cocycle f' obtained from another section s' of π is cohomologous to f.

Proof. The computation

$$\pi(\tilde{f}(a, a')) = \pi s(a) \cdot \pi s(a') - \pi s(a \cdot a') = a \cdot a' - a \cdot a' = 0$$

yields $\operatorname{Im} \tilde{f} \subseteq \operatorname{Ker} \pi = \operatorname{Im} \iota$ (by the definition of a short exact sequence). Hence the map $f : A \times A \to \Gamma$ can be defined by the formula $f = \iota^{-1} \tilde{f}$. It remains to check relations (3-1)-(3-2) for this map. The linearity of s and of the left translations

$t_b : a \mapsto b \cdot a$ gives

$$\begin{aligned}\tilde{f}(a, b+c) &= s(a)\cdot s(b+c) - s(a\cdot(b+c))\\ &= s(a)\cdot(s(b)+s(c)) - s(a\cdot b + a\cdot c)\\ &= s(a)\cdot s(b) + s(a)\cdot s(c) - s(a\cdot b) - s(a\cdot c) = \tilde{f}(a,b) + \tilde{f}(a,c).\end{aligned}$$

hence $f(a, b+c) = f(a,b) + f(a,c)$, by the linearity of ι. Similarly, one has

$$\begin{aligned}\tilde{f}(a+b, c) &= s(a+b)\cdot s(c) - s((a+b)\cdot c)\\ &= (s(a)+s(b))\cdot s(c) - s((a\cdot b)\cdot(a\cdot c))\\ &= (s(a)\cdot s(b))\cdot(s(a)\cdot s(c)) + \tilde{f}(a\cdot b, a\cdot c) - s(a\cdot b)\cdot s(a\cdot c)\\ &= \tilde{f}(a\cdot b, a\cdot c) + (\tilde{f}(a,b) + s(a\cdot b))\cdot(s(a)\cdot s(c)) - s(a\cdot b)\cdot s(a\cdot c)\\ &\overset{(1)}{=} \tilde{f}(a\cdot b, a\cdot c) + s(a\cdot b)\cdot(s(a)\cdot s(c)) - s(a\cdot b)\cdot s(a\cdot c)\\ &= \tilde{f}(a\cdot b, a\cdot c) + s(a\cdot b)\cdot(s(a)\cdot s(c) - s(a\cdot c))\\ &= \tilde{f}(a\cdot b, a\cdot c) + s(a\cdot b)\cdot \tilde{f}(a,c)\\ &\overset{(2)}{=} \tilde{f}(a\cdot b, a\cdot c) + \tilde{f}(a,c).\end{aligned}$$

In (1) we got rid of $\tilde{f}(a,b) \in \iota(\Gamma)$ since the centrality of $\iota(\Gamma)$ yields

$$(\tilde{f}(a,b)+x)\cdot y = (\tilde{f}(a,b)\cdot x)\cdot(\tilde{f}(a,b)\cdot y) = x\cdot y$$

for all $x, y \in E$. This centrality was also used in (2). The relation $f(a+b, c) = f(a\cdot b, a\cdot c) + f(a,c)$ is now obtained from the corresponding relation for $\tilde{f}$ by applying ι^{-1}.

We will next show that the linear map $\varphi : \Gamma \oplus_f A \to E,\ \gamma \oplus a \mapsto \iota(\gamma) + s(a)$ yields an equivalence of extensions. It is bijective, the inverse given by the map $x \mapsto \iota^{-1}(x - s\pi(x)) \oplus \pi(x)$ (this map is well defined since $x - s\pi(x) \in \operatorname{Ker}\pi = \operatorname{Im}\iota$). Let us check that φ intertwines the cycle set operations. One has

$$\begin{aligned}\varphi((\gamma\oplus a)\cdot(\gamma'\oplus a')) &= \varphi((\gamma' + f(a,a'))\oplus a\cdot a') = \iota(\gamma' + f(a,a')) + s(a\cdot a')\\ &= \iota(\gamma') + \tilde{f}(a,a') + (s(a)\cdot s(a') - \tilde{f}(a,a'))\\ &= \iota(\gamma') + s(a)\cdot s(a') = s(a)\cdot\iota(\gamma') + s(a)\cdot s(a')\\ &= s(a)\cdot(\iota(\gamma') + s(a')) = (\iota(\gamma)+s(a))\cdot(\iota(\gamma')+s(a'))\\ &= \varphi(\gamma\oplus a)\cdot\varphi(\gamma'\oplus a').\end{aligned}$$

We use the centrality of $\iota(\gamma')$ and $\iota(\gamma)$. The commutativity of the diagram (3-5) is obvious, and completes the proof.

Suppose now that the reduced cocycles f and f' are obtained from the sections s and s' respectively. Put $\tilde{\theta} = s - s' : A \to E$. This is a linear map with its image contained in $\operatorname{Ker}\pi = \operatorname{Im}\iota$. Hence it defines a linear map $\theta : A \to \Gamma$. To show that f and f' are cohomologous, we establish the property $f' - f = \partial^1\theta$ by computing

$$\begin{aligned}
(\tilde{f} - \tilde{f}')(a, a') &= \tilde{f}(a, a') - \tilde{f}'(a, a') \\
&= s(a) \cdot s(a') - s(a \cdot a') - s'(a) \cdot s'(a') + s'(a \cdot a') \\
&= s(a) \cdot s(a') - s(a) \cdot s'(a') - \tilde{\theta}(a \cdot a') \\
&= s(a) \cdot (s(a') - s'(a')) - \tilde{\theta}(a \cdot a') \\
&= s(a) \cdot \tilde{\theta}(a') - \tilde{\theta}(a \cdot a') \overset{(1)}{=} \tilde{\theta}(a') - \tilde{\theta}(a \cdot a'),
\end{aligned}$$

and applying ι^{-1}, where we use the centrality of $\tilde{\theta}(a')$ in (1). □

We now compare extensions constructed out of different 2-cocycles.

Lemma 3.10. *Let* $(A, \cdot, +)$ *be a linear cycle set, let* Γ *be an abelian group, and let* $f, f' \in RZ^2(A; \Gamma)$ *be two reduced* 2*-cocycles. The linear cycle set extensions* $\Gamma \oplus_f A$ *and* $\Gamma \oplus_{f'} A$ *are equivalent if and only if* f *and* f' *are cohomologous.*

Proof. Suppose that a linear map $\varphi : \Gamma \oplus_f A \to \Gamma \oplus_{f'} A$ provides an equivalence of extensions. The commutativity of the diagram (3-5) forces it to be of the form $\varphi(\gamma \oplus a) = (\gamma + \theta(a)) \oplus a$ for some linear map $\theta : A \to \Gamma$. Further, one computes

$$\begin{aligned}
\varphi((\gamma \oplus a) \cdot (\gamma' \oplus a')) &= \varphi((\gamma' + f(a, a')) \oplus a \cdot a') \\
&= (\gamma' + f(a, a') + \theta(a \cdot a')) \oplus a \cdot a', \\
\varphi(\gamma \oplus a) \cdot \varphi(\gamma' \oplus a') &= ((\gamma + \theta(a)) \oplus a) \cdot ((\gamma' + \theta(a')) \oplus a') \\
&= (\gamma' + \theta(a') + f'(a, a')) \oplus a \cdot a'.
\end{aligned}$$

Thus the map φ entwines the cycle set operations if and only if $f' - f$ is the coboundary $\partial^1\theta$.

In the opposite direction, take cohomologous cocycles f and f'. This means that the relation $f' - f = \partial^1\theta$ holds for a linear map $\theta : A \to \Gamma$. Repeating the arguments above, one verifies that the map $\varphi(\gamma \oplus a) = (\gamma + \theta(a)) \oplus a$ is an equivalence of extensions $\Gamma \oplus_f A \to \Gamma \oplus_{f'} A$. □

Put together, the preceding lemmas yield:

Theorem 3.11. *Let* $(A, \cdot, +)$ *be a linear cycle set and* Γ *be an abelian group. The construction from Lemma 3.9 yields a bijective correspondence*

$$\mathrm{CTExt}(A, \Gamma) \overset{1:1}{\longleftrightarrow} RH^2(A; \Gamma).$$

We finish this section by observing that in degree 2, the normalization brings nothing new to the reduced LCS cohomology theory:

Proposition 3.12. *In a linear cycle set* $(A, \cdot, +)$, *every reduced* 2*-cocycle is normalized. Moreover*, *one has an isomorphism in cohomology*:

$$RH^2(A; \Gamma) \cong RH^2_{\mathrm{N}}(A; \Gamma).$$

Proof. Putting $a = b = 0$ in the defining relation (3-2) for a reduced 2-cocycle f, and using the properties of the element 0 from Lemma 2.2, $f(0, c) = 0$ for all $c \in A$. Moreover, $f(c, 0) = 0$ by linearity. So f is normalized, hence the identification

$$RZ^2(A; \Gamma) = RZ^2_{\mathrm{N}}(A; \Gamma).$$

In degree 1 the normalized and usual complexes coincide, yielding the desired cohomology group isomorphism in degree 2. □

4. Full linear cycle set cohomology

The previous section treated linear cycle set extensions of the form $\Gamma \oplus_f A$. They can be thought of as the direct product $\Gamma \oplus A$ of LCS with the cycle set operation $\cdot$ deformed by f. From now on we will handle a more general situation: the additive operation $+$ on $\Gamma \oplus A$ will be deformed as well. Most proofs in this general case are analogous to but more technical than those from the previous sections.

Take a linear cycle set $(A, \cdot, +)$ and an abelian group Γ. For $i \geqslant 0$, $j \geqslant 1$, let $\mathrm{ShC}_{i,j}(A; \Gamma)$ be the abelian subgroup of $\Gamma^{(A^{\times(i+j)})}$, generated by the *partial shuffles*

$$\sum_{\sigma \in \mathrm{Sh}_{r,j-r}} (-1)^{\sigma} \gamma(a_1, \dots, a_i, a_{i+\sigma^{-1}(1)}, \dots, a_{i+\sigma^{-1}(j)}) \tag{4-1}$$

taken for all $1 \leqslant r \leqslant j-1$, $a_k \in A$, $\gamma \in \Gamma$. Here $\mathrm{Sh}_{r,j-r}$ is the subset of all the permutations σ of j elements satisfying $\sigma(1) \leqslant \cdots \leqslant \sigma(r)$, $\sigma(r+1) \leqslant \cdots \leqslant \sigma(j)$. The term *shuffle* is used when $i = 0$. Put

$$C_{i,j}(A; \Gamma) = \Gamma^{(A^{\times(i+j)})} / \mathrm{ShC}_{i,j}(A; \Gamma).$$

Recall the notation

$$\partial_{n;0}(a_1, \dots, a_n) = (a_1 \cdot a_2, \dots, a_1 \cdot a_n), \tag{4-2}$$

$$\partial_{n;i}(a_1, \dots, a_n) = (a_1, \dots, a_i + a_{i+1}, \dots, a_n), \quad 1 \leqslant i \leqslant n-1, \tag{4-3}$$

from the proof of Proposition 2.1, and consider the coordinate omitting maps

$$\partial'_{n;i}(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n), \quad 1 \leqslant i \leqslant n. \tag{4-4}$$

Combine (the linearizations of) these maps into what we will show to be horizontal and vertical differentials of a bicomplex:

$$\partial^h_{i,j} = \partial_{i+j;0} + \sum_{k=1}^{i-1} (-1)^k \partial_{i+j;k} + (-1)^i \partial'_{i+j;i}, \qquad i \geqslant 1,\ j \geqslant 1; \tag{4-5}$$

$$-\partial^v_{i,j} = \partial'_{i+j;i+1} + \sum_{k=1}^{j-1} (-1)^k \partial_{i+j;i+k} + (-1)^j \partial'_{i+j;i+j}, \quad i \geqslant 0,\ j \geqslant 2. \tag{4-6}$$

Here the empty sums are zero by convention. As before, $C^{\mathrm{D}}_{i,j}(A;\Gamma)$ denotes the abelian subgroup of $\Gamma^{(A^{\times(i+j)})}$ generated by the degenerate $(i+j)$-tuples, and $C^{\mathrm{N}}_{i,j}(A;\Gamma)$ is the quotient $\Gamma^{(A^{\times(i+j)})}/(C^{\mathrm{D}}_{i,j}(A;\Gamma) + \mathrm{ShC}_{i,j}(A;\Gamma))$.

Dually, for $f \in \mathrm{Fun}(A^{\times(i+j)}, \Gamma)$, put

$$\partial_h^{i,j} f = f \circ \partial^h_{i+1,j} \quad \text{and} \quad \partial_v^{i,j} f = f \circ \partial^v_{i,j+1},$$

where $i \geqslant 0$, $j \geqslant 1$, and f is extended to $\mathbb{Z}^{(A^{\times(i+j)})}$ by linearity. Let $C^{i,j}(A;\Gamma)$ be the abelian group of maps $A^{\times(i+j)} \to \Gamma$ whose linearization vanishes on all partial shuffles (4-1) (with γ omitted), and let $C_{\mathrm{N}}^{i,j}(A;\Gamma) \subseteq C^{i,j}(A;\Gamma)$ comprise the maps which are moreover zero on all the degenerate $(i+j)$-tuples.

We now assemble these data into both chain and cochain bicomplex structures with normalization.

Theorem 4.1. *Let $(A, \cdot, +)$ be a linear cycle set and Γ be an abelian group.*

(1) *The abelian groups $\Gamma^{(A^{\times(i+j)})}$, $i \geqslant 0$, $j \geqslant 1$, together with the linear maps $\partial^h_{i,j}$ and $\partial^v_{i,j}$ above, form a chain bicomplex. In other words, the following relations are satisfied:*

$$\partial^h_{i-1,j}\,\partial^h_{i,j} = 0, \qquad i \geqslant 2,\ j \geqslant 1; \tag{4-7}$$

$$\partial^v_{i,j-1}\,\partial^v_{i,j} = 0, \qquad i \geqslant 0,\ j \geqslant 3; \tag{4-8}$$

$$\partial^h_{i,j-1}\,\partial^v_{i,j} = \partial^v_{i-1,j}\,\partial^h_{i,j}, \quad i \geqslant 1,\ j \geqslant 2. \tag{4-9}$$

Moreover, these maps restrict to the subgroups $\mathrm{ShC}_{i,j}(A;\Gamma)$ *and* $C^{\mathrm{D}}_{i,j}(A;\Gamma)$, *and thus induce chain bicomplex structures on* $C_{i,j}(A;\Gamma)$ *and* $C^{\mathrm{N}}_{i,j}(A;\Gamma)$.

(2) *The linear maps $\partial_h^{i,j}$ and $\partial_v^{i,j}$ yield a cochain bicomplex structure for the abelian groups* $\mathrm{Fun}(A^{\times(i+j)},\Gamma)$, $i \geqslant 0$, $j \geqslant 1$. *This structure restricts to* $C^{i,j}(A;\Gamma)$ *and further to* $C_{\mathrm{N}}^{i,j}(A;\Gamma)$.

We abusively denote the induced or restricted maps from the theorem by the same symbols $\partial^h_{\bullet}$, $\partial^v_{\bullet}$, etc.

The proof of the theorem relies on the following interpretation of our bicomplex. Its j-th row is almost the complex from Proposition 2.1, with a slight modification:

the last entry in an n-tuple, to which the $\partial_{n;i}$ with $i>0$ did nothing and on which $\partial_{n;0}$ acted by a left translation $a_n \mapsto a_1 \cdot a_n$, is replaced with the j-tuple of last elements behaving in the same way. In the i-th column, the first i entries of $A^{\times(i+\bullet)}$ are never affected; on the remaining entries the vertical differentials $\partial^v_{i,\bullet}$ act as the differentials from Proposition 2.1 computed for the trivial cycle set operation $a \cdot b = b$. Alternatively, the i-th column can be seen as the Hochschild complex for $(A,+)$ with coefficients in $A^{\times i}$, on which A acts trivially on both sides. Modding out $\mathrm{ShC}_{i,j}(A;\Gamma)$ means passing from the Hochschild to the Harrison complex in each column.

Proof. As usual, it suffices to treat only the homological statements.

Due to the observation preceding the proof, the horizontal relation (4-7) and the vertical relation (4-8) follow from Proposition 2.1. For the mixed relation (4-9), note that the horizontal and vertical differentials involved affect, respectively, the first i and the last j entries of an $(i+j)$-tuple, with the exception of the $\partial_{n;0}$ component of ∂^h. However, this component also commutes with ∂^v because of the linearity (with respect to $+$) of the left translation $a_1 \cdot -$ involved.

Applying a left translation $a \cdot -$ to each entry of a partial shuffle (4-1), one still gets a partial shuffle. Consequently, the horizontal differentials ∂^h restrict to $\mathrm{ShC}_{i,j}(A;\Gamma)$. In order to show that the ∂^v restrict to $\mathrm{ShC}_{i,j}(A;\Gamma)$ as well, it suffices to check that the expression

$$\begin{aligned}\sum_{\sigma\in \mathrm{Sh}_{r,j-r}} &(-1)^{\sigma}(a_{\sigma^{-1}(2)},\dots,a_{\sigma^{-1}(j)})\\ &+\sum_{k=1}^{j-1}(-1)^k \sum_{\sigma\in \mathrm{Sh}_{r,j-r}}(-1)^{\sigma}(a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(k)}+a_{\sigma^{-1}(k+1)},\dots,a_{\sigma^{-1}(j)})\\ &\qquad\qquad+(-1)^j\sum_{\sigma\in \mathrm{Sh}_{r,j-r}}(-1)^{\sigma}(a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(j-1)})\end{aligned}$$

is a linear combination of shuffles for all $j \geqslant 1$, $1 \leqslant r \leqslant j-1$, $a_k \in A$. Let S_1, S_2, and S_3 denote the three sums above, and consider the classical notation

$$\sqcup\!\sqcup_{r,j-r}(a_1,\dots,a_j)=\sum_{\sigma\in \mathrm{Sh}_{r,j-r}}(-1)^{\sigma}(a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(j)})$$

for shuffles, and the convention $\sqcup\!\sqcup_{0,j} = \sqcup\!\sqcup_{j,0} = \mathrm{Id}$. Recall also notation (4-3)–(4-4). The sums S_i then rewrite as

$$\begin{aligned}S_1 &= \sqcup\!\sqcup_{r-1,j-r}\partial'_{j;1}+(-1)^r\sqcup\!\sqcup_{r,j-r-1}\partial'_{j;r+1},\\ S_3 &= (-1)^r\sqcup\!\sqcup_{r-1,j-r}\partial'_{j;r}+(-1)^j\sqcup\!\sqcup_{r,j-r-1}\partial'_{j;j},\\ S_2 &= \sum_{k=1}^{r-1}(-1)^k\sqcup\!\sqcup_{r-1,j-r}\partial_{j;k}+\sum_{k=r+1}^{j-1}(-1)^k\sqcup\!\sqcup_{r,j-r-1}\partial_{j;k},\end{aligned}$$

with empty sums declared to be zero. The decomposition for S_1 follows from the analysis of the two possibilities for $\sigma^{-1}(1)$ with $\sigma \in \mathrm{Sh}_{r,j-r}$, namely, $\sigma^{-1}(1)=1$ and $\sigma^{-1}(1)=r+1$. The decomposition for S_3 corresponds to the dichotomy $\sigma^{-1}(j)=r$ or $\sigma^{-1}(j)=j$. In S_3, the summands with $\sigma^{-1}(k)=u \leqslant r < v=\sigma^{-1}(k+1)$ and $\sigma^{-1}(k)=v, \sigma^{-1}(k+1)=u$ appear with opposite signs and can therefore be discarded. The remaining ones can be divided into two classes: those with $\sigma^{-1}(k) < \sigma^{-1}(k+1) \leqslant r$ and those with $r < \sigma^{-1}(k) < \sigma^{-1}(k+1)$, giving the decomposition above. Our S_i are thus signed sums of shuffles, with the exception of the cases $r \in \{1, j-1\}$. For $r=1$, the nonshuffle terms $\partial'_{j;1}$ and $-\partial'_{j;1}$ appear in S_1 and S_3 respectively; they annihilate each other in the total sum. The case $r=j-1$ is treated similarly.

The possibility to restrict all the ∂^h and ∂^v to $C^{\mathrm{D}}_{i,j}(A;\Gamma)$ is taken care of, as usual, by Lemma 2.2. As a consequence, one obtains a chain bicomplex structure on $C^{\mathrm{N}}_{i,j}(A;\Gamma)$. □

We are now in a position to define the full (co)homology of a linear cycle set:

Definition 4.2. The *cycles, boundaries, homology groups of a linear cycle set* $(A,\cdot,+)$ with coefficients in an abelian group Γ are those of the total chain complex

$$\Big(C_n(A;\Gamma)=\bigoplus_{i+j=n} C_{i,j}(A;\Gamma),\ \partial_n|_{C_{i,j}}=\partial^h_{i,j}+(-1)^i\partial^v_{i,j}\Big)$$

of the bicomplex above. Dually, the *cocycles, coboundaries, cohomology groups of* $(A,\cdot,+)$ are those of the complex

$$\Big(C^n(A;\Gamma)=\bigoplus_{i+j=n} C^{i,j}(A;\Gamma),\ \partial^n=\partial^*_{n+1}\Big).$$

In the *normalized* case, one uses the complexes

$$\Big(C^{\mathrm{N}}_n(A;\Gamma)=\bigoplus_{i+j=n} C^{\mathrm{N}}_{i,j}(A;\Gamma),\ \partial_n\Big) \quad \text{and} \quad \Big(C^n_{\mathrm{N}}(A;\Gamma)=\bigoplus_{i+j=n} C^{i,j}_{\mathrm{N}}(A;\Gamma),\ \partial^n\Big).$$

We use the usual notations $Q_n(A;\Gamma)$, etc., where Q is one of the letters Z, B, or H.

Remark 4.3. In fact our (co)chain bicomplex constructions can be refined into bisimplicial ones.

Remark 4.4. Instead of considering the total complex of our bicomplex, one could start by, say, computing the homology $H^v_{i,\bullet}$ of each column. The horizontal differentials then induce a chain complex structure on each row $H^v_{\bullet,j}$. Observe that the first row is precisely the complex from Proposition 2.1. Its homology is then the reduced homology of our linear cycle set.

5. General linear cycle set extensions

Our next step is to describe what a 2-cocycle looks like for the full version of linear cycle set cohomology theory. Such a 2-cocycle consists of two components $f, g : A \times A \to \Gamma$, seen as elements of $C^{1,1}(A; \Gamma) = \mathrm{Fun}(A \times A, \Gamma)$ and $C^{0,2}(A; \Gamma) = \mathrm{Sym}(A \times A, \Gamma)$, respectively. Here Sym denotes the abelian group of *symmetric* maps, i.e., satisfying

$$g(a, b) = g(b, a). \tag{5-1}$$

These maps should satisfy three identities, one for each component of

$$C^3(A; \Gamma) = C^{2,1}(A; \Gamma) \oplus C^{1,2}(A; \Gamma) \oplus C^{0,3}(A; \Gamma).$$

Explicitly, these identities read

$$f(a + b, c) = f(a \cdot b, a \cdot c) + f(a, c), \tag{5-2}$$

$$f(a, b + c) - f(a, b) - f(a, c) = g(a \cdot b, a \cdot c) - g(b, c), \tag{5-3}$$

$$g(a, b) + g(a + b, c) = g(b, c) + g(a, b + c). \tag{5-4}$$

In particular, f is a 2-cocycle of the cycle set $(A, \cdot)$, and g is a symmetric 2-cocycle of the group $(A, +)$. The reduced cocycles are precisely those with $g = 0$. Further, the 2-coboundaries are couples of maps

$$f(a, b) = \theta(a \cdot b) - \theta(b), \tag{5-5}$$

$$g(a, b) = \theta(a + b) - \theta(a) - \theta(b) \tag{5-6}$$

for some $\theta : A \to \Gamma$.

We next give some elementary properties of 2-cocycles and 2-coboundaries.

Lemma 5.1. *Let (f, g) be a 2-cocycle of a linear cycle set $(A, \cdot, +)$ with coefficients in an abelian group Γ.*

(1) *For all $x \in A$,*
$$f(0, x) = f(x, 0) = 0,$$
$$g(0, x) = g(x, 0) = g(0, 0).$$

(2) *The 2-cocycle (f, g) is normalized if and only if $g(0, 0) = 0$.*

Proof. Let us prove the first claim. The relation $f(0, x) = 0$ follows from (5-2) by choosing $a = 0$. Similarly, the relation $f(x, 0) = 0$ is (5-3) specialized at $b = c = 0$. Substitutions $b = 0$ and either $a = 0$ or $c = 0$ in (5-4) yield the last relation. Now the second claim directly follows from the previous point. $\square$

Lemma 5.2. *Let $(A, \cdot, +)$ be a linear cycle set, let Γ be an abelian group, and let $f, g : A \times A \to \Gamma$ be two maps. Then the set $\Gamma \times A$ with the operations*

$$(\gamma, a) + (\gamma', a') = (\gamma + \gamma' + g(a, a'),\ a + a'),$$
$$(\gamma, a) \cdot (\gamma', a') = (\gamma' + f(a, a'),\ a \cdot a')$$

for $\gamma, \gamma' \in \Gamma$, $a, a' \in A$, is a linear cycle set if and only if (f, g) is a 2-cocycle, i.e., $(f, g) \in Z^2(A; \Gamma)$.

Notation 5.3. The LCS from the lemma is denoted by $\Gamma \oplus_{f,g} A$.

Proof. The left translation invertibility for $\Gamma \oplus_{f,g} A$ follows from the same property for A. Properties (1-3) and (1-4) for $\Gamma \oplus_{f,g} A$ are equivalent to, respectively, properties (5-3) and (5-2) for (f, g). The associativity and the commutativity of $+$ on $\Gamma \oplus_{f,g} A$ are encoded by property (5-4) for $\Gamma \oplus_{f,g} A$ and the symmetry of g respectively. Finally, if (f, g) is a 2-cocycle, then Lemma 5.1 implies that $(-g(0, 0), 0)$ is the zero element for $(\Gamma \oplus_{f,g} A,\ +)$, and the opposite of (γ, a) is $(-\gamma - g(0, 0) - g(a, -a),\ -a)$. □

As we did in Lemma 3.5, we now translate Lemma 5.2 into the language of braces.

Lemma 5.4. *Let $(A, \circ, +)$ be a brace, Γ be an abelian group, and $f, g : A \times A \to \Gamma$ be two maps. Then the set $\Gamma \times A$ with the operations*

$$(\gamma, a) + (\gamma', a') = (\gamma + \gamma' + g(a, a'),\ a + a'),$$
$$(\gamma, a) \circ (\gamma', a') = (\gamma + \gamma' + f(a, a'),\ a \circ a')$$

for $\gamma, \gamma' \in \Gamma$, $a, a' \in A$, is a brace if and only if for the corresponding linear cycle set $(A, \cdot, +)$, the maps

$$\bar{f}(a, b) = -f(a, a \cdot b) + g(a, b) \tag{5-7}$$

and g form a 2-cocycle $(\bar{f}, g) \in Z^2(A; \Gamma)$.

Proof. Recall the correspondence $a \cdot b = a^{-1} \circ (a + b)$ between the corresponding brace and LCS operations. It can also be rewritten as $a \circ b = a + a * b$, where the map $a \mapsto a * b$ is the inverse of the left translation $a \mapsto a \cdot b$.

Now, given any $(\bar{f}, g) \in Z^2(A; \Gamma)$, the formulas from Lemma 5.2 describe an LCS structure on $\Gamma \times A$. Its operation $*$ reads

$$(\gamma, a) * (\gamma', a') = (\gamma' - \bar{f}(a, a * a'),\ a * a').$$

The operations

$$(\gamma, a) + (\gamma', a') = (\gamma + \gamma' + g(a, a'),\ a + a'),$$

and

$$\begin{aligned}(\gamma, a) \circ (\gamma', a') &= (\gamma, a) + (\gamma, a) * (\gamma', a')\\ &= (\gamma + \gamma' - \bar{f}(a, a * a') + g(a, a * a'),\ a \circ a')\end{aligned}$$

then yield a brace structure on $\Gamma \times A$. These formulas have the desired form, with

$$f(a, a') = -\bar{f}(a, a * a') + g(a, a * a'),$$

which, through the substitution $b = a * a'$, is equivalent to (5-7).

Conversely, starting from a brace structure on $\Gamma \times A$ of the desired form, one sees that its associated LCS structure is as described in Lemma 5.2 with some $(\bar{f}, g) \in Z^2(A; \Gamma)$. Repeating the argument above, one obtains the relation (5-7) connecting f, $\bar{f}$, and g. □

Definition 5.5. A *central extension* of a linear cycle set $(A, \cdot, +)$ by an abelian group Γ is the datum of a short exact sequence of linear cycle sets

$$0 \to \Gamma \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0, \tag{5-8}$$

where Γ is endowed with the trivial cycle set structure, and its image $\iota(\Gamma)$ is central in E (in the sense of Definition 3.6). The notion of *equivalence* for central cycle-type LCS extensions (Definition 3.8) transports verbatim to these general extensions. The set of equivalence classes of central extensions of A by Γ is denoted by $\operatorname{Ext}(A, \Gamma)$.

The LCS $\Gamma \oplus_{f,g} A$ from Lemma 5.2 is an extension of A by Γ in the obvious way. We now show that this example is essentially exhaustive.

Lemma 5.6. *Let $\Gamma \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A$ be a central LCS extension, and let $s : A \to E$ be a set-theoretic section of π.*

(1) *The maps $\tilde{f}, \tilde{g} : A \times A \to E$ defined by*

$$\begin{aligned}\tilde{f} &: (a, a') \mapsto s(a) \cdot s(a') - s(a \cdot a'),\\ \tilde{g} &: (a, a') \mapsto s(a) + s(a') - s(a + a')\end{aligned}$$

both take values in $\iota(\Gamma)$ and determine a cocycle $(f, g) \in Z^2(A; \Gamma)$.

(2) *The cocycle above is normalized if and only if s is such, in the sense of $s(0) = 0$.*

(3) *Extensions E and $\Gamma \oplus_{f,g} A$ are equivalent.*

(4) *A cocycle (f', g') obtained from another section s' of π is cohomologous to (f, g). If both cocycles are normalized, then they are cohomologous in the normalized sense.*

Lemma 5.7. *Let $(A, \cdot, +)$ be a linear cycle set, let Γ be an abelian group, and let $(f, g), (f', g') \in Z^2(A; \Gamma)$ be 2-cocycles. The linear cycle set extensions $\Gamma \oplus_{f,g} A$ and $\Gamma \oplus_{f',g'} A$ are equivalent if and only if the cocycle $(f, g) - (f', g')$ is a normalized 2-coboundary.*

Recall that a normalized 2-coboundary is a couple of maps of the form $\partial^1\theta$, where the map $\theta : A \to \Gamma$ is normalized, in the sense of $\theta(0) = 0$.

The proof of these lemmas is technical but conceptually analogous to the proofs of Lemmas 3.9 and 3.10, and will therefore be omitted.

Put together, the preceding lemmas prove:

Theorem 5.8. *Let $(A, \cdot, +)$ be a linear cycle set and Γ be an abelian group. The construction from Lemma 5.6 yields a bijective correspondence*

$$\mathrm{Ext}(A, \Gamma) \xleftrightarrow{1:1} H^2_{\mathrm{N}}(A; \Gamma).$$

In other words, the central extensions of LCS (and thus of braces) are completely determined by their second normalized cohomology groups.

Acknowledgments

The work of L. Vendramin is partially supported by CONICET, PICT-2014-1376, MATH-AmSud, and ICTP. V. Lebed thanks the program ANR-11-LABX-0020-01 and Henri Lebesgue Center (University of Nantes) for support. The authors are grateful to the reviewer for useful remarks and interesting suggestions for a further development of the subject.

References

[Bachiller 2015a] D. Bachiller, "Classification of braces of order p^3", *J. Pure Appl. Algebra* **219**:8 (2015), 3568–3603. MR Zbl

[Bachiller 2015b] D. Bachiller, "Examples of simple left braces", preprint, 2015. arXiv

[Bachiller et al. 2015a] D. Bachiller, F. Cedo, and E. Jespers, "Solutions of the Yang–Baxter equation associated with a left brace", preprint, 2015. arXiv

[Bachiller et al. 2015b] D. Bachiller, F. Cedo, E. Jespers, and J. Okninski, "A family of irretractable square-free solutions of the Yang–Baxter equation", preprint, 2015. arXiv

[Ben David and Ginosar 2016] N. Ben David and Y. Ginosar, "On groups of I-type and involutive Yang–Baxter groups", *J. Algebra* **458** (2016), 197–206. MR

[Catino and Rizzo 2009] F. Catino and R. Rizzo, "Regular subgroups of the affine group and radical circle algebras", *Bull. Aust. Math. Soc.* **79**:1 (2009), 103–107. MR Zbl

[Catino et al. 2015] F. Catino, I. Colazzo, and P. Stefanelli, "On regular subgroups of the affine group", *Bull. Aust. Math. Soc.* **91**:1 (2015), 76–85. MR Zbl

[Catino et al. 2016] F. Catino, I. Colazzo, and P. Stefanelli, "Regular subgroups of the affine group and asymmetric product of radical braces", *J. Algebra* **455** (2016), 164–182. MR Zbl

[Cedó et al. 2010a] F. Cedó, E. Jespers, and J. Okniński, "Retractability of set theoretic solutions of the Yang–Baxter equation", *Adv. Math.* **224**:6 (2010), 2472–2484. MR Zbl

[Cedó et al. 2010b] F. Cedó, E. Jespers, and Á. del Río, "Involutive Yang–Baxter groups", *Trans. Amer. Math. Soc.* **362**:5 (2010), 2541–2558. MR Zbl

[Cedó et al. 2014] F. Cedó, E. Jespers, and J. Okniński, "Braces and the Yang–Baxter equation", *Comm. Math. Phys.* **327**:1 (2014), 101–116. MR Zbl

[Chouraqui 2010] F. Chouraqui, "Garside groups and Yang–Baxter equation", *Comm. Algebra* **38**:12 (2010), 4441–4460. MR Zbl

[Dehornoy 2015] P. Dehornoy, "Set-theoretic solutions of the Yang–Baxter equation, RC-calculus, and Garside germs", *Adv. Math.* **282** (2015), 93–127. MR Zbl

[Etingof et al. 1999] P. Etingof, T. Schedler, and A. Soloviev, "Set-theoretical solutions to the quantum Yang–Baxter equation", *Duke Math. J.* **100**:2 (1999), 169–209. MR Zbl

[Gateva-Ivanova 2004] T. Gateva-Ivanova, "A combinatorial approach to the set-theoretic solutions of the Yang–Baxter equation", *J. Math. Phys.* **45**:10 (2004), 3828–3858. MR Zbl

[Gateva-Ivanova 2015] T. Gateva-Ivanova, "Set-theoretic solutions of the Yang–Baxter equation, Braces, and Symmetric groups", preprint, 2015. arXiv

[Gateva-Ivanova and Cameron 2012] T. Gateva-Ivanova and P. Cameron, "Multipermutation solutions of the Yang–Baxter equation", *Comm. Math. Phys.* **309**:3 (2012), 583–621. MR Zbl

[Gateva-Ivanova and Majid 2008] T. Gateva-Ivanova and S. Majid, "Matched pairs approach to set theoretic solutions of the Yang–Baxter equation", *J. Algebra* **319**:4 (2008), 1462–1529. MR Zbl

[Gateva-Ivanova and Van den Bergh 1998] T. Gateva-Ivanova and M. Van den Bergh, "Semigroups of I-type", *J. Algebra* **206**:1 (1998), 97–112. MR Zbl

[Jespers and Okniński 2005] E. Jespers and J. Okniński, "Monoids and groups of I-type", *Algebr. Represent. Theory* **8**:5 (2005), 709–729. MR Zbl

[Lebed and Vendramin 2015] V. Lebed and L. Vendramin, "Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation", preprint, 2015. arXiv

[Lu et al. 2000] J.-H. Lu, M. Yan, and Y.-C. Zhu, "On the set-theoretical Yang–Baxter equation", *Duke Math. J.* **104**:1 (2000), 1–18. MR Zbl

[Rump 2005] W. Rump, "A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation", *Adv. Math.* **193**:1 (2005), 40–55. MR Zbl

[Rump 2007] W. Rump, "Braces, radical rings, and the quantum Yang–Baxter equation", *J. Algebra* **307**:1 (2007), 153–170. MR Zbl

[Rump 2008] W. Rump, "Semidirect products in algebraic logic and solutions of the quantum Yang–Baxter equation", *J. Algebra Appl.* **7**:4 (2008), 471–490. MR Zbl

[Rump 2014] W. Rump, "The brace of a classical group", *Note Mat.* **34**:1 (2014), 115–144. MR Zbl

[Smoktunowicz 2015a] A. Smoktunowicz, "A note on set-theoretic solutions of the Yang–Baxter equation", preprint, 2015. arXiv

[Smoktunowicz 2015b] A. Smoktunowicz, "On Engel groups, nilpotent groups, rings, braces and the Yang–Baxter equation", preprint, 2015. arXiv

[Soloviev 2000] A. Soloviev, "Non-unitary set-theoretical solutions to the quantum Yang–Baxter equation", *Math. Res. Lett.* **7**:5-6 (2000), 577–596. MR Zbl

[Vendramin 2016] L. Vendramin, "Extensions of set-theoretic solutions of the Yang–Baxter equation and a conjecture of Gateva–Ivanova", *J. Pure Appl. Algebra* **220**:5 (2016), 2064–2076. MR Zbl

Received February 2, 2016. Revised March 22, 2016.

VICTORIA LEBED
LABORATOIRE DE MATHÉMATIQUES JEAN LERAY
UNIVERSITÉ DE NANTES
2 RUE DE LA HOUSSINIÈRE
BP 92208
44322 NANTES CEDEX 3
FRANCE
victoria.lebed@univ-nantes.fr

LEANDRO VENDRAMIN
DEPARTAMENTO DE MATEMÁTICA, FCEN
UNIVERSIDAD DE BUENOS AIRES
PABELLÓN 1
1428 BUENOS AIRES
ARGENTINA
lvendramin@dm.uba.ar

PACIFIC JOURNAL OF MATHEMATICS
Vol. 284, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.284.213

NONCOMMUTATIVE DIFFERENTIALS ON POISSON–LIE GROUPS AND PRE-LIE ALGEBRAS

SHAHN MAJID AND WEN-QING TAO

We show that the quantisation of a connected simply connected Poisson–Lie group admits a left-covariant noncommutative differential structure at lowest deformation order if and only if the dual of its Lie algebra admits a pre-Lie algebra structure. As an example, we find a pre-Lie algebra structure underlying the standard 3-dimensional differential structure on $\mathbb{C}_q[SU_2]$. At the noncommutative geometry level we show that the enveloping algebra $U(\mathfrak{m})$ of a Lie algebra $\mathfrak{m}$, viewed as quantisation of $\mathfrak{m}^*$, admits a connected differential exterior algebra of classical dimension if and only if $\mathfrak{m}$ admits a pre-Lie algebra structure. We give an example where $\mathfrak{m}$ is solvable and we extend the construction to tangent and cotangent spaces of Poisson–Lie groups by using bicross-sum and bosonisation of Lie bialgebras. As an example, we obtain a 6-dimensional left-covariant differential structure on the bicrossproduct quantum group $\mathbb{C}[SU_2]\blacktriangleright\!\!\triangleleft U_\lambda(su_2^*)$.

1. Introduction

It is well-known following [Drinfeld 1987] that the semiclassical objects underlying quantum groups are Poisson–Lie groups. This means a Lie group together with a Poisson bracket such that the group product is a Poisson map. The infinitesimal notion of a Poisson–Lie group is a Lie bialgebra, meaning a Lie algebra $\mathfrak{g}$ equipped with a "Lie cobracket" $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ forming a Lie 1-cocycle and such that its adjoint is a Lie bracket on $\mathfrak{g}^*$. Of the many ways of thinking about quantum groups, this is a "deformation" point of view in which the coordinate algebra on a group is made noncommutative, with commutator controlled at lowest order by the Poisson bracket.

In recent years, the examples initially provided by quantum groups have led to a significant "quantum groups approach" to noncommutative differential geometry in

Majid was on leave at the Mathematical Institute, Oxford, during 2014 when this work was completed. Tao was supported by the China Scholarship Council.
MSC2010: 17D25, 58B32, 81R50.
Keywords: noncommutative geometry, quantum group, left-covariant, differential calculus, bicovariant, deformation, Poisson–Lie group, pre-Lie algebra, (co)tangent bundle, bicrossproduct, bosonisation.

which the next layers of geometry beyond the coordinate algebra are considered, and often classified with the aid of quantum group symmetry. The most important of these is the *differential structure* (also known as the *differential calculus*) on the coordinate algebra, expressed normally as the construction of a bimodule Ω^1 of "1-forms" over the (possibly noncommutative) coordinate algebra A and a map $\mathrm{d}: A \to \Omega^1$ (called the *exterior derivation*) satisfying the Leibniz rule. Usually, Ω^1 is required to be spanned by elements of the form $a\,\mathrm{d}b$, where $a, b \in A$. This is then typically extended to a *differential graded algebra* (DGA) (Ω, d) of all degrees where Ω is formulated as a graded algebra $\Omega = \bigoplus_{i\geq 0} \Omega^i$ generated by $\Omega^0 = A$, Ω^1, and d is a degree-one map such that $\mathrm{d}^2 = 0$ and the "super-Leibniz rule" holds, namely $\mathrm{d}(\xi\eta) = (\mathrm{d}\xi)\eta + (-1)^n \xi\,\mathrm{d}\eta$ for all $\xi \in \Omega^n$, $\eta \in \Omega$. The semiclassical version of what this data means at the Poisson level is known to be a *Poisson-compatible preconnection* (or "Lie–Rinehart connection"; see Remark 2.2). The systematic analysis in [Beggs and Majid 2006] found, in particular, a no-go theorem proving the nonexistence of a left and right translation-covariant differential structure of classical dimension on standard quantum group coordinate algebras $\mathbb{C}_q[G]$ when G is the connected and simply connected Lie group of a complex semisimple Lie algebra $\mathfrak{g}$. Beggs and Majid [2010] had a similar result for the nonexistence of ad-covariant differential structures of classical dimension on enveloping algebras of semisimple Lie algebras. Such results tied in with experience at the algebraic level, where one often has to go to higher-dimensional Ω^1, and [Beggs and Majid 2006; 2010] also provided an alternative, namely to consider nonassociative exterior algebras corresponding to preconnections with curvature. This has been taken up further in [Beggs and Majid 2014b].

The present paper revisits the analysis focussing more clearly on the Lie algebraic structure. For left-covariant differentials on a connected and simply connected Poisson–Lie group, we find (Corollary 4.2) that the semiclassical data exists if and only if the dual Lie algebra $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ admits a so-called pre-Lie structure $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$. Here a pre-Lie structure is a product obeying certain axioms such that the commutator is a Lie algebra, such objects also being called left-symmetric or Vinberg algebras; see [Cartier 2009] and [Burde 2006] for two reviews. Our result has no contradiction to $\mathfrak{g}$ being semisimple and includes quantum groups such as $\mathbb{C}_q[SU_2]$, where we exhibit the pre-Lie structure that corresponds to its known 3-dimensional calculus in [Woronowicz 1989].

Even better, the duals $\mathfrak{g}^*$ for all the quantum groups $\mathbb{C}_q[G]$ are known to be solvable [Majid 1990a] and it may be that all solvable Lie algebras admit pre-Lie algebra structures, a question posed by Milnor; see [Burde 2006]. This suggests for the first time a systematic route to the construction of left-covariant differential calculi for all $\mathbb{C}_q[G]$, currently an unsolved problem. We build on the initial analysis of this example in [Beggs and Majid 2006]. Next, for the calculus to be both left and right covariant (i.e., bicovariant), we find an additional condition (4-6) on Ξ which we relate to infinitesimal or Lie-crossed modules with the coadjoint action; see Theorems 3.1 and 4.1.

The paper also covers in detail the important case of the enveloping algebra $U(\mathfrak{m})$ of a Lie algebra $\mathfrak{m}$, viewed as a quantisation of $\mathfrak{m}^*$. This is a Hopf algebra so, trivially, a quantum group, and our theory applies with $\mathfrak{g} = \mathfrak{m}^*$ an abelian Poisson–Lie group with its Kirillov–Kostant Poisson bracket. In fact our result in this example turns out to extend canonically to all orders in deformation theory, not just the lowest semiclassical order. We show (Proposition 4.4) that $U(\mathfrak{m})$ admits a connected bicovariant differential exterior algebra of classical dimension if and only if $\mathfrak{m}$ admits a pre-Lie structure. The proof builds on results in [Majid and Tao 2015b]. We do not require ad-invariance but the result excludes the case that $\mathfrak{m}$ is semisimple since semisimple Lie algebras do not admit pre-Lie structures [Burde 1994]. The $\mathfrak{m}$ that are allowed do, however, include solvable Lie algebras of the form $[x_i, t] = x_i$, which have been extensively discussed for the structure of "quantum spacetime" (here x_i and t are now viewed as space and time coordinates, respectively), most recently in [Beggs and Majid 2014a]. In the 2-dimensional case we use the known classification of 2-dimensional pre-Lie structures over $\mathbb{C}$ in [Burde 1998] to classify all possible left-covariant differential structures of classical dimension. This includes the standard calculus previously used in [Beggs and Majid 2014a] as well as some other differential calculi in the physics literature [Meljanac et al. 2012]. The 4-dimensional case and its consequences for quantum gravity are explored in our related paper [Majid and Tao 2015a].

We then apply our theory to the quantisation of the tangent bundle and cotangent bundle of a Poisson–Lie group. In Section 5, we recall the use of the Lie bialgebra $\mathfrak{g}$ of a Poisson–Lie group G to construct the tangent bundle as a bicrossproduct of Poisson–Lie groups and its associated "bicross-sum" of Lie bialgebras [Majid 1995]. Our results (see Theorem 5.6) then suggest a full differential structure, not only at semiclassical level, on the associated bicrossproduct quantum groups $\mathbb{C}[G] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g}^*)$ in [Majid 1990a; 1990b; 1995]. We prove this in Proposition 5.7 and give $\mathbb{C}[\mathrm{SU}_2] \blacktriangleright\!\!\triangleleft U_\lambda(\mathrm{su}_2^*)$ in detail. Indeed, these bicrossproduct quantum groups were exactly conceived in the 1980s as quantum tangent spaces of Lie groups. In Section 6, we use a pre-Lie structure on $\mathfrak{g}^*$ to make $\mathfrak{g}$ into a braided-Lie biaglebra [Majid 2000] (see Lemma 6.1). The Lie bialgebra of the cotangent bundle becomes a "bosonisation" in the sense of [Majid 2000] and we construct in some cases a natural preconnection for the semiclassical differential calculus. As before, we cover abelian Lie groups with the Kirillov–Kostant Poisson bracket and a restricted class of quasitriangular Poisson–Lie groups as examples.

Most of the work in the paper is at the semiclassical level but occasionally we have results about differentials at the Hopf algebra level as in [Woronowicz 1989], building on [Majid and Tao 2015b]. We recall that a Hopf algebra A is an algebra equipped with a compatible coalgebra and an "antipode" S in the role of inverse. We denote the coproduct $\Delta : A \to A \otimes A$ by the Sweedler notation $\Delta a = a_{(1)} \otimes a_{(2)}$. A differential calculus (Ω^1, d) on a Hopf algebra is called *left-covariant* if Ω^1 is a left A-comodule with coaction $\Delta_L : \Omega^1 \to A \otimes \Omega^1$ satisfying $\Delta_L(a\,\mathrm{d}b) = a_{(1)}b_{(1)} \otimes a_{(2)}\mathrm{d}b_{(2)}$ for all $a, b \in A$. Similarly for a *right-covariant* calculus with structure map $\Delta_R : \Omega^1 \to \Omega^1 \otimes A$ satisfying $\Delta_R(a\mathrm{d}b) = a_{(1)}\mathrm{d}b_{(1)} \otimes a_{(2)}b_{(2)}$. A calculus is *bicovariant* if it is both left- and right-covariant. A left-covariant calculus can always be put in the form $\Omega^1 = A \otimes \Lambda^1$ as a left A-module, where Λ^1 is the space of invariants under the left coaction, and in the bicovariant case extends canonically to a differential graded algebra Ω [Woronowicz 1989].

2. Preliminaries

2A. ***Deformation of noncommutative differentials.*** We follow the setting given in [Beggs and Majid 2006]. Let M be a smooth manifold and consider the deformation of the coordinate algebra $C^\infty(M)$ by replacing the usual commutative point-wise multiplication (usually omitted) with a new multiplication $\bullet$ of the form $a \bullet b = ab + O(\lambda)$ for all $a, b \in C^\infty(M)$. The noncommutativity of the new product can be expressed in a bracket $\{\,,\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ defined by $[a, b]_\bullet = a \bullet b - b \bullet a = \lambda\{a, b\} + O(\lambda^2)$. We assume that we are working in a deformation setting where we can equate order by order in λ. Then it is well-known that the new product $\bullet$ is associative up to order $O(\lambda^2)$ if and only if the bracket $\{\,,\}$ is a Poisson bracket. We denote the associated bivector by π, so $\{a, b\} = \pi(\mathrm{d}a, \mathrm{d}b)$.

In the same spirit, however, one can likewise consider the deformation of differential forms. The n-forms $\Omega^n(M)$ and exterior algebra $\Omega(M)$ are identified with their classical counterparts as vector spaces. But now $\Omega^1(M)$ is equipped with new left/right actions $a \bullet \tau = a\tau + O(\lambda)$ and $\tau \bullet a = \tau a + O(\lambda)$. The deformed derivation $\mathrm{d}_\bullet : \Omega^n(M) \to \Omega^{n+1}(M)$ is of the form $\mathrm{d}_\bullet a = \mathrm{d}a + O(\lambda)$. Define a linear map $\gamma : C^\infty(M) \otimes \Omega^1(M) \to \Omega^1(M)$ by

$$a \bullet \tau - \tau \bullet a = [a, \tau]_\bullet = \lambda\gamma(a, \tau) + O(\lambda^2).$$

It was shown in [Hawkins 2004; Beggs and Majid 2006] that for $\Omega^1(M)$ with new left/right actions to be a $(C^\infty(M), \bullet)$-bimodule up to order $O(\lambda^2)$ requires the associated map γ to satisfy

$$\gamma(ab, \tau) = \gamma(a, \tau)b + a\gamma(b, \tau), \tag{2-1}$$

$$\gamma(a, b\tau) = b\gamma(a, \tau) + \{a, b\}\tau. \tag{2-2}$$

If $\mathrm{d}_\bullet$ is a derivation up to order $O(\lambda^2)$, then γ should also satisfy

$$\mathrm{d}\{a, b\} = \gamma(a, \mathrm{d}b) - \gamma(b, \mathrm{d}a), \tag{2-3}$$

where $\mathrm{d}: C^\infty(M) \to \Omega^1(M)$ is the usual exterior derivation.

Definition 2.1. Any map $\gamma: C^\infty(M)\otimes\Omega^1(M) \to \Omega^1(M)$ satisfying (2-1) and (2-2) is called a *preconnection* on M. A preconnection γ is said to be *Poisson-compatible* if (2-3) also holds.

Such preconnections can arise by pullback along the map that associates a Hamiltonian vector fields $\hat{a} = \{a, -\}$ to a function $a \in C^\infty(M)$, i.e., $\gamma(a, -) = \nabla_{\hat{a}}$ for a covariant derivative defined at least along Hamiltonian vector fields, in which case the remaining (2-3) appears as a constraint on its torsion. From the analysis above, we see that a Poisson-compatible preconnection controls the noncommutativity of functions and 1-forms, and thus plays a vital role in deforming a differential graded algebra $\Omega(M)$ at lowest order, parallel to the Poisson bracket for $C^\infty(M)$ at lowest order.

Remark 2.2. As pointed out by the referee, a Poisson-compatible preconnection in Definition 2.1 can be seen as an example of a Lie–Rinehart connection; cf. [Huebschmann 1990]. If M is a Poisson manifold then the pair $(C^\infty(M), \Omega^1(M))$ forms a Lie–Rinehart algebra with $\Omega^1(M)$ a Lie algebra by $[\mathrm{d}a, \mathrm{d}b] = \mathrm{d}\{a, b\}$ for all $a, b \in C^\infty(M)$, where $(\Omega^1(M), [\ ,\ ])$ acts on $C^\infty(M)$ by $(\mathrm{d}a)\triangleright b = \pi(\mathrm{d}a, \mathrm{d}b) = \{a, b\}$ for all $a, b \in C^\infty(M)$. In this context we can consider a Poisson-compatible preconnection as a covariant derivative ∇_η along 1-forms $\eta \in \Omega^1$ by $\nabla_{\mathrm{d}a} = \gamma(a, -)$ extended $C^\infty(M)$-linearly, i.e., a Lie–Rinehart connection in this context. Here (2-2) appears as the connection property $\nabla_\eta(a\tau) = \pi(\eta, \mathrm{d}a)\tau + a\nabla_\eta\tau$ while (2-3) appears as the further property $[\eta, \tau] = \nabla_\eta\tau - \nabla_\tau\eta$ for all $\eta, \tau \in \Omega^1(M)$.

2B. ***Poisson–Lie groups and Lie bialgebras.*** Throughout the paper, we mainly work over a Poisson–Lie group G and its Lie bialgebra $\mathfrak{g}$. By definition, the Poisson bracket $\{\ ,\ \}: C^\infty(G)\otimes C^\infty(G) \to C^\infty(G)$ is determined uniquely by a so-called Poisson bivector $\pi = \pi^{(1)}\otimes\pi^{(2)}$, i.e., $\{a, b\} = \pi^{(1)}(\mathrm{d}a)\pi^{(2)}(\mathrm{d}b)$. Then $\mathfrak{g}$ is a Lie bialgebra with Lie cobracket $\delta: \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ given by

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}t}\pi^{(1)}(g)g^{-1}\otimes\pi^{(2)}(g)g^{-1}\Big|_{t=0},$$

where $g = \exp tx \in G$ for any $x \in \mathfrak{g}$. The map δ is a Lie 1-cocycle with respect to the adjoint action, and extends to group 1-cocycles $D(g) = (R_{g^{-1}})_*\pi(g)$ with respect to the left adjoint action and $D^\vee(g) = (L_{g^{-1}})_*\pi(g)$ with respect to the right adjoint action, respectively. Here $D^\vee$ and D are related by $D^\vee(g) = \mathrm{Ad}_{g^{-1}} D(g)$ and thus are equivalent. We recall that a left group cocycle means

$$D(uv) = D(u) + \mathrm{Ad}_u(D(v)) \quad \text{for all } u, v \in G,\ D(e) = 0.$$

When G is connected and simply connected, one can recover D for a given δ by solving

$$\mathrm{d}D(\tilde{x})(g) = \mathrm{Ad}_g(\delta x), \quad D(e) = 0,$$

where $\tilde{x}$ is the left-invariant vector field corresponding to $x \in \mathfrak{g}$. We then recover the Poisson bracket by $\pi(g) = R_{g*}(D(g))$ for all $g \in G$. These notions are due to Drinfeld and an introduction can be found in [Majid 1995].

For convenience, we recall that a *left* $\mathfrak{g}$*-module* over a Lie algebra $\mathfrak{g}$ is a vector space V together with a linear map $\triangleright : \mathfrak{g} \otimes V \to V$ such that $[x, y] \triangleright v = x \triangleright (y \triangleright v) - y \triangleright (x \triangleright v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Dually, a *left* $\mathfrak{g}$*-comodule* over a Lie coalgebra $(\mathfrak{g}, \delta)$ is a vector space V together with a linear map $\alpha : V \to \mathfrak{g} \otimes V$ such that $(\delta \otimes \mathrm{id}) \circ \alpha = ((\mathrm{id} - \tau) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \circ \alpha$. Over a Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$, a *left* $\mathfrak{g}$*-crossed module* $(V, \triangleright, \alpha)$ is both a left $\mathfrak{g}$-module $(V, \triangleright)$ and a left $\mathfrak{g}$-comodule (V, α) such that

$$\alpha(x \triangleright v) = ([x,\ ] \otimes \mathrm{id} + \mathrm{id} \otimes x \triangleright)\alpha(v) + \delta(x) \triangleright v$$

for any $x \in \mathfrak{g}$, $v \in V$. When $\mathfrak{g}$ is finite-dimensional, the notion of a left $\mathfrak{g}$-crossed module is equivalent to a left $\mathfrak{g}$-module $(V, \triangleright)$ that admits a left $\mathfrak{g}^{*\mathrm{op}}$-action $\triangleright'$ satisfying

$$\phi_{(1)} \triangleright' v \langle \phi_{(2)}, x \rangle + x_{(1)} \triangleright v \langle \phi, x_{(2)} \rangle = x \triangleright (\phi \triangleright' v) - \phi \triangleright' (x \triangleright v) \tag{2-4}$$

for any $x \in \mathfrak{g}$, $\phi \in \mathfrak{g}^*$ and $v \in V$, where the left $\mathfrak{g}^{*\mathrm{op}}$-action $\triangleright'$ corresponds to the left $\mathfrak{g}$-coaction α above via $\phi \triangleright' v = \langle \phi, v^{(1)} \rangle v^{(2)}$ with $\alpha(v) = v^{(1)} \otimes v^{(2)}$. Therefore, a left $\mathfrak{g}$-crossed module is precisely a left $D(\mathfrak{g})$-module, where $D(\mathfrak{g})$ is the Drinfeld double of $\mathfrak{g}$; see [Majid 1995]. For brevity, we call a left $\mathfrak{g}$-module V with linear map $\triangleright' : \mathfrak{g}^* \otimes V \to V$ (not necessarily an action) such that (2-4) holds a *left almost* $\mathfrak{g}$*-crossed module*.

2C. ***Left-covariant preconnections.*** The algebra of functions on a Poisson–Lie group G typically deforms to a noncommutative Hopf algebra A and a semiclassical analysis of the covariance of a differential structure was initiated in [Beggs and Majid 2006] in terms of preconnection γ. By definition, a preconnection γ is said to be *left-covariant* (*right-covariant*, or *bicovariant*) if the associated differential calculus on $(C^\infty(G), \bullet)$ is left-covariant (right-covariant, or bicovariant) over $(C^\infty(G), \bullet)$ up to $O(\lambda^2)$. [Beggs and Majid 2006, Lemma 4.3] gives a precise characterisation of this in terms of a map Ξ as follows.

We first explain the notations used in [Beggs and Majid 2006]. We recall that there is a one-to-one correspondence between 1-forms $\Omega^1(G)$ and $C^\infty(G, \mathfrak{g}^*)$, the set of smooth sections of the trivial $\mathfrak{g}^*$ bundle. For any 1-form τ, define $\tilde{\tau} \in C^\infty(G, \mathfrak{g}^*)$ by letting $\tilde{\tau}_g = L_g^*(\tau_g)$. Conversely, any $s \in C^\infty(G, \mathfrak{g}^*)$ defines an 1-form (denoted by $\hat{s}$) by setting $\hat{s}_g = L_{g^{-1}}^*(s(g))$. In particular, we know $\mathrm{d}a \in \Omega^1(G)$ and $\widetilde{\mathrm{d}a} \in C^\infty(G, \mathfrak{g}^*)$ for any $a \in C^\infty(G)$. Denote $\widetilde{\mathrm{d}a}$ by $\hat{L}_a$, then

$$\langle \hat{L}_a(g), v \rangle = \langle \widetilde{\mathrm{d}a}(g), v \rangle = \langle L_g^*((\mathrm{d}a)_g), v \rangle = \langle (\mathrm{d}a)_g, (L_g)_* v \rangle = (L_g)_*(v)a,$$

which is the directional derivation of a with respect to $v \in \mathfrak{g}$ at g.

Using the above notations, a preconnection γ can now be rewritten on $\mathfrak{g}^*$-valued functions as $\tilde{\gamma} : C^\infty(G) \times C^\infty(G, \mathfrak{g}^*) \to C^\infty(G, \mathfrak{g}^*)$ by letting $\tilde{\gamma}(a, \tilde{\tau}) = \widetilde{\gamma(a, \tau)}$. Note that for any $\phi, \psi \in \mathfrak{g}^*$ and $g \in G$, there exist $a \in C^\infty(G)$, $s \in C^\infty(G, \mathfrak{g}^*)$ such that $\hat{L}_a(g) = \phi$ and $s(g) = \psi$. One can define a map $\widetilde{\Xi} : G \times \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ by

$$\tilde{\gamma}(a, s)(g) = \{a, s\}(g) + \widetilde{\Xi}(g, \hat{L}_a(g), s(g)).$$

For brevity, the notation for the Poisson bracket is extended to include $\mathfrak{g}^*$-valued functions on one side.

Beggs and Majid [2006, Proposition 4.5] show that a preconnection γ is left-covariant if and only if $\widetilde{\Xi}(gh, \phi, \psi) = \widetilde{\Xi}(h, \phi, \psi)$ for any $g, h \in G$ and $\phi, \psi \in \mathfrak{g}^*$. Hence for a left-covariant preconnection the map $\widetilde{\Xi}$ defines a map $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ by $\Xi(\phi, \psi) = \widetilde{\Xi}(e, \phi, \psi)$ and conversely, given $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$,

$$\tilde{\gamma}(a, s)(g) = \{a, s\}(g) + \Xi(\hat{L}_a(g), s(g)) \tag{2-5}$$

defines the corresponding left-covariant preconnection γ. In addition, Beggs and Majid [2006, Proposition 4.6] show that a left-covariant preconnection is Poisson-compatible if and only if the corresponding Ξ obeys

$$\Xi(\phi, \psi) - \Xi(\psi, \phi) = [\phi, \psi]_{\mathfrak{g}^*} \tag{2-6}$$

for all $\phi, \psi \in \mathfrak{g}^*$.

Based on these results, we can write down a formula for the preconnection γ in coordinates. Let $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{f^i\}$ be the dual basis of $\mathfrak{g}^*$. Let $\{\omega^i\}$ be the basis of left-invariant 1-forms that is dual to $\{\partial_i\}$ the left-invariant vector fields (generated by $\{e_i\}$) of G. Then the Maurer–Cartan form is

$$\omega = \sum_i \omega^i e_i \in \Omega^1(G, \mathfrak{g}).$$

For any $\eta = \sum_i \eta_i \omega^i \in \Omega^1(G)$ with $\eta_i \in C^\infty(G)$, we know η corresponds to $\tilde{\eta} = \sum_i \eta_i f^i \in C^\infty(G, \mathfrak{g}^*)$. On the other hand, any $s = \sum_i s_i f^i \in C^\infty(G, \mathfrak{g}^*)$ with $s_i \in C^\infty(G)$ corresponds to $\hat{s} = \sum_i s_i \omega^i \in \Omega^1(G)$. In particular, $\widetilde{\mathrm{d}a} = \sum_i (\partial_i a) f^i$.

For any $a \in C^\infty(G)$ and $\tau = \sum_i \tau_i \omega^i \in \Omega^1(G)$, we know $\{a, \tilde{\tau}\} = \sum_i \{a, \tau_i\} f^i$ and

$$\begin{aligned}
\Xi(\widetilde{\mathrm{d}a}(g), \tilde{\tau}(g)) &= \Xi\Big(\sum_i (\partial_i a)(g) f^i, \sum_j \tau_j(g) f^j\Big) \\
&= \sum_{i,j} (\partial_i a)(g) \tau_j(g) \Xi(f^i, f^j) \\
&= \sum_{i,j,k} (\partial_i a)(g) \tau_j(g) \langle \Xi(f^i, f^j), e_k \rangle f^k,
\end{aligned}$$

so

$$\widetilde{\gamma}(a,\widetilde{\tau}) = \sum_k \Big(\{a, \tau_k\} + \sum_{i,j} (\partial_i a)\tau_j \langle \Xi(f^i, f^j), e_k\rangle \Big) f^k.$$

If we write $\Xi^{ij}_k = \langle \Xi(f^i, f^j), e_k\rangle$ (or $\Xi(f^i, f^j) = \sum_k \Xi^{ij}_k f^k$) for any i, j, k, then we have

$$\gamma(a,\tau) = \sum_k \Big(\{a, \tau_k\} + \sum_{i,j} \Xi^{ij}_k (\partial_i a)\tau_j \Big)\omega^k. \tag{2-7}$$

In particular, we have a more handy formula,

$$\gamma(a,\omega^j) = \sum_{i,k} (\partial_i a)\langle \Xi(f^i, f^j), e_k\rangle \omega^k = \sum_{i,k} \Xi^{ij}_k (\partial_i a)\omega^k \quad \text{for all } j. \tag{2-8}$$

3. Bicovariant preconnections

Beggs and Majid [2006, Theorem 4.14] show that γ is bicovariant at the Poisson–Lie group level if and only if

$$\Xi(\phi,\psi) - \mathrm{Ad}^*_{g^{-1}} \Xi(\mathrm{Ad}^*_g \phi, \mathrm{Ad}^*_g \psi) = \phi(g^{-1}\pi^{(1)}(g))\, \mathrm{ad}^*_{g^{-1}\pi^{(2)}(g)} \psi \tag{3-1}$$

for all $g \in G$ and $\phi, \psi \in \mathfrak{g}^*$. We now give a new characterisation in terms of Lie bialgebra-level data.

Theorem 3.1. *Let G be a connected and simply connected Poisson–Lie group. A left-covariant preconnection on G determined by $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is bicovariant if and only if* $(\mathrm{ad}^*, -\Xi)$ *makes $\mathfrak{g}^*$ into a left almost $\mathfrak{g}$-crossed module*, *or explicitly*,

$$\mathrm{ad}^*_x \Xi(\phi,\psi) - \Xi(\mathrm{ad}^*_x \phi, \psi) - \Xi(\phi, \mathrm{ad}^*_x \psi) = \phi(x_{(1)})\, \mathrm{ad}^*_{x_{(2)}}(\psi) \tag{3-2}$$

for all $x \in \mathfrak{g}$ and $\phi, \psi \in \mathfrak{g}^*$, where $\delta(x) = x_{(1)} \otimes x_{(2)}$. This is equivalent to

$$\delta_{\mathfrak{g}^*} \Xi(\phi,\psi) - \Xi(\phi_{(1)}, \psi) \otimes \phi_{(2)} - \Xi(\phi, \psi_{(1)}) \otimes \psi_{(2)} = \psi_{(1)} \otimes [\phi, \psi_{(2)}]_{\mathfrak{g}^*} \tag{3-3}$$

for all $\phi, \psi \in \mathfrak{g}^*$.

Proof. We first show the "only if" part. To obtain the corresponding formula at the Lie algebra level for (3-1), we substitute g with $\exp tx$ and differentiate at $t = 0$. Notice that $\mathrm{d}\,\mathrm{Ad}^*(\exp tx)/\mathrm{d}t|_{t=0} = \mathrm{ad}^*_x$ and $\mathrm{Ad}^*(\exp tx)|_{t=0} = \mathrm{id}_{\mathfrak{g}^*}$. This gives (3-2) as stated, where $\delta(x) = x_{(1)} \otimes x_{(2)} = \mathrm{d}g^{-1}P(g)/\mathrm{d}t|_{t=0}$ when $g = \exp tx$. Now denote ad^*_x by $x\triangleright$ and let $-\Xi(\phi,\ ) = \phi\triangleright$, the left $\mathfrak{g}^{*\mathrm{op}}$-action; then the left-hand side of (3-2) becomes

$$-x \triangleright (\phi \triangleright \psi) + \phi_{(1)} \triangleright \psi \langle \phi_{(2)}, x\rangle + \phi \triangleright (x \triangleright \psi),$$

while the right-hand side is

$$\phi(x_{(1)})\,\mathrm{ad}^*_{x_{(2)}}(\psi) = -\phi(x_{(2)})\,\mathrm{ad}^*_{x_{(1)}}(\psi) = -x_{(1)} \triangleright \psi\langle \phi, x_{(2)}\rangle.$$

Hence (3-2) is the content of

$$\phi_{(1)} \triangleright \psi\langle \phi_{(2)}, x\rangle + x_{(1)} \triangleright \psi\langle \phi, x_{(2)}\rangle = x \triangleright (\phi \triangleright \psi) - \phi \triangleright (x \triangleright \psi)$$

in our case, i.e., that $\mathfrak{g}^*$ is a left almost $\mathfrak{g}$-crossed module under $(\mathrm{ad}^*, -\Xi)$.

Conversely, we can exponentiate x near zero, and solve the ordinary differential equation (3-2) near $g = e$. It has a unique solution (3-1) near the identity. Since the Lie group G is connected and simply connected, one can show that (3-1) is valid on the whole group.

Notice that $\mathrm{ad}^*_x \phi = \phi_{(1)}\langle \phi_{(2)}, x\rangle$ for any $x \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$, so the left-hand side of (3-2) becomes

$$-\Xi(\phi, \psi)_{(1)}\langle \Xi(\phi, \psi)_{(2)}, x\rangle - \Xi(\phi_{(1)}, \psi)\langle \phi_{(2)}, x\rangle - \Xi(\phi, \psi_{(1)})\langle \psi_{(2)}, x\rangle,$$

while the right-hand side of (3-2) is

$$\phi(x_{(1)})\psi_{(1)}\langle \psi_{(2)}, x_{(2)}\rangle = \psi_{(1)}\langle [\phi, \psi_{(2)}]_{\mathfrak{g}^*}, x\rangle,$$

thus (3-2) is equivalent to (3-3) by using the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. □

4. Flat preconnections

As in [Beggs and Majid 2006], the *curvature* of a preconnection γ is defined on Hamiltonian vector fields $\hat{x} = \{x, -\}$ by

$$R(x, y)\tau = \gamma(x, \gamma(y, \tau)) - \gamma(y, \gamma(x, \tau)) - \gamma(\{x, y\}, \tau) \quad \text{for all } \tau \in \Omega^1(G),$$

which agrees with the covariant derivative curvature along Hamiltonian vector fields $\hat{x}$, $\hat{y}$ when this applies, on noting that $[\hat{x}, \hat{y}] = \widehat{\{x, y\}}$. The curvature of a preconnection reflects the obstruction to the Jacobi identity on any functions x, y and 1-form τ up to third order, namely

$$[x, [y, \tau]_\bullet]_\bullet + [y, [\tau, x]_\bullet]_\bullet + [\tau, [x, y]_\bullet]_\bullet = \lambda^2 R(\hat{x}, \hat{y})(\tau) + O(\lambda^3).$$

This is the deformation-theoretic meaning of curvature in this context. We say a preconnection is *flat* if its curvature is zero. This takes a similar form in terms of $\tilde{\gamma}$, namely

$$\tilde{\gamma}(x, \tilde{\gamma}(y, s)) - \tilde{\gamma}(y, \tilde{\gamma}(x, s)) - \tilde{\gamma}(\{x, y\}, s) = 0 \tag{4-1}$$

for all $x, y \in C^\infty(G)$ and $s \in C^\infty(G, \mathfrak{g}^*)$.

Theorem 4.1. *Let G be a connected and simply connected Poisson–Lie group with Lie algebra $\mathfrak{g}$ and γ a Poisson-compatible left-covariant preconnection.*

(i) γ *is flat if and only if the corresponding map* $-\Xi$ *is a right* $\mathfrak{g}^*$*-action* (*or left* $\mathfrak{g}^{*\mathrm{op}}$*-action*) *on* $\mathfrak{g}^*$,

(4-2) $\Xi([\phi, \psi]_{\mathfrak{g}^*}, \zeta) = \Xi(\phi, \Xi(\psi, \zeta)) - \Xi(\psi, \Xi(\phi, \zeta))$ *for all* $\phi, \psi, \zeta \in \mathfrak{g}^*$.

(ii) γ *is bicovariant and flat if and only if* $(\mathrm{ad}^*, -\Xi)$ *makes* $\mathfrak{g}^*$ *a left* $\mathfrak{g}$*-crossed module.*

Proof. Let γ be a Poisson-compatible left-covariant preconnection on a Poisson–Lie group G. Firstly, we can rewrite formula (4-1) in terms of $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$. By definition, the three terms in (4-1) become

$$\begin{aligned}\tilde{\gamma}(x, \tilde{\gamma}(y, s))(g) = \{x, \{y, s\}\}(g) + \{x, \Xi(\hat{L}_y(g), s(g))\}\\ + \Xi(\hat{L}_x(g), \{y, s\}(g)) + \Xi\big(\hat{L}_x(g), \Xi(\hat{L}_y(g), s(g))\big),\end{aligned}$$

$$\begin{aligned}\tilde{\gamma}(y, \tilde{\gamma}(x, s))(g) = \{y, \{x, s\}\}(g) + \{y, \Xi(\hat{L}_x(g), s(g))\}\\ + \Xi(\hat{L}_y(g), \{x, s\}(g)) + \Xi\big(\hat{L}_y(g), \Xi(\hat{L}_x(g), s(g))\big),\end{aligned}$$

and

$$\tilde{\gamma}(\{x, y\}, s)(g) = \{\{x, y\}, s\}(g) + \Xi(\hat{L}_{\{x,y\}}(g), s(g)).$$

Cancelling terms involving the Jacobi identity of a Poisson bracket, formula (4-1) becomes

$$\begin{aligned}\{x, \Xi(\hat{L}_y(g), s(g))\} + \Xi(\hat{L}_x(g), \{y, s\}(g)) + \Xi\big(\hat{L}_x(g), \Xi(\hat{L}_y(g), s(g))\big)\\ - \{y, \Xi(\hat{L}_x(g), s(g))\} - \Xi(\hat{L}_y(g), \{x, s\}(g)) - \Xi\big(\hat{L}_y(g), \Xi(\hat{L}_x(g), s(g))\big)\\ = \Xi(\hat{L}_{\{x,y\}}(g), s(g)).\end{aligned}$$

Note that since γ is Poisson-compatible, this implies

$$\begin{aligned}\hat{L}_{\{x,y\}}(g) &= \tilde{\gamma}(x, \hat{L}_y)(g) - \tilde{\gamma}(y, \hat{L}_x)(g)\\ &= \{x, \hat{L}_y\}(g) + \Xi(\hat{L}_x(g), \hat{L}_y(g)) - \{y, \hat{L}_x\}(g) - \Xi(\hat{L}_y(g), \hat{L}_x(g)).\end{aligned}$$

and $\{x, \Xi(\hat{L}_y(g), s(g)\} = \Xi(\{x, \hat{L}_y\}(g), s(g)) + \Xi(\hat{L}_y(g), \{x, s\}(g))$ by the derivation property of $\{x, -\}$. In this case (4-1) is equivalent to

(4-3) $$\begin{aligned}\Xi(\hat{L}_x(g), \Xi(\hat{L}_y(g), s(g))) - \Xi\big(\hat{L}_y(g), \Xi(\hat{L}_x(g), s(g))\big)\\ = \Xi\big(\Xi(\hat{L}_x(g), \hat{L}_y(g)) - \Xi(\hat{L}_y(g), \hat{L}_x(g)), s(g)\big)\end{aligned}$$

for all $x, y \in C^\infty(G)$ and $s \in C^\infty(G, \mathfrak{g}^*)$.

Now if γ is flat, we can evaluate this equation at the identity e of G, and for any $\phi, \psi, \zeta \in \mathfrak{g}^*$, set $\phi = \hat{L}_x(e)$, $\psi = \hat{L}_y(e)$ and $\zeta = s(e)$ for some $x, y \in C^\infty(G)$ and $s \in C^\infty(G, \mathfrak{g}^*)$. Then (4-3) becomes

$$\Xi(\Xi(\phi, \psi) - \Xi(\psi, \phi), \zeta) = \Xi(\phi, \Xi(\psi, \zeta)) - \Xi(\psi, \Xi(\phi, \zeta)).$$

Using compatibility again, we get (4-2) as displayed. This also shows Ξ is a left $\mathfrak{g}^*$-action on itself, or $\mathfrak{g}^*$ is a left $\mathfrak{g}^{*\mathrm{op}}$-module via $-\Xi$.

Conversely, if $\mathfrak{g}^*$ is a left $\mathfrak{g}^{*\mathrm{op}}$-module via $\triangleright : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ and such that $-\phi\triangleright\psi+\psi\triangleright\phi=[\phi, \psi]_{\mathfrak{g}^*}$, i.e., (4-2) holds. This implies (4-3) for any $x, y \in C^\infty(G)$, $s \in C^\infty(G, \mathfrak{g}^*)$, which is equivalent to (4-1).

The second part of the theorem combines the first part with Theorem 3.1. □

4A. ***Preconnections and pre-Lie algebras.*** Now we recall the notion of a *left pre-Lie algebra* (also known as a *Vinberg algebra* or *left symmetric algebra*). An algebra $(A, \circ)$, not necessarily associative, with product $\circ : A \otimes A \to A$ is called a (*left*) *pre-Lie algebra* if the identity

$$(x \circ y) \circ z - (y \circ x) \circ z = x \circ (y \circ z) - y \circ (x \circ z) \tag{4-4}$$

holds for all $x, y, z \in A$. From the definition, every associative algebra is a pre-Lie algebra and meanwhile every pre-Lie algebra $(A, \circ)$ admits a Lie algebra structure (denoted by $\mathfrak{g}_A$) with Lie bracket given by

$$[x, y]_{\mathfrak{g}_A} := x \circ y - y \circ x \tag{4-5}$$

for all $x, y \in A$. The Jacobi identity of $[\ ,\ ]_{\mathfrak{g}_A}$ holds automatically due to (4-4). With this in mind, we can rephrase Theorems 3.1 and 4.1 as follows.

Corollary 4.2. *A connected and simply connected Poisson–Lie group G with Lie algebra $\mathfrak{g}$ admits a Poisson-compatible left-covariant flat preconnection if and only if $(\mathfrak{g}^*, [\ ,\ ]_{\mathfrak{g}^*})$ admits a pre-Lie structure Ξ. Moreover, this left-covariant preconnection is bicovariant if and only if Ξ in addition obeys*

$$\delta_{\mathfrak{g}^*}\Xi(\phi, \psi) - \Xi(\phi, \psi_{(1)}) \otimes \psi_{(2)} - \psi_{(1)} \otimes \Xi(\phi, \psi_{(2)}) \\ = \Xi(\phi_{(1)}, \psi) \otimes \phi_{(2)} - \psi_{(1)} \otimes \Xi(\psi_{(2)}, \phi) \tag{4-6}$$

for all $\phi, \psi \in \mathfrak{g}^*$.

Proof. The first part is shown by (2-6) and (4-2). For the bicovariant case, the additional condition on Ξ is (3-3). Using compatibility and rearranging terms, we know that (3-3) is equivalent to (4-6) as displayed. □

Example 4.3. Let $\mathfrak{m}$ be a finite-dimensional Lie algebra and $G = \mathfrak{m}^*$ be an abelian Poisson–Lie group with its Kirillov–Kostant Poisson–Lie group structure $\{x, y\} = [x, y]$ for all $x, y \in \mathfrak{m} \subset C^\infty(\mathfrak{m}^*)$ or $S(\mathfrak{m})$ in an algebraic context. By Corollary 4.2, this admits a Poisson-compatible left-covariant flat preconnection if and only if $\mathfrak{m}$ admits a pre-Lie algebra structure $\circ$. This preconnection is always bicovariant as (4-6) vanishes when Lie algebra $\mathfrak{m}^*$ is abelian ($\delta_{\mathfrak{m}} = 0$). Then (2-7) with $\Xi = \circ$ implies

$$\gamma(x, \mathrm{d}y) = \mathrm{d}(x \circ y) \quad \text{for all } x, y \in \mathfrak{m}.$$

(Note that $\widetilde{\mathrm{d}y}$ is a constant-valued function in $C^\infty(G, \mathfrak{m})$, so $\{x, \widetilde{\mathrm{d}y}\} \equiv 0$ and $\tilde{\gamma}(x, \widetilde{\mathrm{d}y}) = \Xi(x, y)$.)

In fact the algebra and its calculus in this example work to all orders. Thus the quantisation of $C^\infty(\mathfrak{m}^*)$ is $U_\lambda(\mathfrak{m})$, defined as a version of the enveloping algebra with relations $xy - yx = \lambda[x, y]$ for all $x, y \in \mathfrak{m}$, where we introduce a deformation parameter. If $\mathfrak{m}$ has an underlying pre-Lie structure then the above results lead to relations

$$[x, \mathrm{d}y] = \lambda\, \mathrm{d}(x \circ y) \quad \text{for all } x, y \in \mathfrak{m},$$

and one can check that this works exactly and not only to order λ precisely as a consequence of the pre-Lie algebra axiom. The full result here is:

Proposition 4.4. *Let* $\mathfrak{m}$ *be a finite-dimensional Lie algebra over a field k of characteristic zero. Then connected bicovariant calculi* Ω^1 *of classical dimension* (*i.e.*, $\dim \Lambda^1 = \dim \mathfrak{m}$) *on the enveloping algebra* $U(\mathfrak{m})$ *are in one-to-one correspondence with pre-Lie structures on* $\mathfrak{m}$.

Proof. A differential calculus is said to be *connected* if $\ker \mathrm{d} = k1$ (as for a connected manifold classically). It is clear from [Majid and Tao 2015b, Propositions 2.11 and 4.7] that a bicovariant differential graded algebra on $U(\mathfrak{m})$ with left-invariant 1-forms $\mathfrak{m}$ as a vector space corresponds to a 1-cocycle $Z^1_\triangleleft(\mathfrak{m}, \mathfrak{m})$ that extends to a surjective right $\mathfrak{m}$-module map $\omega : U(\mathfrak{m})^+ \to \mathfrak{m}$. Here the derivation

$$\mathrm{d} : U(\mathfrak{m}) \to \Omega^1(U(\mathfrak{m})) = U(\mathfrak{m}) \otimes \mathfrak{m}$$

is given by $\mathrm{d}a = a_{(1)} \otimes \omega(\pi(a_{(2)}))$ for any $a \in U(\mathfrak{m})$. Suppose that ω is such a map; we take $\zeta = \omega|_{\mathfrak{m}} \in Z^1_\triangleleft(\mathfrak{m}, \mathfrak{m})$. For any $x \in \mathfrak{m}$ such that $\zeta(x) = 0$, we have $\mathrm{d}x = 1 \otimes \omega(x) = 0$, then $\ker \mathrm{d} = k1$ implies $x = 0$, so ζ is an injection, hence a bijection as $\mathfrak{m}$ is finite-dimensional. Now we can define a product $\circ : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ by $x \circ y = -\zeta^{-1}(\zeta(y) \triangleleft x)$. The 1-cocycle property $\zeta([x, y]) = \zeta(x) \triangleleft y - \zeta(y) \triangleleft x$ implies $[x, y] = \zeta^{-1}(\zeta(x) \triangleleft y - \zeta(y) \triangleleft x) = -y \circ x + x \circ y$ for all $x, y \in \mathfrak{m}$. Hence this makes $\mathfrak{m}$ into a left pre-Lie algebra as

$$\begin{aligned}[x, y] \circ z &= -\zeta^{-1}(\zeta(z) \triangleleft [x, y]) \\ &= \zeta^{-1}((\zeta(z) \triangleleft y) \triangleleft x) - \zeta^{-1}((\zeta(z) \triangleleft x) \triangleleft y) \\ &= x \circ (y \circ z) - y \circ (x \circ z).\end{aligned}$$

Conversely, if $\mathfrak{m}$ admits a left pre-Lie structure $\circ$, then $y \triangleleft x = -x \circ y$ makes $\mathfrak{m}$ into a right $\mathfrak{m}$-module and $\zeta = \mathrm{id}_{\mathfrak{m}}$, the identity map, becomes a bijective 1-cocycle in $Z^1_\triangleleft(\mathfrak{m}, \mathfrak{m})$. The extended map $\omega : U(\mathfrak{m})^+ \to \mathfrak{m}$ and the derivation $\mathrm{d} : U(\mathfrak{m}) \to U(\mathfrak{m}) \otimes \mathfrak{m}$ are given by $\omega(x_1 x_2 \cdots x_n) = ((x_1 \triangleleft x_2) \triangleleft \cdots \triangleleft x_n)$ for any

$x_1x_2\cdots x_n \in U(\mathfrak{m})^+$ and

$$\mathrm{d}(x_1x_2\cdots x_n) = \sum_{p=0}^{n-1} \sum_{\sigma\in \mathrm{Sh}(p,n-p)} x_{\sigma(1)}\cdots x_{\sigma(p)} \otimes \omega(x_{\sigma(p+1)}\cdots x_{\sigma(n)})$$

for any $x_1x_2\cdots x_n \in U(\mathfrak{m})$, respectively. We need to show that $\ker \mathrm{d} = k1$. On the one hand, $k1 \subseteq \ker \mathrm{d}$, as $\mathrm{d}(1) = 0$. On the other hand, denote by $U_n(\mathfrak{m})$ the subspace of $U(\mathfrak{m})$ generated by the products $x_1x_2\cdots x_p$, where $x_1,\ldots,x_p \in \mathfrak{m}$ and $p \le n$. Clearly, $U_0 = k1$, $U_1(\mathfrak{m}) = k1 \oplus \mathfrak{m}$, $U_p(\mathfrak{m})U_q(\mathfrak{m}) \subseteq U_{p+q}(\mathfrak{m})$ and thus $(U_n(\mathfrak{m}))_{n\ge 0}$ is a filtration of $U(\mathfrak{m})$. In order to show $\ker \mathrm{d} \subseteq k1$, it suffices to show that the intersection

$$(\ker \mathrm{d}) \cap U_n(\mathfrak{m}) = k1 \quad \text{for any integer } n \ge 0.$$

We prove this by induction on $n \ge 0$. It is obvious for $n = 0$, and true for $n = 1$ as, for any $v = \sum_i x_i \in (\ker \mathrm{d}) \cap \mathfrak{m}$, $0 = \mathrm{d}v = \sum_i 1 \otimes \omega(x_i) = \sum_i 1 \otimes x_i$ implies $v = \sum_i x_i = 0$. Suppose that $(\ker \mathrm{d}) \cap U_{n-1}(\mathfrak{m}) = k1$ for $n \ge 2$. For any $v \in (\ker \mathrm{d}) \cap U_n(\mathfrak{m})$, without loss of generality we can write $v = \sum_i x_{i_1}x_{i_2}\cdots x_{i_n} + v'$, where $x_{i_j} \in \mathfrak{m}$ and v' is an element in $U_{n-1}(\mathfrak{m})$. We have

$$\begin{aligned}\mathrm{d}v = \sum_i 1 \otimes \omega(x_{i_1}\cdots x_{i_n}) + \sum_i \sum_{j=1}^{n} x_{i_1}\cdots x_{i_{(j-1)}} \hat{x}_{i_j} x_{i_{(j+1)}} \cdots x_{i_n} \otimes x_{i_j} \\ + \sum_i \sum_{r=1}^{n-2} \sum_{\sigma\in \mathrm{Sh}(r,n-r)} x_{i_{\sigma(1)}}\cdots x_{i_{\sigma(r)}} \otimes \omega(x_{i_{\sigma(r+1)}}\cdots x_{i_{\sigma(n)}}) + \mathrm{d}v'.\end{aligned}$$

We denote the elements

$$u_{i_j} := x_{i_1}\cdots x_{i_{(j-1)}} \hat{x}_{i_j} x_{i_{(j+1)}} \cdots x_{i_n} \in U_{n-1}(\mathfrak{m})$$

for any i, $1 \le j \le n$. Except the term $\sum_i \sum_{j=1}^n u_{i_j} \otimes x_{i_j}$, all the summands in $\mathrm{d}v$ lie in $U_{n-2}(\mathfrak{m}) \otimes \mathfrak{m}$, thus $\sum_i \sum_{j=1}^n u_{i_j} \otimes x_{i_j}$ also lies in $U_{n-2}(\mathfrak{m}) \otimes \mathfrak{m}$ as $\mathrm{d}v = 0$. This implies $\sum_i \sum_{j=1}^n u_{i_j} x_{i_j} = v''$ for some element $v'' \in U_{n-1}(\mathfrak{m})$. Rearrange this and add $n-1$ copies of $\sum_i u_{i_n} x_{i_n} = \sum_i x_{i_1}x_{i_2}\cdots x_{i_n}$ on both sides; we get

$$n \sum_i x_{i_1}\cdots x_{i_n} = \sum_i \sum_{j=1}^{n-1} x_{i_1}\cdots x_{i_{(j-1)}} [x_{i_j}, x_{i_{j+1}}\cdots x_{i_n}] + v'';$$

therefore,

$$v = \frac{1}{n} \sum_i \sum_{j=1}^{n-1} x_{i_1}\cdots x_{i_{(j-1)}} [x_{i_j}, x_{i_{j+1}}\cdots x_{i_n}] + \frac{1}{n} v'' + v' \in U_{n-1}(\mathfrak{m}).$$

Thus, we see that v actually lies in $(\ker \mathrm{d}) \cap U_{n-1}(\mathfrak{m})$, hence $v \in k1$ by assumption. Hence $(\ker \mathrm{d}) \cap U_n(\mathfrak{m}) = k1$ for any $n \ge 0$, which completes the proof. □

We apply this to $U_\lambda(\mathfrak{m})$. Because the Hopf algebra here is cocommutative, the canonical extension to a DGA is by the classical exterior or Grassmann algebra on $\Lambda^1=\mathfrak{m}$ with $\mathrm{d}\Lambda^1=0$. To make contact with real classical geometry in the rest of the paper, the standard approach in noncommutative geometry is to work over $\mathbb{C}$ with complexified differential forms and functions and to remember the "real form" by means of a $*$-involution. We recall that a differential graded algebra over $\mathbb{C}$ is called a $*$-DGA if it is equipped with a conjugate-linear map $*:\Omega\to\Omega$ such that

$$*^2=\mathrm{id},\quad (\xi\wedge\eta)^*=(-1)^{|\xi||\eta|}\eta^*\wedge\xi^*,\quad \mathrm{d}(\xi^*)=(\mathrm{d}\xi)^*$$

for any $\xi,\eta\in\Omega$. Let $\mathfrak{m}$ be a real pre-Lie algebra, i.e., there is a basis $\{e_i\}$ of $\mathfrak{m}$ with real structure coefficients. Then this is also a real form for $\mathfrak{m}$ as a Lie algebra. In this case, $e_i^*=e_i$ extends complex-linearly to an involution $*:\mathfrak{m}\to\mathfrak{m}$, which then makes $\Omega(U_\lambda(\mathfrak{m}))$ a $*$-DGA if $\lambda^*=-\lambda$, i.e., if λ is imaginary. If we want λ real then we should take $e_i^*=-e_i$.

Example 4.5. Let $\mathfrak{b}$ be the 2-dimensional complex nonabelian Lie algebra defined by $[x,t]=x$. It admits five families of mutually nonisomorphic pre-Lie algebra structures over $\mathbb{C}$ [Burde 1998], which are

$$\begin{array}{llll}
\mathfrak{b}_{1,\alpha}: & t\circ x=-x, & t\circ t=\alpha t, & \\
\mathfrak{b}_{2,\beta\neq 0}: & x\circ t=\beta x, & t\circ x=(\beta-1)x, & t\circ t=\beta t,\\
\mathfrak{b}_3: & t\circ x=-x, & t\circ t=x-t, & \\
\mathfrak{b}_4: & x\circ x=t, & t\circ x=-x, & t\circ t=-2t,\\
\mathfrak{b}_5: & x\circ t=x, & t\circ t=x+t, &
\end{array}$$

where $\alpha,\beta\in\mathbb{C}$. (Here $\mathfrak{b}_{1,0}\cong\mathfrak{b}_{2,0}$, so we let $\beta\neq 0$.) Thus there are five families of bicovariant differential calculi over $U_\lambda(\mathfrak{b})$:

$$\begin{array}{llll}
\Omega^1(U_\lambda(\mathfrak{b}_{1,\alpha})): & [t,\mathrm{d}x]=-\lambda\,\mathrm{d}x, & [t,\mathrm{d}t]=\lambda\alpha\,\mathrm{d}t; & \\
\Omega^1(U_\lambda(\mathfrak{b}_{2,\beta\neq 0})): & [x,\mathrm{d}t]=\lambda\beta\,\mathrm{d}x, & [t,\mathrm{d}x]=\lambda(\beta-1)\,\mathrm{d}x, & [t,\mathrm{d}t]=\lambda\beta\,\mathrm{d}t;\\
\Omega^1(U_\lambda(\mathfrak{b}_3)): & [t,\mathrm{d}x]=-\lambda\,\mathrm{d}x, & [t,\mathrm{d}t]=\lambda\,\mathrm{d}x-\lambda\,\mathrm{d}t; & \\
\Omega^1(U_\lambda(\mathfrak{b}_4)): & [x,\mathrm{d}x]=\lambda\,\mathrm{d}t, & [t,\mathrm{d}x]=-\lambda\,\mathrm{d}x, & [t,\mathrm{d}t]=-2\lambda\,\mathrm{d}t;\\
\Omega^1(U_\lambda(\mathfrak{b}_5)): & [x,\mathrm{d}t]=\lambda\,\mathrm{d}x, & [t,\mathrm{d}t]=\lambda\,\mathrm{d}x+\lambda\,\mathrm{d}t. &
\end{array}$$

All these examples are $*$-DGAs with $x^*=x$ and $t^*=t$ when $\lambda^*=-\lambda$ as $\{x,t\}$ is a real form of the relevant pre-Lie algebra. We also need for this that α and β are real. The further noncommutative geometry of $\mathfrak{b}_{1,\alpha}$ and $\mathfrak{b}_{2,\beta}$ in 4-dimensional cases is studied in [Majid and Tao 2015a].

Example 4.6. For $q \in \mathbb{C}$, $q \neq 0$, we recall that the Hopf algebra $\mathbb{C}_q[\mathrm{SL}_2]$ is, as an algebra, a quotient of a free algebra $\mathbb{C}\langle a, b, c, d\rangle$ modulo relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad bc = cb,$$
$$ad - da = (q^{-1} - q)bc, \quad ad - q^{-1}bc = 1.$$

Writing the generators a, b, c, d as a single matrix, the coproduct, counit and antipode of $\mathbb{C}_q[\mathrm{SL}_2]$ are given by

$$\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \epsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad S\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a\end{pmatrix},$$

where we understand $\Delta(a) = a \otimes a + b \otimes c$, $\epsilon(a) = 1$, $S(a) = d$, etc. By definition, the quantum group $\mathbb{C}_q[\mathrm{SU}_2]$ is Hopf algebra $\mathbb{C}_q[\mathrm{SL}_2]$ with q real and $*$-structure

$$\begin{pmatrix} a^* & b^* \\ c^* & d^*\end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a\end{pmatrix}.$$

We use the conventions of [Majid 1995] and refer there for the history, which is related both to [Woronowicz 1989] and the Drinfeld theory [1987].

On $\mathbb{C}_q[\mathrm{SU}_2]$, there is a connected left-covariant calculus $\Omega^1(\mathbb{C}_q[\mathrm{SU}_2])$ in [Woronowicz 1989] with basis, in our conventions,

$$\omega^0 = d\,\mathrm{d}a - qb\,\mathrm{d}c, \quad \omega^+ = d\,\mathrm{d}b - qb\,\mathrm{d}d, \quad \omega^- = qa\,\mathrm{d}c - c\,\mathrm{d}a$$

of left-invariant 1-forms which is dual to the basis $\{\partial_0, \partial_\pm\}$ of left-invariant vector fields generated by the Chevalley basis $\{H, X_\pm\}$ of su_2 (so that $[H, X_\pm] = \pm 2X_\pm$ and $[X_+, X_-] = H$). The first-order calculus is generated by $\{\omega^0, \omega^\pm\}$ as a left module while the right module structure is given by the bimodule relations

$$\omega^0 f = q^{2|f|} f\omega^0, \quad \omega^\pm f = q^{|f|} f\omega^\pm$$

for homogeneous f of degree $|f|$, where $|a| = |c| = 1$, $|b| = |d| = -1$, and with exterior derivatives

$$\mathrm{d}a = a\omega^0 + q^{-1}b\omega^+, \quad \mathrm{d}b = -q^{-2}b\omega^0 + a\omega^-,$$
$$\mathrm{d}c = c\omega^0 + q^{-1}d\omega^+, \quad \mathrm{d}d = -q^{-2}d\omega^0 + c\omega^-.$$

These extend to a differential graded algebra $\Omega(\mathbb{C}_q[\mathrm{SU}_2])$ that has same dimension as classically. Moreover, it is a $*$-DGA with

$$\omega^{0*} = -\omega^0, \quad \omega^{+*} = -q^{-1}\omega^-, \quad \omega^{-*} = -q\omega^+.$$

Since $\mathbb{C}_q[\mathrm{SU}_2]$ and $\Omega(\mathbb{C}_q[\mathrm{SU}_2])$ are q-deformations, from Corollary 4.2 these must be quantised from some pre-Lie algebra structure of su_2^*, which we now compute. Let

$$q = e^{\imath\lambda/2} = 1 + \frac{\imath}{2}\lambda + O(\lambda^2)$$

for imaginary λ. The Poisson bracket from the algebra relations is

$$\{a,b\}=-\frac{\imath}{2}ab,\quad \{a,c\}=-\frac{\imath}{2}ac,\quad \{a,d\}=-\imath bc,\quad \{b,c\}=0,$$
$$\{b,d\}=-\frac{\imath}{2}bd,\quad \{c,d\}=-\frac{\imath}{2}cd.$$

The reader should not be alarmed by the $\imath$ as this is a "complexified" Poisson bracket on $C^\infty(\mathrm{SU}_2,\mathbb{C})$ and is a real Poisson bracket on $C^\infty(\mathrm{SU}_2,\mathbb{R})$ when we choose real-valued functions instead of complex-valued functions a, b, c, d here.

As $\mathrm{d}x=\sum_i(\partial_i x)\omega^i$, we know, in the classical limit,

$$\partial_0\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&-b\\c&-d\end{pmatrix},\quad \partial_+\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}b&0\\d&0\end{pmatrix},\quad \partial_-\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}0&a\\0&c\end{pmatrix}.$$

From $a\omega^0-\omega^0a=(1-q^2)a\omega^0=-\imath\lambda a\omega^0+O(\lambda^2)$, we know that $\gamma(a,\omega^0)=-\imath a\omega^0$. Likewise, we can get

$$\gamma\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\omega^i\right)=\frac{1}{2}t_i\begin{pmatrix}a&-b\\c&-d\end{pmatrix}\omega^i\quad\text{for all } i\in\{0,\pm\},\ t_0=-2\imath,\ t_\pm=-\imath.$$

Now we can compute the pre-Lie structure $\Xi:\mathrm{su}_2^*\otimes\mathrm{su}_2^*\to\mathrm{su}_2^*$ by comparing with (2-8), namely

$$\gamma\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\omega^j\right)=\sum_{i,k\in\{0,\pm\}}\Xi^{ij}_k\left(\partial_i\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)\omega^k$$

tells us that the only nonzero coefficients are

$$\Xi^{00}_0=-\imath,\quad \Xi^{0+}_+=-\frac{\imath}{2},\quad \Xi^{0-}_-=-\frac{\imath}{2}.$$

Then

$$\Xi(\phi,\phi)=-\imath\phi,\quad \Xi(\phi,\psi_+)=-\frac{\imath}{2}\psi_+,\quad \Xi(\phi,\psi_-)=-\frac{\imath}{2}\psi_-,$$

and Ξ is zero on other terms, where $\{\phi,\psi_\pm\}$ is the dual basis of su_2^* to $\{H,X_\pm\}$. Thus the corresponding pre-Lie structure of su_2^* is

$$\Xi(\phi,\phi)=-\imath\phi,\quad \Xi(\phi,\psi_\pm)=-\frac{\imath}{2}\psi_\pm\quad\text{and zero otherwise.}$$

Letting $t=-2\imath\phi$, $x_1=\imath(\psi_++\psi_-)$, $x_2=\psi_+-\psi_-$, we have a real pre-Lie structure for $\mathrm{su}_2^*=\mathrm{span}\{t,x_1,x_2\}$:

$$t\circ t=-2t,\quad t\circ x_i=-x_i\quad\text{for all } i=1,2.$$

This is a 3-dimensional version of $\mathfrak{b}_{1,-2}$.

Example 4.7. Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra with r-matrix

$$r=r^{(1)}\otimes r^{(2)}\in\mathfrak{g}\otimes\mathfrak{g}.$$

Then $\mathfrak{g}$ acts on its dual $\mathfrak{g}^*$ by coadjoint action ad^* and by [Majid 2000, Lemma 3.8], $\mathfrak{g}^*$ becomes a left $\mathfrak{g}$-crossed module with $-\Xi$, where Ξ is the left $\mathfrak{g}^*$-action

$$\Xi(\phi, \psi) = -\langle \phi, r^{(2)} \rangle \, \mathrm{ad}^*_{r^{(1)}} \psi.$$

To satisfy the Poisson-compatibility (2-6), $(\mathfrak{g}, r)$ is required to obey

$$r^{(1)} \otimes [r^{(2)}, x] + r^{(2)} \otimes [r^{(1)}, x] = 0, \quad \text{i.e., } r_+ \triangleright x = 0, \quad \text{for all } x \in \mathfrak{g}, \tag{4-7}$$

where $r_+ = \frac{1}{2}(r + r_{21})$ is the symmetric part of r and the second factor of r_+ acts on x via adjoint action of $\mathfrak{g}$. In this case $\mathfrak{g}^*$ has a pre-Lie algebra structure with $\Xi(\phi, \psi) = -\langle \phi, r^{(2)} \rangle \, \mathrm{ad}^*_{r^{(1)}} \psi$ by Corollary 4.2. We see in particular that every finite-dimensional cotriangular Lie bialgebra $\mathfrak{g}^*$ is canonically a pre-Lie algebra. More generally, if the centre $Z(\mathfrak{g})$ is nontrivial then any nonzero $r_+ \in Z(\mathfrak{g})^{\otimes 2}$ combined with a triangular structure r_- gives a strictly quasitriangular $r = r_- + r_+$ obeying (4-7). This is the full content of (4-7) since this requires that the image of r_+ regarded as a map $\mathfrak{g}^* \to \mathfrak{g}$ lies in $Z(\mathfrak{g})$ and r_+ is symmetric. On the other hand, δ and Ξ are the same as computed from r_-, so we may as well take $r_+ = 0$ as far as our present applications are concerned.

5. Quantisation of the tangent bundle $TG = G \triangleright\!\!\!< \underline{\mathfrak{g}}$

We will be interested in quantisation of the tangent bundle TG of a Poisson–Lie group G, with natural noncommutative coordinate algebra in this case provided by a bicrossproduct [Majid 1990b; 1995].

5A. ***Review of bicrossproduct Hopf algebras.*** We start with the notions of double cross-sum and bicross-sum of Lie bialgebras [Majid 1995, Chapter 8]. We say $(\mathfrak{g}, \mathfrak{m}, \triangleleft, \triangleright)$ forms a right–left matched pair of Lie algebras if $\mathfrak{g}$ and $\mathfrak{m}$ are both Lie algebras and $\mathfrak{g}$ right acts on $\mathfrak{m}$ via $\triangleleft$, $\mathfrak{m}$ left acts on $\mathfrak{g}$ via $\triangleright$ with

$$[\phi, \psi] \triangleleft \xi = [\phi \triangleleft \xi, \psi] + [\phi, \psi \triangleleft \xi] + \phi \triangleleft (\psi \triangleright \xi) - \psi \triangleleft (\phi \triangleright \xi),$$
$$\phi \triangleright [\xi, \eta] = [\phi \triangleright \xi, \eta] + [\xi, \phi \triangleright \eta] + (\phi \triangleleft \xi) \triangleright \eta - (\phi \triangleleft \eta) \triangleright \xi,$$

for any $\xi, \eta \in \mathfrak{g}$, $\phi, \psi \in \mathfrak{m}$. Given such a matched pair, one can define the "double cross-sum Lie algebra" $\mathfrak{g} \bowtie \mathfrak{m}$ as the vector space $\mathfrak{g} \oplus \mathfrak{m}$ with the Lie bracket

$$[(\xi, \phi), (\eta, \psi)] = ([\xi, \eta] + \phi \triangleright \eta - \psi \triangleright \xi, [\phi, \psi] + \phi \triangleleft \eta - \psi \triangleleft \xi).$$

In addition, if both $\mathfrak{g}$ and $\mathfrak{m}$ are now Lie bialgebras with $\triangleright$ and $\triangleleft$ making $\mathfrak{g}$ a left $\mathfrak{m}$-module Lie coalgebra and $\mathfrak{m}$ a right $\mathfrak{g}$-module Lie coalgebra, such that

$$\phi \triangleleft \xi_{(1)} \otimes \xi_{(2)} + \phi_{(1)} \otimes \phi_{(2)} \triangleright \xi = 0$$

for all $\xi \in \mathfrak{g}$, $\phi \in \mathfrak{m}$, then the direct sum Lie coalgebra structure makes $\mathfrak{g} \bowtie \mathfrak{m}$ into a Lie bialgebra, *the double cross-sum Lie bialgebra*.

Next, if $\mathfrak{g}$ is finite-dimensional, the matched pair of Lie bialgebras $(\mathfrak{g}, \mathfrak{m}, \triangleleft, \triangleright)$ equivalently defines a *right–left bicross-sum Lie bialgebra* $\mathfrak{m} \triangleright\!\!\blacktriangleleft \mathfrak{g}^*$ built on $\mathfrak{m} \oplus \mathfrak{g}^*$ with

$$[(\phi, f), (\psi, h)] = ([\phi, \psi]_{\mathfrak{m}}, [f, h]_{\mathfrak{g}^*} + f \triangleleft \psi - h \triangleleft \phi), \tag{5-1}$$

$$\delta\phi = \delta_{\mathfrak{m}}\phi + (\mathrm{id} - \tau)\beta(\phi), \quad \delta f = \delta_{\mathfrak{g}^*} f, \tag{5-2}$$

for any $\phi, \psi \in \mathfrak{m}$ and $f, h \in \mathfrak{g}^*$, where the right action of $\mathfrak{m}$ on $\mathfrak{g}^*$ and the left coaction of $\mathfrak{g}^*$ on $\mathfrak{m}$ are induced from $\triangleleft$ and $\triangleright$ by

$$\langle f \triangleleft \phi, \xi \rangle = \langle f, \phi \triangleright \xi \rangle, \quad \beta(\phi) = \sum_i f^i \otimes \phi \triangleleft e_i,$$

for all $\phi \in \mathfrak{m}$, $f \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $\{e_i\}$ is a basis of $\mathfrak{g}$ with dual basis $\{f^i\}$. We refer to [Majid 1995, Section 8.3] for the proof.

Now let $(\mathfrak{g}, \mathfrak{m}, \triangleleft, \triangleright)$ be a matched pair of Lie algebras and M be the connected and simply connected Lie group associated to $\mathfrak{m}$. The Poisson–Lie group $M \triangleright\!\!\blacktriangleleft \mathfrak{g}^*$ associated to the bicross-sum $\mathfrak{m} \triangleright\!\!\blacktriangleleft \mathfrak{g}^*$ is the semidirect product $M \ltimes \mathfrak{g}^*$ (where $\mathfrak{g}^*$ is regarded as an abelian group) equipped with Poisson bracket

$$\{f, g\} = 0, \quad \{\xi, \eta\} = [\xi, \eta]_{\mathfrak{g}}, \quad \{\xi, f\} = \alpha_{*\xi}(f),$$

for all functions f, g on M and linear functions ξ, η on $\mathfrak{g}^*$, where $\alpha_{*\xi}$ is the vector field for the action of $\mathfrak{g}$ on M. See [Majid 1995, Proposition 8.4.7] for the proof. Note that here $\mathfrak{g}, \mathfrak{m}$ are both viewed as Lie bialgebras with zero cobracket, so the Lie bracket and Lie cobracket of the bicross-sum Lie bialgebra $\mathfrak{m} \triangleright\!\!\blacktriangleleft \mathfrak{g}^*$ is now given by (5-1) and (5-2) but with $[\ ,\ ]_{\mathfrak{g}^*} = 0$, $\delta_{\mathfrak{m}} = 0$.

More precisely, let $(\mathfrak{g}, \mathfrak{m}, \triangleleft, \triangleright)$ be a matched pair of Lie algebras, with the associated connected and simply connected Lie groups G acting on $\mathfrak{m}$ and M acting on $\mathfrak{g}$. The action $\triangleleft$ can be viewed as Lie algebra cocycle $\triangleleft \in Z^1_{\triangleright^* \otimes - \mathrm{id}}(\mathfrak{m}, \mathfrak{g}^* \otimes \mathfrak{m})$ and under some assumptions can then be exponentiated to a group cocycle

$$a \in Z^1_{\triangleright^* \otimes \mathrm{Ad}_R}(M, \mathfrak{g}^* \otimes \mathfrak{m}),$$

which defines an infinitesimal action of $\mathfrak{g}$ on M. Hence, by evaluation of the corresponding vector fields, a defines a left action of the Lie algebra $\mathfrak{g}$ on $C^\infty(M)$ [Majid 1990a]:

$$(\tilde{\xi} f)(s) = \tilde{a}_\xi(f)(s) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(s \exp(t a_\xi(s))\big)\Big|_{t=0} \quad \text{for all } f \in C^\infty(M),\ \xi \in \mathfrak{g}. \tag{5-3}$$

We also note that $\mathfrak{m}$ acts on M by a left-invariant vector field:

$$(\widetilde{\phi} f)(s) = \frac{\mathrm{d}}{\mathrm{d}t} f(s \exp(t\phi))\Big|_{t=0}$$

for any $\phi \in \mathfrak{m}$, $f \in C^\infty(M)$, and these two actions fit together to an action of $\mathfrak{g} \bowtie \mathfrak{m}$ on $C^\infty(M)$.

Finally, we can explain the bicrossproduct $\mathbb{C}[M] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g})$ based on a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{m}, \triangleleft, \triangleright)$, where $\mathbb{C}[M]$ is an algebraic model of functions on M. The algebra of $\mathbb{C}[M] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g})$ is the cross product defined by the action (5-3). Its coalgebra, on the other hand, is the cross coproduct given in reasonable cases by a right coaction (defined by the left action of M on $\mathfrak{g}$)

$$\beta : \mathfrak{g} \to \mathfrak{g} \otimes \mathbb{C}[M], \quad \beta(\xi)(s) = s \triangleright \xi \quad \text{for all } \xi \in \mathfrak{g},\ s \in M.$$

The map β is extended to products of the generators of $U_\lambda(\mathfrak{g})$ to form a bicrossproduct $\mathbb{C}[M] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g})$ as in [Majid 1995, Theorem 6.2.2].

The Poisson–Lie group $M \triangleright\!\!\blacktriangleleft \mathfrak{g}^*$ quantises to $\mathbb{C}[M] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g})$ as a noncommutative deformation of the commutative algebra of functions $\mathbb{C}[M \triangleright\!\!\blacktriangleleft \mathfrak{g}^*]$. See [Majid 1995, Section 8.3] for more details. The half-dualisation process we have described at the Lie bialgebra level also works at the Hopf algebra level, at least in the finite-dimensional case. So morally speaking, $U_\lambda(\mathfrak{g}) \bowtie U(\mathfrak{m})$ half-dualises in a similar way to the bicrossproduct Hopf algebra $\mathbb{C}[M] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g})$. If one is only interested in the algebra and its calculus, we can extend to the cross product $C^\infty(M) \rtimes U_\lambda(\mathfrak{g})$.

5B. ***Poisson–Lie group structures on the tangent bundle $G \ltimes \underline{\mathfrak{g}}$.*** Let G be a Lie group with Lie algebra $\mathfrak{g}$. As a Lie group, the tangent bundle TG of a Lie group G can be identified with the semidirect product of Lie groups $G \ltimes \underline{\mathfrak{g}}$ (by the right adjoint action of G on $\mathfrak{g}$) with product

$$(g_1, x)(g_2, y) = (g_1 g_2, \mathrm{Ad}(g_2^{-1})(x) + y) \quad \text{for all } g_1, g_2 \in G,\ x, y \in \mathfrak{g},$$

where $\underline{\mathfrak{g}}$ is $\mathfrak{g}$ but viewed as an abelian Poisson–Lie group under addition. Naturally, the Lie algebra of $G \ltimes \underline{\mathfrak{g}}$ is the semidirect sum Lie algebra $\mathfrak{g} \ltimes \underline{\mathfrak{g}}$ with Lie bracket

$$[\xi, \eta] = [\xi, \eta]_{\mathfrak{g}}, \quad [x, y] = 0, \quad [x, \xi] = [x, \xi]_{\mathfrak{g}} \quad \text{for all } \xi, \eta \in \mathfrak{g},\ x, y \in \underline{\mathfrak{g}}.$$

Keeping in mind the observations in Section 5A, we propose the following construction of a Poisson–Lie structure on the tangent bundle $G \ltimes \underline{\mathfrak{g}}$ via a bicrosssum. In what follows we assume that G is a finite-dimensional connected and simply connected Poisson–Lie group, and $\mathfrak{g}$ is its Lie algebra with the corresponding Lie bialgebra structure. We let

$$\overline{\mathfrak{g}^*} := (\mathfrak{g}^*, [\ ,\ ]_{\mathfrak{g}^*}, \text{zero Lie cobracket}) \quad \text{and} \quad \bar{\mathfrak{g}} := (\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \text{zero Lie cobracket}),$$

where $\overline{\mathfrak{g}^*}$ is the dual of Lie bialgebra $\underline{\mathfrak{g}} = (\mathfrak{g}, \text{zero bracket}, \delta_{\mathfrak{g}})$. One can check that $\overline{\mathfrak{g}^*}$ and $\bar{\mathfrak{g}}$ together form a matched pair of Lie bialgebras with coadjoint actions, i.e.,

$$\xi \triangleleft \phi = -\,\mathrm{ad}^*_\phi\, \xi = \langle \phi, \xi_{(1)} \rangle \xi_{(2)}, \quad \xi \triangleright \phi = \mathrm{ad}^*_\xi\, \phi = \phi_{(1)} \langle \phi_{(2)}, \xi \rangle$$

for any $\phi \in \overline{\mathfrak{g}^*},\ \xi \in \bar{\mathfrak{g}}$.

5B1. *Lie bialgebra level.* The double cross-sum Lie bialgebra $\overline{\mathfrak{g}^*} \bowtie \bar{\mathfrak{g}}$ is then built on $\mathfrak{g}^* \oplus \mathfrak{g}$ as a vector space with Lie bracket

$$[\phi, \psi] = [\phi, \psi]_{\mathfrak{g}^*}, \quad [\xi, \eta] = [\xi, \eta]_{\mathfrak{g}},$$
$$[\xi, \phi] = \xi \triangleleft \phi + \xi \triangleright \phi = \langle \phi, \xi_{(1)} \rangle \xi_{(2)} + \phi_{(1)} \langle \phi_{(2)}, \xi \rangle$$

for all $\phi, \psi \in \overline{\mathfrak{g}^*},\ \xi, \eta \in \bar{\mathfrak{g}}$, and zero Lie cobracket. This is nothing but the Lie algebra of the Drinfeld double $D(\mathfrak{g}) = \mathfrak{g}^* \bowtie \mathfrak{g}$ of $\mathfrak{g}$ with zero Lie-cobracket.

Correspondingly, the right–left bicross-sum Lie bialgebra defined by the matched pair $(\overline{\mathfrak{g}^*}, \bar{\mathfrak{g}}, \triangleleft, \triangleright)$ above is $\bar{\mathfrak{g}} \blacktriangleright\!\!\triangleleft \underline{\mathfrak{g}}$, whose Lie algebra is a semidirect sum $\bar{\mathfrak{g}} \rtimes \underline{\mathfrak{g}}$ and the Lie coalgebra is semidirect cobracket $\bar{\mathfrak{g}} >\!\!\blacktriangleleft \underline{\mathfrak{g}}$, namely

$$[\xi, \eta] = [\xi, \eta]_{\mathfrak{g}}, \quad [x, y] = 0, \quad [x, \xi] = [x, \xi]_{\mathfrak{g}},$$
$$\delta\xi = (\mathrm{id} - \tau)\delta_{\mathfrak{g}}(\xi) = \underline{\xi_{(1)}} \otimes \overline{\xi_{(2)}} - \overline{\xi_{(2)}} \otimes \underline{\xi_{(1)}}, \quad \delta x = \delta_{\mathfrak{g}} x, \tag{5-4}$$

for any $\xi, \eta \in \bar{\mathfrak{g}},\ x, y \in \underline{\mathfrak{g}}$. Here the coaction on $\bar{\mathfrak{g}}$ is the Lie cobracket $\delta_{\mathfrak{g}}$ viewed as a map from $\bar{\mathfrak{g}}$ to $\underline{\mathfrak{g}} \otimes \bar{\mathfrak{g}}$.

5B2. *Poisson–Lie level.* Associated to the right–left bicross-sum Lie bialgebra $\bar{\mathfrak{g}} \blacktriangleright\!\!\triangleleft \underline{\mathfrak{g}}$, the Lie group $G \rtimes \underline{\mathfrak{g}}$ is a Poisson–Lie group (denoted by $\bar{G} \blacktriangleright\!\!\triangleleft \underline{\mathfrak{g}}$) with the Poisson bracket

$$\{f, h\} = 0, \quad \{\phi, \psi\} = [\phi, \psi]_{\mathfrak{g}^*}, \quad \{\phi, f\} = \tilde{\phi} f \tag{5-5}$$

for any $\phi, \psi \in \overline{\mathfrak{g}^*} \subseteq C^\infty(\underline{\mathfrak{g}})$ and $f, h \in C^\infty(\bar{G})$, where $\tilde{\phi}$ denotes the left Lie algebra action of $\overline{\mathfrak{g}^*}$ on $C^\infty(G)$ (viewed as a vector field on G) and is defined by the right action of $\overline{\mathfrak{g}^*}$ on $\bar{\mathfrak{g}}$.

The vector field $\tilde{\phi}$ for any $\phi \in \mathfrak{g}^*$ in this case can be interpreted more precisely. We can view the actions between $\overline{\mathfrak{g}^*}$ and $\bar{\mathfrak{g}}$ as Lie algebra 1-cocycles, namely the right coadjoint action $\triangleleft = -\mathrm{ad}^* : \bar{\mathfrak{g}} \otimes \overline{\mathfrak{g}^*} \to \bar{\mathfrak{g}}$ (of $\overline{\mathfrak{g}^*}$ on $\bar{\mathfrak{g}}$) is viewed as a map $\bar{\mathfrak{g}} \to (\overline{\mathfrak{g}^*})^* \otimes \bar{\mathfrak{g}} = (\underline{\mathfrak{g}})^{**} \otimes \bar{\mathfrak{g}} = \underline{\mathfrak{g}} \otimes \bar{\mathfrak{g}}$. It maps ξ to

$$\sum_i e_i \otimes \xi \triangleleft f^i = \sum_i e_i \otimes \langle f^i, \xi_{(1)} \rangle \xi_{(2)} = \xi_{(1)} \otimes \xi_{(2)},$$

which is nothing but the Lie cobracket $\delta_{\mathfrak{g}}$ of $\mathfrak{g}$. Likewise, the left coadjoint action of $\bar{\mathfrak{g}}$ on $\overline{\mathfrak{g}^*}$ is viewed as the Lie cobracket $\delta_{\mathfrak{g}^*}$ of $\mathfrak{g}^*$. We already know that the Lie 1-cocycle $\delta_{\mathfrak{g}} \in Z^1_{-\mathrm{ad}}(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ exponentiates to a group cocycle

$$D^\vee \in Z^1_{\mathrm{Ad}_R}(G, \mathfrak{g} \otimes \mathfrak{g}),$$

thus

$$\tilde{\phi}_g := (L_g)_*((\phi \otimes \mathrm{id}) D^\vee(g)) \in T_g G, \quad \text{for all } g \in G, \tag{5-6}$$

defines the vector field on G in (5-5).

According to [Majid 1995, Proposition 8.4.7], the Poisson bivector on the tangent bundle $TG = \overline{G} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$ is

$$\pi = \sum_i (\partial_i \otimes \tilde{f}^i - \tilde{f}^i \otimes \partial_i) + \sum_{i,j,k} d_k^{ij} f^k \partial_i \otimes \partial_j, \tag{5-7}$$

where $\{\partial_i\}$ is the basis of left-invariant vector fields generated by the basis $\{e_i\}$ of $\mathfrak{g}$ and $\{f^i\}$ is the dual basis of $\mathfrak{g}^*$. Here

$$P_{KK} = \sum_{i,j,k} d_k^{ij} f^k \partial_i \otimes \partial_j$$

is the known Kirillov–Kostant bracket on $\underline{\mathfrak{g}}$ with

$$\delta_{\mathfrak{g}} e_k = \sum_{ij} d_k^{ij} e_i \otimes e_j.$$

We arrive at the following special case of [Majid 1995, Proposition 8.4.7]:

Lemma 5.1. *Let G be a finite-dimensional connected and simply connected Poisson–Lie group and $\mathfrak{g}$ be its Lie algebra. The tangent bundle $TG = G \rtimes \underline{\mathfrak{g}}$ of G admits a Poisson–Lie structure given by* (5-5) *or* (5-7)*, denoted by $\overline{G} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$. The corresponding Lie bialgebra is $\bar{\mathfrak{g}} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$, given by* (5-4).

5B3. *Bicrossproduct Hopf algebra.* Finally, when the actions and coactions are suitably algebraic, we have a bicrossproduct Hopf algebra $\mathbb{C}[\overline{G}] \blacktriangleright\!\triangleleft U_\lambda(\overline{\mathfrak{g}^*})$ as a quantisation of the commutative algebra of functions $\mathbb{C}[\overline{G} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}]$ on the tangent bundle $\overline{G} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$ of a Poisson–Lie group G. The commutation relations of $\mathbb{C}[\overline{G}] \blacktriangleright\!\triangleleft U_\lambda(\overline{\mathfrak{g}^*})$ are

$$[f, h] = 0, \quad [\phi, \psi] = \lambda[\phi, \psi]_{\mathfrak{g}^*}, \quad [\phi, f] = \lambda\widetilde{\phi} f$$

for any $\phi, \psi \in \overline{\mathfrak{g}^*} \subseteq C^\infty(\underline{\mathfrak{g}})$ and $f, h \in \mathbb{C}[\overline{G}]$. This construction is still quite general but includes a canonical example for all compact real forms $\mathfrak{g}$ of complex simple Lie algebras based in the Iwasawa decomposition to provide the double cross product or "Manin triple" in this case [Majid 1990a]. We start with an even simpler example.

Example 5.2. Let $\mathfrak{m}$ be a finite-dimensional real Lie algebra, viewed as a Lie bialgebra with zero Lie-cobracket. Take $G = \mathfrak{m}^*$, the abelian Poisson–Lie group with Kirillov–Kostant Poisson bracket given by the Lie bracket of $\mathfrak{m}$. Then $\mathfrak{g} = \mathfrak{m}^*$ and $\overline{\mathfrak{g}^*} = \mathfrak{m}$ and $\bar{\mathfrak{g}} = \overline{\mathfrak{m}^*} = \mathbb{R}^n$, where $n = \dim \mathfrak{m}$. Since the Lie bracket of $\mathfrak{m}^*$ is zero, $\overline{\mathfrak{m}^*}$ acts trivially on $\mathfrak{m}$, while $\mathfrak{m}$ acts on $\overline{\mathfrak{m}^*}$ by right coadjoint action $-\mathrm{ad}^*$, namely

$$f \triangleleft \xi = -\mathrm{ad}^*_\xi f, \quad \text{or} \quad \langle f \triangleleft \xi, \eta\rangle = \langle f, [\xi, \eta]_{\mathfrak{m}}\rangle$$

for any $f \in \overline{\mathfrak{m}^*}$, $\xi, \eta \in \mathfrak{m}$. So $(\mathfrak{m}, \overline{\mathfrak{m}^*}, \triangleleft = -\mathrm{ad}^*, \triangleright = 0)$ forms a matched pair.

The double cross-sum of the matched pair $(\mathfrak{m}, \overline{\mathfrak{m}^*})$ is $\mathfrak{m} \bowtie \overline{\mathfrak{m}^*}$, the semidirect sum Lie algebra with coadjoint action of $\mathfrak{m}$ on $\overline{\mathfrak{m}^*}$:

$$[\xi,\eta]=[\xi,\eta]_{\mathfrak{m}},\quad [f,h]=0,\quad [f,\xi]=f\triangleleft\xi=\langle\xi, f_{(1)}\rangle f_{(2)},$$
$$\delta\xi=0,\quad \delta f=0\quad \text{for all } \xi,\eta\in\mathfrak{m},\ f,h\in\overline{\mathfrak{m}^*}.$$

Meanwhile, the right–left bicross-sum of the matched pair $(\mathfrak{m}, \overline{\mathfrak{m}^*})$ is $\overline{\mathfrak{m}^*} \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^*$, the semidirect sum Lie coalgebra

$$[f,h]=0,\quad [\phi,\psi]=0,\quad [\phi,f]=\phi\triangleleft f=0,$$
$$\delta f=(\mathrm{id}-\tau)\beta(f),\quad \delta\phi=\delta_{\mathfrak{m}^*}\phi,$$

for any $f,h\in\overline{\mathfrak{m}^*}$, $\phi,\psi\in\mathfrak{m}^*$, where the left coaction of $\mathfrak{m}^*$ on $\overline{\mathfrak{m}^*}$ is given by

$$\beta:\overline{\mathfrak{m}^*}\to\mathfrak{m}^*\otimes\overline{\mathfrak{m}^*},\quad \beta(f)=\sum_i f^i\otimes f\triangleleft e_i,$$

and $\{e_i\}$ is a basis of $\mathfrak{m}$ with dual basis $\{f^i\}$ of $\mathfrak{m}^*$.

The tangent bundle of $\mathfrak{m}^*$ is the associated Poisson–Lie group of $\overline{\mathfrak{m}^*} \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^*$, which is $\overline{M^*} \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^* = \mathbb{R}^n \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^*$, an abelian Lie group, where we identify the abelian Lie group $\overline{M^*}$ with its abelian Lie algebra $\overline{\mathfrak{m}^*}$. Let $\{x^i\}$ be the coordinate functions on $\mathbb{R}^n$ identified with $\{e_i\}\subset\mathfrak{m}\subseteq C^\infty(\overline{\mathfrak{m}^*})=C^\infty(\mathbb{R}^n)$, as $e_i\big(\sum_j\lambda_j f^j\big)=\lambda_i$. The right action of $\mathfrak{m}$ on $\overline{\mathfrak{m}^*}$ transfers to $\delta_{\mathfrak{m}^*}\in Z^1(\overline{\mathfrak{m}^*}, \underline{\mathfrak{m}^*}\otimes\overline{\mathfrak{m}^*})$. As a Lie group $\overline{M^*}$ is abelian and $\overline{M^*}=\overline{\mathfrak{m}^*}=\mathbb{R}^n$, so the associated group cocycle is identical to $\delta_{\mathfrak{m}^*}$, thus from (5-6) we have

$$\tilde{\xi}_x f=\langle x_{(1)},\xi\rangle x_{(2)x} f=\sum_i\langle x_{(1)},\xi\rangle\langle x_{(2)},e_i\rangle f^i_x f=\sum_i\langle[\xi,e_i]_{\mathfrak{m}},x\rangle\frac{\partial f}{\partial x^i}(x),$$

where we use the Lie cobracket in an explicit notation. This shows that

$$\tilde{\xi}=\sum_{i,j,k}\langle f^i,\xi\rangle c^k_{ij}x^k\frac{\partial}{\partial x^j}\quad \text{for all } \xi\in\mathfrak{m},$$

where c^k_{ij} are the structure coefficients of Lie algebra $\mathfrak{m}$, i.e., $[e_i,e_j]_{\mathfrak{m}}=\sum_k c^k_{ij}e_k$. Therefore the Poisson bracket on $\mathbb{R}^n \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^*$ is given by

$$\{f,h\}=0,\quad \{\xi,\eta\}=[\xi,\eta]_{\mathfrak{m}},\quad \{\xi,f\}=\tilde{\xi}f=\sum_{i,j,k}\langle f^i,\xi\rangle c^k_{ij}x^k\frac{\partial f}{\partial x^j},$$

where $f,h\in C^\infty(\mathbb{R}^n)$ and $\xi,\eta\in\mathfrak{m}$.

The bicrossproduct Hopf algebra $\mathbb{C}[\overline{G}]\blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g}^*)=\mathbb{C}[\mathbb{R}^n]\rtimes U_\lambda(\mathfrak{m})$, as the quantisation of $C^\infty(\mathbb{R}^n \blacktriangleright\!\!\triangleleft\, \mathfrak{m}^*)$, has commutation relations

$$[x^i,x^j]=0,\quad [e_i,e_j]=\lambda\sum_k c^k_{ij}e_k,\quad [e_i,x^j]=\lambda\sum_k c^k_{ij}x^k,$$

where $\{x^i\}$ are coordinate functions of $\mathbb{R}^n = \overline{\mathfrak{m}^*}$, identified via the basis $\{e_i\}$ of $\mathfrak{m}$. As an algebra we can equally well take $C^\infty(\mathbb{R}^n) \rtimes U_\lambda(\mathfrak{m})$, i.e., not limiting ourselves to polynomials. Then $[e_i, f] = \lambda \sum_{j,k} c_{ij}^k x^k \,\partial f/\partial x^j$ more generally for the cross relations.

Example 5.3. We take SU_2 with the standard Drinfeld–Sklyanin Lie bialgebra structure on su_2, where the matched pair comes from the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{C})$ [Majid 1990a]. The bicrossproduct Hopf algebra $\mathbb{C}[\mathrm{SU}_2] \blacktriangleright\!\!\vartriangleleft U_\lambda(\mathrm{su}_2^*)$, as an algebra, is the cross product $\mathbb{C}[\mathrm{SU}_2] \rtimes U_\lambda(\mathrm{su}_2^*)$ with a, b, c, d commuting, $ad - bc = 1$, $[x^i, x^3] = \lambda x^i$ $(i = 1, 2)$ and

$$[x^i, \boldsymbol{t}] = \lambda \boldsymbol{t}[e_i, \boldsymbol{t}^{-1} e_3 \boldsymbol{t} - e_3],\ \ i = 1, 2, 3,$$

that is,

$$\begin{aligned}
&[x^1, \boldsymbol{t}] = -\lambda bc\boldsymbol{t}e_2 + \frac{\lambda}{2}\boldsymbol{t}\ \mathrm{diag}(ac, -bd) + \frac{\lambda}{2}\,\mathrm{diag}(b, -c),\\
&[x^2, \boldsymbol{t}] = \lambda bc\boldsymbol{t}e_1 - \frac{\imath\lambda}{2}\boldsymbol{t}\,\mathrm{diag}(ac, bd) + \frac{\imath\lambda}{2}\,\mathrm{diag}(b, c),\\
&[x^3, \boldsymbol{t}] = -\lambda ad\boldsymbol{t} + \lambda\,\mathrm{diag}(a, d),
\end{aligned} \tag{5-8}$$

where $\boldsymbol{t} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\{e_i\}$ and $\{x^i\}$ are bases of su_2 and su_2^* as the half-real forms of $\mathrm{sl}_2(\mathbb{C})$ and $\mathrm{sl}_2^*(\mathbb{C})$ respectively. The coalgebra of $\mathbb{C}[\mathrm{SU}_2] \blacktriangleright\!\!\vartriangleleft U_\lambda(\mathrm{su}_2^*)$ is the cross coproduct $\mathbb{C}[\mathrm{SU}_2] \blacktriangleright\!\!< U_\lambda(\mathrm{su}_2^*)$ associated with

$$\Delta(x^i) = 1 \otimes x^i - 2\sum_k x^k \otimes \mathrm{Tr}(\boldsymbol{t}e_i\boldsymbol{t}^{-1}e_k), \quad \epsilon(x^i) = 0 \quad \text{for all } i \in \{1, 2, 3\}.$$

The $*$-structure is the known one on $\mathbb{C}[\mathrm{SU}_2]$ with $x^{i*} = -x^i$ for each i.

Proof. We recall that the coordinate algebra $\mathbb{C}[\mathrm{SU}_2]$ is the commutative algebra $\mathbb{C}[a, b, c, d]$ modulo the relation $ad - bc = 1$ with $*$-structure

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

As a Hopf $*$-algebra, the coproduct, counit and antipode of $\mathbb{C}[\mathrm{SU}_2]$ are given by

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let $\{H, X_\pm\}$ and $\{\phi, \psi_\pm\}$ be the dual bases of $\mathrm{sl}_2(\mathbb{C})$ and $\mathrm{sl}_2^*(\mathbb{C})$ respectively, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As the half-real forms of $\mathrm{sl}_2(\mathbb{C})$ and $\mathrm{sl}_2^*(\mathbb{C})$, the Lie algebras su_2 and su_2^* have bases

$$e_1=-\tfrac{1}{2}\imath(X_++X_-),\quad e_2=-\tfrac{1}{2}(X_+-X_-),\quad e_3=-\tfrac{1}{2}\imath H,$$
$$x^1=\psi_++\psi_-,\quad x^2=\imath(\psi_+-\psi_-),\quad x^3=2\phi,$$

respectively. Note that $x^i=-\imath f^i$, where $\{f^i\}$ is the dual basis of $\{e_i\}$.

The Lie brackets and Lie cobrackets of su_2 and su_2^* are given by

$$[e_i,e_j]=\epsilon_{ijk}e_k\quad\text{and}\quad \delta e_i=\imath e_i\wedge e_3\quad\text{for all } i,j,k,$$
$$[x^1,x^2]=0,\quad [x^i,x^3]=x^i,\ i=1,2,\quad \delta x^1=\imath(x^2\otimes x^3-x^3\otimes x^2),$$
$$\delta x^2=\imath(x^3\otimes x^1-x^1\otimes x^3),\quad \delta x^3=\imath(x^1\otimes x^2-x^2\otimes x^1),$$

where ϵ_{ijk} is totally antisymmetric and $\epsilon_{123}=1$. Writing $\xi=\xi^ie_i\in\mathrm{su}_2$ and $\phi=\phi_ix^i\in\mathrm{su}_2^*$ for 3-vectors $\vec{\xi}=(\xi^i)$, $\vec{\phi}=(\phi_i)$, we know that $(\mathrm{su}_2^*,\mathrm{su}_2)$ forms a the matched pair of Lie bialgebras with interacting actions

$$\vec{\xi}\triangleleft\vec{\phi}=(\vec{\xi}\times\vec{e_3})\times\vec{\phi},\quad \vec{\xi}\triangleright\vec{\phi}=\vec{\xi}\times\vec{\phi}.$$

To obtain the action of su_2^* on $\mathbb{C}[\mathrm{SU}_2]$, we need to solve [Majid 1995, Proposition 8.3.14]

$$\frac{\mathrm{d}}{\mathrm{d}t}a_\phi(\mathrm{e}^{t\xi}u)\Big|_{t=0}=\mathrm{Ad}_{u^{-1}}(\xi\triangleleft(u\triangleright\phi)),\quad a_\phi(I_2)=0.$$

Note that SU_2 acts on su_2^* by $u\triangleright\vec{\phi}=\mathrm{Rot}_u\vec{\phi}$, where we view ϕ as an element in su_2 via $\rho(\phi)=\phi_ie_i$. One can check that

$$a_{\vec{\phi}}(u)=\vec{\phi}\times(\mathrm{Rot}_{u^{-1}}(\vec{e_3})-\vec{e_3})$$

is the unique solution to the differential equation. Now we can compute by (5-3)

$$\begin{aligned}(\phi\triangleright \boldsymbol{t}^i_j)(u)&=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{t}^i_j(u\mathrm{e}^{ta_\phi(u)})\Big|_{t=0}\\&=\sum_k\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{t}^i_k(u)\boldsymbol{t}^k_j(\mathrm{e}^{ta_\phi(u)})\Big|_{t=0}\\&=\sum_k u^i_k(a_\phi(u))^k_j\\&=\sum_k u^i_k[\rho(\phi),u^{-1}e_3u-e_3]^k_j,\end{aligned}$$

where $\rho(\phi)=\sum_i\phi_ie_i$. This shows that

$$[x^i,\boldsymbol{t}]=\lambda x^i\triangleright\boldsymbol{t}=\lambda\boldsymbol{t}[e_i,\boldsymbol{t}^{-1}e_3\boldsymbol{t}-e_3],$$

as displayed. For each i, we can work out the terms on the right explicitly (using $ad-bc=1$) as

$$[x^1,\boldsymbol{t}]=-\frac{\lambda}{2}\begin{pmatrix}abd-a^2c-2b, & b^2d-a^2d+a\\ ad^2-ac^2-d, & bd^2-acd+2c\end{pmatrix},$$
$$[x^2,\boldsymbol{t}]=-\frac{\imath\lambda}{2}\begin{pmatrix}a^2c+abd-2b, & a^2d+b^2d-a\\ ac^2+ad^2-d, & bd^2+acd-2c\end{pmatrix},$$
$$[x^3,\boldsymbol{t}]=-\lambda\begin{pmatrix}a^2d-a, & abd\\ acd, & ad^2-d\end{pmatrix}.$$

These can be rewritten as the formulae (5-8) we stated.

For convenience, we use Pauli matrices $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2=\left(\begin{smallmatrix}0&-\imath\\ \imath&0\end{smallmatrix}\right)$, $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Clearly, $e_i=-\frac{1}{2}\imath\sigma_i$ and σ_i obey $\sigma_i\sigma_j=\delta_{ij}I_2+\imath\epsilon_{ijk}\sigma_k$ and $[\sigma_i,\sigma_j]=2\imath\epsilon_{ijk}\sigma_k$.

The coaction of $\mathbb{C}[\mathrm{SU}_2]$ on su_2^* is defined by $\beta(\phi)(u)=u\triangleright\phi=\mathrm{Rot}_u\vec{\phi}$ for any $u\in\mathrm{SU}_2$, $\phi\in\mathrm{su}_2^*$. Again, we view ϕ as an element in su_2, so $\rho(u\triangleright\phi)=u\rho(\phi)u^{-1}$, namely $\sum_i(u\triangleright\phi)_i\sigma_i=\sum_i\phi_iu\sigma_iu^{-1}$. In particular, we have

$$(u\triangleright x^i)_1\sigma_1+(u\triangleright x^i)_2\sigma_2+(u\triangleright x^i)_3\sigma_3=u\sigma_iu^{-1},\quad i=1,2,3.$$

Multiplying by σ_k on the right and then taking the trace of both sides, and using $\mathrm{Tr}(\sigma_i\sigma_j)=2\delta_{ij}$ we have $2(u\triangleright x^i)_k=\mathrm{Tr}(u\sigma_iu^{-1}\sigma_k)$. Therefore

$$u\triangleright x^i=\frac{1}{2}\sum_k\mathrm{Tr}(u\sigma_iu^{-1}\sigma_k)x^k=-2\sum_k\mathrm{Tr}(ue_iu^{-1}e_k)x^k,$$

and thus $\beta(x^i)=\frac{1}{2}\sum_k x^k\otimes\mathrm{Tr}(\boldsymbol{t}\sigma_i\boldsymbol{t}^{-1}\sigma_k)=-2\sum_k x^k\otimes\mathrm{Tr}(\boldsymbol{t}e_i\boldsymbol{t}^{-1}e_k)$. This gives rise to the coproduct of x^i as stated. This example is dual to a bicrossproduct from this matched pair computed in [Majid 1995]. □

5C. *Preconnections on the tangent bundle $\overline{G}\blacktriangleright\!\!\blacktriangleleft\underline{\mathfrak{g}}$.* We use the following lemma to construct left pre-Lie structures on $(\overline{\mathfrak{g}}\triangleright\!\!\blacktriangleleft\underline{\mathfrak{g}})^*=(\overline{\mathfrak{g}})^*\blacktriangleright\!\!\triangleleft(\underline{\mathfrak{g}})^*=\underline{\mathfrak{g}^*}\blacktriangleright\!\!\triangleleft\overline{\mathfrak{g}^*}$, where the Lie bracket is the semidirect sum $\underline{\mathfrak{g}^*}\rtimes\mathfrak{g}^*$ and the Lie cobracket is the semidirect cobracket $\mathfrak{g}^*\blacktriangleright\!\!<\overline{\mathfrak{g}^*}$, namely

$$[\phi,\psi]=0,\quad [f,\phi]=f\triangleright\phi=[f,\phi]_{\mathfrak{g}^*},\quad [f,g]=[f,g]_{\mathfrak{g}^*},$$
$$\delta\phi=\delta_{\mathfrak{g}^*}\phi=\phi_{(1)}\otimes\phi_{(2)},\quad \delta f=\underline{f_{(1)}}\otimes\overline{f_{(2)}}-\underline{f_{(2)}}\otimes\overline{f_{(1)}},$$

for any $\phi,\psi\in\underline{\mathfrak{g}^*}$, $f,g\in\overline{\mathfrak{g}^*}$. For convenience, we denote $f\in\mathfrak{g}^*$ by $\overline{f}$ if viewed in $\overline{\mathfrak{g}^*}$ or $\underline{f}$ if viewed in $\underline{\mathfrak{g}^*}$.

Lemma 5.4. *Let $(A,\circ)$ be a left pre-Lie algebra and $(B,*)$ a left pre-Lie algebra in the category ${}_{\mathfrak{g}_A}\mathcal{M}$ of left $\mathfrak{g}_A$-modules, i.e., there is a left $\mathfrak{g}_A$-action $\triangleright$ on B such that*

$$a\triangleright(x*y)=(a\triangleright x)*y+x*(a\triangleright y)\tag{5-9}$$

for any $a, b \in A,\ x, y \in B$. Then there is a left pre-Lie algebra structure on $B \oplus A$

$$(x, a)\,\tilde{\circ}\,(y, b) = (x * y + a \triangleright y, a \circ b).$$

We denote this pre-Lie algebra by $B \rtimes A$, and have $\mathfrak{g}_{B\rtimes A} = \mathfrak{g}_B \rtimes \mathfrak{g}_A$ for the associated Lie algebras.

Proof. This is a matter of directly verifying according to the axioms of a left pre-Lie algebra. □

Corollary 5.5. *Let $(\mathfrak{m}, \circ)$ be a left pre-Lie algebra. Suppose it admits a (not necessarily unital) commutative associative product $\cdot$ such that*

$$[\xi, x \cdot y]_{\mathfrak{m}} = [\xi, x]_{\mathfrak{m}} \cdot y + x \cdot [\xi, \eta]_{\mathfrak{m}} \quad \textit{for all}\ \ \xi, x, y \in \mathfrak{m},$$

where $[\ ,\ ]_{\mathfrak{m}}$ is the Lie bracket defined by $\circ$. Denote the underlying pre-Lie algebra by $\underline{\mathfrak{m}} = (\mathfrak{m}, \cdot)$. Then $\underline{\mathfrak{m}} \rtimes_{\mathrm{ad}} \mathfrak{m}$ is a left pre-Lie algebra with product

$$(x, \xi)\,\tilde{\circ}\,(y, \eta) = (x \cdot y + [\xi, y]_{\mathfrak{m}}, \xi \circ \eta) \tag{5-10}$$

for any $x, y \in \underline{\mathfrak{m}},\ \xi, \eta \in \mathfrak{m}$.

Proof. Take $(A, \circ) = (\mathfrak{m}, \circ)$ and $(B, *) = (\mathfrak{m}, \cdot)$ in Lemma 5.4. Here $(\mathfrak{m}, \circ)$ left acts on $(\mathfrak{m}, \cdot)$ by the adjoint action and (5-9) is exactly the condition displayed. □

The assumption made in Corollary 5.5 is that $(\mathfrak{m}, \cdot, [\ ,\ ])$ is a (not necessarily unital) Poisson algebra with respect to the Lie bracket, and that the latter admits a left pre-Lie structure $\circ$.

Theorem 5.6. *Let G be a finite-dimensional connected and simply connected Poisson–Lie group with Lie bialgebra $\mathfrak{g}$. Assume that $(\mathfrak{g}^*, [\ ,\ ]_{\mathfrak{g}^*})$ admits a pre-Lie structure $\circ$ and also that $\mathfrak{g}^*$ admits a (not necessarily unital) Poisson algebra structure $(\mathfrak{g}^*, *, [\ ,\ ]_{\mathfrak{g}^*})$*

$$[f, \phi * \psi]_{\mathfrak{g}^*} = [f, \phi]_{\mathfrak{g}^*} * \psi + \phi * [f, \psi]_{\mathfrak{g}^*} \tag{5-11}$$

for any $\phi, \psi \in \underline{\mathfrak{g}^*},\ f \in \mathfrak{g}^*$. Then the semidirect sum $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}^*$ admits a pre-Lie algebra product $\tilde{\circ}$ given by

$$(\phi, f)\,\tilde{\circ}\,(\psi, h) = (\phi * \psi + [f, \psi]_{\mathfrak{g}^*}, f \circ h), \tag{5-12}$$

and the tangent bundle $\overline{G} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$ in Lemma 5.1 admits a Poisson-compatible left-covariant flat preconnection.

Proof. We take $\mathfrak{m} = \mathfrak{g}^*$ in Corollary 5.5. We know $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}^*$ is the Lie algebra of $\underline{\mathfrak{g}^*} \blacktriangleright\!\triangleleft \overline{\mathfrak{g}^*}$, dual to Lie algebra $\bar{\mathfrak{g}} \triangleright\!\blacktriangleleft \underline{\mathfrak{g}}$ of the tangent bundle. Then we apply Corollary 4.2. □

The corresponding preconnection can be computed explicitly from (2-8). For a Poisson–Lie group G, let $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{f^i\}$ the dual basis of $\mathfrak{g}^*$. Denote by $\{\omega^i\}$ the basis of left-invariant 1-forms that is dual to $\{\partial_i\}$ the left-invariant vector fields of G generated by $\{e_i\}$ as before. For the abelian Poisson–Lie group $\underline{\mathfrak{g}}$ with Kirillov–Kostant Poisson bracket, let $\{E_i\}$ be a basis of $\underline{\mathfrak{g}}$ and $\{x^i\}$ the dual basis of $\overline{\mathfrak{g}^*}$. Then $\{\mathrm{d}x^i\}$ is the basis of left-invariant 1-forms that is dual to $\{\partial/\partial x^i\}$, the basis of the left-invariant vector fields on $\underline{\mathfrak{g}}$ generated by $\{E_i\}$. Now we can choose $\{e_i, E_i\}$ to be the basis of $\bar{\mathfrak{g}} \triangleright\!\!\blacktriangleleft \underline{\mathfrak{g}}$, and so $\{f^i, x^i\}$ is the dual basis for $\underline{\mathfrak{g}^*} \blacktriangleright\!\!\triangleleft \overline{\mathfrak{g}^*}$. Denote by $\{\tilde{\partial}_i, D_i\}$ the left-invariant vector fields on $\overline{G} \triangleright\!\!\blacktriangleleft \underline{\mathfrak{g}}$ generated by $\{e_i, E_i\}$, and denote by $\{\tilde{\omega}^i, \widetilde{\mathrm{d}x^i}\}$ the corresponding dual basis of left-invariant 1-forms. By construction, when viewing any $f \in C^\infty(G)$ and $\phi \in \mathfrak{g}^* \subset C^\infty(\mathfrak{g})$ as functions on the tangent bundle, we know

$$\tilde{\partial}_i f = \partial_i f, \quad \tilde{\partial}_i \phi = \mathrm{ad}^*_{e_i} \phi, \quad D_i f = 0, \quad D_i \phi = \frac{\partial}{\partial x^i}\phi.$$

This implies

$$\tilde{\partial}_i = \partial_i + \sum_j (\mathrm{ad}^*_{e_i} x^j)\frac{\partial}{\partial x^j}, \quad D_i = \frac{\partial}{\partial x^i}, \quad \tilde{\omega}^i = \omega^i, \quad \widetilde{\mathrm{d}x^i} = \mathrm{d}x^i - \sum_k (\mathrm{ad}^*_{e_k} x^i)\omega^k.$$

Let $\tilde{\circ}$ be the pre-Lie structure of $\underline{\mathfrak{g}^*} \bowtie \mathfrak{g}^*$ constructed by (5-12) in terms of $*$ and $\circ$ in the setting of Theorem 5.6. The Poisson-compatible left-covariant flat preconnection on the tangent bundle is then, for any function a,

$$\gamma(a, \omega^j) = \sum_{i,k} \tilde{\partial}_i a \langle f^i * f^j, e_k\rangle \omega^k + \sum_{i,k} D_i a \langle [x^i, f^j]_{\mathfrak{g}^*}, e_k\rangle \omega^k,$$
$$\gamma(a, \widetilde{\mathrm{d}x^j}) = \sum_{i,k} D_i a \langle x^i \circ x^j, E_k\rangle \widetilde{\mathrm{d}x^k}.$$

If we write

$$f^i * f^j = \sum_k a^{ij}_k f^k, \quad x^i \circ x^j = \sum_k b^{ij}_k x^k,$$
$$[x^i, f^j]_{\mathfrak{g}^*} = \sum_k \langle [x^i, f^j]_{\mathfrak{g}^*}, e_k\rangle f^k = \sum_{s,k} d^{sj}_k \langle x^i, e_s\rangle f^k,$$

where $[f^i, f^j]_{\mathfrak{g}^*} = d^{ij}_k f^k$, then the left-covariant preconnection on the tangent bundle $\overline{G} \triangleright\!\!\blacktriangleleft \underline{\mathfrak{g}}$ is

$$\gamma(f, \omega^j) = \sum_{i,k} a^{ij}_k (\partial_i f)\omega^k, \quad \gamma(f, \widetilde{\mathrm{d}x^j}) = 0, \quad \gamma(\phi, \widetilde{\mathrm{d}x^j}) = \sum_{i,k} b^{ij}_k \left(\frac{\partial \phi}{\partial x^i}\right) \widetilde{\mathrm{d}x^k},$$
$$\gamma(\phi, \omega^j) = \sum_{i,k} \left(a^{ij}_k \,\mathrm{ad}^*_{e_i} \phi + \sum_s d^{sj}_k \langle x^i, e_s\rangle \left(\frac{\partial \phi}{\partial x^i}\right)\right) \omega^k$$

for any $f \in C^\infty(G)$, $\phi \in \mathfrak{g}^* \subset C^\infty(\mathfrak{g})$.

This result applies, for example, to tell us that we have a left-covariant differential structure on quantum groups such as $\mathbb{C}[\overline{G}] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g}^*)$ at least to lowest order in deformation. In the special case when the product $*$ is zero, there is a natural differential calculus not only at lowest order. Under the notations above, we have:

Proposition 5.7. *Let G be a finite-dimensional connected and simply connected Poisson–Lie group with Lie algebra $\mathfrak{g}$. If the dual Lie algebra $\mathfrak{g}^*$ admits a pre-Lie structure $\circ : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ with respect to its Lie bracket* ($[\ ,\ ]_{\mathfrak{g}^*}$ *determined by $\delta_{\mathfrak{g}}$*), *then the bicrossproduct $\mathbb{C}[\overline{G}] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g}^*)$* (*if it exists*) *admits a left-covariant differential calculus*

$$\Omega^1 = (\mathbb{C}[\overline{G}] \blacktriangleright\!\!\triangleleft U_\lambda(\mathfrak{g}^*)) \rtimes\!\!\!\ltimes \Lambda^1$$

with left-invariant 1-forms Λ^1 spanned by basis $\{\omega^i, \widetilde{\mathrm{d}x^i}\}$, where the commutation relations and the derivatives are given by

$$[f, \omega^i] = 0, \quad [f, \widetilde{\mathrm{d}x^i}] = 0, \quad [x^i, \omega^j] = \sum_k \lambda \langle [x^i, f^j]_{\mathfrak{g}^*}, e_k\rangle \omega^k,$$

$$[x^i, \widetilde{\mathrm{d}x^j}] = \lambda \widetilde{\mathrm{d}(x^i \circ x^j)}, \quad \mathrm{d}f = \sum_j (\partial_j f)\omega^j, \quad \mathrm{d}x^i = \widetilde{\mathrm{d}x^i} + \sum_j (\mathrm{ad}^*_{e_j} x^i)\omega^j$$

for any $f \in \mathbb{C}[\overline{G}]$.

Proof. It is easy to see that we have a bimodule Ω^1. As the notation indicates [Majid and Tao 2015b], the left action on Ω^1 is the product of the bicrossproduct quantum group on itself while the right action is the tensor product of the right action of the bicrossproduct on itself and a right action on Λ^1. The right action of $\mathbb{C}[G]$ here is trivial, namely

$$\omega^j \triangleleft f = f(e)\omega^j, \quad \widetilde{\mathrm{d}x^j} \triangleleft f = f(e)\widetilde{\mathrm{d}x^j};$$

the right actions of x^i are clear from the commutation relations and given (summation understood) by

$$\begin{aligned} \omega^j \triangleleft x^i &= -\lambda \langle [x^i, f^j]_{\mathfrak{g}^*}, e_k\rangle \omega^k = -\lambda d_k^{sj} \langle x^i, e_s\rangle \omega^k, \\ \widetilde{\mathrm{d}x^j} \triangleleft x^i &= -\lambda \widetilde{\mathrm{d}(x^i \circ x^j)} = -\lambda b_k^{ij} \widetilde{\mathrm{d}x^k}. \end{aligned}$$

One can check that these fit together to a right action of the bicrossproduct quantum group by using the Jacobi identity of $\mathfrak{g}^*$, the pre-Lie identity on $\circ$, and the fact that $(\widetilde{x^i} f)(e) = \widetilde{x}^i_e f = 0$ by (5-6).

We check that the Leibniz rule holds. The conditions

$$\mathrm{d}[f, h] = 0 \quad \text{and} \quad \mathrm{d}[x^i, x^j] = \lambda \mathrm{d}[x^i, x^j]_{\mathfrak{g}^*}$$

are easy to check, so we omit these. It remains to check that

$$\mathrm{d}[x^i, f] = \lambda \mathrm{d}(\widetilde{x^i} f) \quad \text{for all } f \in \mathbb{C}[\overline{G}]. \tag{5-13}$$

The right-hand side of (5-13) is

$$\lambda\,\mathrm{d}(\widetilde{x^i}f) = \lambda\partial_j(\widetilde{x^i}f)\omega^j,$$

while the left-hand side of (5-13) is

$$\begin{aligned}
\mathrm{d}[x^i, f] &= \mathrm{d}(x^i f - x^i f) = [\mathrm{d}x^i, f] + [x^i, \mathrm{d}f]\\
&= [\widetilde{\mathrm{d}x^i} + (\mathrm{ad}^*_{e_j}x^i)\omega^j, f] + [x^i, (\partial_j f)\omega^j]\\
&= 0 + [\mathrm{ad}^*_{e_j}x^i, f]\omega^j + [x^i, \partial_j f]\omega^j + (\partial_k f)[x^i, \omega^k]\\
&= [\mathrm{ad}^*_{e_j}x^i, f]\omega^j + [x^i, \partial_j f]\omega^j + \lambda(\partial_k f)\langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle\omega^j\\
&= \big([\mathrm{ad}^*_{e_j}x^i, f] + [x^i, \partial_j f] + \lambda\langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle(\partial_k f)\big)\omega^j\\
&= \lambda\big(\widetilde{\mathrm{ad}^*_{e_j}x^i}\, f + \widetilde{x^i}(\partial_j f) + \langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle(\partial_k f)\big)\omega^j.
\end{aligned}$$

It suffices to show that $\partial_j(\widetilde{x^i}f) = \widetilde{\mathrm{ad}^*_{e_j}x^i}\, f + \widetilde{x^i}(\partial_j f) + \langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle(\partial_k f)$, namely

$$[\partial_j, \widetilde{x^i}] = \widetilde{\mathrm{ad}^*_{e_j}x^i} + \langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle\partial_k.$$

Recall that in the double cross-sum $\mathfrak{g}^* \bowtie \mathfrak{g}$, for any $e_j \in \mathfrak{g}$, $x^i \in \mathfrak{g}^*$,

$$[e_j, x^i] = e_j \triangleleft x^i + e_j \triangleright x^i = \langle [x^i, f^k]_{\mathfrak{g}^*}, e_j\rangle e_k + \mathrm{ad}^*_{e_j}x^i.$$

Therefore the condition left to check is nothing but the Lie bracket of elements e_j and x^i viewed as the infinitesimal action of $\mathfrak{g}^* \bowtie \mathfrak{g}$ on $\mathbb{C}[\overline{G}]$, as explained in the general theory of double cross-sums in Section 5A. □

Now we compute the left-covariant first-order differential calculus on the bicrossproduct quantum group $\mathbb{C}[\mathrm{SU}_2] \blacktriangleright\!\!\triangleleft U_\lambda(\mathrm{su}_2^*)$ constructed in Example 5.3.

Example 5.8. As in Example 4.6, the classical connected left-covariant calculus on $\mathbb{C}[\mathrm{SU}_2]$ has basis of left-invariant 1-forms

$$\omega^0 = d\,\mathrm{d}a - b\,\mathrm{d}c = c\,\mathrm{d}b - a\,\mathrm{d}d, \quad \omega^+ = d\,\mathrm{d}b - b\,\mathrm{d}d, \quad \omega^- = a\,\mathrm{d}c - c\,\mathrm{d}a$$

(corresponding to the Chevalley basis $\{H, X_\pm\}$ of su_2) with exterior derivative

$$\mathrm{d}a = a\omega^0 + b\omega^-, \quad \mathrm{d}b = a\omega^+ - b\omega^0, \quad \mathrm{d}c = c\omega^0 + d\omega^-, \quad \mathrm{d}d = c\omega^+ - d\omega^0.$$

Let $\circ : \mathrm{su}_2^* \otimes \mathrm{su}_2^* \to \mathrm{su}_2^*$ be a left pre-Lie algebra structure of su_2^* with respect to the Lie bracket $[x^1, x^2] = 0$ and $[x^i, x^3] = x^i$ for $i = 1, 2$. Let

$$\{\widetilde{\mathrm{d}x^1}, \widetilde{\mathrm{d}x^2}, \widetilde{\mathrm{d}x^3}\}$$

complete the basis of left-invariant 1-forms on the tangent bundle as explained above. According to Proposition 5.7, this defines a 6-dimensional connected left-covariant differential calculus on the bicrossproduct $\mathbb{C}[\mathrm{SU}_2]\blacktriangleright\!\!\triangleleft U_\lambda(\mathrm{su}_2^*)$ with commutation relations and exterior derivative given by

$$[\boldsymbol{t},\omega^l]=0\quad\text{for all } l\in\{0,\pm\},\qquad [\boldsymbol{t},\widetilde{\mathrm{d}x^i}]=0,$$
$$[x^i,\widetilde{\mathrm{d}x^j}]=\lambda\widetilde{\mathrm{d}(x^i\circ x^j)}\quad\text{for all } i,j\in\{1,2,3\},$$
$$[x^1,\omega^0]=\tfrac{1}{2}\lambda(\omega^++\omega^-),\quad [x^1,\omega^+]=0,\quad [x^1,\omega^-]=0,$$
$$[x^2,\omega^0]=\tfrac{1}{2}\imath\lambda(\omega^+-\omega^-),\quad [x^2,\omega^+]=0,\quad [x^2,\omega^-]=0,$$
$$[x^3,\omega^0]=0,\quad [x^3,\omega^+]=-\lambda\omega^+,\quad [x^3,\omega^-]=-\lambda\omega^-,$$
$$\mathrm{d}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\omega^0&\omega^+\\\omega^-&-\omega^0\end{pmatrix},$$
$$\mathrm{d}x^1=\widetilde{\mathrm{d}x^1}+2\imath x^2\omega^0+x^3\omega^+-x^3\omega^-,\quad \mathrm{d}x^2=\widetilde{\mathrm{d}x^2}-2\imath x^1\omega^0+\imath x^3\omega^++\imath x^3\omega^-,$$
$$\mathrm{d}x^3=\widetilde{\mathrm{d}x^3}-(x^1+\imath x^2)\omega^++(x^1-\imath x^2)\omega^-.$$

Proof. The commutation relations and derivative are computed from the formulae provided in Proposition 5.7. It is useful to also provide an independent, more algebraic proof of the example from [Majid and Tao 2015b, Theorem 2.5], where left-covariant first-order differential calculi Ω^1 over a Hopf algebra A are constructed from pairs (Λ^1,ω) where Λ^1 is a right A-module and $\omega: A^+\to\Lambda^1$ is a surjective right A-module map. Given such a pair, the commutation relation and derivative are given by $[a,v]=av-a_{(1)}v\triangleleft a_{(2)}$ and $\mathrm{d}a=a_{(1)}\otimes\omega\pi_\epsilon(a_{(2)})$ for any $a\in A$ and $v\in\Lambda^1$, where $\pi_\epsilon=\mathrm{id}-1\epsilon$ and ϵ is the counit.

Firstly, the classical calculus on $A:=\mathbb{C}[\mathrm{SU}_2]$ corresponds to a pair (Λ^1_A,ω_A) with $\Lambda^1_A=\mathrm{span}\{\omega^0,\omega^\pm\}$, where the right $\mathbb{C}[\mathrm{SU}_2]$-action on Λ^1_A and the right $\mathbb{C}[\mathrm{SU}_2]$-module surjective map $\omega_A:\mathbb{C}[\mathrm{SU}_2]^+\to\Lambda^1_A$ are given by

$$\omega^j\triangleleft\boldsymbol{t}=\epsilon(\boldsymbol{t})\omega^j,\quad j\in\{0,\pm\},$$
$$\omega_A(\boldsymbol{t}-I_2)=\omega_A\begin{pmatrix}a-1&b\\c&d-1\end{pmatrix}=\begin{pmatrix}\omega^0&\omega^+\\\omega^-&-\omega^0\end{pmatrix}.$$

Meanwhile, the calculus over $H:=U_\lambda(\mathrm{su}_2^*)$ corresponds to a pair (Λ^1_H,ω_H) with

$$\Lambda^1_H=\mathrm{span}\{\widetilde{\mathrm{d}x^1},\widetilde{\mathrm{d}x^2},\widetilde{\mathrm{d}x^3}\},$$

in which the right $U_\lambda(\mathrm{su}_2^*)$-action on Λ^1_H and the right $U_\lambda(\mathrm{su}_2^*)$-module surjective map $\omega_H:U_\lambda(\mathrm{su}_2^*)^+\to\Lambda^1_H$ are given by

$$\widetilde{\mathrm{d}x^j}\triangleleft x^i=-\lambda\widetilde{\mathrm{d}(x^i\circ x^j)}\quad\text{and}\quad \omega_H(x^i)=\widetilde{\mathrm{d}x^i}\quad\text{for all } i,j\in\{1,2,3\}.$$

Next we construct a pair (Λ^1,ω) over $\widetilde{A}=A\blacktriangleright\!\!\triangleleft H$ with direct sum $\Lambda^1=\Lambda^1_A\oplus\Lambda^1_H$. First, it is clear that Λ^1_H is a right $\widetilde{A}$-module with trivial A-action $\mathrm{d}x^j\triangleleft\boldsymbol{t}=\epsilon(\boldsymbol{t})\,\mathrm{d}x^j$, One can see this more generally as

$$v \triangleleft ((h_{(1)} \triangleright a)h_{(2)}) = \epsilon(a) v \triangleleft h = (v \triangleleft h) \triangleleft a = v \triangleleft (ha).$$

Next, we define a right $U_\lambda(\mathrm{su}_2^*)$-action on Λ^1_A by the Lie bracket of su_2^* viewing $\{\omega^0, \omega^\pm\}$ as $\{\phi, \psi_\pm\}$ (the dual basis to $\{H, X_\pm\}$), where

$$\{x^1 = \psi_+ + \psi_-,\ x^2 = \imath(\psi_+ - \psi_-),\ x^3 = 2\phi\}$$

is the basis for the half-real form su_2^* of sl_2^*, namely

$$\begin{aligned} &\omega^0 \triangleleft x^1 = -\tfrac{1}{2}\lambda(\omega^+ + \omega^-), && \omega^+ \triangleleft x^1 = 0, && \omega^- \triangleleft x^1 = 0,\\ &\omega^0 \triangleleft x^2 = -\tfrac{1}{2}\imath\lambda(\omega^+ - \omega^-), && \omega^+ \triangleleft x^2 = 0, && \omega^- \triangleleft x^2 = 0,\\ &\omega^0 \triangleleft x^3 = 0, && \omega^+ \triangleleft x^3 = \lambda\omega^+, && \omega^- \triangleleft x^3 = \lambda\omega^-. \end{aligned} \tag{5-14}$$

This H-action commutes with the original trivial A-action on Λ^1_A, hence Λ^1_A also becomes a right $\widetilde{A}$-module, as does $\Lambda^1_A \oplus \Lambda^1_H$.

We then define the map $\omega : \widetilde{A}^+ \to \Lambda^1_A \oplus \Lambda^1_H$ on generators by

$$\omega(\boldsymbol{t} - I_2) = \omega_A(\boldsymbol{t} - I_2) = \begin{pmatrix} \omega^0 & \omega^+ \\ \omega^- & -\omega^0 \end{pmatrix}, \quad \omega(x^i) = \omega_H(x^i) = \widetilde{\mathrm{d}x^i} \quad \text{for } i \in \{1, 2, 3\}.$$

This extends to the whole of $\widetilde{A}^+$ as a right $\widetilde{A}$-module map. To see that ω is well-defined, it suffices to check

$$\omega(x^i \boldsymbol{t} - \boldsymbol{t} x^i) = \omega([x^i, \boldsymbol{t}]) \quad \text{for all } i \in \{1, 2, 3\},$$

where $[x^i, \boldsymbol{t}]$ are cross relations (5-8) computed in Example 5.3. On the one hand,

$$\begin{aligned} \omega(x^i \boldsymbol{t} - \boldsymbol{t} x^i) &= \omega(x^i \boldsymbol{t} - (\boldsymbol{t} - I_2)x^i - x^i I_2)\\ &= \omega_H(x^i) \triangleleft \boldsymbol{t} - \omega_A(\boldsymbol{t} - I_2) \triangleleft x^i - \omega_H(x^i) I_2\\ &= -\omega_A(\boldsymbol{t} - I_2) \triangleleft x^i, \end{aligned}$$

that is,

$$\omega(x^i \boldsymbol{t} - \boldsymbol{t} x^i) = -\begin{pmatrix} \omega^0 & \omega^+ \\ \omega^- & -\omega^0 \end{pmatrix} \triangleleft x^i. \tag{5-15}$$

Since

$$\begin{aligned} [x^1, \boldsymbol{t}] &= -\lambda bc\boldsymbol{t}e_2 + \frac{\lambda}{2}\boldsymbol{t}\ \mathrm{diag}(ac, -bd) + \frac{\lambda}{2}\mathrm{diag}(b, -c)\\ &= -\lambda bc\boldsymbol{t}e_2 + \frac{\lambda}{2}(\boldsymbol{t} - I_2)\ \mathrm{diag}(ac, -bd) + \frac{\lambda}{2}\mathrm{diag}(ca, -bd) + \frac{\lambda}{2}\mathrm{diag}(b, -c), \end{aligned}$$

we know

$$\begin{aligned}\omega([x^1, \boldsymbol{t}]) &= -\lambda\omega(b)\epsilon(c\boldsymbol{t}e_2) + \frac{\lambda}{2}\omega((\boldsymbol{t} - I_2))\epsilon(\mathrm{diag}(ac, -bd)) \\ &\qquad + \frac{\lambda}{2}\,\mathrm{diag}(\omega(c) \triangleleft a, -\omega(b) \triangleleft d) + \frac{\lambda}{2}\,\mathrm{diag}(\omega(b), -\omega(c)) \\ &= \frac{\lambda}{2}\,\mathrm{diag}(\omega^+ + \omega^-, -\omega^+ - \omega^-),\end{aligned}$$

using $\epsilon(\boldsymbol{t}) = I_2$. Likewise, we have

$$\begin{aligned}\omega([x^1, \boldsymbol{t}]) &= \frac{\lambda}{2}\begin{pmatrix}\omega^+ + \omega^- & 0 \\ 0 & -\omega^+ - \omega^-\end{pmatrix}, \\ \omega([x^2, \boldsymbol{t}]) &= \frac{\imath\lambda}{2}\begin{pmatrix}\omega^+ - \omega^- & 0 \\ 0 & -\omega^+ + \omega^-\end{pmatrix}, \\ \omega([x^3, \boldsymbol{t}]) &= \lambda\begin{pmatrix}0 & -\omega^+ \\ -\omega^- & 0\end{pmatrix}.\end{aligned}$$

Comparing with (5-15), we see that $\omega(x^i\boldsymbol{t} - \boldsymbol{t}x^i) = \omega([x^i, \boldsymbol{t}])$ holds for each $i = 1, 2, 3$ if and only if the right H-action on Λ^1_A is the one defined by (5-14). From the coproduct of x^i given in Example 5.3, we know

$$\mathrm{d}x^i = \widetilde{\mathrm{d}x^i} + \tfrac{1}{2}x^k\omega\big(\pi_\epsilon(\mathrm{Tr}(\boldsymbol{t}\sigma_i\boldsymbol{t}^{-1}\sigma_k))\big).$$

This gives rise to the formulae for derivatives on x^i as displayed. □

We now analyse when a Poisson-compatible left-covariant flat preconnection is bicovariant.

Lemma 5.9. *Let $\mathfrak{g}$ be in the setting of Theorem 5.6. The pre-Lie structure $\tilde{\circ}$ given by* (5-12) *of $\underline{\mathfrak{g}^*} \blacktriangleright\!\triangleleft \overline{\mathfrak{g}^*}$ gives a bicovariant preconnection in Corollary 4.2 if and only if*

$$\delta_{\mathfrak{g}^*}(f \circ g) = 0, \quad f_{(1)} \otimes [f_{(2)}, g]_{\mathfrak{g}^*} = 0, \tag{5-16}$$

$$f_{(1)} \circ g \otimes f_{(2)} = -f \circ g_{(1)} \otimes g_{(2)}, \tag{5-17}$$

$$\delta_{\mathfrak{g}^*}(\phi * \psi) = 0, \quad \phi * f_{(1)} \otimes f_{(2)} = 0, \tag{5-18}$$

for all $\phi, \psi \in \underline{\mathfrak{g}^*}$, $f, g \in \overline{\mathfrak{g}^*}$.

Proof. Since the bicovariance condition (4-6) is bilinear on entries, it suffices to show that $\tilde{\circ}$ obeys (4-6) on any pair of elements (ϕ, ψ), (ϕ, f), (f, ϕ) and (f, g) if and only if all the displayed identities hold for any $\phi, \psi \in \underline{\mathfrak{g}^*}$, $f, g \in \overline{\mathfrak{g}^*}$.

Firstly, for any $f \in \overline{\mathfrak{g}^*}$ and $\phi \in \underline{\mathfrak{g}^*}$, (4-6) on $\tilde{\circ}$ reduces to

$$\delta_{\mathfrak{g}^*}[f, \phi]_{\mathfrak{g}^*} - [f, \phi_{(1)}] \otimes \phi_{(2)} - \phi_{(1)} \otimes [f, \phi_{(2)}] = \underline{f_{(1)}} * \phi \otimes \overline{f_{(2)}} + [\overline{f_{(1)}}, \phi] \otimes f_{(2)}.$$

The only term in the above identity not lying in $\underline{\mathfrak{g}^*} \otimes \underline{\mathfrak{g}^*}$ is $\underline{f_{(1)}} * \phi \otimes \overline{f_{(2)}}$, which hence equals zero. Noting that $\delta_{\mathfrak{g}^*}$ is a 1-cocycle, the remaining terms imply that $f_{(1)} \otimes [f_{(2)}, \phi]_{\mathfrak{g}^*} = 0$. Changing the role of f and ϕ in (4-6) implies $\phi * \underline{f_{(1)}} \otimes \overline{f_{(2)}} = 0$, as required.

Next, for any $f, g \in \overline{\mathfrak{g}^*}$, the condition (4-6) on $\tilde{\circ}$ requires

$$\begin{aligned}\underline{(f\circ g)_{(1)}} \otimes \overline{(f\circ g)_{(2)}} + \overline{(f\circ g)_{(1)}} \otimes \underline{(f\circ g)_{(2)}} - [f, \underline{g_{(2)}}]_{\mathfrak{g}^*} \otimes \overline{g_{(2)}} \\ - f\circ \overline{g_{(1)}} \otimes \underline{g_{(2)}} - \underline{g_{(1)}} \otimes f \circ \overline{g_{(2)}} - \overline{g_{(1)}} \otimes [f, \underline{g_{(2)}}]_{\mathfrak{g}^*} \\ = [\underline{f_{(1)}}, g]_{\mathfrak{g}^*} \otimes \overline{f_{(2)}} + \overline{f_{(1)}} \circ g \otimes \underline{f_{(2)}} - \underline{g_{(1)}} \otimes \overline{g_{(2)}} \circ f.\end{aligned}$$

The terms in the above identity lying in $\overline{\mathfrak{g}^*} \otimes \underline{\mathfrak{g}^*}$ are exactly the condition (4-6) on the pre-Lie structure $\circ$ for $\overline{\mathfrak{g}^*}$. Cancelling this, the remaining terms in $\underline{\mathfrak{g}^*} \otimes \overline{\mathfrak{g}^*}$ reduce to $g_{(1)} \circ f \otimes g_{(2)} + g \circ f_{(1)} \otimes f_{(2)} = 0$, which is equivalent to

$$f_{(1)} \circ g \otimes f_{(2)} + f \circ g_{(1)} \otimes g_{(2)} = 0 \quad \text{for all } f, g \in \mathfrak{g}^*.$$

Combining the above with $f_{(1)} \otimes [f_{(2)}, \phi]_{\mathfrak{g}^*} = 0$, the condition (4-6) on $\circ$ reduces to $\delta_{\mathfrak{g}^*}(f \circ g) = 0$.

Finally, for any $\phi, \psi \in \underline{\mathfrak{g}^*}$, the condition (4-6) on $\tilde{\circ}$ reduces to (4-6) on $*$ for $\underline{\mathfrak{g}^*}$. Since $*$ is commutative, this eventually becomes

$$(\phi * \psi)_{(1)} \otimes (\phi * \psi)_{(2)} = \phi * \psi_{(1)} \otimes \psi_{(2)} + \phi_{(1)} * \psi \otimes \phi_{(2)}.$$

Since $\phi * f_{(1)} \otimes f_{(2)} = 0$, this reduces to $\delta_{\mathfrak{g}^*}(\phi * \psi) = 0$. □

The conditions in Lemma 5.9 all hold when the Lie bracket of $\mathfrak{g}$ (or the Lie cobracket of $\mathfrak{g}^*$) vanishes. Putting these results together we have:

Proposition 5.10. *Let G be a finite-dimensional connected and simply connected Poisson–Lie group with Lie bialgebra $\mathfrak{g}$. Assume that $(\mathfrak{g}^*, [\ ,\ ]_{\mathfrak{g}^*})$ obeys the conditions in Theorem 5.6 and Lemma 5.9. Then the tangent bundle $\overline{G} \triangleright\!\!\blacktriangleleft \underline{\mathfrak{g}}$ in Lemma 5.1 admits a Poisson-compatible bicovariant flat preconnection.*

Example 5.11. In the setting of Example 5.2, we already know from Corollary 4.2 that the abelian Poisson–Lie group $\mathbb{R}^n \,{>\!\!\blacktriangleleft}\, \mathfrak{m}^*$ admits a Poisson-compatible left-covariant (bicovariant) flat preconnection if and only if $(\overline{\mathfrak{m}^*} \,{>\!\!\blacktriangleleft}\, \mathfrak{m}^*)^* = \underline{\mathfrak{m}} \rtimes_{\mathrm{ad}} \mathfrak{m}$ admits a pre-Lie structure.

From Corollary 5.5, we know that such a pre-Lie structure $\tilde{\circ}$ exists and is given by $(x, \xi)\tilde{\circ}(y, \eta) = (x \cdot y + [\xi, y]_{\mathfrak{m}}, \xi \circ \eta)$ if we assume $(\mathfrak{m}, \cdot, [\ ,\ ]_{\mathfrak{m}})$ to be a finite-dimensional (not necessarily unital) Poisson algebra such that $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ admits a pre-Lie structure $\circ : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$. Then the corresponding preconnection is

$$\gamma((x, \xi), \mathrm{d}(y, \eta)) = \mathrm{d}(x \cdot y + [\xi, y]_{\mathfrak{m}}, \xi \circ \eta)$$

for any $x, y \in \underline{\mathfrak{m}}$, $\xi, \eta \in \mathfrak{m}$.

In fact this extends to all orders. Under the assumptions above, according to Proposition 4.4, the noncommutative algebra $U_\lambda(\underline{\mathfrak{m}} \rtimes_{\mathrm{ad}} \mathfrak{m}) = S(\underline{\mathfrak{m}}) \rtimes U_\lambda(\mathfrak{m})$, or the cross product of algebras $\mathbb{C}[\mathbb{R}^n] \rtimes U(\mathfrak{m})$ (as quantisation of $C^\infty(\mathbb{R}^n \blacktriangleright\!\!\blacktriangleleft \mathfrak{m}^*)$), admits a connected bicovariant differential graded algebra

$$\Omega(U_\lambda(\underline{\mathfrak{m}} \rtimes_{\mathrm{ad}} \mathfrak{m})) = (S(\underline{\mathfrak{m}}) \rtimes U_\lambda(\mathfrak{m})) \ltimes \Lambda(\underline{\mathfrak{m}} \rtimes_{\mathrm{ad}} \mathfrak{m})$$

as quantisation. Note that $\mathrm{d}(x, \xi) = 1 \otimes (x, \xi) \in 1 \otimes \Lambda^1$. The commutation relations on generators are

$$[\xi, \eta] = \lambda[\xi, \eta]_{\mathfrak{m}}, \quad [x, y] = 0, \quad [\xi, x] = \lambda[\xi, x]_{\mathfrak{m}},$$
$$[x, \mathrm{d}y] = \lambda \mathrm{d}(x \cdot y), \quad [\xi, \mathrm{d}x] = \lambda \mathrm{d}[\xi, x]_{\mathfrak{m}}, \quad [\xi, \mathrm{d}\eta] = \lambda \mathrm{d}(\xi \circ \eta),$$

for any $x, y \in \underline{\mathfrak{m}},\ \xi, \eta \in \mathfrak{m}$.

6. Semiclassical data on the cotangent bundle $T^*G = \underline{\mathfrak{g}^*} \rtimes G$

In this section, we focus on the semiclassical data for quantisation of the cotangent bundle T^*G of a Poisson–Lie group G. We aim to construct preconnections on T^*G.

As a Lie group, the cotangent bundle T^*G can be identified with the semidirect product of Lie groups $\underline{\mathfrak{g}^*} \rtimes G$ with product given by

$$(\phi, g)(\psi, h) = (\phi + \mathrm{Ad}^*(g)(\psi), gh)$$

for any $g, h \in G,\ \phi, \psi \in \mathfrak{g}^*$. As before, $\underline{\mathfrak{g}^*}$ is $\mathfrak{g}^*$ but viewed as an abelian Lie group under addition. In particular,

$$(\phi, g)^{-1} = (-\mathrm{Ad}^*(g^{-1})(\phi), g^{-1}) \quad \text{and} \quad (0, g)(\phi, e)(0, g)^{-1} = (\mathrm{Ad}^*(g)\phi, e).$$

Here Ad^* is the coadjoint action of G on the dual of its Lie algebra. The Lie algebra of T^*G is then identified with the semidirect sum of Lie algebras $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}$, where the Lie bracket of $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}$ is given by

$$[(\phi, x), (\psi, y)] = (\mathrm{ad}^*_x \psi - \mathrm{ad}^*_y \phi, [x, y]_{\mathfrak{g}}) \tag{6-1}$$

for any $\phi, \psi \in \underline{\mathfrak{g}^*},\ x, y \in \mathfrak{g}$. Here $\underline{\mathfrak{g}^*}$ is $\mathfrak{g}^*$ viewed as abelian Lie algebra and ad^* denotes the usual left coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ (or $\underline{\mathfrak{g}^*}$).

Our strategy to build Poisson–Lie structures on the cotangent bundle here is to construct Lie bialgebra structures on $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}$ via bosonisation of Lie bialgebras. Then we can exponentiate the obtained Lie cobracket of $\underline{\mathfrak{g}^*} \rtimes \mathfrak{g}$ to a Poisson–Lie structure on $\underline{\mathfrak{g}^*} \rtimes G$. We can always do this, as we work in the nice case where the Lie group is connected and simply connected.

6A. ***Lie bialgebra structures on $\underline{\mathfrak{g}^*} \rtimes\!\!\!\triangleleft\, \mathfrak{g}$ via bosonisation.*** Let ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ denote the monoidal category of left Lie $\mathfrak{g}$-crossed modules. A *braided-Lie bialgebra* $\mathfrak{b} \in {}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ is $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \delta_{\mathfrak{b}}, \triangleright, \beta)$ given by a $\mathfrak{g}$-crossed module $(\mathfrak{b}, \triangleright, \beta)$ that is both a Lie algebra $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}})$ and a Lie coalgebra $(\mathfrak{b}, \delta_{\mathfrak{b}})$ living in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$, with the infinitesimal braiding $\Psi : \mathfrak{b}\otimes\mathfrak{b} \to \mathfrak{b}\otimes\mathfrak{b}$ obeying $\Psi(x, y) = \mathrm{ad}_x\, \delta_{\mathfrak{b}} y - \mathrm{ad}_y\, \delta_{\mathfrak{b}} x - \delta_{\mathfrak{b}}([x, y]_{\mathfrak{b}})$ for any $x, y \in \mathfrak{b}$. If $\mathfrak{b}$ is a braided-Lie bialgebra in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$, then the bisum $\mathfrak{b} \rtimes\!\!\!\triangleleft\, \mathfrak{g}$ with semidirect Lie bracket/cobracket is a Lie bialgebra [Majid 2000].

For our purposes, a straightforward solution is to ask for

$$\underline{\mathfrak{g}^*} = (\mathfrak{g}^*,\ [\,,\,] = 0,\ \delta_{\mathfrak{g}^*},\ \mathrm{ad}^*,\ \alpha)$$

to be a braided-Lie algebra in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ for some left $\mathfrak{g}$-coaction α on $\mathfrak{g}^*$.

Lemma 6.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra and suppose there is a linear map $\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ such that* (2-6) *holds. Then*

$$\underline{\mathfrak{g}^*} = (\mathfrak{g}^*,\ [\,,\,] = 0,\ \delta_{\mathfrak{g}^*},\ \mathrm{ad}^*,\ \alpha)$$

is a braided-Lie bialgebra in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ if and only if Ξ is a pre-Lie structure on $\mathfrak{g}^*$ such that Ξ is covariant under the Lie cobracket $\delta_{\mathfrak{g}^*}$, in the sense that

$$\Xi(\phi, \psi)_{(1)} \otimes \Xi(\phi, \psi)_{(2)} = \Xi(\phi, \psi_{(1)}) \otimes \psi_{(2)} + \psi_{(1)} \otimes \Xi(\phi, \psi_{(2)}) \tag{6-2}$$

and

$$\Xi(\phi_{(1)}, \psi) \otimes \phi_{(2)} = \psi_{(1)} \otimes \Xi(\psi_{(2)}, \phi) \tag{6-3}$$

for any $\phi, \psi \in \mathfrak{g}^*$. Here the left $\mathfrak{g}$-coaction α and the left pre-Lie product Ξ of $\mathfrak{g}^*$ are mutually determined via

$$\langle \alpha(\phi), \psi \otimes x\rangle = -\Xi(\psi, \phi)(x) \tag{6-4}$$

for any $\phi, \psi \in \mathfrak{g}^*,\ x \in \mathfrak{g}$. In this case, the bisum $\underline{\mathfrak{g}^*} \rtimes\!\!\!\triangleleft\, \mathfrak{g}$ is a Lie bialgebra with Lie bracket given by (6-1) *and Lie cobracket given by*

$$\delta(\phi, X) = \delta_{\mathfrak{g}} X + \delta_{\mathfrak{g}^*}\phi + (\mathrm{id} - \tau)\alpha(\phi) \tag{6-5}$$

for any $\phi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$.

Proof. Since the Lie bracket is zero, by definition, the question amounts to finding a left $\mathfrak{g}$-coaction α on $\mathfrak{g}^*$ such that (1) (ad^*, α) makes $\underline{\mathfrak{g}^*}$ into a left $\mathfrak{g}$-crossed module; (2) $\delta_{\mathfrak{g}^*}$ is a left $\mathfrak{g}$-comodule map under α; and (3) the infinitesimal braiding Ψ on $\underline{\mathfrak{g}^*}$ is trivial, i.e.,

$$\Psi(\phi, \psi) = \mathrm{ad}^*_{\psi^{(1)}}\, \phi \otimes \psi^{(2)} - \mathrm{ad}^*_{\phi^{(1)}}\, \psi \otimes \phi^{(2)} - \psi^{(2)} \otimes \mathrm{ad}^*_{\psi^{(1)}}\, \phi + \phi^{(2)} \otimes \mathrm{ad}^*_{\phi^{(1)}}\, \psi \tag{6-6}$$

is zero for any $\phi, \psi \in \underline{\mathfrak{g}^*}$, where we write $\alpha(\phi) = \phi^{(1)} \otimes \phi^{(2)}$.

Clearly, α is a left $\mathfrak{g}$-coaction on $\mathfrak{g}^*$ if and only if Ξ defines a left $\mathfrak{g}^*$ action on itself, since α and Ξ are adjoint to each other by (6-4), thus if and only if Ξ is left pre-Lie structure, due to (2-6). Next, the condition that the Lie cobracket $\delta_{\mathfrak{g}^*}$ is a left $\mathfrak{g}$-comodule map under α means $\delta_{\mathfrak{g}^*}$ is a right $\mathfrak{g}^*$-module map under $-\Xi$. This is exactly the assumption (6-2) on Ξ. In this case, the cross condition (3-2) or (4-6) (using compatibility) for making $\mathfrak{g}^*$ a left $\mathfrak{g}$-crossed module under (ad^*, α) becomes (6-3).

It suffices to show that the infinitesimal braiding Ψ on $\mathfrak{g}^*$ is trivial on $\underline{\mathfrak{g}^*}$. By construction,

$$\langle \alpha(\phi), \varphi \otimes x\rangle = -\Xi(\varphi, \phi)(x),$$

so

$$\mathrm{ad}^*_{\psi^{(1)}} \phi \otimes \psi^{(2)} = \phi_{(2)} \otimes \Xi(\phi_{(1)}, \psi),$$

where

$$\alpha(\phi) = \phi^{(1)} \otimes \phi^{(2)} \quad \text{and} \quad \delta_{\mathfrak{g}^*}\phi = \phi_{(1)} \otimes \phi_{(2)}.$$

Thus, using (6-3),

$$\begin{aligned}
\Psi(\phi, \psi) &= \mathrm{ad}^*_{\psi^{(1)}} \phi \otimes \psi^{(2)} - \mathrm{ad}^*_{\phi^{(1)}} \psi \otimes \phi^{(2)} - \psi^{(2)} \otimes \mathrm{ad}^*_{\psi^{(1)}} \phi + \phi^{(2)} \otimes \mathrm{ad}^*_{\phi^{(1)}} \psi \\
&= \Xi(\psi_{(1)}, \phi) \otimes \psi_{(2)} - \psi_{(2)} \otimes \Xi(\psi_{(1)}, \phi) \\
&\qquad\qquad - \Xi(\phi_{(1)}, \psi) \otimes \phi_{(2)} + \phi_{(2)} \otimes \Xi(\phi_{(1)}, \psi) \\
&= \Xi(\psi_{(1)}, \phi) \otimes \psi_{(2)} + \psi_{(1)} \otimes \Xi(\psi_{(2)}, \phi) \\
&\qquad\qquad - \Xi(\phi_{(1)}, \psi) \otimes \phi_{(2)} - \phi_{(1)} \otimes \Xi(\phi_{(2)}, \psi) \\
&= 0. \qquad \square
\end{aligned}$$

Example 6.2. Let $\mathfrak{m}$ be a pre-Lie algebra with product $\circ : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{m}^*$ with zero Lie bracket as in Example 4.3. This meets the conditions in Lemma 6.1 and we have a Lie bialgebra $\underline{\mathfrak{g}^*} \blacktriangleright\!\!\blacktriangleleft \mathfrak{g} = \underline{\mathfrak{m}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{m}^*$ with zero Lie bracket and with Lie cobracket

$$\delta\phi = \delta_{\mathfrak{m}^*}\phi \quad \text{and} \quad \delta x = (\mathrm{id} - \tau)\alpha(x) \quad \text{for all } \phi \in \mathfrak{m}^*,\ x \in \mathfrak{m},$$

where α is given by the pre-Lie algebra structure $\circ$ on $\mathfrak{m}$, i.e., $\langle x \otimes \phi, \alpha(y)\rangle = -\langle \phi, x \circ y\rangle$. The Lie bialgebra here is the dual of the semidirect sum Lie algebra $\tilde{\mathfrak{m}} = \mathfrak{m}^* \rtimes \mathfrak{m}$ (viewed as a Lie bialgebra with zero Lie cobracket), where $\mathfrak{m}$ acts on $\mathfrak{m}^*$ by the adjoint to the action of $\mathfrak{m}$ on $\mathfrak{m}$ given by $\circ$, i.e., $\langle x \triangleright \phi, y\rangle = -\phi(x \circ y)$,

$$[x, y] = [x, y]_{\mathfrak{m}}, \quad [x, \phi] = x \triangleright \phi \quad \text{and} \quad [\phi, \psi] = 0 \quad \text{for all } x, y \in \mathfrak{m},\ \phi, \psi \in \mathfrak{m}^*.$$

The Poisson bracket on $\tilde{\mathfrak{m}}^* = \underline{\mathfrak{m}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{m}^*$ is then the Kirillov–Kostant one for $\tilde{\mathfrak{m}}$, i.e., given by this Lie bracket.

Example 6.3. Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra with r-matrix

$$r = r^{(1)} \otimes r^{(2)} \in \mathfrak{g} \otimes \mathfrak{g}$$

such that $r_+ \triangleright X = 0$ for all $X \in \mathfrak{g}$. As in Example 4.7, $\mathfrak{g}^*$ is a pre-Lie algebra with product $\Xi(\phi, \psi) = -\langle \phi, r^{(2)} \rangle \operatorname{ad}^*_{r^{(1)}} \psi$. Direct computation shows Ξ satisfies (6-2)–(6-3) without any further requirement. So $\underline{\mathfrak{g}^*} = (\mathfrak{g}^*, [\,,\,] = 0, \delta_{\mathfrak{g}^*}, \operatorname{ad}^*, \alpha)$ is a braided-Lie bialgebra in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ with $\alpha(\phi) = r^{(2)} \otimes \operatorname{ad}^*_{r^{(1)}} \phi$. Hence, from Lemma 6.1, $\underline{\mathfrak{g}^*} \bowtie \mathfrak{g}$ is a Lie bialgebra with Lie bracket given by (6-1) and Lie cobracket given by (6-5), i.e.,

$$\delta(\phi, X) = \delta_{\mathfrak{g}} X + \delta_{\mathfrak{g}^*}\phi + (\mathrm{id} - \tau)(r^{(2)} \otimes \operatorname{ad}^*_{r^{(1)}} \phi). \tag{6-7}$$

Note that if $\mathfrak{g}$ is a quasitriangular Lie bialgebra, Majid [2000, Corollary 3.2, Lemma 3.4] shows that $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ is a braided-Lie bialgebra with Lie bracket given by

$$[\phi, \psi] = 2\langle \phi, r_+^{(1)} \rangle \operatorname{ad}^*_{r_+^{(2)}} \psi = 0$$

in our case, so in this example $\underline{\mathfrak{g}^*}$ in Lemma 6.1 agrees with a canonical construction. On the other hand, this class of examples is more useful in the case where $\mathfrak{g}$ is triangular.

6B. *Poisson–Lie structures on $\underline{\mathfrak{g}^*} \rtimes G$ induced from $\underline{\mathfrak{g}^*} \bowtie \mathfrak{g}$.* Next we exponentiate our Lie bialgebra structure $\underline{\mathfrak{g}^*} \bowtie \mathfrak{g}$ constructed by Lemma 6.1 to a Poisson–Lie structure on the cotangent bundle. As usual this is done by exponentiating δ to a group 1-cocycle D.

Proposition 6.4. *Let G be a connected and simply connected Poisson–Lie group. If its Lie algebra $\mathfrak{g}$ with a given coaction α meets the conditions of Lemma 6.1 then $\underline{\mathfrak{g}^*} \rtimes G$ is a Poisson–Lie group with*

$$D(\phi, g) = \operatorname{Ad}_\phi D(g) + \delta_{\mathfrak{g}^*}\phi + (\mathrm{id} - \tau)(\phi^{(1)} \otimes \phi^{(2)} - \tfrac{1}{2} \operatorname{ad}^*_{\phi^{(1)}} \phi \otimes \phi^{(2)}),$$

where $\alpha(\phi) = \phi^{(1)} \otimes \phi^{(2)}$.

Proof. Because of the cocycle condition, it suffices to find $D(\phi) := D(\phi, e)$ and $D(g) := D(e, g)$; then

$$D(\phi, g) = D(\phi) + \operatorname{Ad}_\phi D(g) \quad \text{for all } (\phi, g) \in \mathfrak{g}^* \rtimes G,$$

where

$$\operatorname{Ad}_\phi(X) = X - \operatorname{ad}^*_X \phi \quad \text{for all } X \in \mathfrak{g} \subset \mathfrak{g}^* \bowtie \mathfrak{g},\ \phi \in \mathfrak{g}^*.$$

We require

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t\phi) = \operatorname{Ad}_{t\phi}(\delta\phi),$$

which we solve writing

$$D(\phi) = \delta_{\mathfrak{g}^*}\phi + Z(\phi),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t\phi) = \mathrm{Ad}_{t\phi}((\mathrm{id}-\tau)\circ\alpha(\phi)) = (\mathrm{id}-\tau)\circ\alpha(\phi) - t(\mathrm{id}-\tau)(\mathrm{ad}^*_{\phi^{(1)}}\phi\otimes\phi^{(2)}),$$
$$Z(0)=0.$$

Integrating this to

$$Z(t\phi) = t(\mathrm{id}-\tau)\circ\alpha(\phi) - \tfrac{1}{2}t^2(\mathrm{id}-\tau)(\mathrm{ad}^*_{\phi^{(1)}}\phi\otimes\phi^{(2)}),$$

we obtain

$$D(\phi) = \delta_{\mathfrak{g}^*}\phi + (\mathrm{id}-\tau)\big(\phi^{(1)}\otimes\phi^{(2)} - \tfrac{1}{2}\,\mathrm{ad}^*_{\phi^{(1)}}\phi\otimes\phi^{(2)}\big),$$

where $\alpha(\phi)=\phi^{(1)}\otimes\phi^{(2)}$. The general case $\mathrm{d}D(\phi+t\psi)/\mathrm{d}t|_{t=0}=\mathrm{Ad}_\phi(\delta\psi)$ amounts to the vanishing of the expression (6-6), which we saw holds under our assumptions in the proof of Lemma 6.1. □

Example 6.5. In the setting of Example 6.3 with $(\mathfrak{g}, r)$ quasitriangular such that $r_+\triangleright X = 0$ for all $X\in\mathfrak{g}$, we know that $\mathfrak{g}^*\rtimes\!\!\!\lhd G$ is a Poisson–Lie group with

$$D(\phi, g) = \delta_{\mathfrak{g}^*}\phi + \mathrm{Ad}_{(\phi,g)}(r) - r + 2r_+\triangleright\phi - r_+\triangleright(\phi\otimes\phi),$$

where $\triangleright$ denotes the coadjoint action ad^*. As $\alpha(\phi)=r_{21}\triangleright\phi$, direct computation shows that $D(\phi)=\delta_{\mathfrak{g}^*}(\phi)+(\mathrm{id}-\tau)r_{21}\triangleright\phi + r_-\triangleright(\phi\otimes\phi)$. Since the differential equation for $D(g)$ is the usual one on G for $\mathfrak{g}$ quasitriangular, $D(g)=\mathrm{Ad}_g(r)-r$ and we obtain the stated result. Note that

$$\begin{aligned}\mathrm{Ad}_\phi(r) &= (r^{(1)} - r^{(1)}\triangleright\phi)\otimes(r^{(2)}-r^{(2)}\triangleright\phi)\\ &= r + r\triangleright(\phi\otimes\phi) - r^{(1)}\triangleright\phi\otimes r^{(2)} - r^{(1)}\otimes r^{(2)}\triangleright\phi.\end{aligned}$$

The differential equation $\mathrm{d}D(\phi+t\psi)/\mathrm{d}t|_{t=0}=\mathrm{Ad}_\phi(\delta\psi)$ amounts to

$$r_+\triangleright(\mathrm{id}-\tau)(\phi\otimes\psi)=0,$$

which is guaranteed by $r_+\triangleright X=0$ for all $X\in\mathfrak{g}$.

Note that we can view $r\in(\underline{\mathfrak{g}}^*\rtimes\!\!\!\lhd\mathfrak{g})^{\otimes 2}$, where it will obey the the classical Yang–Baxter equation and, in our case, $\mathrm{ad}_\phi(r_+)=0$ as $r_+\triangleright\phi=0$ on $\mathfrak{g}^*$ under our assumptions. In this case $\underline{\mathfrak{g}}^*\rtimes\!\!\!\lhd\mathfrak{g}$ is quasitriangular with the same r, with Lie cobracket

$$\delta_r(\phi) = \mathrm{ad}_\phi(r) = -r^{(1)}\triangleright\phi\otimes r^{(2)} - r^{(1)}\otimes r^{(2)}\triangleright\phi = (\mathrm{id}-\tau)r_{21}\triangleright\phi$$

at the Lie algebra level (differentiating the above $\mathrm{Ad}_{t\phi}$) and with δX as before. In our case the cobracket has an additional $\delta_{\mathfrak{g}^*}\phi$ term reflected also in D.

6C. ***Preconnections on the cotangent bundle $\underline{\mathfrak{g}^*}{>\!\!\!\triangleleft} G$.*** Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra and suppose that its dual $\mathfrak{g}^*$ admits a pre-Lie structure

$$\Xi : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$$

such that (6-2) and (6-3) hold as in the setting of Lemma 6.1. Then the dual of the Lie bialgebra $\underline{\mathfrak{g}^*} {>\!\!\!\triangleleft} \mathfrak{g}$ is $\bar{\mathfrak{g}} {>\!\!\!\triangleleft} \mathfrak{g}^*$, with Lie bracket the semidirect sum $\mathfrak{g} {>\!\!\!\triangleleft} \mathfrak{g}^*$ and Lie cobracket the semidirect cobracket $\bar{\mathfrak{g}} {>\!\!\!\blacktriangleleft} \mathfrak{g}^*$, that is,

$$[x, y] = [x, y]_{\mathfrak{g}}, \quad [\phi, x] = \phi \triangleright x, \quad [\phi, \psi] = [\phi, \psi]_{\mathfrak{g}^*},$$
$$\delta x = (\mathrm{id} - \tau)\beta(x), \quad \delta\phi = \delta_{\mathfrak{g}^*}\phi,$$

for any $x, y \in \mathfrak{g}$, $\phi, \psi \in \mathfrak{g}^*$. Here the left action and coaction of $\mathfrak{g}^*$ on $\mathfrak{g}$ are given by

$$\langle \phi \triangleright x, \psi \rangle = -\Xi(\phi, \psi)(x) \quad \text{and} \quad \langle \beta(x), y \otimes \phi \rangle = \langle \phi, [x, y] \rangle, \tag{6-8}$$

respectively.

Here again, we use Lemma 5.4 to construct pre-Lie algebra structures on the semidirect sum $\bar{\mathfrak{g}} {>\!\!\!\triangleleft} \mathfrak{g}^*$.

Theorem 6.6. *Let G be a connected and simply connected Poisson–Lie group with Lie bialgebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ admit two pre-Lie structures Ξ and $\circ$, with Ξ obeying* (6-2) *and* (6-3) *as in the setting of Lemma 6.1. Let $\mathfrak{g}$ also admit a pre-Lie structure $*$ such that*

$$\phi \triangleright (x * y) = (\phi \triangleright x) * y + x * (\phi \triangleright y), \tag{6-9}$$

for all $x, y \in \mathfrak{g}$, $\phi \in \mathfrak{g}^*$, where $\triangleright$ is defined by (6-8)*. Then the Lie algebra $\bar{\mathfrak{g}} {>\!\!\!\triangleleft} \mathfrak{g}^*$ admits a pre-Lie structure $\tilde{\circ}$:*

$$(x, \phi)\, \tilde{\circ}\, (y, \psi) = (x * y + \phi \triangleright y, \phi \circ \psi), \tag{6-10}$$

and the cotangent bundle $\underline{\mathfrak{g}^*} {>\!\!\!\triangleleft} G$ admits a Poisson-compatible left-covariant flat preconnection.

Proof. Since $(\mathfrak{g}, \Xi)$ is in the setting of Lemma 6.1, the left $\mathfrak{g}^*$-action in the semidirect sum $\bar{\mathfrak{g}} {>\!\!\!\triangleleft} \mathfrak{g}^*$ is the one defined in (6-8). The rest is immediate from Lemma 5.4 and Corollary 4.2. $\square$

To construct a bicovariant preconnection, the pre-Lie structure constructed in Theorem 6.6 must satisfy the bicovariance condition (4-6).

Proposition 6.7. *In the setting of Theorem 6.6, the pre-Lie structure $\tilde{\circ}$ of $\bar{\mathfrak{g}} {>\!\!\!\triangleleft} \mathfrak{g}^*$ defined by* (6-10) *obeys the bicovariance condition if and only if $\circ$ obeys* (4-6), *$*$ is* *associative and*

$$[x, y] * z = [y, z] * x, \tag{6-11}$$

$$((\mathrm{ad}^*_x \psi) \circ \phi)(y) + \Xi(\mathrm{ad}^*_y \phi, \psi)(x) = 0, \tag{6-12}$$

$$\Xi(\phi, \psi)([x, y]_{\mathfrak{g}}) = \Xi(\phi, \mathrm{ad}^*_y \psi)(x) - (\phi \circ \mathrm{ad}^*_x \psi)(y), \tag{6-13}$$

for any $x, y, z \in \mathfrak{g}$ and $\phi, \psi \in \mathfrak{g}^*$. The associated preconnection is then bicovariant.

Proof. Since (4-6) is bilinear, it suffices to show that (4-6) holds on any pair of elements (x, y), (x, ϕ), (ϕ, x) and (ϕ, ψ) if and only if all the conditions and displayed identities hold. Here we write $\beta(x) = x^1 \otimes x_2 \in \mathfrak{g}^* \otimes \mathfrak{g}$, so we know

$$\langle x^1, y\rangle x_2 = [x, y]_{\mathfrak{g}}, \quad x^1 \langle x_2, \phi\rangle = -\,\mathrm{ad}^*_x \phi.$$

Firstly, for any $\phi, \psi \in \mathfrak{g}^*$, the condition (4-6) for $\tilde{\circ}$ reduces to (4-6) on the pre-Lie structure $\circ$ for $\mathfrak{g}^*$.

Secondly, for any $x, y \in \bar{\mathfrak{g}}$, the condition (4-6) requires

$$(x * y)^1 \otimes (x * y)_2 - (x * y)_2 \otimes (x * y)^1 - x^1 \triangleright y \otimes x_2 + x_2 * y \otimes x^1 + x * y_2 \otimes y^1 = y^1 \otimes [x, y_2]_{\mathfrak{g}} + y_2 \otimes y^1 \triangleright x.$$

The terms lying in $\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}$ on both sides should be equal, i.e., $-x^1 \triangleright y \otimes x_2 = y_2 \otimes y^1 \triangleright x$, which is equivalent to $-\Xi(\mathrm{ad}^*_x \psi, \phi)(y) = \Xi(\mathrm{ad}^*_y \phi, \psi)(x)$. This is true from our assumption (6-3) on Ξ. The terms in $\bar{\mathfrak{g}} \otimes \mathfrak{g}^*$ give $[x * y, z] = [x, z] * y + x * [y, z]$, i.e., $*$ is associative. The terms in $\mathfrak{g}^* \otimes \bar{\mathfrak{g}}$ give $(x * y)^1 \otimes (x * y)_2 = y^1 \otimes [x, y_2]_{\mathfrak{g}}$ and, applying the first factor to $z \in \mathfrak{g}$, we obtain $[x * y, z] = [x, [y, z]]$, which is equivalent to $[x, z] * y = [z, y] * x$.

Now, for any $x \in \mathfrak{g}$, $\phi \in \mathfrak{g}^*$, the condition (4-6) reduces to

$$0 = x^1 \circ \phi \otimes x_2 - \phi_{(1)} \otimes \phi_{(2)} \triangleright x.$$

Applying $y \otimes \psi$, this becomes $-\Xi(\mathrm{ad}^*_y \phi, \psi)(x) = ((\mathrm{ad}^*_x \psi) \circ \phi)(y)$.

Finally, for any $\phi \in \mathfrak{g}^*$, $x \in \bar{\mathfrak{g}}$, the condition (4-6) requires

$$(\phi \triangleright x)^1 \otimes (\phi \triangleright x)_2 - (\phi \triangleright x)_2 \otimes (\phi \triangleright x)^1 - \phi \circ x^1 \otimes x_2 + \phi \triangleright x_2 \otimes x^1 - x^1 \otimes \phi \triangleright x_2 + x_2 \otimes \phi \circ x^1 = \phi_{(1)} \triangleright x \otimes \phi_{(2)} + x_2 \otimes x^1 \circ \phi.$$

The terms lying in $\mathfrak{g}^* \otimes \bar{\mathfrak{g}}$ give

$$(\phi \triangleright x)^1 \otimes (\phi \triangleright x)_2 - \phi \circ x^1 \otimes x_2 - x^1 \otimes \phi \triangleright x_2 = 0.$$

Applying $y \otimes \psi$, this is equivalent to

$$-\Xi(\phi, \mathrm{ad}^*_y \psi)(x) + (\phi \circ \mathrm{ad}^*_x \psi)(y) + \Xi(\phi, \psi)([x, y]_{\mathfrak{g}}) = 0.$$

Applying $\psi \otimes y$ to the terms lying in $\bar{\mathfrak{g}} \otimes \mathfrak{g}^*$, after cancelling the identity just obtained, we have $((\mathrm{ad}^*_x \psi) \circ \phi)(y) + \Xi(\mathrm{ad}^*_y \phi, \psi)(x) = 0$. □

For simplicity, one can choose $\Xi = \circ$ in Theorem 6.6 and Proposition 6.7:

Corollary 6.8. *Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra. Assume that $\mathfrak{g}^*$ admits a pre-Lie structure Ξ such that* (6-2) *and* (6-3) *hold. Also assume that $\mathfrak{g}$ admits a pre-Lie structure $*$ such that* (6-9) *holds, where the action is defined by* (6-8) *from Ξ. Then*

$$(x, \phi) \,\tilde{\circ}\, (y, \psi) = (x * y + \phi \triangleright y,\ \Xi(\phi, \psi))$$

defines a pre-Lie structure for the Lie algebra $\bar{\mathfrak{g}} \bowtie \mathfrak{g}^*$, and thus provides a Poisson-compatible left-covariant flat preconnection on the cotangent bundle $\mathfrak{g}^* \bowtie G$. Moreover, if $*$ is associative and obeys (6-11), *then the pre-Lie structure $\tilde{\circ}$ obeys* (4-6) *and the corresponding preconnection is bicovariant.*

Proof. Clearly, there is no further condition on $\circ$ in the case $\circ = \Xi$ in Theorem 6.6. In the bicovariant case, the further conditions on $\circ$ in Proposition 6.7 are (4-6), (6-12) and (6-13). These all can be proven from the assumptions (6-2) and (6-3) we already made on Ξ. In particular, (6-3) shows that (6-12) is true, and (6-2) is simply a variation of (6-13) when $\circ = \Xi$. The only conditions left in Proposition 6.7 are that $*$ is associative and (6-11). □

Example 6.9. In the easier case of Example 6.2, we already know the answer: a Poisson-compatible bicovariant flat preconnection on $\tilde{\mathfrak{m}}^* = \underline{\mathfrak{m}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{m}^*$ corresponds to a pre-Lie algebra structure on $\tilde{\mathfrak{m}} = \mathfrak{m}^* \bowtie \mathfrak{m}$.

Assume $\tilde{\circ}$ is such a pre-Lie structure, and also assume $\tilde{\circ}$ is such that $\tilde{\circ}(\mathfrak{m} \otimes \mathfrak{m}) \subseteq \mathfrak{m}$, $\tilde{\circ}(\mathfrak{m}^* \otimes \mathfrak{m}^*) \subseteq \mathfrak{m}^*$, $\tilde{\circ}(\mathfrak{m} \otimes \mathfrak{m}^*) \subseteq \mathfrak{m}^*$ and that the restriction of $\tilde{\circ}$ on the other subspace is zero. Directly from the definition of pre-Lie structure, one can show $\circ := \tilde{\circ}|_{\mathfrak{m}\otimes\mathfrak{m}}$ also provides a pre-Lie structure for $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$, while $* := \tilde{\circ}|_{\mathfrak{m}^*\otimes\mathfrak{m}^*}$ provides a pre-Lie structure for $(\mathfrak{m}^*, [\ ,\ ]_{\mathfrak{m}^*} = 0)$, thus $*$ is associative and (6-11) holds automatically. Meanwhile, $\triangleright := \tilde{\circ}|_{\mathfrak{m}\otimes\mathfrak{m}^*}$ can be shown to be a left $\mathfrak{m}$-action on $\mathfrak{m}^*$, which is exactly the adjoint to the left $\mathfrak{m}$-action on $\mathfrak{m}$ given by the pre-Lie structure $\circ$ on $\mathfrak{m}$. Applying $\tilde{\circ}$ to any $x \in \mathfrak{m}$, $\phi, \psi \in \mathfrak{m}^*$, one has $x \triangleright (\phi * \psi) = (x \triangleright \phi) * \psi + \phi * (x \triangleright \psi)$, i.e., (6-9). The analysis above shows that $\circ, *, \triangleright$ corresponds to the data in Corollary 6.8. So this example agrees with our construction of Poisson-compatible bicovariant flat preconnections on $\underline{\mathfrak{g}}^* \bowtie \mathfrak{g} = \underline{\mathfrak{m}} \blacktriangleright\!\!\blacktriangleleft \mathfrak{m}^*$ in the case of $\mathfrak{g} = (\mathfrak{m}^*, [\ ,\ ]_{\mathfrak{m}^*} = 0)$ in Corollary 6.8.

We already know how to quantise the algebra $C^\infty(\tilde{\mathfrak{m}}^*)$ or $S(\tilde{\mathfrak{m}})$ and its differential graded algebra as in Example 4.3. More precisely, the quantisation of $S(\tilde{\mathfrak{m}})$ is the noncommutative algebra $U_\lambda(\tilde{\mathfrak{m}})$ with relations $xy - yx = \lambda[x, y]$ for all $x, y \in \tilde{\mathfrak{m}}$, so

$$U_\lambda(\tilde{\mathfrak{m}}) = U_\lambda(\mathfrak{m}^* \bowtie \mathfrak{m}) = S(\mathfrak{m}^*) \bowtie U_\lambda(\mathfrak{m})$$

with cross relations $x\phi - \phi x = \lambda x \triangleright \phi$ for all $x \in \mathfrak{m}$, $\phi \in \mathfrak{m}^*$. Meanwhile, as in Example 4.3 and Proposition 4.4, the preconnection on $\tilde{\mathfrak{m}}^* = \underline{\mathfrak{m}} \blacktriangleright\!\!\!\blacktriangleleft \mathfrak{m}^*$ is given by

$$\gamma((\phi, x), \mathrm{d}(\psi, y)) = \mathrm{d}((\phi, x)\,\tilde{\circ}\,(\psi, y)) = \mathrm{d}(\phi * \psi + x \triangleright \phi, x \circ y).$$

Thus, the quantised differential calculus is

$$\Omega(U_\lambda(\tilde{\mathfrak{m}})) = U_\lambda(\tilde{\mathfrak{m}}) \ltimes \Lambda(\tilde{\mathfrak{m}}) = (S(\mathfrak{m}^*) \rtimes U_\lambda(\mathfrak{m})) \ltimes \Lambda(\mathfrak{m}^* \oplus \mathfrak{m})$$

with bimodule relations

$$[(\phi, x), \mathrm{d}(\psi, y)] = \lambda\, \mathrm{d}(\phi * \psi + x \triangleright \phi, x \circ y)$$

for all $(\phi, x), (\psi, y) \in \tilde{\mathfrak{m}} \subset U_\lambda(\tilde{\mathfrak{m}})$, where $\Lambda(\mathfrak{m}^* \oplus \mathfrak{m})$ denotes the usual exterior algebra on the vector space $\mathfrak{m}^* \oplus \mathfrak{m}$ and $\mathrm{d}(\psi, y) = 1 \otimes (\psi + y) \in 1 \otimes \Lambda$.

For a concrete example, we take $\mathfrak{m}$ the 2-dimensional complex nonabelian Lie algebra defined by $[x, y] = x$ and for $\mathfrak{m}^*$ the 2-dimensional abelian Lie algebra with its five families of pre-Lie structures [Burde 1998]. Among many choices of pairs of pre-Lie structures for $\mathfrak{m}$ and $\mathfrak{m}^*$, there are two pairs which meet our condition (6-9) and provide a pre-Lie structure for $\tilde{\mathfrak{m}} = \mathfrak{m}^* \rtimes \mathfrak{m}$, namely

(1) $\quad y \circ x = -x, \quad y^2 = -\frac{1}{2}y, \quad Y * Y = X,$
$\quad x \triangleright X = 0, \quad x \triangleright Y = 0, \quad y \triangleright X = X, \quad y \triangleright Y = \frac{1}{2}Y;$

(2) $\quad y \circ x = -x, \quad X * Y = X, \quad Y * X = X, \quad Y * Y = Y,$
$\quad x \triangleright X = 0, \quad y \triangleright Y = 0, \quad y \triangleright X = X, \quad y \triangleright Y = 0,$

where $\{X, Y\}$ is chosen to be the basis of $\mathfrak{m}^*$ dual to $\{x, y\}$. By Theorem 6.6 and the general analysis earlier, we know that $\Omega(U_\lambda(\tilde{\mathfrak{m}})) = U_\lambda(\tilde{\mathfrak{m}}) \ltimes \Lambda(\mathfrak{m}^* \oplus \mathfrak{m})$ is a bicovariant differential graded algebra. In particular,

$$\Omega^1(U_\lambda(\tilde{\mathfrak{m}})) = U_\lambda(\tilde{\mathfrak{m}})\, \mathrm{d}x \oplus U_\lambda(\tilde{\mathfrak{m}})\, \mathrm{d}y \oplus U_\lambda(\tilde{\mathfrak{m}})\, \mathrm{d}X \oplus U_\lambda(\tilde{\mathfrak{m}})\, \mathrm{d}Y.$$

The bimodule relations for case (1) are

$$[y, \mathrm{d}x] = -\lambda\, \mathrm{d}x, \quad [y, \mathrm{d}y] = -\tfrac{1}{2}\lambda\, \mathrm{d}y, \quad [Y, \mathrm{d}Y] = \lambda\, \mathrm{d}X,$$
$$[y, \mathrm{d}X] = \lambda\, \mathrm{d}X, \quad [y, \mathrm{d}Y] = \tfrac{1}{2}\lambda\, \mathrm{d}Y.$$

For case (2), we have

$$[y, \mathrm{d}x] = -\lambda\, \mathrm{d}x, \quad [X, \mathrm{d}Y] = \lambda\, \mathrm{d}X, \quad [Y, \mathrm{d}X] = \lambda\, \mathrm{d}X, \quad [Y, \mathrm{d}Y] = \lambda\, \mathrm{d}Y,$$
$$[y, \mathrm{d}X] = \lambda\, \mathrm{d}X.$$

Example 6.10. Suppose that $\mathfrak{g}$ is quasitriangular with $r_+ \triangleright x = 0$ for all $x \in \mathfrak{g}$ as in Example 6.3. According to Corollary 6.8, if $\mathfrak{g}$ admits a pre-Lie product $*$ such that

$$[r^{(1)}, x * y] \otimes r^{(2)} = [r^{(1)}, x] * y \otimes r^{(2)} + x * [r^{(1)}, y] \otimes r^{(2)}, \tag{6-14}$$

from (6-9), then $\bar{\mathfrak{g}} \bowtie \mathfrak{g}^*$ in Example 6.3 admits a pre-Lie structure $\tilde{\circ}$

$$x\tilde{\circ}y = x * y, \quad \phi\tilde{\circ}x = \phi \triangleright x = -\langle\phi, r^{(2)}\rangle[r^{(1)}, x], \quad \phi\tilde{\circ}\psi = -\langle\phi, r^{(2)}\rangle \operatorname{ad}^*_{r^{(1)}} \psi,$$

and thus determines a Poisson-compatible left-covariant flat preconnection on the cotangent bundle $\underline{\mathfrak{g}^*} \bowtie G$. Such a preconnection is bicovariant if $*$ is associative and (6-11) holds, and in this case condition (6-9) vanishes. Recall that we cannot take $\mathfrak{g}$ semisimple here since it will not then admit a pre-Lie structure.

For a concrete example, we take $\mathfrak{g}$ to again be the 2-dimensional Lie algebra $[x, t] = x$ as in Example 4.5 but with $\delta x = 0$ and $\delta t = x \otimes t - t \otimes x$ as a triangular Lie bialgebra with $r = t \otimes x - x \otimes t$. If $\{X, T\}$ is the dual basis to $\{x, t\}$ then the pre-Lie algebra structure $\circ$ of $\mathfrak{g}^*$ determined by r is

$$T \circ X = -T, \quad X \circ X = -X,$$

and otherwise zero, which is isomorphic to $\mathfrak{b}_{2,1}$ listed in Example 4.5. On the other hand, computation shows that among all the possible pre-Lie algebra structures for $\mathfrak{g}$ listed in Example 4.5, precisely $\mathfrak{b}_{1,-1}$ and $\mathfrak{b}_{2,1}$ satisfy condition (6-14), giving us two pre-Lie algebra structures on $\bar{\mathfrak{g}} \bowtie \mathfrak{g}^*$ by our construction, namely

(1) $$t * x = -x, \quad t * t = -t, \quad T \circ X = -T, \quad X \circ X = -X,$$
$$X \triangleright x = x, \quad T \triangleright t = x;$$

(2) $$x * t = x, \quad t * t = t, \quad T \circ x = -T, \quad X \circ X = -X,$$
$$X \triangleright x = x, \quad T \triangleright t = x.$$

These determine two Poisson-compatible left-covariant flat preconnections on the cotangent bundle $\underline{\mathfrak{g}^*} \bowtie G$. In case (1) this is also bicovariant as $*$ is associative and satisfies (6-11), which can be checked directly.

References

[Beggs and Majid 2006] E. J. Beggs and S. Majid, "Semiclassical differential structures", *Pacific J. Math.* **224**:1 (2006), 1–44. MR Zbl

[Beggs and Majid 2010] E. J. Beggs and S. Majid, "Quantization by cochain twists and nonassociative differentials", *J. Math. Phys.* **51**:5 (2010), art. ID 053522. MR Zbl

[Beggs and Majid 2014a] E. J. Beggs and S. Majid, "Gravity induced from quantum spacetime", *Classical Quantum Gravity* **31**:3 (2014), art. ID 035020. MR Zbl

[Beggs and Majid 2014b] E. J. Beggs and S. Majid, "Semiquantisation functor and Poisson–Riemannian geometry, I", preprint, 2014. arXiv

[Burde 1994] D. Burde, "Left-symmetric structures on simple modular Lie algebras", *J. Algebra* **169**:1 (1994), 112–138. MR Zbl

[Burde 1998] D. Burde, "Simple left-symmetric algebras with solvable Lie algebra", *Manuscripta Math.* **95**:3 (1998), 397–411. MR Zbl

[Burde 2006] D. Burde, "Left-symmetric algebras, or pre-Lie algebras in geometry and physics", *Cent. Eur. J. Math.* **4**:3 (2006), 323–357. MR Zbl

[Cartier 2009] P. Cartier, "Vinberg algebras and combinatorics", preprint (IHES/M/09/34), Institut des Hautes Études Scientifiques, 2009, available at http://goo.gl/H74Hzz.

[Drinfeld 1987] V. G. Drinfeld, "Quantum groups", pp. 798–820 in *Proceedings of the International Congress of Mathematicians* (Berkeley, CA, 1986), vol. 1, edited by A. M. Gleason, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Hawkins 2004] E. Hawkins, "Noncommutative rigidity", *Comm. Math. Phys.* **246**:2 (2004), 211–235. MR Zbl

[Huebschmann 1990] J. Huebschmann, "Poisson cohomology and quantization", *J. Reine Angew. Math.* **408** (1990), 57–113. MR Zbl

[Majid 1990a] S. Majid, "Matched pairs of Lie groups associated to solutions of the Yang–Baxter equations", *Pacific J. Math.* **141**:2 (1990), 311–332. MR Zbl

[Majid 1990b] S. Majid, "Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicrossproduct construction", *J. Algebra* **130**:1 (1990), 17–64. MR Zbl

[Majid 1995] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995. MR Zbl

[Majid 2000] S. Majid, "Braided-Lie bialgebras", *Pacific J. Math.* **192**:2 (2000), 329–356. MR Zbl

[Majid and Tao 2015a] S. Majid and W.-Q. Tao, "Cosmological constant from quantum spacetime", *Phys. Rev. D* **91**:12 (2015), 124028, 10. MR

[Majid and Tao 2015b] S. Majid and W.-Q. Tao, "Duality for generalised differentials on quantum groups", *J. Algebra* **439** (2015), 67–109. MR Zbl

[Meljanac et al. 2012] S. Meljanac, S. Krešić-Jurić, and R. Štrajn, "Differential algebras on κ-Minkowski space and action of the Lorentz algebra", *Internat. J. Modern Phys. A* **27**:10 (2012), art. ID 1250057. MR Zbl

[Woronowicz 1989] S. L. Woronowicz, "Differential calculus on compact matrix pseudogroups (quantum groups)", *Comm. Math. Phys.* **122**:1 (1989), 125–170. MR Zbl

Received February 5, 2015. Revised March 3, 2016.

SHAHN MAJID
SCHOOL OF MATHEMATICAL SCIENCES
QUEEN MARY UNIVERSITY OF LONDON
MILE END ROAD
LONDON
E1 4NS
UNITED KINGDOM
s.majid@qmul.ac.uk

WEN-QING TAO
SCHOOL OF MATHEMATICS AND STATISTICS
HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
WUHAN, 430074
CHINA
wqtao@hust.edu.cn

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 284 No. 1 September 2016

0030-8730(201609)284:1;1-U